A WALRASIAN THEORY OF MONEY AND BARTER*

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We study a barter economy in which each good is produced in two qualities and no trader can distinguish between the qualities of those goods he neither consumes nor produces. We show that in competitive equilibrium there exists a (unique) good—the one for which the discrepancy between qualities is smallest—that serves as the medium of exchange: this good mediates every trade. Equilibrium is inefficient because production of the medium would be lower if it were not for its mediating role. Introducing fiat money enhances welfare by eliminating this distortion. However, high inflation drives traders back to the commodity medium.

I. INTRODUCTION

Money has always been something of an embarrassment to economic theory. Everyone agrees that it is important; indeed, much of macroeconomic policy discussion makes no sense without reference to money. Yet, for the most part theory fails to provide a good account for it. Indeed, in the best developed model of a competitive economy—the Arrow-Debreu [1954] framework—there is no role for money at all. Rather than there being a medium of exchange, prices are quoted in terms of a fictitious unit of account, agents trade at those prices, and that is the end of the story.

One important exception to the rule that money plays no essential part in theory is the overlapping generations consumption-loan model [Samuelson 1958]. In that model, on which there

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is a considerable literature, the introduction of fiat money permits borrowing and lending across generations and so can dramatically alter the nature of equilibrium. But even there money is indispensable only because, by assumption, there are no other durable assets. Once we admit other assets (e.g., land) that survive over time (and so can serve as a store of value), fiat money loses its central purpose.\(^1\)

Of course, we know from everyday experience that money acts not only as an intergenerational store of value but as a medium of exchange. As Jevons [1875] pointed out, it eliminates the need for a "double coincidence of wants." If I have apples but want bananas, then, in a barter economy, I must wait until I can find someone willing to give up bananas for apples in order to trade, and this delay may be costly. By contrast, in a monetized economy the trader who buys my apples need not be the same as the one who sells me bananas, and this decoupling of identities relaxes the constraints on trade.

But there are at least two reasons why this contrast does not completely clinch the case against barter. First, just as in a well-organized economy there is a place where one can go to sell apples and a place where one can buy bananas, we can, in principle, imagine a place where one could go to exchange apples for bananas. If such a place were to exist, there would be no delay in finding trading partners, and so one important argument against barter would collapse. Of course, if similar provisions were made for all pairs of goods, our economy would require \(n(n - 1)/2\) markets (assuming \(n\) goods), instead of the usual \(n\), and one might object that this proliferation of markets would itself be costly. But, second, even if there were only \(n\) markets, we could still ask why, if I want bananas, I cannot simply go to the banana market and pay for bananas with apples. After all, even if the banana-seller does not want the apples herself, she can always sell them again. In other words, barter seems no worse than monetary exchange if apples can serve as media of exchange.\(^2\)

1. In an economy with uncertainty it may still have value in helping to diversify portfolios, but in this respect it is not distinguished from many nonmonetary assets.

2. This position was challenged by Ostroy and Starr [1974], who argued that one cost of barter is the long chain of exchange transactions that must occur before an economy clears. However, their perspective conceives of a barter economy as one without well-organized markets, and regards the cost of transaction as proportional to the length of the chain. If, as we have tried to do, one instead assumes that markets are well organized, and measures cost as proportional to the length of an individual trader's transaction chain, the distinction between barter and
The point of view we take in this paper is that the reason the banana-seller may not accept apples is that she cannot properly evaluate them. That is, if she does not know much about apples, she may not be able to discern the value of the apples she is presented with. There are at least two ways in which we can interpret this failure of discernment. The orange-seller may simply be unable to appreciate the apples' physical characteristics, e.g., perhaps she cannot distinguish between ripe and rotten fruit. Alternatively, she may be ignorant of their market characteristics. That is, she may not know how much she can resell them for later on. (The difficulties that ignorance of market characteristics pose for barter have been emphasized by Friedman [1960]). The banana-seller's ignorance about apples might not matter much if she could be sure that I am as ignorant as she. If we were both approximately risk-neutral, we could set the banana/apple exchange rate in terms of the expected price of apples. But if she suspects that I know more about apples than she does, she may worry that I will take advantage of her. That is (if the reader will excuse the mélange of fruit), she may fear that I will try to stick her with a "lemon" [Akerlof 1971]. By postulating such an asymmetry, we are trying to capture an elementary fact of economic life. As traders we are reasonably familiar with the physical properties and prices of the goods we buy and sell on a regular basis. But for each of us there is a vast array of goods with which we have little experience. Moreover, for different traders the sets of unfamiliar goods are quite different, so that if someone tries to sell us something that we do not know much about, we become wary of being exploited.

From this perspective, the role of money becomes clear. Money is simply a good whose physical characteristics can be reasonably well discerned by every trader, and whose current and future market prices are known to the trader in terms of the goods that he buys and sells frequently. Thus, it is a device for overcoming the adverse selection problem that arises in barter. In other words, in Jevons' phrase, money is identifiable. It is this attribute of money, namely the ability to overcome asymmetric

monetary exchange, we argue, largely vanishes (see the discussion following Proposition 5).

3. The problem confronting the banana-seller would be underscored more dramatically if instead of attempting to use apples as payment I offered, say, an Impressionist painting, for which the difficulty of evaluation would presumably be particularly acute. Hence, an early title for this paper was "A Monet Theory of Money."
information problems, which, according to Alchian [1977], is the principal advantage of monetary exchange over barter.

The model we develop below is an attempt to bring out this point formally. We stay as close to a standard Walrasian model as possible. Thus, we assume that (i) trade is anonymous (so that traders cannot trust one another to keep promises about quality or delivery, nor can such promises be enforced by third parties); (ii) markets are well organized but decentralized (for each good there is a known location where that good can be bought and sold, but there is no central clearinghouse); and (iii) markets are competitive, i.e., each trader takes all prices as given. The major point of departure is that we suppose that each good comes in two qualities—say, high and low—and that, although producers and consumers of a given good can distinguish between its qualities, other traders cannot. Thanks to the lemons problem, these uninformed traders, when presented with this good in a transaction, will naturally presume that it is of low quality. They will, therefore, be willing to pay the low-quality but not the high-quality price for the good. We deduce that a version of Gresham’s law pertains to our model: only low-quality versions of goods are candidates to be media of exchange, since only low-quality goods can be properly priced. This already makes a significant contrast with the Arrow-Debreu setting, where any good can serve as a medium of exchange.

We will show, however, that not all low-quality goods can function as media of exchange. In fact, generically, the medium is unique. Moreover, it corresponds to (the low-quality version of) the good for which the discrepancy (in a well-defined sense) between high and low qualities is smallest.

This finding seems to accord with the evolution of gold and certain other metals—particularly when used for coins—as widespread media of exchange. Historically, two innovations were important to these metals’ success: Archimedes’ specific gravity test and the serrated edge. Both inventions, in effect, reduced variation in unobservable quality: the specific gravity test by making it hard to pass off base metal as gold or silver, and the serrated edge by defeating the practice of “coin-clipping.” Thus, our theoretical finding can be thought of as a formal explanation for the historical prevalence of gold as a medium of exchange.

4. The ban on contracts (third party enforcement of promises) is a stronger assumption than invoked by many Walrasian models. However, it corresponds well to many everyday trades in which money (as opposed to credit) is used.
Another implication of our model is a relation quite similar to the fundamental equation of the classical quantity theory of money:

\[ PQ = MV, \]

where \( P \) is the price level, \( Q \) is output, \( M \) is the money supply, and \( V \) is the velocity of money. It is notable that velocity turns out to be a well-defined and meaningful concept in our framework. As formula (*) suggests, moreover, we can show that if velocity is somehow exogenously increased the equilibrium quantity of money (the good that functions as the medium of exchange) falls. However, in contrast to the standard quantity theory, these shifts are not allocation-neutral in our model. This is because the need for money creates a production distortion. Specifically, in our model a low-quality good is produced and consumed only because it can serve as money. In the absence of an informational asymmetry, it is too poor a substitute for the high-quality good to persist in positive quantities. Thus, an increase in velocity (which means that the same quantity of money circulates more frequently, thereby mediating more exchanges) enables the economy to get by with a lower quantity of the low-quality good, a desirable thing. To return to our historical analogy, gold is hardly the most intrinsically useful commodity. Although it serves an ornamental function and has certain industrial uses, most of its value throughout history has derived from its central monetary role. Thus, one can plausibly argue that, relative to an economy with no informational constraints, too much of it is produced, where the distortionary costs are the resources devoted to prospecting, mining, and refining it.

From this standpoint, the great virtue of fiat money is its capacity to function as money while being essentially costless to produce. That is, unlike gold, fiat money creates no real distortion in the economy. This perspective helps explain why inflation is costly: a major cost is the risk of “demonetizing” the economy—of driving traders back to using gold. We examine these ideas more carefully below.

Our approach to studying the limitations of barter and the value of money is not the only possible one. Following the seminal paper by Kiyotaki and Wright [1989], a sizable literature has developed that uses a search framework5 to address some of the

5. We are using the term “search” to denote a class of decentralized trading models, including some in which there is no active search. The use of such models to study money goes back to Diamond [1984].
questions discussed here. One significant difference between the two approaches is that in a search model such as theirs, where there are significant strategic complementarities, one need not appeal to informational asymmetry for there to be costs of barter. The mere fact that a segment of the population is unwilling to accept good A in exchange for other goods will make every other trader also unwilling to accept good A (because he will have a hard time getting rid of it). As a result, certain types of barter may simply be impossible in the particular equilibrium being played. This kind of coordination failure cannot occur in our framework because of the price-taking assumptions we make. Similarly, the strategic complementarities of the search equilibrium framework mean that there are normally multiple equilibria: a variety of goods can serve as the medium of exchange. In contrast, our Walrasian setup permits a sharper prediction about which good will act as money. Indeed, as we have noted, one of our primary theoretical results (Proposition 1) establishes the (generic) uniqueness of equilibrium (and hence of the medium of exchange).

Besides its weaker predictions the search framework has several other drawbacks from our standpoint. First, although search is indeed important in certain markets (such as the labor market), the markets for many other goods are, and historically have been, relatively well integrated and seem to fit the Walrasian framework better. Second, many of the broad macro/policy implications of these models are quite different from those of a Walrasian model. Therefore, one may run into trouble when basing the theory of money on search frictions while relying on the Walrasian model for one’s other macroeconomic intuitions. Finally, partly because of the inevitable complexities of search models, the early Kiyotaki and Wright papers did not provide a theory of how the money supply affects the price level. Given that this is a central question of monetary policy, the omission seems important. More recent papers in this tradition—Hayashi and Matsui [1996], Shi [1994, 1995], Trejos and Wright [1995], and Green and Zhou [1995], for example—have gone further toward developing a theory of the price level, but they all either rule out barter or rely in important ways on indivisibilities. By contrast, a Walrasian framework permits us to avoid such constraints.


