Stock Market Mean Reversion and the Optimal Equity Allocation of a Long-Lived Investor

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First draft: September 1997
This version: April 2001

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Abstract

This paper solves numerically the intertemporal consumption and portfolio choice problem of an infinitely-lived investor who faces a time-varying equity premium. The solutions we obtain are very similar to the approximate analytical solutions of Campbell and Viceira (1999), except at the upper extreme of the state space where both the numerical consumption and portfolio rules flatten out. We also consider a constrained version of the problem in which the investor faces borrowing and short-sales restrictions. These constraints bind when the equity premium moves away from its mean in either direction, and are particularly severe for risk-tolerant investors. The constraints have substantial effects on optimal consumption, but much more modest effects on optimal portfolio choice in the region of the state space where they are not binding.

JEL classification: G12.

Keywords: Hedging demand, intertemporal portfolio choice, mean reversion.

1 Introduction

This paper solves numerically the intertemporal consumption and portfolio choice problem of an infinitely-lived investor who faces a time-varying equity premium. It has been understood for many years that, in principle, the solution to this problem can be very different from the solution to a familiar static optimization problem. The classic papers of Merton (1969, 1971) and Samuelson (1969) showed that time variation in investment opportunities affects the portfolio choice of power-utility investors unless they have log utility. More generally, if investors have Epstein-Zin utility (Epstein and Zin (1989), Weil (1990)) with separate coefficients governing risk aversion and the elasticity of intertemporal substitution in consumption, Giovannini and Weil (1989) have shown that time-variation in investment opportunities affects portfolio choice unless the coefficient of relative risk aversion equals one.

More recently, two lines of empirical research have made it clear that this is an important issue in practice. First, many authors have documented evidence that the excess return on stocks over Treasury bills is predictable (see Campbell 1987, Campbell and Shiller 1988, Fama and French 1988, 1989, Hodrick 1992, or the textbook treatment in Campbell, Lo, and MacKinlay 1997, Chapter 7). Second, the large literature on the equity premium puzzle finds that average excess stock returns are too high to be consistent with a representative-investor model with unit relative risk aversion (see Campbell 1996, Cecchetti, Lam, and Mark 1994, Cochrane and Hansen 1992, Hansen and Jagannathan 1991, Kocherlakota 1996, Mehra and Prescott 1985, or the textbook treatment in Campbell, Lo, and MacKinlay 1997, Chapter 8). These results suggest that it is important for economists to have a good understanding of the microeconomic problem of optimal consumption and portfolio choice when there is time variation in the investment opportunity set and investors have relative risk aversion greater than one. Only when we understand this problem can we hope to build general equilibrium models that are consistent with the observed behavior of asset markets.

The problem, however, is not trivial analytically. Nonlinearities in both the Euler equations and the intertemporal budget constraint make it extremely hard to find exact analytical solutions. Recently a few special cases have been solved. In a continuous-time model with a constant riskless interest rate and a single risky asset whose expected return follows a mean-reverting AR(1) process, for example, the model can be solved if long-lived investors have power utility defined over terminal wealth (Kim and Omberg 1996), or if investors have power utility defined over consumption and the innovation to the expected asset return is perfectly correlated with the innovation to the unexpected return, making the asset market effectively complete

(Wachter 1999), or if the investor has Epstein-Zin utility with intertemporal elasticity of substitution restricted to equal one (Campbell and Viceira 1999, Schroder and Skiadas 1999). Campbell and Viceira (1999, henceforth CV) also present an approximate analytical solution for a discrete-time model with general Epstein-Zin utility. Their approach is to log-linearize the Euler equations and the intertemporal budget constraint around the exact solution that applies if the intertemporal elasticity of substitution is one. They obtain approximate portfolio and consumption rules that are, respectively, linear and quadratic in the single state variable of the problem.

In this paper we consider the same discrete-time model as CV; that is, we assume a constant riskless interest rate and an AR(1) process for the risky asset return, but we do not assume perfect correlation between innovations to the expected and unexpected return, and we allow the investor to have general Epstein-Zin utility defined over consumption. We make two contributions relative to CV.

First, we use a numerical solution method that allows us to evaluate the accuracy of CV's approximate solution. We discretize the state-space and approximate the distribution for the innovations in the random variables using Gaussian quadrature. The solution algorithm assumes a portfolio allocation rule which is a p'th order polynomial in the state-variable and uses a variant of the Newton-Raphson algorithm to optimize over the coefficients of this polynomial. We use the Den Haan-Marcet (DHM) statistic (Den Haan and Marcet 1994) to choose the optimal value for p, and to check the accuracy of the numerical solution. Consistent with CV, we find that the portfolio rule depends strongly on risk aversion but hardly at all on the elasticity of intertemporal substitution, and that the portfolio rule is approximately linear in the state variable while the log consumption-wealth ratio is approximately quadratic. However these approximations break down at the upper extreme of the state space, where both the numerical portfolio and consumption rules flatten out in a way that is not captured by CV. We also use the DHM statistic to test the accuracy of the CV solution.

Second, our numerical solution method allows us to consider a constrained version of the CV problem in which the investor is not allowed to borrow at the riskless interest rate or to short-sell the risky asset. Such constraints are realistic, and they affect the form of the solution since the investor's optimal plans take account of the possibility that the constraints may bind in the future, even if they are not binding today. Relatively little is known about the effects of such constraints on optimal portfolios, because the constraints make it hard to find analytical solutions except in special cases with constant portfolio rules such as Campbell and Viceira (2001). In our model we find substantial effects of the constraints on optimal consumption, but relatively small effects on portfolio choice in the region of the state space in which the constraints are not binding.

This paper extends the results in Campbell and Koo (1997, henceforth CK). CK take the agent's portfolio at each point in time as given and assume a time-series process for the return on this portfolio. They solve numerically for the optimal allocation of the investor's wealth between consumption and savings. Here we solve numerically for the investor's optimal portfolio as well. This allows us to assess the accuracy of log-linear approximations in problems involving both portfolio and consumption choice.

A number of other recent papers have also presented numerical solutions to intertemporal consumption and portfolio choice problems. Important examples include Balduzzi and Lynch (1999), Barberis (2000), Brandt (1999), Brennan, Schwartz, and Lagnado (1996, 1997), Cocco, Gomes, and Maenhout (1998), and Lynch (2001). These papers concentrate on problems with a finite horizon and power utility, typically defined over wealth, whereas we consider an infinite horizon and Epstein-Zin utility over consumption. Our paper also differs in its careful comparison of analytical and numerical solutions, and of constrained and unconstrained solutions.

The structure of the paper is as follows. In section 2 we present the problem we would like to solve, following CV. Section 3 describes the solution algorithm as well as the DHM test used to evaluate its accuracy. In Section 4 we present and discuss the numerical solutions for the unconstrained and constrained optimization problems. Section 5 concludes.

2 The Model

We consider an infinitely-lived investor with recursive preferences described by:

$$U(C_t, \mathrm{E}_t\, U_{t+1}) = \left\{ (1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(\mathrm{E}_t\, U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}, \tag{1}$$

where C_t is the investor's period t consumption, $\delta < 1$ is the discount factor, $\gamma > 0$ is the coefficient of relative risk aversion and $\psi > 0$ is the elasticity of intertemporal substitution. These preferences were proposed by Epstein and Zin (1989) and Weil (1990) as a generalization of power utility that disentangles risk aversion from the elasticity of intertemporal substitution. Power utility is the special case where $\psi = 1/\gamma$.

