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A DIFFERENTIAL APPROACH TO DOMINANT STRATEGY MECHANISMS[1]

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This paper shows how a number of questions about dominant strategy mechanisms in models with public goods can be conveniently formulated as systems of partial differential equations. The question of the existence of dominant strategy mechanisms with given desirable properties becomes equivalent to the integrability of these equations.

1. INTRODUCTION

FOLLOWING Vickrey [**11**], Clarke [**27**], and Groves [**6**] a number of papers[2] have explored the properties of dominant strategy mechanisms, in particular in the framework of the so called "free rider problem," where elicitation of truthful evaluation of public goods is sought.

This paper shows how a number of questions about dominant strategy mechanisms in models with public goods can be conveniently formulated as systems of partial differential equations. The question of the existence of dominant strategy mechanisms with such desirable properties becomes equivalent to the *integrability* of these equations.[3]

In addition to enabling us to derive rapidly and strengthen a variety of known results on incentives, our approach permits us to develop certain new theorems and to provide an insight into the common mathematical structure of several apparently different questions.

In Section 2, we establish notation and in the following section substantially strengthen the Green–Laffont [**5**] characterization theorem. Section 4 contains a condition on the class of utility functions which is necessary and sufficient for the existence of balanced incentive compatible mechanisms. This condition is used to prove an impossibility theorem for two-agent models and some possibility and impossibility results for models with more than two agents. Finally in Section 5 we consider coalition incentive compatibility. Our results are essentially negative, centering around a general necessary condition for the existence of mechanisms which are incentive compatible for a class of coalitions.

2. THE MODEL

We consider an economy with $n (n \geqslant 2)$ consumers (indexed by $i = 1, \ldots, n$) and two commodities, one public and one private.

The utility function of consumer i, $u_i(K, x_i)$, is additively separable between the public good K and the private good x_i, $i = 1, \ldots, n$, and without loss of generality

[1] Maskin's research was supported, in part, by the National Science Foundation and Laffont's research was supported by Cordes n. 136–77.

[2] See, for example, Green and Laffont [**3**, **5**].

[3] In Laffont and Maskin [**10**], we apply this approach to mechanisms for which the solution concept is taken to be the maximization of expected utility.

we write it as:

$$u_i(K, x_i) = v_i(K, \theta_i) + x_i$$

where θ_i, lying in a space Θ_i, is a parameter of the valuation functions and where for simplicity we assume that the quantity K of public good ranges in the set $\mathring{R}_+ =]0, \infty[$.[4]

The decision maker is supposed to know the *functions* $v_i(\cdot, \cdot)$ (possibly identical for all consumers) but is ignorant of the true value $\hat{\theta}_i$ of the parameter θ_i which identifies agent i's tastes for the public good. The purpose of a mechanism is to choose an optimal level of the public good in this framework of imperfect information. More formally, a *mechanism* is a mapping, $f(\cdot)$, from the "*strategy spaces*"[5] $\Theta = \Pi_{i=1}^n \Theta_i$ into $\mathring{R}_+ \times R^n$, composed of a decision function, $d(\cdot)$ from Θ into $\mathring{R}_+$, and of an n-tuple of transfer functions $t(\cdot) = [t_1(\cdot), \dots, t_n(\cdot 4)]$ from Θ into R. $d(\cdot)$ associates to any n-tuple θ of announced parameters a quantity $d(\theta)$ of public good, while $t_i(\theta)$ is a transfer of the private good to agent i, $i = 1, \dots, n$. The mechanism is said to be *continuously differentiable* or C^1 when the function $f(\cdot)$ is continuously differentiable.

Since the endowments of the private good play no role in the paper we will write the utility function of an agent i faced with a mechanism $(d(\cdot), t(\cdot))$ as:

$$v_i(d(\theta), \hat{\theta}_i) + t_i(\theta) \qquad (i = 1, \dots, n).$$

A mechanism $f(\cdot) = [d(\cdot), t(\cdot)]$ is said to be *strongly individually incentive compatible* (s.i.i.c), if the truth is a dominant strategy for each consumer; that is, if for any i, any $\theta \in \Theta$[6]

$$v_i(d(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i) + t_i(\hat{\theta}_i, \theta_{-i})$$
$$\geqslant v_i(d(\theta_i, \theta_{-i}), \hat{\theta}_i) + t_i(\theta_i, \theta_{-i}).$$

ASSUMPTION 1: For $i = 1, \dots, n$, let Θ_i be an open interval in R and $v_i : \mathring{R}_+ \times \Theta_i \to R$ be a continuously differentiable function such that for any $\theta \in \Theta = \Pi_{i=1}^n \Theta_i$, there exists $K^*(\theta) \in \mathring{R}_+$ for which (i) $\Sigma_{i=1}^n v_i(K^*(\theta), \theta_i) = \max_{K>0} \Sigma_{i=1}^n v_i(K, \theta_i)$, (ii $K^*(\theta)$ is continuously differentiable.[7]

[4] Generalizations to multidimensional project spaces are straightforward.