7. For example, in India everyone knows where to go to buy or sell fish.
There is a broader literature on the role of money in settings with imperfect information. Examples of such papers include Jones [1976], King and Plosser [1986], Bernhardt and Engineer [1987], Smith [1986], Townsend [1989], and Williamson [1990]. All these papers concentrate on issues quite different from the ones we treat. Williamson and Wright [1994], which introduces informational asymmetries about the goods traded into a Kiyotaki-Wright model, is the paper that comes closest to our work (see also Cuadras-Morato [1994]). However, like the Kiyotaki and Wright papers, that article focuses on the indeterminacy of the medium of exchange and the possibility of multiple Pareto-ranked equilibria in this kind of environment. While we agree that this indeterminacy is of interest, we also feel that these results are best viewed against a Walrasian benchmark, such as the one provided by our model. In this sense, we view our efforts as complementary.

In Section II we lay out the basic, finite-period model. In Section III we characterize the (essentially) unique equilibrium of this model and show that it entails the existence of a unique medium of exchange. The connection with the quantity theory of money is also drawn. We consider two straightforward extensions in Section IV: a positive discount rate and differential durability of goods. Then, in Section V we embed our finite-period model in an infinite-horizon framework in order to discuss steady states. One virtue of an infinite horizon is that it enables us to explore the role of fiat money. Specifically, in Section IV we consider the welfare effects of money and inflation. We conclude in Section VII with some comments on future directions for research.

II. The Model

II.1. An Informal Description

It may be helpful to begin with an informal description of the physical setting we have in mind. This will enable us to motivate the formal assumptions that follow in subsections II.2 and II.3.

Imagine an economy with no fiat money in which there are many goods and where the markets for different goods are geographically dispersed. Thus, there is an apple region, a banana region, and so on. There is no central clearinghouse where one can go to buy or sell all these goods.

Each trader in the economy produces a single kind of good—apples, bananas, or whatever—and there are many traders producing each kind of good. Every apple-producer divides his
production between high- and low-quality apples (where the mar-
ginal benefit and cost of the former are higher). Just as he is
single-minded about production, every trader consumes only a
single kind of good, but not the same kind as he produces; i.e.,
traders are not self-sufficient.

Because a trader has dual roles as producer and consumer,
it is convenient to think of his having two identities. If he is an
apple-producer, one incarnation stays at home in the apple region
and sells apples in his own shop. (Thus, there will be many small
shops in each region, one for each trader.) The other goes out to
buy the good the trader consumes (say, bananas) or, in principle,
other goods to be resold (for bananas) at a later date (as we will
see, however, this latter sort of arbitrage does not actually occur
in equilibrium).

Trade is bilateral. That is, to buy bananas, a buyer goes to a
banana shop and transacts directly with the banana-seller there.
Just as there is no central clearinghouse for the whole economy,
neither are there clearinghouses within individual regions.

Trade takes place in a finite number of discrete trading peri-
ods. The geographic dispersion of markets means that a trader
can visit only one region per period. However, once in a region he
can visit whichever shop he wishes; this keeps the shops within
the region competitive. Moreover, the fact that he is free to choose
which region to visit in any given period serves to eliminate arbi-
trage opportunities across regions. Hence, although there is no
Walrasian auctioneer, prices will nevertheless be Walrasian.

Exchanges between the buyer and seller in the banana shop
are unmonitorable by third parties. This has two important im-
lications. First, it means that the buyer must pay for the ba-
nanas on the spot. Any kind of credit or deferred payment would
be infeasible because, ultimately, such arrangements rely on a
court or some other authority being able to ascertain whether or
not a particular transaction took place (otherwise, a buyer who
gave a seller an IOU for some bananas could later claim that he
never received them or that the IOU was not his).8 Futures con-
tracts and short sales are ruled out for exactly the same reason.
Second, because there is no fiat money, it means that the buyer
must pay for his bananas with physical goods (either the good

8. If the buyer and seller had an on-going relationship, then conceivably the
unmonitorability of their transactions by others might not be essential. Each
could “punish” the other for any violations of their implicit agreement, as in the
repeated games literature. However, such a “self-enforcing” arrangement nor-
mally requires an indefinite time horizon (otherwise, the scheme would unravel
that he produces himself, or other goods that he has acquired in previous transactions). The fact that traders are constrained in this way to barter transactions is central to the major task of this paper: to show how the institution of “money” emerges endogenously in a barter economy.

We come to the only departure from an otherwise fairly standard competitive framework: the restrictions on traders’ information. A trader who produces apples and consumes bananas can distinguish between high- and low-quality apples and between high- and low-quality bananas (he is informed with respect to apples and bananas) but not between the qualities of any other good. The most important consequence of this shortcoming, in combination with the bilateral trade and unmonitorability assumption, is that he cannot execute any trade involving the high-quality variety of any good other than apples and bananas. This implication is crucial to our results below, and so it is worthwhile to examine why it is so.

Consider a sequence of trades by which the trader comes into possession of high-quality coconuts, which he cannot distinguish from their low-quality counterpart (he is an uninformed trader with respect to coconuts). This can be traced back to an exchange in which some (possibly different) uninformed trader buys high-quality coconuts from an informed trader (since only informed traders produce coconuts). But the latter would be foolish to supply anything but low-quality coconuts if he could get away with it. And the former’s ignorance, together with the absence of monitoring, ensures that the informed trader can get away with it. This is just the “lemons” problem referred to in the introduction.

We conclude therefore that, except for exchanges in which a double coincidence of wants obtains, at least one side of any exchange must involve a low-quality good. Note that the restric-
tion that trade be bilateral (rather than multilateral) figures in this conclusion. If there were a central meeting point where traders of all kinds could come, then in principle it might be possible to organize a multilateral exchange of purely high-quality goods among traders without any double coincidences at all. To see how this could be done even in the absence of outside monitoring, see footnote 20.

II.2. Production, Preferences, and Trade

Let us lay out the model more precisely. Although we have in mind an economy with many goods, for the purpose of formal analysis, we shall suppose that there are just three types of goods, A, B, and C. Each type \( X \in (A,B,C) \) comes in two qualities: \( X_1 \) (high quality) and \( X_2 \) (low quality). Goods are perfectly divisible.

There are also three types of traders, again labeled A, B, and C, according to the type of good they produce. Hence, \( X \)-traders produce only goods of type \( X \). More specifically, an \( X \)-trader is endowed with one unit of labor, which can be applied to a linear production technology: good \( X_1 \) requires two units of labor per unit of output, whereas \( X_2 \) requires one unit of labor per unit.\(^{11}\) An \( X \)-trader can allocate his labor endowment in any way he chooses between \( X_1 \)- and \( X_2 \)-production. We suppose that there are large but finite (and, for symmetry, equal) numbers of each of A-, B-, and C-traders, so that assuming that traders take prices as given makes sense.

Just as production is linear, we suppose that preferences are linear. A-traders consume only goods of type B, B-traders only type C, and C-traders only type A. Notice that this assumption means that, whenever two traders exchange the goods they produce, there cannot be a double coincidence of wants; one of the traders must accept goods that he does not consume. An A-trader's preferences can be represented by the utility function,

\[
k^B b_1 + b_2,
\]

where, for \( i = 1,2 \), \( b_i \) is consumption of good \( B_i \), and \( k^B \) is a scalar coefficient. Analogously, B- and C-traders' preferences are also linear, with coefficients \( k^C \) and \( k^A \), respectively.

\(^{11}\) Except for our discussion of welfare (see subsection III.4), disutility of labor plays no role in the model. To avoid keeping track of it, therefore, let us suppose that it is small enough so that a trader always supplies his entire endowment of labor at equilibrium prices.
We are particularly interested in the production distortions induced by our informational constraints. To highlight these distortions, we shall assume that

\[ k^X > 2 \quad \text{for all } X = A, B, C. \]

That is, the marginal rate of substitution between high- and low-quality goods exceeds the corresponding marginal rate of transformation. Condition (**) means that, in the absence of any informational imperfection, only high-quality goods are efficient to produce. In particular, no low-quality goods will be produced in the perfect-information Walrasian equilibrium. Contrapositively, any low-quality production that occurs once the informational constraints are imposed can be attributed directly to those constraints.

As discussed in subsection II.1, trade is restricted to be bilateral: each exchange involves just two parties. Because exchanges are also unmonitorable by third parties, any sort of credit, short sales, or futures trading is ruled out. We are left only with barter—direct swaps of physical goods—which require no contractual agreement. Let us assume that all barterers consist of one single good being exchanged for another.\(^{12}\)

We suppose that an A-trader can distinguish between A\(_1\) and A\(_2\) (since he produces goods of type A) and between B\(_1\) and B\(_2\) (since he consumes type B goods), but that he cannot distinguish between C\(_1\) and C\(_2\) (which he neither produces nor consumes). Similarly, B- and C-traders cannot distinguish between qualities of type A and type B goods, respectively. As mentioned earlier, this informational restriction together with our other assumptions implies that an A-trader cannot execute any trade involving C\(_1\), a B-trader any trade involving A\(_1\), and a C-trader any trade involving B\(_1\). Let us suppose that traders of the same type never trade with one another.\(^{13}\) Then our bilateral trade and informa-

\(^{12}\) In principle, we could imagine a trader exchanging one good for two others, so that three goods all told would be involved in the trade. However, to simplify matters, we assume for now that each trade entails only two goods. In subsection II.4 we will show that this assumption can be invoked without loss of generality.

\(^{13}\) We can impose this prohibition without loss of generality. To see this, suppose that, say, one A-trader exchanges A\(_2\) for C\(_2\) with another A-trader in equilibrium. Because the two traders are ex ante identical, each would be as well off if this transaction were not made, as long as the former carried out all subsequent transactions that the latter would have made with the A\(_2\), and the latter carried out all subsequent transactions that the former would have carried out with the C\(_2\). Moreover, no other trader would be affected by the fact that the identity of his trading partner may thereby switch from one A-trader to the other. Hence, in the
tional restrictions also imply that any trade must involve at least one low-quality good (For example, the only high-quality that both an A- and a B-trader can distinguish is $B_1$, and so no other high-quality good can be involved in a trade between them.)

Even without the informational restrictions, our assumptions imply that if there is to be any equilibrium trade, the model must have multiple trading periods. In a one-period model an A-trader will wish to exchange the type A goods he produces for type B goods. Given that only B-traders produce B goods and that all trade is bilateral, the exchange must be with a B-trader. But the B-trader will not be happy about receiving type A goods, which he can neither consume nor—in the absence of a subsequent trading period—resell. Thus, the value of an intertemporal trading framework is that it permits reselling.