Each period the investor must decide how much to consume out of her wealth and how to allocate the remaining wealth between two tradable assets: A risky asset (asset 1) with one-period log return given by $r_{1,t+1}$ and a riskless asset (asset f) with constant log return given by r_f.

The model assumes that the expectation of the log excess return $(r_{1,t+1} - r_f)$ on the risky asset is state-dependent,

$$\mathrm{E}_t \left[r_{1,t+1} - \overline{r}_f \right] = x_t, \tag{2}$$

where x_t is the single state variable of the model that follows an AR(1) process:

$$x_{t+1} = \mu + \phi(x_t - \mu) + \eta_{t+1}. \tag{3}$$

The unexpected log excess return is denoted by u_{t+1}. The random variables u_{t+1} and η_{t+1} are jointly normal and conditionally homoskedastic, with variances σ_u^2 and σ_η^2, respectively. We also allow for correlation between the unexpected log excess return and innovations in the state variable, and denote their covariance by $\sigma_{u\eta}$.

So far the setup is very similar to the one in CK. They however take the rate of return on the portfolio as given, whereas we consider the investor's optimal portfolio as well and thus introduce an additional choice variable. The one-period return on the portfolio from time t to time $t+1$ is

$$R_{p,t+1} = \alpha_t (R_{1,t+1} - R_f) + R_f, \tag{4}$$

where $R_{1,t+1} = \exp\{r_{1,t+1}\}$, $R_f = \exp\{r_f\}$ and α_t is the proportion of total wealth invested in the risky asset at time t.

We want to solve for the intertemporal consumption and portfolio policies that maximize (1) subject to the budget constraint

$$W_{t+1} = R_{p,t+1}\left(W_t - C_t\right), \tag{5}$$

where W_t is total wealth at the beginning of time t and $R_{p,t+1}$ is the return on wealth (4).

The investor's objective function (1) has been normalized so that the value function is homogeneous of degree one. Therefore, we can solve for the optimal consumption-wealth ratio and portfolio allocation rule when the investor has wealth equal to one. Following CK we simplify notation by defining $x \equiv x_t$, $y \equiv x_{t+1}$, $R_1 \equiv R_{1,t+1}$ and $R_p \equiv R_{p,t+1}$. We also denote the unit wealth indirect value function by $V(x)$, the consumption-wealth ratio by $c(x)$, and the fraction of wealth invested in the risky asset by $\alpha(x)$. The problem we solve is:

$$V(x) = \max_{0\le c(x)\le 1,\alpha(x)} \left\{ (1-\delta)c(x)^{1-\frac{1}{\psi}} + \delta(1-c(x))^{1-\frac{1}{\psi}} \left\{ \mathrm{E}\left[V(y)^{1-\gamma}R_p^{1-\gamma}|x\right]\right\}^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{6}$$

Epstein and Zin (1989) show that the Euler equations for consumption and portfolio choice for this problem are given by

$$c(x) = \frac{C}{1+C} \tag{7}$$

where

$$C = \left(\frac{1-\delta}{\delta}\right)^{\psi} \left\{ \mathrm{E}\left[V(y)^{1-\gamma}R_p^{1-\gamma}|x\right]\right\}^{\frac{1-\psi}{1-\gamma}},$$

$$\mathrm{E}\left[V(y)^{1-\gamma}R_p^{-\gamma}(R_1 - \overline{R}_f)|x\right] = 0, \tag{8}$$

and

$$R_p = \alpha(x)(R_1 - R_f) + R_f. \tag{9}$$

Unfortunately, it is not possible to write $\alpha(x)$ explicitly as a function of $V(y)$ and $c(x)$. Therefore a fully numerical solution to this problem would involve solving the non-linear equation (8) for $\alpha(x)$. Instead we assume that $\alpha(x)$ is a p'th order polynomial in the state-variable:

$$\alpha(x) = a_0 + a_1 x + a_2 x^2 + ... + a_p x^p \tag{10}$$

and optimize with respect to the coefficients of the polynomial. The approximate analytical solution proposed by CV is a particular case of (10), with $a_j = 0$ for $j \geq 2$.

3 Numerical Solution Method

3.1 Finding the solution

We discretize the state space and approximate it with 35 equally spaced grid points, centered at the unconditional mean of the state variable. We set the distance between any two points to 0.25%. We approximate the distribution for the innovations to the state variable using Gaussian quadrature methods with 9 quadrature points.

We use the Newton-Raphson algorithm to optimize over the coefficients of the policy function for $\alpha(x)$, allowing for either a third or a fourth order polynomial in the state variable. The approximate analytical solution in CV involves a portfolio allocation rule which is linear in the state variable. Since the log-linear solution is likely to be close to the optimal solution we use its coefficients as our initial values, with the coefficients on the higher order terms initialized at zero. This makes divergence of our numerical algorithm less likely and reduces the time needed to obtain convergence.

We initialize the value function by setting it equal to a constant. The results obtained are not sensitive to the value of this constant in a wide range around the mean value implied by the CV log-linear solution. More details of the numerical solution algorithm are given in the Appendix.[2]

For some parameter values the exact analytical solution is known for part of the problem (Giovannini and Weil 1989). In particular, when $\psi = 1$ the investor consumes in each period a fixed fraction of his wealth, and when $\gamma = 1$ the optimal portfolio rule is the same as with log utility so there is no intertemporal hedging demand for the risky asset. For these special cases we change our numerical algorithm by imposing the known exact analytical solution for the consumption and portfolio rules, respectively.

3.2 Evaluating solution accuracy

To test the accuracy of the numerical solution we use the Den Haan-Marcet (DHM) statistic (Den Haan and Marcet 1994).[3] The DHM statistic does not require knowledge of the true solution and it can be used to select between different functional forms for the policy functions. We based our choice between a third and a fourth order polynomial for $\alpha(x)$ on the values obtained for the DHM statistics.

[2] The program used to solve this model, written in Gauss, is available for download at www.london.edu/faculty/fgomes.

[3] The corresponding Gauss code is also available for download at www.london.edu/faculty/fgomes.

In order to implement this test we first simulate different time series of realizations for the exogenous variables, u and η. Given these realizations and the optimal numerical consumption and portfolio allocation rules, we compute the expectational errors (ε_{t+1}) implied by the Euler equations of the model, (7) and (8). These expectational errors should be orthogonal to the previous period's information set:

$$\mathrm{E}[\varepsilon_{t+1} \otimes h(z_t)] = 0, \tag{11}$$

for any function $h(.)$ and any variable z_t belonging to the information set available at time t. In our application we choose a vector of instruments $z_t = (1, x_t, x_t^2)$, and set $h(.)$ equal to the identity function.