[5] Because only agent i may know $\hat{\theta}_i$, he is of course not constrained to reveal his true parameter.

[6] $\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_n)$ and $\theta = (\theta_i, \theta_{-i})$.

[7] Several possible alternative sets of postulates on the $v(\cdot)$ functions imply (i) and (ii). For example, (ii) is obtained with $v(\cdot, \theta)$ strictly concave in K (as a consequence of the implicit function theorem). To infer (i) one may assume that, for any θ in an open interval of R, there exists $\bar{K}_1(\theta)$, $\bar{K}_2(\theta)$ such that

$$0 < \bar{K}_1(\theta) < \bar{K}_2(\theta) \qquad \text{and}$$

$$\frac{\partial v_i}{\partial K}(K, \theta) < 0 \qquad \text{for any} \quad K \geqslant \bar{K}_2(\theta),$$

$$\frac{\partial v_i}{\partial K}(K, \vartheta) > 0 \qquad \text{for any} \quad K \leqslant \bar{K}_1(\theta).$$

Under Assumption 1, strong individual incentive compatibility implies

$$\frac{\partial v_i}{\partial K}(d(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)\frac{\partial d}{\partial \theta_i}(\hat{\theta}_i, \theta_{-i}) + \frac{\partial t_i}{\partial \theta_i}(\hat{\theta}_i, \theta_{-i}) = 0.$$

Since we require this equality for any $\hat{\theta}_i$ in Θ_i we deduce the identity:

$$(1) \qquad \frac{\partial t_i}{\partial \theta_i}(\theta) \equiv -\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d}{\partial \theta_i}(\theta).$$

A mechanism $f(\cdot) = [d(\cdot), t(\cdot)]$ attains *success* if

$$\sum_{i=1}^{n} v_i(d(\hat{\theta}), \hat{\theta}_i) = \sum_{i=1}^{n} v_i(K^*(\hat{\theta}), \hat{\theta}_i),$$ [8]

implying, with differentiability,

$$(2) \qquad \sum_{i=1}^{n} \frac{\partial v_i}{\partial K}(d(\theta), \theta_i) \equiv 0.$$

Implicitly differentiating (2) yields

$$(3) \qquad \frac{\partial d(\theta)}{\partial \theta_i} = \frac{\partial^2 v_i/\partial K \partial \theta_i}{\sum_{j=1}^{n} (\partial^2 v_j/\partial K^2)}, \qquad \text{if defined.}$$

Finally, a mechanism is said to be *satisfactory* if it is both successful and s.i.i.c. Clearly, (1) and (2) apply to a satisfactory mechanism.

3. CHARACTERIZATION OF SATISFACTORY CONTINUOUSLY DIFFERENTIABLE MECHANISMS

We begin with a characterization theorem of all satisfactory C^1-mechanisms for a given family of admissible valuation functions.

$$V = \{v_1(\cdot, \theta_1), \ldots, v_n(\cdot, \theta_n)/\theta \in \Theta\}.$$

THEOREM 3.1.: *Let V be an admissible family which satisfies Assumption 1. Then (a) there exist satisfactory C^1-mechanisms, (b) a C^1-mechanism $(d(\cdot), t(\cdot))$ is satisfactory if and only if*

$$(4) \qquad \sum_{i=1}^{n} v_i(d(\hat{\theta}), \hat{\theta}_i) = \sum_{i=1}^{n} v_i(K^*(\hat{\theta}), \hat{\theta}_i) \quad \textit{for all} \quad \hat{\theta} \in \Theta$$

and

$$(5) \qquad t_i(\theta) = \sum_{j \neq i} v_j(d(\theta), \theta_j) + h_i(\theta_{-i}) \qquad (i = 1, \ldots, n)$$

where $h_i(\cdot)$ is an arbitrary C^1-function from $\Pi_{j \neq i}\Theta_j$ into R.

[8] Without a lower bound on the consumption of the private good, this is a necessary condition for Pareto optimality.

PROOF: That a C^1-mechanism $(d(\cdot), t(\cdot))$ satisfying (4) and (5) is satisfactory can be immediately verified, establishing the sufficiency of (b).

Choose $d(\cdot)$ which is C^2 and satisfies (4). The C^1-mechanism $(d(\cdot), t^*(\cdot))$, where for all i, $t_i^*(\theta) = \Sigma_{j\neq i}(d(\theta), \theta_j)$, satisfies (5) and therefore is satisfactory, establishing (a).