In our basic model we assume that there are a finite number $T$ of discrete trading periods indexed by $t = 1, \ldots, T$. We take $T$ to be exogenous (but in subsection III.4 explore the implications of different values of $T$). As discussed in subsection II.1, it is conceptually helpful to think of a trader as comprising two individuals: a buyer and a seller. From this perspective, a trader executes (at most) two transactions each period, one in each capacity. In view of the absence of credit, he cannot sell a quantity of any good that was not in his possession at the beginning of the period. For simplicity, we assume that all goods not yet consumed disappear after period $T$.

Until Section IV we will suppose that traders do not discount the future at all. That is, as long as consumption of a given good occurs sometimes within the $T$ trading periods, a trader is indifferent about exactly when it occurs. In Section IV we show that our qualitative findings extend to discounting, provided that the discount rate is not too high.

Although there are no futures markets, we assume that each good is tradable for other contemporaneous goods. As noted
above, traders are price-takers. For any two goods, $X_i$ and $Y_j$, let $p_t(X_i, Y_j)$ be the relative price in period $t$ of $X_i$ in terms of $Y_j$, i.e., how much a $Y_j$ trader must sell in order to buy one unit of $X_i$. (Hence $p_t(X_i, Y_j) = 1/p_t(Y_j, X_i)$.) Notice that by expressing prices in this way, we implicitly assume that they are independent of the quantities traded (i.e., that they are “linear”). In fact, this linearity follows immediately from arbitrage.\(^{16}\)

II.3. Individual Choice

We now formulate a $Y$-trader’s decision problem, for $Y \in \{A, B, C\}$. In each period $t$ he selects a set $\mathcal{T}_t$ of (informationally feasible) bilateral transactions that are executed simultaneously. If we strictly followed the story in subsection II.1, $\mathcal{T}_t$ would consist of (at most) two transactions: one corresponding to the trader’s role as buyer, and one to that as seller. Nothing in the formal analysis, however, depends on $\mathcal{T}_t$ being limited to only two transactions. Each transaction $\tau \in \mathcal{T}_t$ specifies the pair of goods $\mathcal{X}(t) = \{X_i, X_j\}$ that the $Y$-trader exchanges and the quantities exchanged, $q^\tau(X_i)$ and $q^\tau(X_j)$ (where a positive quantity denotes a purchase and a negative quantity a sale). The constraint that $\tau$ be informationally feasible can be expressed as

\[
\begin{align*}
q^\tau(A_i) &= 0 \quad \text{and} \quad q^\tau(B_j)q^\tau(C_i) = 0, \quad \text{if } Y = B \\
q^\tau(B_i) &= 0 \quad \text{and} \quad q^\tau(A_j)q^\tau(C_i) = 0, \quad \text{if } Y = C \\
q^\tau(C_i) &= 0 \quad \text{and} \quad q^\tau(A_i)q^\tau(B_i) = 0, \quad \text{if } Y = A.
\end{align*}
\]

To understand (1), note that a $B$-trader cannot trade $A_i$; hence $q^\tau(A_i) = 0$. Moreover, as noted in subsection II.2, he cannot make a trade involving more than one high-quality good; hence $q^\tau(B_i)q^\tau(C_i) = 0$. The other two lines follow similarly.

Because there is no credit, the net value of each transaction executed must be zero; i.e., for all $\tau \in \mathcal{T}_t$, if $\mathcal{X}(t) = \{X_i, X_j\}$, then

16. Suppose, to the contrary, that there were two different relative prices $p_t(X_i, Y_j)$ and $p'_t(X_i, Y_j)$ in equilibrium, according to whether a trader bought, say, two or three units of $X_i$, respectively. Let $p''_t(X_i, Y_j)$ be the equilibrium price of buying one unit of $X_i$ in terms of $Y_j$. Then either (i) $p''_t(X_i, Y_j) < p_t(X_i, Y_j)$, or (ii) $p''_t(X_i, Y_j) < p'_t(X_i, Y_j)$. But in the former case the trader would be better off buying one unit of $X_i$ in two separate transactions than two units in a single transaction, and in the latter the trader would be better off buying one unit of $X_i$ in three separate transactions than three units in a single transaction. Hence, at least one of the prices $p_t(X_i, Y_j)$ and $p'_t(X_i, Y_j)$ cannot prevail in equilibrium. (We are perhaps belaboring this point because as footnote 18 will illustrate there are some (standard) arbitrage arguments that do not go through in our model.)
Let $e_t(X_i)$ be the amount of good $X_i$ that the $Y$-trader consumes (eats) in period $t$ (of course, if, say, $Y = A$, then $e_t(X_i)$ will be positive only if $X_i$ is $B_1$ or $B_2$), and let $z_t(X_i)$ be the quantity of good $X_i$ that he has at the end of period $t$ (Thus, $z_0(X_i)$ can be interpreted as the quantity of good $X_i$ that the trader produces.) The trader's holdings of good $i$ at the end of period $t$ equal what he began with plus what he acquired (a sale counts as a negative acquisition) minus what he ate. That is, for all $t = 1, \ldots, T$,

$$z_t(X_i) = z_{t-1}(X_i) + \sum_{\tau \in \mathcal{F}_t} q^\tau(X_i) - e_t(X_i).$$

The constraint that the trader can sell only goods in his possession is formalized by the requirement,

$$z_{t-1}(X_i) + \sum_{\tau \in \mathcal{F}_t} q^\tau(X_i) - e_t(X_i) > 0 \quad \text{for all } X_i \text{ and } t,$$

where $\mathcal{F}_t(X_i) = \{ \tau \in \mathcal{F}_t \mid q^\tau(X_i) < 0 \}$. Hence, given prices $\{p_t(\cdot, \cdot)\}_{t=1}^T$, a $Y$-trader chooses a production/trade/consumption plan $\alpha = \{\mathcal{F}_t, e_t(\cdot), z_0(\cdot)\}_{t=1}^T$ to maximize

$$\sum_{t=1}^T (k^{Y+1}e_t((Y + 1)_1) + e_t((Y + 1)_2),$$

(***)

(where $Y + 1 = B$ if $Y = A$, etc.) such that, for all $t$, constraints (1)–(4) hold,

$$e_t(\cdot) \text{ and } z_t(\cdot) \text{ are nonnegative for all } t,$$

and productive feasibility is satisfied:

$$2z_0(Y_1) + z_0(Y_2) \leq 1, \text{ and } z_0(X_i) = 0 \text{ if } X_i \notin \{Y_1, Y_2\}.$$
and

\[(ii) \quad (1 - \lambda)p_t(X_i, X_k) = 1.\]

Note first that \(p_t(X_i, X_j) > 1.\) Otherwise, our trader's partner would be better off selling \(p_t(X_i, X_j')\) units of \(X_j'\) (instead of one unit each of \(X_j'\) and \(X_k'\) for one unit of \(X_j\)), a contradiction of equilibrium. Hence, we can find \(\lambda \in (0, 1)\) satisfying (i). Now if, say, \((1 - \lambda)p_t(X_i, X_k') > 1,\) our trader would be better off than in equilibrium by selling \(\lambda\) units of \(X_i\) for 1 unit of \(X_j'\) and \(1 - \lambda\) units of \(X_i\) for \((1 - \lambda)p_t(X_i, X_k')\) units of \(X_k'.\) Similarly, the trader's partner could find a pair of better-than-equilibrium trades if \((1 - \lambda)p_t(X_i, X_k) < 1.\) Hence, (ii) must hold after all. We conclude that the transaction in which one unit of \(X_i\) is exchanged for one unit each of \(X_j'\) and \(X_k'\) can be thought of as two exchanges: one in which \(\lambda\) units of \(X_i\) are exchanged for 1 unit of \(X_j',\) and the other in which \(1 - \lambda\) units of \(X_i\) are exchanged for 1 of \(X_k'.\)

\[\text{III. Equilibrium}\]

\[\text{III.1. Definition of Equilibrium}\]

In the Walrasian tradition an equilibrium for this model consists of prices \(\{p_t(X_i, Y_j)\}\) for all periods \(t\) and all pairs of goods \((X_i, Y_j)\); and production/trade/consumption plans \(\{\hat{\epsilon}^h(\cdot)\}\), one for each trader \(h,\) such that (i) each trader is optimizing, i.e., if trader \(h\) is of type \(Y,\) then

\[(7) \quad \hat{\epsilon}^h \text{ maximizes } (***) \text{ subject to } (1) - (6);\]

and (ii) all markets clear, i.e., each trader \(h\) can find a trading partner for each of his transactions \(\tau:\)

\[\text{for all } t, \text{ there is a one-to-one correspondence between}\]
\[\text{the set of period } t \text{ transactions } \hat{\mathcal{T}}_t \equiv \bigcup_h \hat{\mathcal{T}}^h_t \text{ and itself in}\]
\[\text{which each transaction } \tau \in \hat{\mathcal{T}}_t \text{ is paired with its}\]
\[\text{complement } \tau^c. 17\]

17. The complement \(\tau^c\) of transaction \(\tau\) is the same transaction with the trading partners' roles reversed. Hence, \(\tau^c\) is defined so that

\[q^{\tau}(X_i) = -q^{\tau^c}(X_i)\]

for all \(X_i \in X(\tau).\)
Because of the no-credit and informational constraints on the feasible set, our model does not exactly fit the conventional Walrasian framework. Indeed, these constraints preclude the existence of equilibrium in the case $T = 1$. To see this, note that in equilibrium all prices must be strictly positive. Otherwise, excess demand will be unbounded. This means that in a one-period model a $B$-trader, say, will be unwilling to buy a positive quantity of $A_2$; he cannot resell the $A_2$; and so he is better off using his $B$-goods to buy $C$-goods. Hence, excess demand by $A$-traders for $B$-goods is necessarily strictly positive (since in a one-period model $A$-traders must obtain $B$-goods from $B$-traders using $A_2$), a violation of equilibrium.

Nevertheless, our model is conventional enough so that for $T \geq 2$ equilibrium does exist. We next exhibit an equilibrium for the case $T = 5$ (see Proposition 6 for a demonstration by construction that equilibrium exists for general $T \geq 5$). We construct this equilibrium so that, in every equilibrium transaction, good $A_2$ is exchanged for some other good $X_i$. Hence, in exhibiting the equilibrium in Table I, we report only the prices $\hat{p}(X_i,A_2)$. For each $X_j \neq A_2$, we take $k_c(X_i,X_j) = \hat{p}(X_i,A_2)/\hat{p}(X_j,A_2)$.\(^{18}\)

### III.2. An Example

Assume that $k^C > k^B > k^A$. We shall exhibit an equilibrium for the case $T = 5$. This equilibrium is symmetric in the sense that all traders of a given type behave identically. Thus, we may speak of a "representative $X$-trader" for $X = A, B, C$.