Given these instruments we can test the accuracy of our numerical solution by computing:

$$B_t \equiv \frac{\sum_{t=1}^{T} \varepsilon_{t+1} \otimes h(z_t)}{T}, \tag{12}$$

and testing whether B_T is statistically different from zero. The significance of deviations from zero is evaluated by using the DHM test statistic

$$TB_T' A_T^{-1} B_T, \tag{13}$$

where

$$A_T \equiv \frac{\sum_{t=1}^{T} [\varepsilon_{t+1} \otimes h(z_t)][\varepsilon_{t+1} \otimes h(z_t)]'}{T}. \tag{14}$$

Under the null hypothesis that the numerical solution is correct the asymptotic distribution (as $T \to \infty$) of this test statistic is a χ^2_{qm} where m is the number of Euler equations being tested (two in the general case and one in the special cases where part of the solution is known) and q is the dimension of h (three in our application with three instruments where h is the identity function).

By comparing the result of (13) with the relevant critical values we can obtain evidence on the validity of the numerical solution. In order to reduce (and in the limit eliminate) the possibility of type I errors we performed a large number of simulations and computed the percentage of test results in the upper and lower 5% critical values of a χ^2_{qm}. In particular, we used 2000 time-series of 400 observations each, giving us 100 years of data as in CK (1997).

4 Optimal Portfolio Choice

4.1 Parameter values

We calibrate the model using quarterly US data from the Center for Research in Securities Prices (CRSP) over the period 1926.1–1999.4. We take the risky asset to be an aggregate stock market index and the riskless asset to be a short-term Treasury bill. (Of course the real return on a nominal Treasury bill is not literally riskless, but following a long tradition in empirical finance we ignore the small amount of uncertainty caused by short-term inflation risk.) The single state variable of the problem is the log dividend-price ratio, $(d_t - p_t)$. We obtain the parameters that define the stochastic structure of the model by estimating of the following restricted VAR(1) model:

$$\begin{pmatrix} r_{1,t+1} - r_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \beta_0 \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \beta_1 \end{pmatrix} (d_t - p_t) + \begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix}, \tag{15}$$

where $(\varepsilon_{1,t+1}, \varepsilon_{2,t+1}) \sim N(0, \Omega)$. CV show how to solve for the parameters of the theoretical model given the estimated parameters of (15). Table 1 reports both sets of parameters in natural quarterly units.

In our 1926-1999 sample period, the average real return on Treasury bills was extremely low at 44 basis points per year, and the average equity premium was extremely high at 9.2% per year. One could argue, following Campbell and Shiller (1998) and Fama and French (2000), that the historical equity premium greatly overstates prospects for stock returns in the future. However we do not make any adjustment to the sample average, merely noting that it implies extremely large average stockholdings for investors with moderate risk aversion coefficients; this is a manifestation of the equity premium puzzle in our microeconomic model with exogenous asset returns and endogenous portfolios.

The predictability of stock returns from the dividend-price ratio is modest in this sample period. The t-statistic for the dividend-price ratio is only about 1.2, and the R^2 statistic in the quarterly return regression is only 0.5%. The evidence for predictability is stronger if one considers only the postwar period, or ends the sample before the extraordinary runup of the US stock market during the late 1990's. Once again, however, we make no adjustment to the sample estimates but merely calculate the portfolio implications of our point estimates.

Two other characteristics of the estimated model are worth noting. The log dividend-price ratio is extraordinarily persistent, with a quarterly serial correlation of 0.97, and its innovations have an almost perfect negative correlation of -0.95 with stock returns. Both the persistence and the negative correlation with returns are increased by the inclusion of the late 1990's in the sample period.[4] They have the effect of increasing the importance of intertemporal hedging demand, counteracting the effect of reduced short-term return predictability.

The discount factor under time-additive utility δ is set to 0.94 in annual terms. We solve the model for coefficients of relative risk aversion γ of 1, 2, 4, 10, and 20, and for elasticities of intertemporal substitution ψ of 1, 0.5, 0.25, 0.1, and 0.05. Each of these values is the reciprocal of one of our risk aversion values, so we consider the power utility cases for which $\psi = 1/\gamma$. We emphasize values for $\psi \leq 1$ since time-series studies of representative-agent models suggest that ψ is well below one and may be close to zero (Hall 1988, Campbell and Mankiw 1989).

4.2 Unconstrained portfolio choice

In this section we present our numerical results for the unconstrained problem in which the investor can short-sell either of the assets. We report results when the portfolio allocation rule is modelled as a third-order polynomial in the state variable. We choose a third-order polynomial because convergence for a fourth-order polynomial is much slower while the DHM statistics are little better than for a third-order polynomial.

Figures 1a and 2a plot the optimal portfolio rule as a function of the state variable. For comparison purposes the log-linear (CV) solution is also included. Figure 1a plots the portfolio allocation rules for $\gamma = 4$ and 20 and $\psi = 1/.75$, while Figure 2a plots those rules for the same coefficients of relative risk aversion and $\psi = 1/4$. The solid vertical lines in the figures are drawn two standard deviations above and below the unconditional mean of the state variable.

Several interesting results are visible in these figures. First, as in CV the optimal portfolio allocations to stocks are positive even at the point of the state space where the expected gross excess stock return is zero—that is, even when the stock market is pure risk with no reward. Short-term investors would invest nothing in a risky asset unless induced to do so by a positive expected excess return. Long-term investors, however, know that on average they hold profitable long positions in stocks. An increase in the expected return is therefore welfare-enhancing, while a decrease is welfare-reducing. The negative correlation of the realized stock return with the dividend-price ratio implies that stock prices move opposite expected returns, and

[4] CV report a much smaller negative correlation of -0.74 for the period 1947.1-1995.4. However this is due to a timing error in the construction of stock returns. Campbell and Viceira (2000) correct the error and report a correlation of -0.96. They also report complete corrected solutions to the consumption and portfolio choice problem.

therefore can be used to hedge the risk of random variation in the expected return. For this reason long-term investors with relative risk aversion greater than one hold stocks even when there is no current reward for doing so.

Second, the portfolio rules are almost identical in Figures 1a and 2a. This shows that the elasticity of intertemporal substitution has almost no effect on optimal portfolio choice, while the coefficient of relative risk aversion has a large effect. This is another property of the CV solution that is verified by the numerical solution.

Third, the log-linear solution and the numerical solution are very close in a large interval around the mean of the state-space, where both rules are approximately linear with almost exactly the same level and slope. However the numerical solution flattens out towards the upper extreme of the state-space.

Figures 1b and 2b plot the optimal consumption-wealth ratio as a function of the state variable, using the same preference parameters as Figures 1a and 2a. Again the solution in CV is also reported for comparison. Instead of postulating a functional form for the numerical optimal consumption rule, we used (7) to obtain it; thus the only approximations involved in our solution method are the discretization of the state-space, the 9-point Gaussian quadrature approximation of the distribution of the random variables, and the restriction of the functional form for $\alpha(x)$.

The consumption rules depend sensitively on both the elasticity of intertemporal substitution and the coefficient of relative risk aversion. Risk aversion matters because it affects the expected return on the portfolio and thus the income that is available for consumption. The elasticity of intertemporal substitution matters because changing expected returns have opposing income and substitution effects. As the expected portfolio return increases, it is possible to consume more out of wealth (the income effect), but there is an incentive to consume less in order to profit from favorable investment opportunities (the substitution effect). The former effect dominates if the elasticity of substitution is less than one as in Figure 2b, while the latter dominates if the elasticity is greater than one as in Figure 1b. The consumption-wealth ratio is a constant if the elasticity of intertemporal substitution is exactly equal to one.