Finally, consider a satisfactory C^1-mechanism $(d(\cdot), t(\cdot))$. From s.i.i.c. of $(d(\cdot), t(\cdot))$ and $(d(\cdot), t^*(\cdot))$ we have

$$\text{(6)} \qquad \frac{\partial t_i}{\partial \theta_i}(\theta_i, \theta_{-i}) \equiv -\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_i} \equiv \frac{\partial t_i^*}{\partial \theta_i}(\theta_i, \theta_{-i}).$$

Integrating (6) yields

$$t_i(\theta_i, \theta_{-i}) = t_i^*(\theta_i, \theta_{-i}) + h_i(\theta_{-i}) \qquad (i = 1, \ldots, n).$$

Q.E.D.

The mechanisms satisfying (4) and (5) have been called Groves mechanisms in Green and Laffont [**5**] (see Groves [**6**]). In Green and Laffont [**5**], a similar characterization is given in the discrete and continuous cases. In the continuous case, no restriction beyond continuity is imposed on the admissible family of valuation functions.

Theorem 3.1 shows that if one restricts the admissible to the class of differentiable functions indexed by a single parameter θ in an open interval, no satisfactory mechanisms beyond the Groves class can be found. The theorem applies, for example, to the family of quadratic functions

$$V(K, \theta_i) = \theta_i K - K^2/2,$$

where θ_i belongs to a given open interval of R, $i = 1, \ldots, n$. The theorem strengthens Theorem 3 in Green and Laffont [**5**].

The second principal merit of the approach taken here is its constructive character, which, given an admissible family of valuation functions, enables us to construct the transfer functions explicitly. This feature is particularly useful in the sections to follow where additional constraints such as balance or coalition incentive compatibility[9] are imposed on the mechanisms.

To illustrate this constructive character, let us consider the quadratic case and obtain the associated transfer functions. We have

$$d(\theta) = \frac{\sum_{i=1}^{n} \theta_i}{n} \quad \text{and} \quad \frac{\partial d(\theta)}{\partial \theta_i} = \frac{1}{n}$$

[9] The approach taken here sheds some light on the mathematical origin of the arbitrary functions $h_i(\cdot)$ which here become simply constants of integration.

and the differential equation (1) becomes:

$$\frac{\partial t_i}{\partial \theta_i}(\theta) = -\frac{1}{n}\left(\theta_i - \frac{\sum_{j=1}^{n} \theta_j}{n}\right) = -\frac{1}{n}\left(1-\frac{1}{n}\right)\theta_i + \frac{1}{n^2}\sum_{j\neq i}\theta_j.$$

Hence,

$$t_i(\theta_i, \theta_{-i}) = \frac{1}{2n}\left(1-\frac{1}{n}\right)\theta_i^2 + \frac{1}{n^2}\left(\sum_{j\neq i}\theta_j\right)\theta_i + h_i(\theta_{-i}).$$

It is then merely a matter of calculation to check that this expression differs from

$$\sum_{j\neq i} v_j(d(\theta), \theta_j) = \sum_{j\neq i}\left[\theta_i\left(\frac{\sum_{j=1}^{n}\theta_j}{n}\right) - \frac{1}{2n^2}\left(\sum_{j\neq i}\theta_j\right)^2\right]$$

by a function of θ_{-i}.

One might believe that the smaller the admissible set, the stronger the characterization theorem; when agents have less opportunity for misrepresenting their preferences, there are ostensibly fewer constraints on potential mechanisms. As a referee pointed out to us, however, this intuitive argument is incorrect in general because as the admissible set shrinks, the set of possible mechanisms itself shrinks. Indeed, it is possible (as the referee has shown) to give examples of domains V_1 and V_2, with V_2 a proper subset of V_1, such that there exist successful, non-Groves mechanisms for V_2 but not for V_1. Of course, as Theorem 3.1 demonstrates, such examples are impossible if V_2 consists of differentiable functions indexed by a parameter whose domain is an interval. Independent of our work, Holmstrom [**8**] established the more general result that every successful mechanism on a smoothly connected admissible set is in the Groves class. Therefore, his theorem generalizes Theorem 3.1, although his proof is nonconstructive.

4. BALANCED MECHANISMS

A well known deficiency of Groves mechanisms is the fact that they are not in general balanced; that is the sum of the prescribed transfers, $\Sigma_{i=1}^{n} t_i(\theta)$, is not identically zero[10] over the range of parameters $\theta \in \Theta$.

By restricting the space of admissible functions one might hope to be able to obtain balance through an appropriate choice of the arbitrary functions $h_i(\cdot)$ in the Groves mechanisms. Indeed, Groves and Loeb [**7**] have shown that, for $n \geq 3$, there exists a balanced satisfactory mechanism for the quadratic family, $V_Q = \{\theta_1 K - (K^2/2), \ldots, \theta_n K - (K^2/2), \theta \in \Theta\}$.