Define $a_i$ to be the average production of good $A_i$ by an $A$-trader in equilibrium. Similarly, let $b_i$ and $c_i$ be the equilibrium per capita production levels of goods $B$ and $C$, respectively. Take $a_1 = [(k^A/2)^{1/2} - 1]/(k^A - 2)$, $a_2 = 1 - 2a_1$, $b_1 = c_1 = \frac{1}{2}$, and $b_2 = c_2 = 0$. (The logic behind these choices will be given in subsection III.4.) Equilibrium transactions and prices are described by Table I. In each period, the entire produced quantity of $A_2$ (i.e., $a_2$ per capita) is traded. Hence, the table completely describes aggregate

\(^{18}\) In a more standard Walrasian framework this equation would follow automatically from arbitrage. For example, if the left-hand side were greater than the right, a trader interested in selling $X'_j$ to buy $X'_i$ would be better off selling $X'_j$ to buy $A_2$ and using the $A_2$ to buy $X'_i$ than in exchanging $X'_j$ for $X'_i$ directly, and so $\hat{p}(X_i,X_j)$ could not be an equilibrium price. However, in our model the $A_2$ obtained from selling $X'_j$ could not be used to purchase $X'_i$ until the next period, at which point relative prices might have changed. Therefore, the equation need not hold.
TABLE I

<table>
<thead>
<tr>
<th>Periods</th>
<th>Traders</th>
<th>Prices ((p_t(X_t, A_2)))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(A)</td>
<td>(B)</td>
</tr>
<tr>
<td>1</td>
<td>(-A_2, + B_1)</td>
<td>(-B_1, + A_2)</td>
</tr>
<tr>
<td>2</td>
<td>(-A_2, + C_1)</td>
<td>(-C_1, + A_2)</td>
</tr>
<tr>
<td>3</td>
<td>(-A_1, + A_2)</td>
<td>(-A_2, + A_1)</td>
</tr>
<tr>
<td>4</td>
<td>(-A_2, + B_1)</td>
<td>(-B_1, + A_2)</td>
</tr>
<tr>
<td>5</td>
<td>(-A_2, + C_1)</td>
<td>(-C_1, + A_2)</td>
</tr>
</tbody>
</table>

production and trade in equilibrium. That this in fact constitutes an equilibrium can be verified mechanically.

The feature to emphasize first about this example is that there is a unique medium of exchange, namely, good \(A_2\). That is, as already discussed, \(A_2\) is on one side of every transaction. We will show below (Propositions 1–3) that equilibrium in this example is (essentially) unique, so that \(A_2\) in fact necessarily functions as the medium of exchange in the economy. Notice that in the course of five periods good \(A_2\) makes almost two complete cycles through the economy. It begins with the representative \(A\)-trader (who produces it), then moves successively to the \(B\)-, \(C\)-, and \(A\)-, and \(B\)-traders, and finally to the \(C\)-trader (who consumes it). We will see below (Proposition 3) that as the number of trading periods \(T\) increases, the number of cycles that \(A_2\) makes rises correspondingly. Thus, \(T\) can be viewed as a measure of the velocity of money.

Another property to note is that the prices of \(A_1\), \(B_1\), \(C_1\) (in terms of \(A_2\)) increase over time (since \(q < 1\)). The value of \(A_2\) derives from its dual roles as consumption good and medium of exchange. But as the last period (period 5) approaches, the mediating function becomes less and less important because there are fewer future trading opportunities left. Hence a decline in the relative price of \(A_2\) (i.e., an increase in the relative prices of \(A_1\), \(B_1\), and \(C_1\)) is to be expected. This property will also be shown to generalize (Proposition 2).

The final thing to observe is that, relative to the first-best (an economy without informational constraints), equilibrium in this example is inefficient: the fact that a positive quantity of \(A_2\) is produced entails a loss of welfare.
III.3. Two Implications of Arbitrage

As our comments about the above example suggest, equilibrium in the model of Section II is (generically) unique, not in the sense of individuals’ trading patterns (where there is considerable indeterminacy\textsuperscript{19}) but in the aggregate quantities produced, traded, and consumed. It turns out that these aggregate equilibrium quantities are completely determined by two arbitrage relations. Let $T^*$ be the greatest integer less than or equal to $(T + 1)/3$, and for $i = 1,2$, define $a_i$, $b_i$, and $c_i$ to be per capita production of $A_i$, $B_i$, and $C_i$, respectively, as in the example of subsection III.2.

**Lemma 1.** The inequality,

\begin{equation}
\frac{k^X}{2} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{T^*} \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{T^*} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{T^*} \geq 1,
\end{equation}

holds for $X = A, B, C$, in equilibrium.

**Proof.** The argument relies entirely on arbitrage. Consider an $A$-trader. One option he has is to produce one unit of $A_2$ and sell it to a $B$-trader in period 1 in exchange for $B_1$. By doing so, he obtains $p_1(A_2, B_1)$ units of $B_1$. Thus, we infer that

\begin{equation}
p_1(A_2, B_1) \leq \frac{(k^B b_1 + b_2)}{k^B}.
\end{equation}

(The expression $k^B b_1 + b_2$ is the $A$-trader’s equilibrium utility. Thus, if the left-hand side of (11) were greater than the right-hand side, he could obtain higher than equilibrium utility, a contradiction.)

Next consider a $B$-trader. One strategy he could follow would be to produce $1/2$ unit of $B_1$, sell it to an $A$-trader in period 1 for $(1/2) p_1(B_1, A_2)$ units of $A_2$, and then sell the $A_2$ to a $C$-trader in period 2 for $(1/2) p_1(B_1, A_2) p_2(A_2, C_1)$ units of good $C_1$. This would result in utility:

\begin{equation}
\frac{(k^C/2)}{2} p_1(B_1, A_2) p_2(A_2, C_1).
\end{equation}

Following the above logic, we conclude that

\begin{equation}
p_1(B_1, A_2) p_2(A_2, C_1) \leq 2(k^C c_1 + c_2)/k^C.
\end{equation}

\textsuperscript{19} For instance, in the example of subsection III.2, a $B$-trader is indifferent about whether he exchanges $B_1$ for $A_2$ in period 1 or 4, and so any shift on his part between those two periods can be made consistent with equilibrium by correspondingly adjusting the trades of other $B$-traders and their partners.
Similarly, a $C$-trader who sells $1/2$ unit of $C_1$ for $A_2$ in period 2 and then uses this to buy $A_1$ in period 3 obtains utility:

$$\left( \frac{k^A}{2} \right) p_2(C_1, A_2) p_3(A_2, A_1),$$

and so

$$p_2(C_1, A_2) p_3(A_2, A_1) \leq \frac{2(k^A a_1 + a_2)}{k^A}.$$  \hspace{1cm} (13)

Continuing in the same way, we obtain

$$p_3(A_1, A_2) p_4(A_2, B_1) \leq \frac{2(k^B b_1 + b_2)}{k^B}.$$  \hspace{1cm} (14)

Finally, in period $T$, a $C$-trader could sell $1/2$ unit of $C_1$ to a $B$-trader for $(1/2) p_T(C_1, A_2)$ units of $A_2$. Hence,

$$p_T(C_1, A_2) \leq \frac{2(k^A a_1 + a_2)}{k^A}.$$  \hspace{1cm} (15)

Multiplying the expressions in (11)–(15) together, we find that all prices cancel, and we are left with

$$1 \leq \left[ \frac{k^B b_1 + b_2}{k^B} \right]^{T^*_B} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{T^*_C} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{T^*_A}.$$  \hspace{1cm} (16)

Rewriting (16), we obtain (10) when $X = A$. The argument is entirely symmetric when $X = B$ and $X = C$.

QED

The above argument invoking (11)–(16) places an upper bound on the utilities attainable, given equilibrium prices, from feasible (but not necessarily optimal) transactions involving $A_2$, namely, the equilibrium utilities. Now, of course, it may not even be optimal for an $A$-trader to produce $A_2$ in equilibrium, in which case we would expect the inequality in (10) to be strict. However, suppose that producing $A_2$ were optimal; i.e., that the level of $A_2$ were positive in equilibrium. Intuitively, one would anticipate the upper bound to be attained for equilibrium sequences involving
In other words, (10) should hold with equality when \( X = A_2 \). Formally, we have

**Lemma 2.** If, for given \( X \in \{A, B, C\} \), the equilibrium production of \( X_2 \) is positive, then (10) holds with equality for that type of good \( X \).

**Proof.** We present the argument for the case \( T = 2 \) (the general proof is relegated to the Appendix). Suppose that the equilibrium production of \( A_2 \) is positive. Then there must be an exchange in which a \( C \)-trader acquires some \( A_2 \) for consumption. Given the informational constraints, the \( C \)-trader must sell \( A_1, B_2, C_2 \), or \( C_1 \) to obtain this \( A_2 \).

Now, the case in which the \( C \)-trader sells \( A_1 \) can be ruled out easily. Note that \( A_1 \) could be bought only by an \( A \)- or a \( C \)-trader. The latter possibility is eliminated because we have assumed (without loss of generality) that two \( C \)-traders never trade. But the former possibility is also ruled out since the exchange must occur in the second period (how else could the \( C \)-trader have acquired the \( A_1 \) he is selling?), and so the \( A \)-trader must consume the \( A_1 \) he buys, an impossibility.

Next suppose that the \( C \)-trader sells \( B_2 \). This must occur in the second period (since the \( B_2 \) must have been acquired in the first). Hence, the \( B_2 \) must be sold to an \( A \)-trader (since only \( A \)-traders consume \( B_2 \)). The \( A_2 \) that the \( A \)-trader sells must have been produced by him. He could not have acquired it by selling \( A_1 \) in the first period (since that would have entailed trade between two \( A \)-traders). Hence, following the logic of the proof of Lemma 1,

\[
p_2(A_2, B_2) = k^B b_1 + b_2.
\]

Now, the \( C \)-trader must sell \( C_1 \) or \( C_2 \) in the first period to acquire his \( B_2 \). Assume that he sells \( C_1 \) (the argument is very similar in the \( C_2 \) case). Then,

\[
(1/2) p_1(C_1, B_2) p_2(B_2, A_2) = k^A a_1 + a_2.
\]

The \( C_1 \) sold by the \( C \)-trader must be purchased by a \( B \)-trader (since an \( A \)-trader cannot buy \( C_1 \)). Moreover, the \( B \)-trader cannot resell it for \( C_2 \) in period 2, since that would entail trade between two \( B \)-traders (only a \( B \)-trader would buy \( C_1 \) in period 2). Hence,

\[
(1/2) k^C p_1(B_2, C_1) = k^C c_1 + c_2.
\]
Multiplying (17)–(19) together, we obtain
\[
1 = \frac{k^B k^A}{2} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right] \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right] \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right],
\]
and so
\[
(20) \quad 1 > \frac{k^A}{2} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right] \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right] \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right],
\]
which contradicts Lemma 1. A very similar contradiction follows if the C-trader sells $C_2$ to obtain $A_2$.