The consumption rules are nonlinear and roughly quadratic. CV give the intuition for this. As the expected stock return increases, the investor earns a higher return on given stockholdings, and also increases the portfolio weight on stocks; thus the expected portfolio return increases nonlinearly, and this feeds through into the consumption rule whenever the elasticity of intertemporal substitution is different from one.

Like the portfolio rules, the optimal numerical and log-linear consumption rules are very close in most of the state space, except at the upper extreme, where the numerical solution flattens out. It is interesting to note that the optimal numerical and log-linear rules tend to be closer over a larger portion of the state space when the coefficient of relative risk aversion is large than when it is small. One might have thought that the greater curvature of the utility function implied by large risk aversion would negatively affect the accuracy of the approximation; but in this model greater curvature actually helps the approximation because it limits the extent to which the investor wants to take advantage of the predictability in excess returns.

Table 2 presents the mean optimal numerical percentage allocation to stocks and the optimal numerical and log-linear percentage allocation to stocks at the mean of the state-space. The difference between these two values is the result of Jensen's inequality; there is no difference in the approximate solution, since it is linear in the state variable. For $\gamma = 1$ portfolio choice is myopic, and therefore the log-linear and numerical results coincide (and do not depend on ψ). In general, the average optimal allocation to stocks varies significantly with γ, but little with ψ, a pattern already identified by CV.

We do not present results for $\gamma < 1$ because in this region of the parameter space we encounter difficult numerical convergence problems. The value function tends to increase without limit at the extremes of the state space, because the investor's low risk aversion combined with a high and time-varying equity premium lead the individual to aggressively time the stock market and achieve very high or even unbounded utility. One way to assist convergence is to consider a lower δ, but even with this shift in parameters the DHM statistics indicate that the numerical solution is not very accurate.

As we increase risk aversion above one in Table 2, we see that average allocations to stocks fall less than proportionately with risk aversion. The investor with $\gamma = 1$ holds on average a leveraged portfolio with a 200% allocation to stocks. The investor with $\gamma = 4$ holds on average about 75% in stocks, rather than the 50% that would be implied by a static portfolio rule; and the investor with $\gamma = 20$ holds on average about 20% in stocks, rather than the 10% that would be implied by a static rule. The increase in average stock allocations is the result of positive intertemporal hedging demand for stocks by conservative investors. Even with the modest predictability in returns that we have estimated using a single state variable over the period 1926–1999, conservative investors' hedging demand can be important relative to their static demand. Substantially larger hedging demands are reported by Campbell and Viceira (2000) and Campbell, Chan, and Viceira (2001) who look at alternative sample periods and consider multiple state variables.

Table 2 shows that the mean optimal log-linear percentage allocation to stocks is slightly larger than the one obtained numerically. This is due to the fact that the numerical optimal portfolio allocation rule flattens out towards the upper extreme of the state-space whereas the approximate analytical rule keeps on increasing linearly. However, this happens mostly for values of the state-variable for which the unconditional probability is very small, and therefore the mean optimal log-linear and numerical allocations are very close.

Table 3 reports the standard deviations of the optimal numerical and log-linear portfolio rules. The fact that the optimal numerical portfolio rule is approximately linear around the unconditional mean of the state-variable but flattens out towards the upper extreme of the state-space, whereas the log-linear portfolio rule is linear throughout, explains why the latter has a higher standard deviation. Again, the difference between the two standard deviations is small because of the low unconditional probability of the portion of the state space where the flattening occurs. The standard deviation of the optimal percentage allocation to stocks decreases with γ since risk-averse individuals are less aggressive market timers.

Tables 4 through 6 present analogous calculations for the consumption-wealth ratio. In interpreting these tables one should remember that the optimal log-linear consumption policy is quadratic in the state-variable. This is why the optimal log-linear percentage consumption-wealth ratio at the mean of the state-space and the mean optimal log-linear consumption wealth ratio do not coincide. Also, for $\psi = 1$ the exact analytical consumption-wealth ratio is known and therefore the log-linear and numerical solutions coincide (and do not depend on γ).

The sensitivity of the consumption-wealth ratio to the parameters γ and ψ is similar for the log-linear and numerical solutions. For values of $\psi < 1$ (> 1) the mean optimal percentage consumption-wealth ratio is increasing (decreasing) in γ. For low (high) values of γ the mean optimal percentage consumption-wealth ratio is increasing (decreasing) in ψ. The intuition is the same as in CV. Highly risk-averse investors, at the bottom of the table, hold most of their wealth in the riskless asset and hence earn a low return. If this return is below the rate of time preference, investors that are unwilling to substitute consumption intertemporally ($\psi < 1$) choose to consume more out of wealth than investors who are willing to substitute consumption intertemporally ($\psi > 1$). The income effect of a negative time-preference-adjusted rate of return on saving dominates for investors with $\psi < 1$, while the substitution effect dominates for investors with $\psi > 1$. This pattern reverses for risk-tolerant investors, in the upper part of the table, because these investors hold risky portfolios whose expected rate of return is larger than their rate of time preference.

Table 6 presents the standard deviations of the optimal log-linear and numerical consumption-wealth ratios. In general the log-linear standard deviations are larger. The reason is that our numerical policy functions are approximately quadratic at the unconditional mean of the state-space (and very similar to the log-linear policy functions) but flatten out towards the upper extreme of the state-space as shown in Figures 1b and 2b. This pattern is similar to the one we obtained for the portfolio rule in Table 3.

Table 7 presents DHM tests of the accuracy of our numerical solutions. If our solutions were exact, the DHM statistics would give us 5% of errors in both the lower and upper critical tails. The numbers reported in Table 7 lie between 2% and 10% and generally cluster around 5%, implying that our numerical solutions are quite accurate. CK obtained better DHM test results, but their model has only one decision variable and one Euler equation. We note that we also obtain DHM statistics closer to 5% in the special cases ($\gamma = 1$ or $\psi = 1$) for which we have only one decision variable.

We also used DHM statistics to evaluate the accuracy of the log-linear (CV) solution. In order to do this we calculated numerically the value function implied by the log-linear portfolio and consumption rules, using stage 2 of our numerical algorithm. The log-linear solution dramatically fails the DHM test, giving 0% of errors in the lower tail and 100% of errors in the upper tail. Evidently the inaccuracy at the upper extreme of the state space, shown in Figures 1 and 2, has a large impact on the DHM statistics. These findings also illustrate the power of DHM statistics to detect small deviations from the optimal solution.

Finally, we use DHM statistics to explore the importance of nonlinearities in the portfolio rule. We compare the DHM statistics for numerical solutions based on polynomial portfolio rules that are third-order, second-order, first-order, and first-order restricted to have the slope implied by the log-linear CV solution. In each case we calculate the optimal portfolio rule, consumption rule, and value function using our standard numerical algorithm. The results are not reported in a table, but can be summarized as follows. Comparing the second-order with the third-order polynomial rule, there is moderate deterioration in the DHM statistics to about 10% with the deterioration being worse for high values of gamma. The deterioration is much worse for the first-order polynomial rules especially when gamma is 10 or higher. In the extreme, when the slope is restricted to the CV value and gamma is high, 100% of the DHM statistics fall in the upper 5% critical tail.