First, we give a necessary and sufficient condition for an admissible family of valuation functions to admit a balanced satisfactory mechanism.

[10] See Green and Laffont [**3**], Hurwicz [**9**], Walker [**12**] for different proofs of the nonexistence of balanced Groves mechanisms.

THEOREM 4.1: *Under Assumption* 1 *there exists a balanced satisfactory C^n-mechanism for the class of admissible C^n-functions*

$$\mathcal{V}=\{v_1(\cdot,\theta_1),\dots,v_n(\cdot,\theta_n)/\theta\in\Theta\}$$

if and only if

$$\sum_{i=1}^{n}\frac{\partial^{n-1}}{\partial\theta_{-i}}\left[\frac{\partial v_i}{\partial K}\cdot\frac{\partial d}{\partial\theta_i}\right]\equiv 0^{11} \qquad \textit{for some } d(\cdot) \textit{ satisfying } (4). \tag{7}$$

PROOF: Suppose there exists a balanced satisfactory C^n-mechanism $(d(\cdot), t(\cdot))$ with respect to $\mathcal{V}$. Then,

$$\sum_{i=1}^{n} t_i(\theta)\equiv 0$$

or, from (5),

$$\sum_{i=1}^{n}\left[-\int\frac{\partial v_i}{\partial K}\cdot\frac{\partial d}{\partial\theta_i}\,d\theta_i+h_i(\theta_{-1})\right]\equiv 0. \tag{8}$$

Differentiating (8) with respect to $\theta_1,\dots,\theta_n$, we obtain

$$\sum_{i=1}^{n}\frac{\partial^{n-1}}{\partial\theta_{-i}}\cdot\frac{\partial v_i}{\partial K}\cdot\frac{\partial d(\theta)}{\partial\theta_i}\equiv 0 \tag{9}$$

which establishes necessity.

Reintegrating (9) successively with respect to $\theta_1,\dots,\theta_n$ regenerates (8). Hence sufficiency. *Q.E.D.*

The main interest of Theorem 4.1 is to permit a direct check of possibility of balance for a given admissible family without actually attempting to construct the transfers.

We next prove an impossibility theorem in the case of $n=2$, with an assumption of sufficient richness of the admissible class.

We first observe that there are rather trivial but large admissible families with respect to which balance is possible. For example, choose $\bar{K}\in\mathring{R}_+$, let $\bar{\Theta}=\Theta_1=\Theta_2=\cdots=\Theta_n$, and take

$$\mathcal{F}=\{v(\cdot)|v:\mathring{R}_+\times\bar{\Theta}\to R,\ v\text{ is }C^1,\quad\text{and}\quad v(\bar{K},\bar{\theta})=\max_{K\in\mathring{R}_+}v(K,\bar{\theta})\quad\text{for all }\bar{\theta}\in\bar{\Theta}\}.$$

[11] The operator $\partial^{n-1}/\partial\theta_{-i}$ is defined as $\partial^{n-1}/\partial\theta_1\dots\partial\theta_{i-1}\partial\theta_{i+1}\dots\partial\theta_n$.

Let

$$V_T = \{v_1(\cdot, \theta_1), \ldots, v_n(\cdot, \theta_n) | \theta_i \in \bar{\Theta}, \quad v_i \in \mathscr{F}\}.$$

Then the mechanism which always chooses $\bar{K}$ and makes no transfers is obviously satisfactory and balanced for any number of agents. To rule out such degeneracies, we must invoke a requirement of richness for our admissible family.

ASSUMPTION 2: $\mathscr{V}$ is such that (a) for all $C^1 d(\theta)$ satisfying (4) $(\partial d(\theta)/\partial \theta_i) \neq 0$ for each i and each θ; (b) for all $\theta \in \Theta$, $\Sigma(\partial^2 v_i(d(\theta), \theta)/\partial K^2) \neq 0$ for all $d(\cdot)$ satisfying (4).

Intuitively, Assumption 2(a) says that an agent's strategy space is sufficiently varied so that by slightly changing his announced parameter he can always affect the chosen supply of the public good. Assumption 2(b) is made only to avoid the unlikely event that the sum of the second derivatives vanishes along the optimal $d(\theta)$ locus. For strictly concave functions such an event is generically impossible.