Finally, suppose that the C-trader sells $C_1$. We have
\[
(21) \quad (1/2)p_2(C_1, A_2) = k^A a_1 + a_2.
\]
Moreover, as above, the buyer must be a B-trader. Since the B-trader sells $A_2$ in this exchange, the trade must occur in period 2. Thus, the B-trader acquires $A_2$ (from an A-trader) by selling $B_1$ or $B_2$ in period 1. Suppose first that he sells $B_2$. We have
\[
(22) \quad p_1(B_2, A_2)p_2(A_2, C_1) = (k^C c_1 + c_2)/k^C.
\]
The A-trader who buys $B_2$ must consume it (if he sold it for $B_1$ in period 2, the exchange could only be with another A-trader, which we have ruled out). Hence,
\[
(23) \quad p_1(A_2, B_1) = k^b b_1 + b_2.
\]
Multiplying (21)–(23) together, we again obtain (20), leading to the same contradiction as before. Suppose therefore that the B-trader sells $B_1$. This implies that
\[
(24) \quad (1/2)p_1(B_1, A_2)p_2(A_2, C_1) = (k^C c_1 + c_2)/k^C.
\]
Because the A-trader who buys the $B_1$ consumed it, we have
\[
(25) \quad p_1(A_2, B_1) = (k^b b_1 + b_2)/k^b.
\]
Multiplying (21), (24), and (25) together, we obtain
\[
1 = \frac{k^A}{2} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right] \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right] \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right],
\]
as was to be shown. QED
III.4. Properties of Equilibrium

Armed with Lemmas 1 and 2, we can now readily characterize equilibrium. As the introduction suggests, the informational feasibility constraint (1) by itself implies something akin to Gresham’s law: any exchange between traders not of the same type must involve a low-quality good. That is, only low-quality goods are media of exchange. In turn, this implies that equilibrium production of at least one low-quality good must be positive, since traders do not consume the goods they produce. The next result establishes that, generically, the medium of exchange and, in fact, the equilibrium quantities of all goods are unique.

PROPOSITION 1. Suppose without loss of generality that

\[ k^A \leq k^B \leq k^C. \]

If the first inequality is strict (which occurs for a generic choice of \( k^X \)'s satisfying (26)), then, in any equilibrium, \( a_2 (= 1 - 2a_1) > 0, b_2 = c_2 = 0, b_1 = c_1 = 1/2, \) and \( a_1 \) satisfies

\[ \frac{k^A}{2} \left[ \frac{2(k^A a_1 + 1 - 2a_1)}{k^A} \right]^{t^*} = 1. \]

Proof. From Gresham’s law we know that at least one of \( a_2, b_2, \) and \( c_2 \) is positive. If \( b_2 > 0, \) then from Lemma 2, (10) holds with equality when \( X = B. \) But then if \( k^A < k^B, \) (10) is violated for \( X = A, \) a contradiction of Lemma 1. We conclude that \( b_2 = 0 \) and, similarly, that \( c_2 = 0. \) Hence, \( a_2 > 0. \) Applying Lemma 2 again and the production constraint \( 2a_1 + a_2 = 1, \) we obtain (27).

QED

If \( k^A < k^B \leq k^C \) as hypothesized by Proposition 1, then the discrepancy between high and low quality (as measured by the deviation of the marginal rate of substitution from one) is smallest for goods of type \( A. \) This is the sense in which our model provides a theoretical explanation for the pervasive use of gold as a medium of exchange: gold is a good for which variations in quality that are undetectable to “uninformed” traders are particularly small.

The fact that the equilibrium medium of exchange minimizes distortion, however, does not directly imply that equilibrium is Pareto-efficient in any standard sense. Indeed, as we have already noted, equilibrium is clearly not first-best efficient; positive
production of low-quality goods rules that out. The more pertinent question, however, is whether equilibrium is efficient relative to the informational, bilateral, and unmonitorability constraints that we have imposed on trade. Here the answer is not so clear. We believe that it is yes but do not have a formal proof.

The subtlety of the issue has mainly to do with the unmonitorability constraint. For example, consider the following simple scheme, which attains a first-best allocation. All traders produce only high-quality goods; each A-trader gives his $A_1$ to a C-trader; each C-trader gives his $C_1$ to a B-trader; and finally each B-trader gives his $B_1$ to an A-trader. Such behavior obviously satisfies the informational and bilateral constraints. However, it does not pass muster with unmonitorability. The problem is that, if exchanges cannot be monitored, there is nothing to prevent, say, a C-trader from collecting some A-trader's supply of $A_1$ without bothering to produce any $C_1$ himself. In this way he could avoid incurring any disutility of labor (this is the only point where we invoke disutility of labor).

We see then that in a world of unmonitorable trade an important virtue of Walrasian trade is to provide a natural way of identifying those who are “entitled” to others' goods. Specifically, a trader is so entitled if he himself has money or goods that he can offer in exchange. Indeed, one reason why we conjecture that Walrasian equilibrium is efficient is simply that it is so difficult to think of alternative identification schemes. (For an identification scheme akin to that provided by fiat money, see footnote 21.)

In the example of subsection III.2, we noted that prices in terms of money rise over time. We attributed this trend to the decline in value of money's mediating role as the last period of exchange nears. It is easy to see that the property of rising prices is a general feature of equilibrium.

**Proposition 2.** Suppose that the hypotheses of Proposition 1 hold. Then in any equilibrium

\[
\begin{align*}
2 &= q^r p_{1+3r}(B_1, A_2) \\
2 &= q^r p_{2+3r}(C_1, A_2) \\
2 &= q^r p_{3r}(A_1, A_2) \quad \text{for all } r,
\end{align*}
\]

where $q = (2(k^Aa_1 + a_2))/K^A$. Moreover, for all $r$, $t$, and $t'$, with $t < t'$,
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\[
\begin{align*}
&\begin{cases}
p_t(B_1, A_2) \leq 2/q' \leq p_t'(B_1, A_2), & \text{if } t \leq 1 + 3r \leq t' \\
p_t(C_1, A_2) \leq 2/q' \leq p_t'(C_1, A_2), & \text{if } t \leq 2 + 3r \leq t' \\
p_t(A_1, A_2) \leq 2/q' \leq p_t'(A_1, A_2), & \text{if } t \leq 3r \leq t'.
\end{cases}
\end{align*}
\] (29)

Proof. From Lemma 2, (10) holds with equality when \(X = A\). Hence, (11)-(15) all hold with equality. We infer that (28) holds.

As for (29) note that, because (11)-(15) hold with equality, a \(C\)-trader can attain his equilibrium utility level by selling \(C_1\) for \(A_2\) in periods 2, 5, 8, etc., and buying \(A_1\) with \(A_2\) in periods 3, 6, 9, etc. Similarly, an \(A\)-trader can attain his equilibrium utility level by selling \(A_1\) for \(A_2\) in periods 3, 6, 9, etc. and buying \(B_1\) with \(A_2\) in periods 1, 4, 7, etc. Suppose, contrary to (29), that

\[
p_t'(A_1, A_2) < p_{3r}(A_1, A_2)
\] (30)

for some \(t'\) and \(r\) with \(t' > 3r\). But then a \(C\)-trader does better to buy \(A_1\) in period \(t'\) than in period \(3r\), contrary to our observation that the latter behavior attains the \(C\)-trader's equilibrium utility. Hence, (30) is impossible. Alternatively, suppose that

\[
p_t(A_1, A_2) > p_{3r}(A_1, A_2)
\] (31)

for some \(t\) and \(r\) with \(t < 3r\). Then an \(A\)-trader does better to sell \(A_1\) in period \(t\) than in period \(3r\), again contrary to our preceding analysis. Thus, (31) is also impossible, and we conclude that the third pair of inequalities in (29) holds. That the first and second pairs hold follows similarly.

QED

We saw that, in the example of subsection III.2, money made two cycles through the economy. It is natural to expect that, as \(T\) increases, money will circulate correspondingly more times. This can be put more precisely as follows.

Define a standard \(B_1\)-exchange to be an exchange in which an \(A\)-trader buys \(B_1\) from a \(B\)-trader for \(A_2\). Call a standard \(B_1\)-exchange regular if it occurs in some period 1, 4, \ldots, or 1 + 3\((T^* - 1)\). Similarly, a standard \(C_1\)-exchange entails a \(B\)-trader buying \(C_1\) from a \(C\)-trader for \(A_2\) (and is deemed regular if it occurs in one of periods 2, 5, \ldots, 2 + 3\((T^* - 1)\)), and a standard \(A_1\)-exchange involves a \(C\)-trader buying \(A_1\) from an \(A\)-trader for \(A_2\) (it is regular if it occurs in period 3, 6, \ldots, or 3 + 3\((T^* - 2)\)). Collectively, the standard \(A_1\)-, \(B_1\)-, and \(C_1\)-exchanges constitute the standard exchanges.
PROPOSITION 3. Suppose that the hypotheses of Proposition 1 hold. In any equilibrium all exchanges are regular and standard. Furthermore, in any period the entire quantity produced of $A_2$ is exchanged for high-quality output.

*Proof.* See the Appendix.

From Proposition 3 we can trace exactly how money moves through the economy in equilibrium. Specifically, in period 1, $A$-traders buy $B_1$ from $B$-traders using $A_2$. In period 2, $B$-traders buy $C_1$ from $C$-traders using the $A_2$ they acquired in period 1. In period 3, $C$-traders buy $A_1$ from $A$-traders using the $A_2$ they acquired in period 2. In period 4, $A$-traders again buy $B_1$, and so on until period $3T^* - 1$, when $B$- and $C$-traders exchange $A_2$ for $C_1$ and each group consumed what it has received.

Thus, given $T$, money makes $T^*$ cycles through the economy, and so $T$ is a measure of the velocity of money. If we think of subdividing a given interval of time more finely so as to increase the number of trading periods, money will circulate correspondingly more times in that interval. In line with the quantity theory of money, moreover, a given quantity of money can mediate more exchanges as velocity increases, and so less money is needed.

PROPOSITION 4. Under the hypotheses of Proposition 1, the equilibrium per capita quantity of money $a_2$ is a decreasing function of $T$. Moreover, in the limit as $T \to \infty$, $a_2$ tends to 0.

*Proof.* This follows directly from (27).