4.3 Constrained portfolio choice

In this section we present numerical results for the constrained problem in which the investor is not allowed to borrow at the riskfree rate or to short-sell the risky asset, so the portfolio share $\alpha(x)$ is constrained to lie in the unit interval. Our main focus in this section is to compare the constrained and unconstrained numerical solutions.

Our solution algorithm for the constrained problem is a slight variation of the one presented above for the unconstrained problem. At each iteration of the algorithm the portfolio allocation rule is given by:

$$\alpha_k^{const}(x) = \text{Min}[\text{Max}(a_0^k + a_1^k x + a_2^k x^2 + a_3^k x^3, 0), 1] \tag{16}$$

where k refers to the iteration. The results for the constrained problem are presented in Tables 8 and 9. Figures 3 and 4 plot the constrained and unconstrained optimal portfolio and consumption rules for $\gamma = 4$ and 20 and $\psi = 1/4$.

The constrained portfolio rules, like the unconstrained rules, are nondecreasing in the state variable. Thus the zero constraint binds when the state variable is low, and the unit constraint binds when the state variable is high. This means that if the constrained model is a reasonable approximation to reality investors should be out of the stock market at a time when the log dividend-price ratio is low. Figure 3 illustrates that the area over which the portfolio rules are constrained decreases as the coefficient of relative risk aversion increases. This is of course a consequence of the fact that risk-averse investors have lower average stock allocations and are less aggressive market timers. The figure also shows that the slope of the optimal rule in the unconstrained region may be affected by the existence of the constrained region; nonetheless the unconstrained solution, with the constraints imposed, appears to be a good first approximation to the constrained solution.

The presence of binding constraints implies that the standard deviation of the optimal constrained percentage allocation to stocks (Table 8) is smaller than the standard deviation of the unconstrained allocation (Table 3). The constraints reduce the average allocation to stocks for investors with low coefficients of risk aversion, since these investors more often wish to leverage their stockholdings than to sell stocks short; the constraints have no detectable effect on the average allocation for highly risk-averse investors.

The portfolio constraints also affect the consumption-investment decision as shown in Table 9. For example, when the state variable is sufficiently high the investor would like to hold a leveraged position in the risky asset, which he is not allowed to do. Instead he holds an unleveraged position, and earns a smaller return than he would otherwise. This affects the steady-state value of his consumption-wealth ratio. Comparing Tables 5 and 9 we see that for $\psi < 1$ the mean optimal percentage consumption-wealth ratio with constrained portfolio choice is smaller than the one with unconstrained portfolio choice. This result can be understood by considering the income and substitution effects of portfolio constraints. Portfolio constraints reduce the average portfolio return, which reduces consumption through the income effect but increases consumption through the substitution effect. When $\psi < 1$ the income effect dominates, but when $\psi > 1$ the substitution effect dominates and the portfolio constraints increase average consumption.

Comparing Tables 6 and 9 we see that portfolio constraints reduce the standard deviation of the optimal consumption-wealth ratio. An intuitive way of understanding this result is to note that portfolio constraints make the several states look more alike. For example, even if the log dividend-price ratio is very high, the investor is limited in his ability to exploit this unusual investment opportunity. This effect is stronger for investors with low risk aversion, who are constrained over a larger region of the state space.

5 Conclusion

This paper has analyzed the implications of stock return predictability for the portfolio and consumption decisions of long-lived investors. We have studied investors who consume out of their financial wealth and have recursive Epstein-Zin-Weil utility, a generalization of power utility that enables us to distinguish between the coefficient of relative risk aversion and the elasticity of intertemporal substitution. We assume that these investors choose in discrete time between a riskless asset with a constant return, and a risky asset with constant return variance whose expected log return follows an AR(1) process. We have calibrated the asset return processes to fit the behavior of quarterly US stock returns over the period 1926–1999, using the log dividend-price ratio as a proxy for the expected log stock return.

We have used numerical techniques to obtain the optimal policies. We find that the optimal portfolio rule is approximately linear and the log consumption-wealth ratio approximately quadratic in the state variable over a wide interval around the unconditional mean of the state variable. However, both policy functions flatten out towards the upper extreme of the state space. The investor responds less aggressively to movements in the log dividend-price ratio when the ratio is already extremely high.

We have also calculated the optimal policies for investors who face borrowing and short-sales constraints. This is probably a more realistic problem for many investors. We find that the constrained optimal portfolio rules are close to the unconstrained optimal rules with constraints imposed, and are particularly close for investors with high risk aversion. The presence of constraints has important effects on both the level and variability of optimal consumption.

We have compared our numerical solution with the approximate analytical solution proposed by Campbell and Viceira (1999), in which the portfolio rule is globally linear and the consumption rule is globally quadratic. For the parameter values considered here, the approximate analytical solution is very close to the numerical solution provided that the state variable is no more than two standard deviations above its mean. However the analytical solution cannot capture the flattening out of the policy functions at high values of the state variable.

Den Haan-Marcet (1994) statistics are sensitive indicators of errors in candidate solutions to dynamic optimization problems. These statistics indicate that our numerical solution is quite accurate, but they are able to pick up the approximation errors in the analytical solution at the upper end of the state space.

This paper has the important limitation that it studies a partial equilibrium model. We have solved the microeconomic problem of an investor facing exogenous asset returns, but we do not show how these asset returns could be consistent with general equilibrium. The difficulty is particularly severe because we find that all investors should change their portfolio allocations in the same direction as the expected stock return, regardless of their preferences. That is, all investors should buy and sell assets at the same time. This cannot be consistent with a general equilibrium model that makes realistic assumptions about asset supplies.

One possible resolution of this difficulty is that the representative investor has different preferences from those assumed here, perhaps the habit-formation preferences of Campbell and Cochrane (1999) that can generate shifts in risk aversion and hence changing risk premia with a constant riskless interest rate. Under this interpretation the portfolio rules described in this paper should be used only by investors with constant risk aversion, who cannot be typical of the market as a whole.

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Appendix: Numerical Solution Algorithm

We use Newton-Raphson's algorithm to optimize over the coefficients of the policy function for $\alpha(x)$. We discretize the state space and approximate it with 35 equally spaced grid points, centered at the unconditional mean of the state variable. We set the distance between any two points to 0.25%. We approximate the distribution for the innovations to the state variable using Gaussian quadrature methods with 9 quadrature points.

Since the log-linear solution in Campbell and Viceira (CV, 1999) is likely to be close to the optimal solution we use its coefficients as our initial values, with the coefficients in the higher order terms initialized at zero. This makes divergence of our numerical algorithm less likely and reduces the time needed to obtain convergence. We initialize the value function by setting it equal to a constant. The results obtained are not sensitive to the value of this constant in a wide range around the mean value implied by the CV log-linear solution.