COROLLARY 4.1: *Under Assumptions 1 and 2 there exists no balanced satisfactory C^2-mechanism for $n = 2$.*

PROOF: A simple application of Theorem 4.2. A direct proof can also be given using Young's theorem.[12] *Q.E.D.*

As a further example of the use of Theorem 4.1. we show:

COROLLARY 4.2: *For $n \geq 3$, (a) there exists a balanced satisfactory C^n-mechanism for the quadratic class; (b) there exists no balanced satisfactory C^n-mechanism for the admissible class*

$$\mathscr{V}_R = \{v(\cdot, \theta_i) = K - \theta_i K^2, \quad \theta_i \in \Theta_i, \quad i = 1, \ldots, n\}.$$[13]

[12] Young's theorem says that for a twice differentiable function $f(x, y)$,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

[13] The reader can also verify the nonexistence of balanced Groves mechanisms for the admissible classes:

$$v_1 = \{\theta_i K - \log K, \theta_i \in \Theta_i\}, \qquad v_2 = \{\theta_i \log K - K, \theta_i \in \Theta_i\},$$

$$v_3 = \{\theta_i e^{-K} + K, \theta_i \in \Theta_i\}, \qquad v_4 = \{\theta_i \log K - K^2, \theta_i \in \Theta_i\},$$

$$v_5 = \left\{\theta_i K - \frac{K^3}{3}, \theta_i \in \Theta_i\right\}, \quad \text{etc.}$$

Our conjecture is that the quadratic case is the only nontrivial case which permits balance.

PROOF: (a) In the quadratic case:

$$\frac{\partial v_i}{\partial K}\cdot\frac{\partial d(\theta)}{\partial \theta_i}=-\frac{1}{n}\left(1-\frac{1}{n}\right)\theta_i+\frac{1}{n^2}\sum_{j\neq i}\theta_j.$$

Clearly, for $n>2$

$$\frac{\partial^{n-1}}{\partial \theta_{-i}}\frac{\partial v_i}{\partial K}\cdot\frac{\partial d(\theta)}{\partial \theta_i}\equiv 0 \qquad \text{for any } i. \tag{10}$$

Hence the result by Theorem 4.1.

Observe that, for $n=2$, (10) is not satisfied (cf. Corollary 4.1).

(b) In the case of $\mathscr{V}_R$:

$$\frac{\partial v_i}{\partial K}\cdot\frac{\partial d(\theta)}{\partial \theta_i}=-\left(1-\frac{\theta_i n}{\sum_{j=1}^{n}\theta_j}\right)\frac{n}{2}\left(\sum_{j=1}^{n}\theta_j\right)^{-2}.$$

So

$$\sum_{i=1}^{n}\frac{\partial^{n-1}}{\partial \theta_{-i}}\left(\frac{\partial v_i}{\partial K}\cdot\frac{\partial (d(\theta))}{\partial \theta_i}\right)=\frac{(-1)^n\cdot n\cdot n!}{2}\left(\sum_{j=1}^{n}\theta_j\right)^{-(n+1)}\not\equiv 0. \qquad \textit{Q.E.D.}$$

5. COALITION INCENTIVE COMPATIBILITY

The lack of robustness of Groves mechanisms with respect to manipulations by coalitions is a well known fact. Green and Laffont [4] prove that, in the case of a $\{0, 1\}$ project space, no coalition incentive compatible dominant strategy mechanism exists even if one restricts allowable coalitions[14] to a single coalition of any size, when no restriction is imposed on preferences.

In this section we give a similar impossibility result for differentiable satisfactory mechanisms and then give a necessary condition for coalition incentive compatibility for a restricted class of coalitions.

A mechanism $(d(\theta), t(\theta))$ is said to be *strongly coalitionally incentive compatible* (s.c.i.c.) for a class $\mathscr{C}$ of coalitions with respect to an admissible class of valuation functions

$$\mathscr{V}=\{v_1(\cdot\,,\theta_1),\ldots,v_n(\cdot\,,\theta_n)/\theta\in\Theta\}$$

[14] The approach taken in Green and Laffont [4] is then to show that, for small coalitions, the probability of a large per capita gain through misrepresentation of preferences is as small as desired for a large economy. Therefore, if the formation of coalitions is costly, small coalitions will not form. Large coalitions can be excluded on other grounds, like the free rider problem of their own. Bennett and Conn [1] prove that coalition incentive compatibility breaks down if coalitions of size two are allowed.

iff: $\forall C \in \mathscr{C}$, $\forall(\theta_C, \theta_{-C}) \in \Theta$,[15] $\forall \hat{\theta}_C \in \Pi_{i \in C}\Theta_i$,

$$\sum_{i \in C}[v_i(d(\hat{\theta}_C, \theta_{-C}), \hat{\theta}_i) + t_i(\hat{\theta}_C, \theta_{-C})] \geqslant \sum_{i \in C}[v_i(d(\theta_C, \theta_{-C}), \hat{\theta}_i) + t_i(\theta_C, \theta_{-C}). \tag{11}$$

Under Assumption 1 and with the same reasoning as in Section 2, these conditions can be translated into a system of differential equations:

$$\sum_{i \in C}\left(\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_j} + \frac{\partial t_i(\theta)}{\partial \theta_j}\right) \equiv 0 \tag{12}$$

for all $j \in C$, and for all C in $\mathscr{C}$.