Although the inverse relation between velocity and quantity is entirely classical, there is an important way in which our model deviates from orthodoxy. Namely, a fall in the quantity of money is not welfare neutral. Indeed, as $a_2$ falls, welfare rises (more precisely, the welfare of $C$-traders rises; that of $A$- and $B$-traders remains the same), as equilibrium production of $A_1$ increases.

PROPOSITION 5. Under the hypotheses of Proposition 1, equilibrium utility of $C$-traders is an increasing function of $T$ (and that of $A$- and $B$-traders is independent of $T$). In the limit of $T \to \infty$, $C$-trader utility tends to the first-best level.

*Proof.* This follows immediately from Proposition 1 and formula (27).
The fact that an increase in $T$ leads to welfare improvement underscores the fact that in a model like ours counting the number of transactions needed before the economy clears is an inappropriate measure of the inefficiency of the economy (see footnote 2). Indeed, as $T \to \infty$, Proposition 3 shows that the number of transactions also tends to infinity, and yet, from Proposition 5, welfare converges to the first-best level. However, note that from Proposition 2 there is no reason for an individual trader ever to make more than two transactions: selling the good he produces for $A_2$ and then reselling the $A_2$ for the good he consumes. In fact, this is the same number of transactions he would execute even if the economy were monetized.

Propositions 1–5 are devoted to characterizing equilibrium. For completeness we will also confirm existence. The example of subsection III.2 exhibits an equilibrium when $T = 5$. We now turn to arbitrary $T$. We do this by exhibiting a symmetric equilibrium explicitly.

**Proposition 6.** Under the hypotheses of Proposition 1, there exists a symmetric equilibrium in which, for $i = 1, 2, a_i, b_i, c_i$ satisfy the formulae of Proposition 1; trade is characterized by Proposition 3; for all $t$ and $r$, prices of high-quality goods satisfy

\[
p_t(B_1, A_2) = \frac{2}{q_r}, \quad \text{if } 3r \leq t \leq 3r + 2,
\]
\[
p_t(C_1, A_2) = \frac{2}{q_r}, \quad \text{if } 1 + 3r \leq t \leq 3r + 3,
\]
\[
p_t(A_1, A_2) = \frac{2}{q_r}, \quad \text{if } 3r - 1 \leq t \leq 3r + 1,
\]

and prices of low-quality goods satisfy $p_t(B_2, A_2) = p_t(C_2, A_2) = q$ for all $t$, where $q = 2(k^a a_1 + 1 - 2a_1)/k^a$, and $a_1$ satisfies (27).

**Proof.** Merely a matter of mechanical verification.

**III.5. The Bilateral Trade and Unmonitorability Assumptions**

Let us return to the bilateral trade and unmonitorability assumptions introduced in subsection II.1. As we noted, the former assumption gets at the idea that trade is decentralized, i.e., that markets, although well organized in the sense that traders know where to find the goods they want, are geographically dispersed rather than centrally located. This means that if an $A$-trader wishes to buy $B_1$ he can readily find the market where it is sold but, once there, he is unlikely to find a $C$-trader, who could com-
plete a three-way exchange of $A_1$ for $B_1$.\footnote{If a C-trader could be found, then in principle it would be possible to conduct a three-way exchange even in the absence of outside monitoring. For example, consider the following stylized scheme. Three traders—one each of types $A$, $B$, and $C$—sit in a circle around a rotating table. Each party puts half a unit of the high-quality good he produces on the table directly in front of him, and all parties have the opportunity to inspect one another's goods. Thus, the A-trader can scrutinize the $B_1$, etc. Once a party is satisfied, he can press a button, and when everyone has done this, the table rotates 120 degrees, so that the A-trader gets the $B_1$, the $B$-trader gets the $C_1$, and the $C$-trader gets the $A_1$. Unless everyone presses the button, the table does not rotate.} Therefore, the A-trader will not be able to buy the $B_1$ using $A_2$.

This discussion should make clear that what is important about the bilateral trade assumption is not that there literally be just two parties to every trade but simply that trading circles as in footnote 20 be too costly to arrange. If, as in subsection II.1, there were many more than three goods, such circles would typically have to be quite large, and so ruling them out would be a relatively weak assumption.

As for the unmonitorability assumption, it seems very much in the spirit of the anonymity that most everyday cash transactions entail. To understand the role that it plays, consider the model of Section II, but let us now relax the assumption that a good must be physically in a trader's possession in order for him to sell it; i.e., let us drop constraint (4). Then for $T = 1$, we claim that the following is an equilibrium. Let the price of all low-quality goods be 1 and that of all high-quality goods be 2. Suppose that each A-trader exchanges 1 unit of $A_2$ for 1/2 unit of $B_1$ with some $B$-trader; each $B$-trader exchanges 1 unit of $A_2$ for 1/2 unit of $C_1$ with some $C$-trader; and each $C$-trader trades 1 unit of $A_2$ for 1/2 of $A_1$ with some A-trader.\footnote{A similar sort of trading arrangement is the following "money-lending" mechanism. The money-lender (who could be either one of the traders or some outside authority) first issues each trader with one unit of "money" and imposes the requirement that it be repaid after the final period. Each $A$-trader uses his money to buy 1/2 unit of $B_1$ from a $B$-trader, each $B$-trader buys 1/2 unit of $C_1$ from a $C$-trader with his money, and each $C$-trader purchases 1/2 of $A_1$ from an $A$-trader. Each trader thus ends up with one unit of money, which he then repays to the money-lender. Because of the requirement of repayment, this scheme also falls afoul of the unmonitorability constraint. But it is very similar to the model of fiat money (which does not violate unmonitorability but requires an infinite horizon) presented in Section VI.}

Note first that such behavior does indeed violate constraint (4) because, in a one-period model, $B$- and $C$-traders do not actually have the $A_2$ that they are trading. However, observe that good $A_2$ is in zero net demand by all traders, i.e., every trader both buys and sells one unit of it. Thus, no physical production...
of $A_2$ is actually required, and therefore, the outcome we have described is first-best efficient.

The idea that $A_2$ can serve as a medium of exchange without actually being produced may seem somewhat paradoxical until it is remembered that it is not really $A_2$ but only promises to deliver $A_2$ (IOU's) that are being traded. These promises therefore constitute money in much the same sense that gold- or silver-certificates formerly did. The only difference between certificates and IOU's is that the former are public promises (i.e., promises by government), whereas the latter are private (but presumably enforced by a public court).

IV. TWO EXTENSIONS: DISCOUNTING AND DURABILITY

The analysis so far has presumed that traders are indifferent about when they consume within the $T$ trading periods and that all goods survive to period $T$. We now briefly examine what happens when these presumptions are dropped.

Let us first introduce a discount factor $\delta$ common to all traders. If we continue to assume that $k^A < k^B \leq k^C$, then, by analogy with the proofs of Lemmas 1 and 2, we can show that

$$\frac{\delta^{3r} p_{3r}(A_1, A_2)}{p_{3r+1}(B_1, A_2)} = \frac{2(k^B b + b_2)}{k^B}, \quad 1 \leq 3r \leq T - 1$$

$$\frac{\delta^{3r+1} p_{3r+1}(B_1, A_2)}{p_{3r+2}(C_1, A_2)} = \frac{2\delta(k^Cc + c_2)}{k^C}, \quad 1 \leq 3r + 1 \leq T - 1$$

and

$$\frac{\delta^{3r+2} p_{3r+2}(C_1, A_2)}{p_{3r+3}(A_1, A_2)} = \frac{2\delta^2(k^A a + a_2)}{k^A}, \quad 1 \leq 3r + 2 \leq T - 1,$$

and so if the analogue of Proposition 1 holds,

$$\frac{\delta^{T^*} k^A}{2} \left[ \frac{2(k^A a + a_1)}{k^A} \right]^{T^*} = \delta^{(T+2)(T-1)/2}. \quad (32)$$

But we can find $a_1$ and $a_2$ satisfying (32) if and only if

$$\left[ \frac{2}{k^A} \right]^{T^*-1} < \delta^{(T^2-T-4)/2}. \quad (33)$$

It can be shown that if (33) holds and $T \geq 5$, equilibrium quantities are unique and satisfy $a_2 > 0, b_2 = c_2 = 0, b_1 = c_1 = 1/2$. 

If $\delta$ is small enough, however, Walrasian equilibrium fails to exist (see the Appendix for a formal demonstration). The difficulty is that everybody will try to consume in the first period, and as we argued in subsection II.2, such intentions cannot be mutually consistent. This illustrates the simple point that a trading arrangement based on delayed gratification cannot work if traders are too impatient.

Next, let us revert to the case of no discounting but now consider a model in which, although goods $A_1, B_1, B_2, C_1,$ and $C_2$ as before endure for $T$ periods, $A_2$ survives for only $\hat{T}$ periods, where $\hat{T} < T$.

Arguments similar to those that gave us Lemma 1 yield

\begin{equation}
\frac{k^A}{2} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{\hat{T}^*} \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{\hat{T}^*} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{\hat{T}^*} \geq 1
\end{equation}

\begin{equation}
\frac{k^B}{2} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{\hat{T}^*} \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{\hat{T}^*} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{\hat{T}^*} \geq 1
\end{equation}

and

\begin{equation}
\frac{k^C}{2} \left[ \frac{2(k^C c_1 + c_2)}{k^A} \right]^{\hat{T}^*} \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{\hat{T}^*} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{\hat{T}^*} \geq 1
\end{equation}

where $\hat{T}^*$ is the greatest integer not bigger than $(\hat{T} + 1)/3$. Moreover, arguments similar to those establishing Lemma 2 imply that (i) if $a_2 > 0$, then (34) holds with equality; (ii) if $b_2 > 0$, then (35) holds with equality; and (iii) if $k^A < k^B < k^C$, then $c_2 = 0$.

Now, if $a_2 > 0$, then (35) and the fact that (34) holds with equality imply

\begin{equation}
\frac{k^B}{k^A} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{\hat{T}^*} \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{\hat{T}^*} \geq 1
\end{equation}

which, from the equality version of (34), implies that

\begin{equation}
\frac{k^A}{2} \left[ \frac{k^A}{k^B} \right]^{\hat{T}^*} \leq 1.
\end{equation}

But if (37) holds strictly, then it is readily shown that (35) holds strictly, and so $b_2 = 0$. Conversely, if (35) fails to hold, then we must have $b_2 > 0$ and $a_2 = 0$. Hence, in this version of the model there is a trade-off between “identifiability” and “durability.” On
the one hand, the smaller is the ratio $k^A/k^B$ (i.e., the more identifiable type $A$-goods are relative to $B$-goods), the more likely it is that $A_2$ will function as the medium of exchange. On the other hand, the smaller is the ratio $\tilde{T}^*/T^* - \tilde{T}^*$ (i.e., the greater is the durability of $B_2$ relative to $A_2$), the more likely it is that $B_2$ will be the medium of exchange.