The algorithm consists of the following steps:

1. Given initial values for the coefficients of the portfolio rule (10) and for the value function, we compute the consumption rule using equation (7):

$$c(x) = \frac{\left(\frac{1-\delta}{\delta}\right)^{\psi} \left\{E\left[V(y)^{1-\gamma} R_p^{1-\gamma} | x\right]\right\}^{\frac{1-\psi}{1-\gamma}}}{1 + \left(\frac{1-\delta}{\delta}\right)^{\psi} \left\{E\left[V(y)^{1-\gamma} R_p^{1-\gamma} | x\right]\right\}^{\frac{1-\psi}{1-\gamma}}} \text{ for each } x \text{, given } V(.). \tag{17}$$

2. We use this consumption rule to compute a new value function using equation (6):

$$V(x) = \left\{(1-\delta)c(x)^{1-\frac{1}{\psi}} + \delta(1-c(x))^{1-\frac{1}{\psi}} \left\{E\left[V(y)^{1-\gamma} R_p^{1-\gamma} | x\right]\right\}^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{18}$$

for each x, given $c(.)$. We iterate on this functional equation until we obtain convergence of the value function at each state. This gives us values for the value function and the CWR which correspond to the parameters of the portfolio rule.

3. We evaluate numerically the gradient (G) of the value function at each state with respect to the coefficients of $\alpha(x)$ as follows. Using 1 and 2 we compute the value function and CWR corresponding to perturbed coefficients. Given these values we compute the gradient for each state:

$$G(x; \alpha^0) = (V(x; \alpha^0 + \varepsilon * I_p) - V(x; \alpha^0 - \varepsilon * I_p))/2\varepsilon, \tag{19}$$

where α^0 denotes the vector of parameters of the portfolio rule, ε is a small increment, and I_p is a unit vector of dimension p.

In order to determine the direction of steepest ascent (G^*) we compute an weighted average of these gradients using the ergodic distribution of the state variable:

$$G^* = \int G(x; \alpha^0) dF(x), \tag{20}$$

where $F(x)$ is the cumulative distribution function corresponding to the ergodic distribution of x.

4. To determine the optimal step size we compute the Hessian (H) of the value function with respect to the coefficients, using the same approach as in the calculation of the gradient. To avoid divergence problems we control the step size. The new coefficients are given by

$$\alpha^{k+1} = \alpha^k - \lambda H^{-1} G^*, \tag{21}$$

where k refers to the iteration, and λ is set initially to 2/3. Given these coefficients the new value function is compared with the previous one and in case there is no improvement λ is reduced (multiplied by 2/3). This is repeated as long as no improvement occurs.

5. With the coefficients obtained in 4 we compute the value function and CWR using 1 and 2.

6. We iterate 3, 4 and 5 until the weighted average value function converges.

Table 1: Estimates of the Stochastic Process for Returns (1926.1 - 1999.4)

(A) Restricted VAR(1):

$$\begin{pmatrix} r_{1,t+1} - r_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} 0.083 \\ (0.057) \\ -0.098 \\ (0.059) \end{pmatrix} + \begin{pmatrix} 0.020 \\ (0.017) \\ 0.970 \\ (0.018) \end{pmatrix} (d_t - p_t) + \begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 11.598E-3 & -11.392E-3 \\ (0.960E-3) & (0.970E-3) \\ -11.392E-3 & 12.503E-3 \\ (0.970E-3) & (1.035E-3) \end{pmatrix} \qquad R^2 = \begin{pmatrix} 0.005 \\ 0.908 \end{pmatrix}$$

(B) Derived model:

$$\begin{aligned} r_{1,t+1} - r_f &= x_t + u_{t+1} \\ x_{t+1} &= \underset{(0.003)}{1.718E-2} + \underset{(0.018)}{0.970} \; (x_t - \mu) + \eta_{t+1} \end{aligned}$$

$$\begin{pmatrix} \sigma_u^2 & \sigma_{u,\eta} \\ \sigma_{u,\eta} & \sigma_\eta^2 \end{pmatrix} = \begin{pmatrix} 11.598E-3 & -0.229E-3 \\ (0.960E-3) & (0.199E-3) \\ -0.229E-3 & 0.005E-3 \\ (0.199E-3) & (0.009E-3) \end{pmatrix}$$

$r_f = 0.11E-2 \qquad \sigma_x^2/\sigma_u^2 = 0.737E-2 \qquad corr(\eta, u) = -0.946$

Table 2: Mean Optimal Numerical Percentage Allocation to Stocks and Optimal Numerical (Log-Linear) Percentage Allocation to Stocks at the Mean of the State-Space

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	198.14	198.14	198.14	198.14	198.14	198.14
	198.14	198.14	198.14	198.14	198.14	198.14
	(198.14)	(198.14)	(198.14)	(198.14)	(198.14)	(198.14)
2.00	128.90	124.00	122.54	121.27	122.43	121.90
	129.62	124.03	122.95	121.31	123.70	123.60
	(130.87)	(129.30)	(127.41)	(126.64)	(126.21)	(126.08)
4.00	76.15	77.67	75.50	76.62	77.14	75.99
	77.29	79.93	76.30	78.14	78.67	77.18
	(77.94)	(78.46)	(79.25)	(79.53)	(79.71)	(79.77)
10.00	32.03	33.20	35.54	36.61	38.28	37.20
	32.59	33.81	36.52	37.63	39.22	37.66
	(34.92)	(36.61)	(39.50)	(41.10)	(42.11)	(42.45)
20.00	16.67	17.46	19.38	20.76	19.86	19.48
	16.67	17.46	19.37	20.75	19.82	19.49
	(18.14)	(19.47)	(22.08)	(23.74)	(24.84)	(25.23)

Table 3: Standard Deviation of the Optimal Numerical (Log-Linear) Percentage Allocation to Stocks

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	77.29	77.29	77.29	77.29	77.29	77.29
	(77.29)	(77.29)	(77.29)	(77.29)	(77.29)	(77.29)
2.00	42.41	45.11	43.27	44.38	41.81	42.63
	(45.28)	(45.08)	(44.83)	(44.73)	(44.67)	(44.65)
4.00	22.66	23.40	21.98	21.75	21.79	21.85
	(25.04)	(25.10)	(25.19)	(25.22)	(25.24)	(25.25)
10.00	8.58	8.42	9.20	8.99	8.26	8.32
	(10.71)	(10.91)	(11.21)	(11.36)	(11.46)	(11.49)
20.00	3.44	3.41	2.66	2.05	3.37	3.86
	(5.48)	(5.63)	(5.89)	(6.04)	(6.12)	(6.15)

Table 4: Optimal Numerical (Log-Linear) Percentage Consumption-Wealth Ratio at the Mean of the State-Space

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	1.17	1.53	2.03	2.26	2.39	2.43
	(1.16)	(1.53)	(2.08)	(2.34)	(2.50)	(2.55)
2.00	1.40	1.53	1.69	1.75	1.78	1.79
	(1.38)	(1.53)	(1.74)	(1.83)	(1.88)	(1.90)
4.00	1.63	1.53	1.39	1.32	1.28	1.27
	(1.60)	(1.53)	(1.45)	(1.41)	(1.39)	(1.38)
10.00	1.84	1.53	1.11	0.91	0.79	0.75
	(1.81)	(1.53)	(1.16)	(1.00)	(0.90)	(0.87)
20.00	1.92	1.53	0.98	0.71	0.55	0.49
	(1.90)	(1.53)	(1.02)	(0.78)	(0.65)	(0.61)