Let us first observe that a satisfactory mechanism is s.c.i.c. for the universal coalition if and only if it is balanced. Suppose we have a balanced mechanism; then,

$$\sum_{i=1}^{n} \frac{\partial t_i}{\partial \theta_j}(\theta) \equiv 0 \qquad j = 1, \ldots, n. \tag{13}$$

Success and (13) imply (12).

Reciprocally, (12) and success imply (13); therefore $\Sigma_{i=1}^{n} t_i(\theta)$ is a constant which can be chosen to be zero.

We first show that under the assumption of sufficient richness of the admissible family, no s.c.i.c. C^2-mechanism exists when all coalitions are allowed.

THEOREM 5.1: *Under Assumptions* 1 *and* 2, *if the valuation functions are* C^2, *there exists no s.c.i.c.* C^2*-mechanism for the class of all coalitions.*

PROOF: The existence question amounts to the possibility of integrating the system (12) when all subsets of $\{1, \ldots, n\}$ are allowed as potential coalitions.

The mechanism is in particular s.c.i.c. for a size 2 coalition; we have for a pair (i, j):

$$\frac{\partial t_i}{\partial \theta_i}(\theta) \equiv -\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_i}; \tag{14}$$

$$\frac{\partial t_j}{\partial \theta_j}(\theta) \equiv -\frac{\partial v_j}{\partial K}(d(\theta), \theta_j)\frac{\partial d(\theta)}{\partial \theta_j};$$

$$\frac{\partial t_i}{\partial \theta_j}(\theta) + \frac{\partial t_j}{\partial \theta_j}(\theta) \equiv -\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_j} - \frac{\partial v_j}{\partial K}(d(\theta), \theta_j)\frac{\partial d(\theta)}{\partial \theta_j}. \tag{15}$$

[15] By analogy, with previous notation:

$\theta_C = \{(\theta_i), i \in C\}$, $\theta_{-C} = \{(\theta_i), i \notin C\}$, $\theta = (\theta_C, \theta_{-C})$.

Equations (14) and (15) give:

$$\frac{\partial t_i}{\partial \theta_j}(\theta) \equiv -\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_j}. \tag{16}$$

From the Young theorem, we must have

$$\frac{\partial}{\partial \theta_i}\left(\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_j}\right) \equiv \frac{\partial}{\partial \theta_j}\left(\frac{\partial v_i}{\partial K}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_i}\right) \tag{17}$$

which reduces to

$$\frac{\partial^2 v_i}{\partial K \partial \theta_i}(d(\theta), \theta_i)\frac{\partial d(\theta)}{\partial \theta_j} \equiv 0. \tag{18}$$

Substituting (3) into (18) we obtain

$$\frac{\partial d(\theta)}{\partial \theta_i} \cdot \frac{\partial d(\theta)}{\partial \theta_j} \sum_{l=1}^{n} \frac{\partial^2 v_l}{\partial K^2}(d(\theta), \theta_l) \equiv 0$$

which is impossible from Assumption 2. *Q.E.D.*

It is clear from the above argument that we do not need all coalitions to obtain nonexistence. In fact, the proof requires only one coalition of size two. The last question we turn to is then, whether we can characterize the classes of coalitions which admit s.c.i.c. mechanisms.

We shall establish a simple general result of which our impossibility theorem, using coalitions of size two, is a special case.

To do this, we must first define what we shall call the *differential incidence matrix* for a class of coalitions $\mathscr{C}$. Divide $\mathscr{C}$ into subclasses $\mathscr{C}_1, \ldots, \mathscr{C}_n$ where $\mathscr{C}_i$ consists of all coalitions in $\mathscr{C}$ containing player i. Obviously, the $\mathscr{C}_i$'s are not in general disjoint. Write $\mathscr{C}_i = \{C_{i1}, \ldots, C_{im_i}\}$, where m_i is the cardinality of $\mathscr{C}_i$. Construct a $\sum_{i=1}^{n} m_i \times n^2$ matrix A so that the entry of row $\sum_{i=1}^{s} m_i + t$ and column $rn + q$ is 1 if individual q is in coalition $C_{s+1,t}$ and $r = s$ and 0 otherwise, where $r, s = 0, 1, \ldots, n-1, t = 1, \ldots, m_{s+1}$, and $q = 1, \ldots, n$. For example, suppose $n = 4$ and $\mathscr{C} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 2, 4\}, \{2, 3\}\}$.