V. STEADY STATES

So far we have considered basically a one-shot economy: production occurs once-and-for-all, trade unfolds over $T$ periods, and then the economy ends. We now show that this economy can be embedded within an infinite-horizon framework for the purpose of examining steady state equilibrium.

Suppose that in each period $t = 1, 2, \ldots$, there is an infusion of $m$ new traders of each type $A$, $B$, and $C$. On their arrival new traders produce according to the linear technology described in Section II. They then trade and consume for $T$ periods, at which point they and any unconsumed goods they have produced disappear. Preferences are the same as in Section II; there is no discounting.

Let us assume that $k_A < k_B \leq k_C$. Then we would expect that good $A_2$ will function as the medium of exchange in steady state equilibrium. At any time $t$ there are, in fact, $T$ different vintages of $A_2$ available, and so each of these must have a different price. (There are different vintages of the high-quality goods available too, but since these goods are consumed as soon as they get into the right hands, they need not be differentially priced.) For $1, \ldots, T$ let $A^t_1$ be $t$-period-old $A_2$ (define $B^t_1$ and $C^t_1$ analogously). Reinterpreting the prices in Proposition 6, we obtain the following:

**Proposition 7.** In the above infinite-horizon model, the prices

\[
\begin{align*}
    p(B^t_1, A^t_2) &= 2/q^r, & \text{if } 3r - 1 < t < 3r + 1, \\
    p(C^t_1, A^t_1) &= 2/q^r, & \text{if } 1 + 3r - 1 < t < 3r + 3, \\
    p(A^t_1, A^t_2) &= 2/q^r, & \text{if } 3r - 1 < t < 3r + 1, \\
    p(A^t_2, A^t_2) &= 1/q^r, & \text{if } 3r - 1 < t < 3r + 3, \\
    p(B^t_2, A^t_2) &= 1/q^{r-1}, & \text{if } 3r - 1 < t < 3r + 1, \\
    p(C^t, A^t_1) &= 1/q^{r-1},
\end{align*}
\]
for all \( t = 1, \ldots, T \) and \( r = 0, \ldots, T^* \), constitute a steady state equilibrium together with the following exchanges: for all \( t \), (i) each \( t \)-year-old \( A \)-trader exchanges \( a_2 \) units of \( A^t_2 \) for \( a_2 p(B^t_1,A^t_2) \) units of \( B^t_1 \) with some \( t \)-period-old \( B \)-trader (provided that \( t - 1 \) is a multiple of 3); (ii) each \( t \)-year-old \( B \)-trader exchanges \( a_2 \) units of \( A^t_2 \) for \( a_2 p(C^t_1,A^t_2) \) units of \( C^t_1 \) with some \( t \)-period-old \( C \)-trader (provided that \( t - 2 \) is a multiple of 3); (iii) each \( t \)-year-old \( C \)-trader exchanges \( a_2 \) units of \( A^t_2 \) for \( a_2 p(A^t_1,A^t_2) \) units of \( A^t_1 \) with some \( t \)-period-old \( A \)-trader (provided that \( t \) is a multiple of 3), where \( a_2 = 1 \) and \( a_1 \) satisfies (27).

**Proof.** The proposition is just a reinterpretation of Propositions 3 and 6. The only thing to check is that the possibility of exchanging \( A^t_2, B^t_2, \) and \( C^t_2 \) of different vintages (which was not possible in the static model) does not create new arbitrage opportunities for any trader. For example, consider an \( A \)-trader. If he trades \( A^t_2 \) for \( A^t_2 \), he obtains \( 1/q^t \) units of the latter per unit of the former (where \( 3r - 1 \leq t \leq 3r + 1 \)). If in the next period he then sells what now is \( A^{t+1}_2 \) for \( B^t_1 \), he obtains \( q^t/2 \) or \( q^{t+1}/2 \) units of the latter for the former. Thus, the \( A \)-trader does no better with these trades than with steady state equilibrium transactions. Similar reasoning applies to \( B \)- and \( C \)-traders.

QED

Proposition 7 exhibits a steady equilibrium in which trade is completely segregated according to cohort: \( t \)-period-old traders transact only with other \( t \)-period-old traders and the \( A^t_2 \) they exchange is only of vintage \( t \). But intercohort steady states are also possible (although they continue to entail the same aggregate production). For example, suppose, for some \( t \) (with \( t - 1 \) divisible by 3), that some \( t \)-period-old \( A \)-trader sells \( A^t_2 \) in period \( t \) to a \( t + 3 \)-period-old \( B \)-trader, and correspondingly, some \( t + 3 \)-period-old \( A \)-trader sells \( A^{t+3}_2 \) to a \( t \)-period-old \( B \)-trader. Then as long as the older \( B \)-trader continues to trade with the \( t \)-cohort (which is possible provided that he sells all his \( B^t_1 \) and buys all his \( C^t_1 \) before period \( T - 3 \)) and the younger \( B \)-trader continues to trade with the \( t + 3 \)-cohort, we still have a steady state equilibrium.

**VI. Fiat Money**

When \( T < \infty \) (i.e., velocity is less than infinite), we have seen that equilibrium is inefficient in the sense that it entails production and consumption of a low-quality good. This suggests that
there is a potentially valuable role in our model for **fiat money** (which can be thought of as a good that is essentially costless to produce, that confers no utility, and whose quality is discernible by everybody). Namely, if fiat money takes over as the medium of exchange, the production distortion we have discussed is eliminated.

To introduce fiat money, of course, we must deal with the end-point problem: who will hold the money in the last period? We shall, therefore, appeal to an infinite horizon, but using a somewhat different framework from that of the preceding section. Recall the model of Section II: production followed by periods of trading. Call this sequence of events an *epoch*. We shall suppose that there is an infinite sequence of epochs indexed by $s = 1, 2, \ldots$. Traders are infinitely lived and have a discount factor $\delta$ across epochs (there is no discounting within an epoch).

Imagine now that into this economy we introduce another good, which is produced at zero cost by a single “producer,” whom we call the government. We shall denote this good—fiat money—by $M$. Suppose that, in the first period of the first epoch, each trader is endowed with a quantity of money $M_0 > 0$. In each period $t$ of each epoch $s$, the government spends a quantity of money $M_{ts}$ on each of the three high-quality goods, where

$$M_{ts} = \begin{cases} mM_{t-1s}, & \text{if } t > 1 \\ mM_{ts-1}, & \text{if } t = 1, \end{cases}$$

and $M_{11} = mM_0$. This setup corresponds well to many analyses of inflation in the macroeconomic literature.

Given prices $\{p_\tau(t, t')\}$, each $Y$-trader in this economy chooses $\{T_{ts}, e_{ts}(t, t'), z_{0s}(t, t')\}_{t,s}$ (which includes trades of money) to maximize

$$\sum_{s=1}^{\infty} \sum_{t=1}^{T} \delta^{s-1} \sum_{t=1}^{T} \left( k^{Y(t)} e_{ts}(t, (Y + 1)_1) + e_{ts}(t, (Y + 1)_2) \right),$$

subject to

$$p_\tau(X_i, X_j)q^t(X_i) + q^t(X_j) = 0,$$

for all $\tau \in T_{ts}$ and $X_i, X_j \in \mathcal{X}(\tau)$;

$$z_{ts}(X_i) = z_{ts-1}(X_i) + \sum_{\tau \in T_{ts}} q^t(X_i) - e_{ts}(X_i),$$

for all $t, z$, and $X_i \in \mathcal{X}(\tau)$;

---

22. A well-known finite-horizon solution to the end-point problem is the device of requiring traders to return the money at the end of the last period (see footnote 21).
\( z_{t-s}(X_i) + \sum_{\tau \in \mathcal{H}(\tau)} q^\tau(X_i) - e_t(X_i) \geq 0, \)

for all \( t, s, \) and \( X_i \in \mathcal{H}(\tau); \)

\( e_t(X_i) \geq 0 \) and \( z_t(X_i) \geq 0, \) for all \( t, s, \) and \( X_i \in \mathcal{H}(\tau); \)

and

\( 2z_{t-s}(Y_1) + z_{t-s}(Y_2) \leq 1, z_{t-s}(X_i) = 0, X_i \notin \{Y_1, Y_2\} \) and all \( s, X_i \in \mathcal{H}(\tau) - \{M\} \) and \( Z_0(M) = M_0. \)

Let \( \mathcal{T}_{ts}^s \) be the set of transactions that the government carries out in period \( t \) of epoch \( s. \) Let \( \hat{\mathcal{T}}_{ts} \) be the union of all traders’ \( ts \)-transactions (including \( \mathcal{T}_{ts}^s \)). Then the market-clearing requirement takes the form,

\[
\text{(45) for all } t \text{ and } s, \text{ there is a one-to-one correspondence between the set of period } ts \text{-transactions } \hat{\mathcal{T}}_{ts} \text{ and itself in which each transaction } \tau \in \hat{\mathcal{T}}_{ts} \text{ is paired with its complement } \tau^c.
\]

One equilibrium satisfying \((38)-(45)\) is the same as that in the model without money. That is, the equilibrium given by Proposition 6 will simply be replicated in every epoch. Under certain conditions, however, there exists another equilibrium. In this other equilibrium, for all goods \( X \) and \( Y, \) all periods \( t, \) and epochs \( s, \) prices are given by

\[
\text{(46) } p_{ts}(X_1, M) = 2(1 + m)^{(s-1)T+t} \cdot M_0 \frac{1 - \delta}{\delta^{(t-1)/T}} (1 - \delta^{1/T})
\]

\[
\text{(47) } p_{ts}(X_2, M) = p_{ts}(X_1, M) / 2,
\]

\[
\text{(48) } p_{ts}(X_1, Y_2) = 2 \left( \frac{1 + m}{\delta^{1/T}} \right)^t.
\]

As for quantities, if \( q_{ts}^Y(X_i) \) is the amount of good \( X_i \) bought by a \( Y \)-trader in period \( t \) of epoch \( s, \)

\[
\text{(49) } q_{ts}^Y(X_i) = \begin{cases} 
-\frac{1}{2} \frac{1 - \delta^{1/T}}{1 - \delta}, & \text{if } X_i = Y_1 \\
-q_{ts}^Y(Y_1) \frac{1 + m}{1 - \delta}, & \text{if } X_i = (Y + 1)_1 \\
0, & \text{otherwise.}
\end{cases}
\]

To understand \((46)\), note that for a \( Y \)-trader to be willing to sell good \( Y_1 \) in each period \( t = 1, \ldots, T \) of epoch \( s \) and then to buy
good \((Y + 1)\), in the following period \(t + 1 = 2, \ldots, T\), and period 1 of the next epoch, prices must satisfy

\[
\frac{p_{ts}(X_1,M)}{p_{ts}(X_1,M)} = \frac{p_{ts}(X_1,M)}{p_{ts}(X_1,M)} = \ldots = \frac{p_{ts}(X_1,M)}{p_{ts}(X_1,M)} = \frac{\delta p_{ts}(X_1,M)}{p_{ts+1}(X_1,M)}
\]

because, in equilibrium, \(p_{ts}(X_1,M) = p_{ts}(Y_1,M) = p_{ts}(Y + 1)_1,M)\).