Table 5: Mean Optimal Numerical (Log-Linear) Percentage Consumption-Wealth Ratio

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	1.16	1.53	2.09	2.37	2.53	2.59
	(1.15)	(1.53)	(2.16)	(2.50)	(2.71)	(2.78)
2.00	1.39	1.53	1.72	1.81	1.85	1.87
	(1.37)	(1.53)	(1.78)	(1.91)	(1.98)	(2.01)
4.00	1.62	1.53	1.41	1.35	1.31	1.30
	(1.59)	(1.53)	(1.47)	(1.44)	(1.43)	(1.43)
10.00	1.83	1.53	1.11	0.91	0.80	0.76
	(1.81)	(1.53)	(1.17)	(1.01)	(0.92)	(0.89)
20.00	1.92	1.53	0.98	0.71	0.55	0.50
	(1.90)	(1.53)	(1.02)	(0.79)	(0.65)	(0.61)

Table 6: Standard Deviation of the Optimal Numerical (Log-Linear) Percentage Consumption-Wealth Ratio

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	0.15	0.00	0.36	0.59	0.73	0.78
	(0.16)	(0.00)	(0.41)	(0.71)	(0.92)	(1.00)
2.00	0.12	0.00	0.22	0.33	0.41	0.44
	(0.13)	(0.00)	(0.25)	(0.40)	(0.50)	(0.54)
4.00	0.08	0.00	0.12	0.17	0.20	0.21
	(0.09)	(0.00)	(0.14)	(0.21)	(0.26)	(0.27)
10.00	0.04	0.00	0.04	0.06	0.07	0.06
	(0.05)	(0.00)	(0.06)	(0.08)	(0.10)	(0.10)
20.00	0.02	0.00	0.02	0.03	0.02	0.02
	(0.03)	(0.00)	(0.03)	(0.04)	(0.04)	(0.04)

Table 7: Percentage of Den Haan-Marcet Statistics in the Lower (Upper) 5% Critical Tail

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	5.25	—	4.50	5.50	3.55	5.00
	(6.10)	(—)	(6.35)	(7.65)	(6.95)	(7.35)
2.00	6.15	5.05	5.10	4.95	5.85	5.20
	(6.70)	(6.90)	(8.20)	(9.10)	(7.65)	(7.25)
4.00	5.00	5.55	5.30	4.15	4.25	4.70
	(6.85)	(5.00)	(6.80)	(7.25)	(7.30)	(8.05)
10.00	4.15	5.95	3.65	2.65	3.15	3.30
	(5.85)	(5.05)	(7.25)	(7.80)	(7.05)	(6.55)
20.00	1.65	4.10	1.70	0.55	3.60	2.60
	(8.75)	(4.85)	(8.50)	(9.50)	(6.40)	(8.75)

Table 8: Constrained Portfolio Choice: Mean Optimal Percentage Allocation to Stocks, Optimal Percentage Allocation to Stocks at the Mean of the State-Space and Standard Deviation of the Optimal Percentage Allocation to Stocks

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	96.04	96.04	96.04	96.04	96.04	96.04
	100.00	100.00	100.00	100.00	100.00	100.00
	(15.53)	(15.53)	(15.53)	(15.53)	(15.53)	(15.53)
2.00	91.70	91.71	91.69	91.30	91.73	91.82
	100.00	100.00	100.00	100.00	100.00	100.00
	(17.50)	(17.45)	(17.51)	(18.17)	(17.45)	(17.34)
4.00	71.65	71.96	72.56	72.74	72.92	72.92
	72.63	72.91	73.34	73.63	73.78	73.85
	(21.07)	(20.81)	(20.22)	(20.20)	(20.03)	(20.10)
10.00	32.03	33.20	35.55	36.62	38.28	37.20
	32.59	33.81	36.52	37.63	39.22	37.66
	(8.57)	(8.42)	(9.18)	(8.98)	(8.26)	(8.32)
20.00	16.67	17.46	19.38	20.76	19.86	19.48
	16.67	17.46	19.37	20.75	19.82	19.49
	(3.44)	(3.41)	(2.66)	(2.05)	(3.37)	(3.86)

Table 9: Constrained Portfolio Choice: Mean Optimal Percentage Consumption-Wealth Ratio, Optimal Percentage Consumption-Wealth Ratio at the Mean of the State-Space and Standard Deviation of the Optimal Percentage Consumption-Wealth Ratio

R.R.A.	E.I.S.					
	1/.75	1	1/2	1/4	1/10	1/20
1.00	1.44	1.53	1.68	1.75	1.79	1.81
	1.44	1.53	1.67	1.74	1.77	1.79
	(0.09)	(0.00)	(0.15)	(0.23)	(0.28)	(0.29)
2.00	1.51	1.53	1.57	1.59	1.60	1.60
	1.51	1.53	1.56	1.57	1.57	1.57
	(0.09)	(0.00)	(0.14)	(0.21)	(0.25)	(0.26)
4.00	1.63	1.53	1.40	1.34	1.30	1.29
	1.63	1.53	1.39	1.31	1.27	1.25
	(0.08)	(0.00)	(0.11)	(0.16)	(0.19)	(0.20)
10.00	1.83	1.53	1.11	0.91	0.80	0.76
	1.84	1.53	1.11	0.91	0.79	0.75
	(0.04)	(0.00)	(0.04)	(0.06)	(0.07)	(0.06)
20.00	1.92	1.53	0.98	0.71	0.55	0.50
	1.92	1.53	0.98	0.71	0.55	0.49
	(0.02)	(0.00)	(0.02)	(0.03)	(0.02)	(0.02)