Then we can write

$$\mathscr{C}_1 = \{\{1\}, \{1, 3\}, \{1, 2, 4\}\},$$

$$\mathscr{C}_2 = \{\{2\}, \{1, 2, 4\}, \{2, 3\}\},$$

$$\mathscr{C}_3 = \{\{3\}, \{1, 3\}, \{2, 3\}\}$$

$$\mathscr{C}_4 = \{\{4\}, \{1, 2, 4\}\}.$$

So

$$
A = \begin{array}{ll}
 & \begin{array}{cccc} \overbrace{}^{\theta_1} & \overbrace{}^{\theta_2} & \overbrace{}^{\theta_3} & \overbrace{}^{\theta_4} \end{array} \\
\mathscr{C}_1 \begin{cases} \{1\} \\ \{1,3\} \\ \{1,2,4\} \end{cases} & \begin{array}{cccc} 1\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 \\ 1\;0\;1\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 \\ 1\;1\;0\;1 & 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 \end{array} \\
\mathscr{C}_2 \begin{cases} \{2\} \\ \{1,2,4\} \\ \{2,3\} \end{cases} & \begin{array}{cccc} 0\;0\;0\;0 & 0\;1\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 \\ 0\;0\;0\;0 & 1\;1\;0\;1 & 0\;0\;0\;0 & 0\;0\;0\;0 \\ 0\;0\;0\;0 & 0\;1\;1\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 \end{array} \\
\mathscr{C}_3 \begin{cases} \{3\} \\ \{1,3\} \\ \{2,3\} \end{cases} & \begin{array}{cccc} 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;1\;0 & 0\;0\;0\;0 \\ 0\;0\;0\;0 & 0\;0\;0\;0 & 1\;0\;1\;0 & 0\;0\;0\;0 \\ 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;1\;1\;0 & 0\;0\;0\;0 \end{array} \\
\mathscr{C}_4 \begin{cases} \{4\} \\ \{1,2,4\} \end{cases} & \begin{array}{cccc} 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;1 \\ 0\;0\;0\;0 & 0\;0\;0\;0 & 0\;0\;0\;0 & 1\;1\;0\;1 \end{array}
\end{array} \tag{19}
$$

Observe that we have labelled blocks of four columns successively θ_1, θ_2, θ_3, and θ_4. This is because A is the matrix of coefficients of the system of differential equations which characterize a mechanism that is s.c.i.c. for the class $\mathscr{C}$. In particular, for this 4 person example, the equations (12) can be written as:

$$At = -Av \tag{20}$$

where

$$v' = (v_{11}, v_{21}, v_{31}, v_{41}, v_{12}, v_{22}, v_{32}, v_{42}, v_{13}, v_{23}, v_{33}, v_{43}, v_{14}, v_{24}, v_{34}, v_{44}),$$

$$v_{ij} = \frac{\partial v_i}{\partial K} \cdot \frac{\partial \bar{K}}{\partial \theta_j},$$

$$t' =$$

$$\left(\frac{\partial t_1}{\partial \theta_1}, \frac{\partial t_2}{\partial \theta_1}, \frac{\partial t_3}{\partial \theta_1}, \frac{\partial t_4}{\partial \theta_1}, \frac{\partial t_1}{\partial \theta_2}, \frac{\partial t_2}{\partial \theta_2}, \frac{\partial t_3}{\partial \theta_2}, \frac{\partial t_4}{\partial \theta_2}, \frac{\partial t_1}{\partial \theta_3}, \frac{\partial t_2}{\partial \theta_3}, \frac{\partial t_3}{\partial \theta_3}, \frac{\partial t_4}{\partial \theta_3}, \frac{\partial t_1}{\partial \theta_4}, \frac{\partial t_2}{\partial \theta_4}, \frac{\partial t_3}{\partial \theta_4}, \frac{\partial t_4}{\partial \theta_4}\right).$$

Observe that, in our proof of the nonexistence of a satisfactory mechanism which is s.c.i.c. for a class $\mathscr{C}$ containing a coalition of cardinality two, the key to the argument was the demonstration that, for some $i \neq j$,

$$\frac{\partial t_i}{\partial \theta_j} = -\frac{\partial v_i}{\partial K} \frac{\partial d(\theta)}{\partial \theta_j}.$$

This idea is the basis of the following theorem which provides a necessary condition for coalitional incentive compatibility.[16]

[16] It should be possible to obtain necessary and sufficient conditions along the lines of the Frobenius theorem.