Figure 1.a: Portfolio Rule When ψ = 1.33

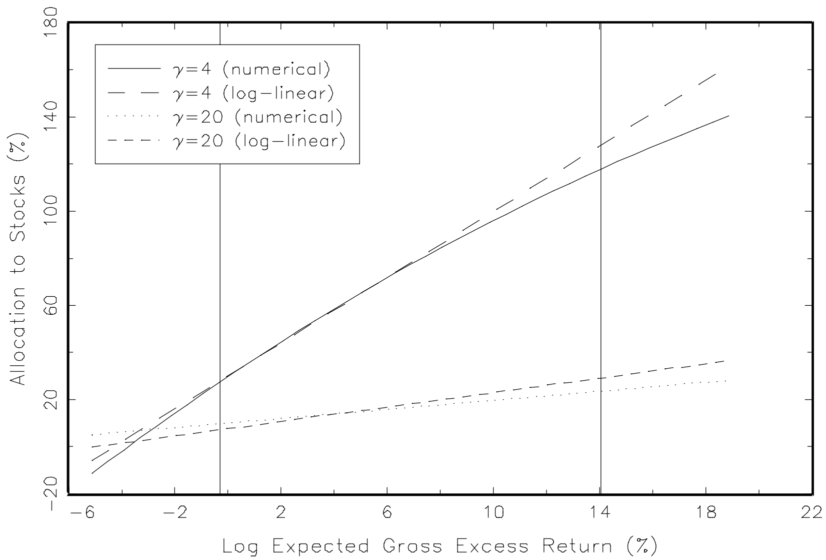

Figure 1.b: Consumption−Wealth Ratio When ψ = 1.33

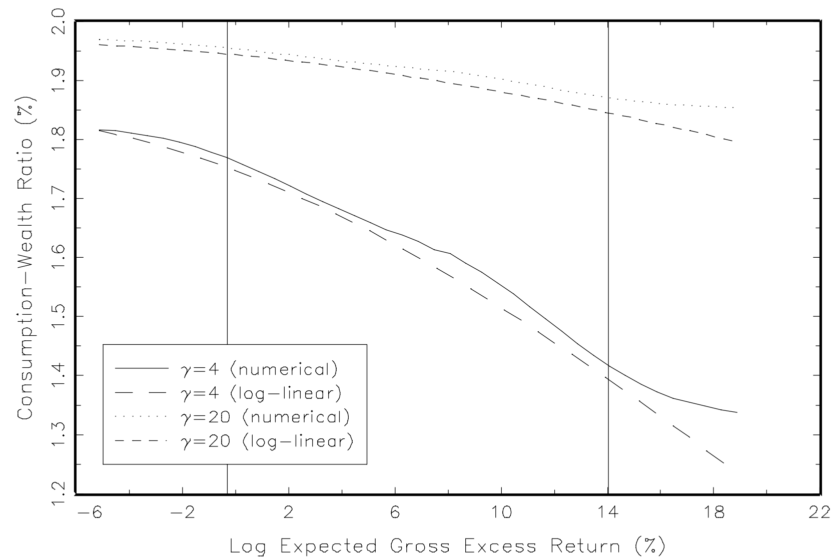

Figure 2.a: Portfolio Rule When ψ = 0.25

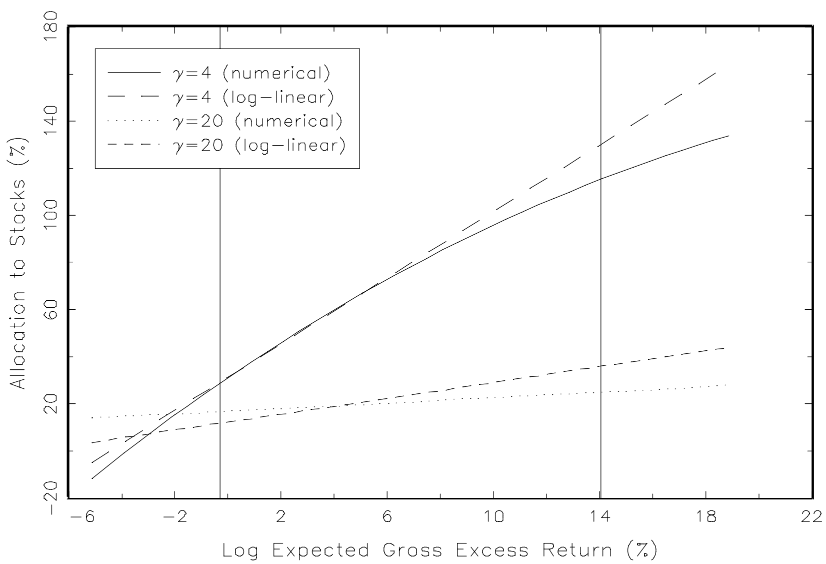

Figure 2.b: Consumption−Wealth Ratio When ψ = 0.25

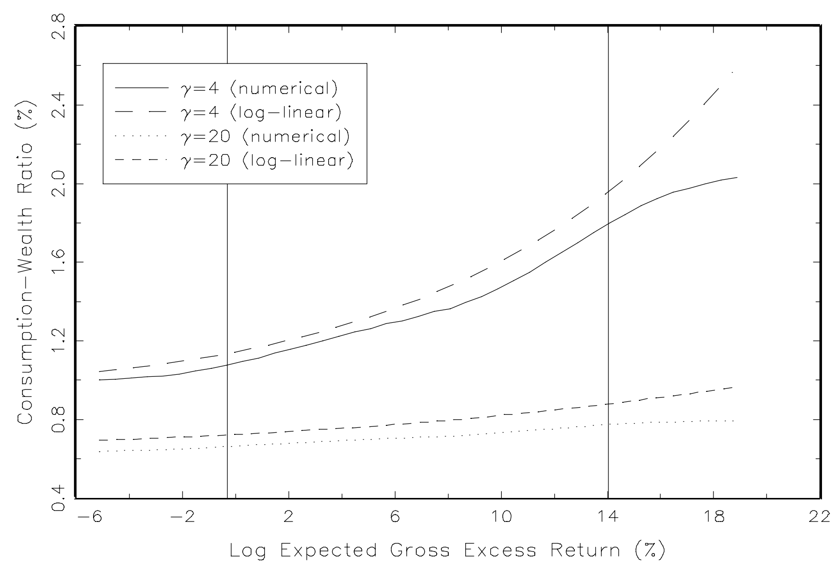

Figure 3: Unconstrained and Constrained Portfolio Rule When ψ = 0.25

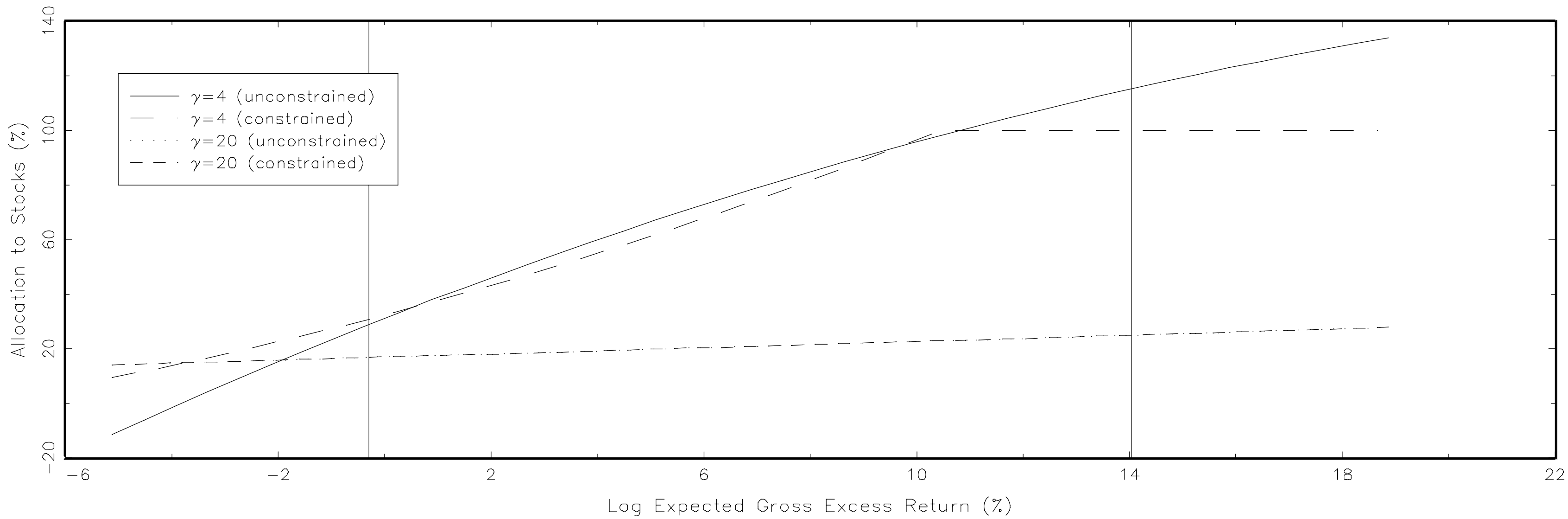

Figure 4: Unconstrained and Constrained Consumption−Wealth Ratio When ψ = 0.25