THEOREM 5.2: *Suppose the admissible family V satisfies Assumptions 1 and 2. If $(d(\cdot), t(\cdot))$ is a satisfactory mechanism with respect to V and is s.c.i.c. for $\mathscr{C}$, then if e_{rn+s} is the $(rn+s)$th unit vector in R^{n^2} $(r=0,1,\ldots,n-1,\ s=1,\ldots,n)$ and $s\neq r+1$, there does not exist $x\in R^{\Sigma_{i=1}^n m_i}$*[17] *such that $x'A=e'_{\cdot rn+t}$*

REMARK: There are, of course, many equivalent ways of stating this theorem. Among them is the statement that the null space (i.e., the kernel) of A is orthogonal to all unit vectors e_{rn+s}, where $s\neq r+1$.

PROOF: Let $(d(\cdot), t(\cdot))$ be as hypothesized and suppose that there exists x such that $x'A=e'_{rn+s}$ for some $s\neq r+1$. Rewriting equations (12) as $At=-Av$ by analogy with our 4 person example above, and premultiplying by x gives

$$(21)\qquad -\frac{\partial t_s}{\partial\theta_{r+1}}(\theta)=x'At=-x'Av$$
$$=-\frac{\partial v_s}{\partial K}(d(\theta),\theta_s)\frac{\partial d(\theta)}{\partial\theta_{r+1}}.$$

But (21) leads to a contradiction as shown in the proof of Theorem 5.1 above. *Q.E.D.*

Theorem 5.2 can be used to derive several corollaries. The first makes use of the special separable form of A.

COROLLARY 5.1: *If V satisfies Assumptions 1 and 2 and the rows of A corresponding to $\mathscr{C}_i$ have rank n for some $i=1,\ldots,n$, then there exists no satisfactory mechanism with respect to V which is s.c.i.c. for $\mathscr{C}$.*

COROLLARY 5.2: *If V satisfies Assumptions 1 and 2 and $\mathscr{C}$ consists of all coalitions of cardinality m for $2\leqslant m\leqslant n-1$, then there does not exist a satisfactory mechanism with respect to V which is s.c.i.c. for $\mathscr{C}$.*

COROLLARY 5.3: *Let V satisfy Assumptions 1 and 2 and suppose $\mathscr{C}$ contains two coalitions C_2 and C_1 such that $C_2=C_1\cup\{i\}$ for some i. Then there exists no satisfactory mechanism with respect to V which is s.c.i.c. for $\mathscr{C}$.*

Note that Corollary 5.3 implies Theorem 5.1.

We conclude by the construction of an example of mechanism which is s.c.i.c for a restricted class of coalitions.

Consider the case

$$v_i(K,\theta_i)=\theta_iK-\frac{K^2}{2}\qquad (i=1,2,3,4)$$

with $\mathscr{C}=\{1\},\{2\},\{3\},\{4\},\{1,2,3\}$.

[17] This necessary condition can be easily tested by a variety of well known methods including the simplex algorithm.

From Section 3 we know that coalitions {1}, {2}, {3}, {4} impose

$$t_i(\theta) = -\frac{\theta_i^2}{2}\frac{(n-1)}{n} + \frac{\theta_i}{n^2}\sum_{j\neq i}\theta_j + h_i(\theta_{-1}) \qquad (i=1,2,3,4)$$

where $n=4$,

Incentive compatibility for coalition {1, 2, 3} requires

$$\sum_{i=1}^{3}\frac{\partial t_i}{\partial\theta_i} = -\sum_{i=1}^{3}\frac{\partial v_i}{\partial K}[d(\theta),\theta_i]\frac{\partial d(\theta)}{\partial\theta_j} \qquad (j=1,2,3),$$

which reduces to

$$\frac{\partial h_2}{\partial\theta_1}+\frac{\partial h_3}{\partial\theta_1} = -\frac{(n-1)(\theta_2+\theta_3)}{n^2}+\frac{2\theta_1}{n^2},$$

$$\frac{\partial h_1}{\partial\theta_2}+\frac{\partial h_3}{\partial\theta_2} = -\frac{(n-1)(\theta_1+\theta_3)}{n^2}+\frac{2\theta_2}{n^2},$$

$$\frac{\partial h_1}{\partial\theta_3}+\frac{\partial h_2}{\partial\theta_3} = -\frac{(n-1)(\theta_1+\theta_2)}{n^2}+\frac{2\theta_3}{n^2}.$$

This differential system is satisfied by

$$h_1(\theta_2,\theta_3) = \frac{\theta_2^2+\theta_3^2}{2n^2} - \theta_2\theta_3\frac{(n-1)}{n^2},$$

$$h_2(\theta_1,\theta_3) = \frac{\theta_1^2+\theta_2^2}{2n^2} - \theta_1\theta_3\frac{(n-1)}{n^2},$$

$$h_3(\theta_1,\theta_2) = \frac{\theta_1^2+\theta_3^2}{2n^2} - \theta_1\theta_2\frac{(n-1)}{n^2}.$$

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