Moreover, to equilibrate supply and demand, we have

\[
\sum_{t=1}^{T} \frac{(m + 1)^{(s-1)T+t} M_0}{p_{ts}(X_1,M)} = \frac{1}{2}
\]

Solving (50) and (51), we get (46). Now,

\[
q^Y(Y_1) = -\frac{(m + 1)^{(s-1)T+t} M_0}{p_t(Y_1,M)}, \quad q^Y((Y + 1)_1) = \frac{(m + 1)^{(s-1)T+t-1} M_0}{p_t(Y_1,M)}
\]

which, in view of (46), gives us (49).

As for (48), observe that we must have

\[
\frac{1}{p_{1s}(X_1,Y_2)} \leq \frac{1}{2} \left( \frac{\delta^{1/T}}{1 + m} \right).
\]

Otherwise, a \(Y\)-trader is better off selling \(Y_2\) for \((Y + 1)_1\) in period 1 than selling \(Y_1\) for \(M\) in period 1 and then buying \((Y + 1)_1\) in period 2. Moreover, we must have

\[
\frac{p_{ts}(X_1,Y_2)}{p_{ts+1}(X_1,Y_2)} \leq \frac{p_{ts}(X_1,M)}{p_{ts-1}(X_1,M)}.
\]

Otherwise, a \(Y\)-trader is better off selling \(Y_1\) for \(Y_2\) in period \(t\) and then reselling the \(Y_2\) for \((Y + 1)_1\) in period \(t + 2\) than selling \(Y_1\) for \(M\) in period \(t\) and then buying \((Y + 1)_1\) in period \(t + 1\). Formula (48) then follows when (53) holds with equality.

Notice that, in the equilibrium described by (46)–(49), welfare converges to the first best (the allocation that would prevail with barter if there were no informational imperfections) as \(\delta\) goes to 1 and \(m\) goes to 0. Thus, for \(\delta\) near 1 and \(m\) near enough 0, the introduction of money can definitely promote a welfare improvement. Note too that prices in (46) and (47) are proportional to the money supply \((1 + m)^{(s-1)T+t} M_0\).

Now, for (46)–(49) to constitute an equilibrium, it cannot be the case that a \(C\)-trader is better off selling \(C_1\) for \(A_2\) than selling
C_1 for M and then buying A_1. Hence, from (46) and (48) we must have

$$\frac{2(1 + m)^{T+2}}{\delta} \leq \frac{K^A \delta^{1/T}}{1 + m};$$

i.e.,

$$(1 + m)^{T+1} \leq \frac{K^A}{2} \delta^{[(T+1)/T]}. \tag{54}$$

We conclude that m must be sufficiently small to satisfy (54) in order to ensure the existence of a monetary equilibrium. For too rapid a monetary expansion—i.e., for m bigger than the critical values at which (54) holds with equality—the only equilibrium is that of Proposition 6. The disappearance of the monetary equilibrium corresponds to a serious and well-recognized historical problem with hyperinflation: the risk that the economy will be demonetized; i.e., agents will fall back on barter as the form of exchange.\textsuperscript{23}

VII. CONCLUSION

In this paper we develop a simple Walrasian general equilibrium model in which there is a role for a medium of exchange. Although we feel our assumptions are naturally motivated, the model is quite special. The demands of tractability have limited us to a single example from a larger class of models. The characterization of the properties of this broader class is clearly a substantial task remaining to be done. We therefore conclude with some of the leading open questions.

First, in the paper we focus on the particular case of linear preference, and the more general case of concave preferences needs to be examined. If preferences are concave, the marginal rates of substitution between the two types of each good will typically depend on how much of each type was being consumed. As a result, no single good need always be more identifiable than all other goods. We conjecture that this may sometimes lead to more than one good being used as a medium of exchange at the same time.

\textsuperscript{23} Looking at data from a variety of hyperinflations, Barro [1972] finds that economies seem to behave as if there is some threshold level of inflation beyond which the inflationary spiral becomes unstable.
Second, our results may depend on the sequencing of trades. The trading periods that are relevant for different people may be very different and allowing for this possibility may significantly alter our results.

Third, as a referee suggested, it would be interesting to investigate the nature of nonsteady state equilibria in the fiat money version of the model. This would help relate our approach to others in the literature.

Finally, we have yet to investigate the welfare implications of our model fully. In particular, we do not know whether the equilibrium in our model is constrained Pareto-optimal in an appropriately defined sense. We suspect that the answer is yes, but confirmation must await future work.

APPENDIX

**Lemma 2.** If the equilibrium production of $X_2$ is positive, then

\[
\frac{k^x}{2} \left[ \frac{2(k^a a_1 + a_2)}{k^A} \right]^{\tau^*} \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{\tau^*} \left[ \frac{2(k^c c_1 + c_2)}{k^C} \right]^{\tau^*} = 1.
\]

**Proof.** Suppose that $a_2 > 0$. We must show that (10) holds when $X = A$.

We first introduce the idea of a *physical portion* of a good. Suppose that we divide a stock of given good in half. We will then have two portions of equal quantity but physically distinct. Much of the following argument relies on tracing a particular physical portion of a good around the economy. Thus, the difference between this concept and that of “quantity” should be borne in mind.

Consider the equilibrium behavior of an $(X - 1)$-trader. Suppose that in some period $t(1)$ he trades a physical portion $\alpha$ of a good $Y(0)$ he has produced (hence, $Y(0)$ is either $(X - 1)_1$ or $(X - 1)_2$) for a physical portion $\beta$ of some other good $Y(1)$. Now, of course, he may retrade (or consume) different subportions of $\beta$ at different times. However, if we divide up $\alpha$ into subportions appropriately, we can ensure that each subportion of $\alpha$ corresponds to a subportion of $\beta$ retraded (or consumed) in its entirety at a single time. For example, suppose that in period 1 the $(X - 1)$-trader sells 1 unit of $(X - 1)_1$ for 3/4 units of $X_2$. Suppose that he resells 1/3 unit of this $X_2$ in period 2, 1/4 unit in period 4,
and consumes the rest in period 2. We can divide up the unit of \((X - 1)\) into physical portions of size \(4/9\), \(1/3\), and \(2/9\), so that the \(4/9\)-unit portion can be thought of as being exchanged for the portion of \(X_2\) resold in period 2, the \(1/3\)-unit portion is exchanged for the portion of \(X_2\) resold in period 4, and the \(2/9\)-unit portion is exchanged for the portion of \(X_2\) that is consumed.

Clearly, this argument generalizes. Indeed (since in equilibrium only finitely many transactions are made),\(^{24}\) if \(\alpha\) is chosen appropriately (i.e., by suitable subdivision), we can guarantee not only that the entire physical portion of \(Y(1)\) which the trader obtains for \(\alpha\) is retraded at a single time, but that the same is true of the physical portion \(Y(2)\) that it is exchanged for, and so on, for \(Y(3), Y(4)\) etc., up to the physical portion of \(Y(l)\) \((Y(l) = X_1\) or \(X_2)\) that the trader ultimately consumes. Define a complete trader-sequence of equilibrium transactions for our \((X - 1)\)-trader to be such a sequence: i.e., one in which he trades a physical portion of \(Y(0)\) for a portion of \(Y(1)\) in period \(t(1)\); he trades the portion of \(Y(1)\) for a physical portion of \(Y(2)\) in period \(t(2)\); and so on, until finally in period \(t(l)\), he trades the portion of \(Y(l - 1)\) acquired in period \(t(l - 1)\) for a physical portion of \(Y(l)\), which he consumes.\(^{25}\)

Following the logic of the proof of Lemma 1, one can readily establish that

\[
p_{t(1)}(Y(0), Y(1))p_{t(2)}(Y(1), Y(2)) \cdots p_{t(l)}(Y(l - 1), Y(l))
\]

\[
(A.1) = \begin{cases} 
 2(k^{x_1}x_1 + x_2)/k^x, & \text{if } Y(0) = X_1 \text{ and } Y(l) = X_1 \\
 2(k^{x_2}x_1 + x_2)/k^x, & \text{if } Y(0) = X_2 \text{ and } Y(l) = X_1 \\
 k^{x_1}x_1 + x_2, & \text{if } Y(0) = X_1 \text{ and } Y(l) = X_2 \\
 k^{x_2}x_1 + x_2, & \text{if } Y(0) = X_2 \text{ and } Y(l) = X_2,
\end{cases}
\]

where \(x_1\) and \(x_2\) are the equilibrium per capita quantities of \(X_1\) and \(X_2\), respectively.

We shall say that a certain physical portion \(\beta\) of good \(Y'\) is first-order related to a certain physical portion \(\alpha\) of good \(Y\) if \(\beta\) and \(\alpha\) each belong to transactions in the same complete trader-sequence. A physical portion \(\beta\) of good \(Y'\) is second-order related

\(^{24}\) There are only finitely many periods and finitely many traders, and each trader executes only finitely many transactions in any period.

\(^{25}\) Notice that by appropriately choosing the physical portions as above, we can ensure that the set of equilibrium transactions executed by our \((X - 1)\)-trader is partitioned by his complete trader sequences. That is, each transaction belongs to a unique complete trader sequence.
to \(\alpha\) if there exists a physical portion \(\gamma\) of some good \(Y''\) such that \(\beta\) is first-order related to \(\gamma\) and \(\gamma\) is first-order related to \(\alpha\). Continuing iteratively, for any \(n\) we can define what it means for a physical portion of good \(Y'\) to be \(n\)th-order related to \(\alpha\). We shall say that a physical portion \(\beta\) of good \(Y'\) is related to \(\alpha\) if, for some \(n\), \(\beta\) is \(n\)th-order related to \(\alpha\).

Fix a physical portion \(\alpha\) of some good \(Y\). Let \(\mathcal{I}(\alpha)\) be the set of all equilibrium transactions \(\tau\) such that \(\tau\) belongs to a complete trader-sequence in which a physical portion \(\beta\) (of some good \(Y'\)) related to \(\alpha\) is traded. Consider a transaction \(\tau \in \mathcal{I}(\alpha)\), in which some trader trades physical portion \(\mu\) of good \(Z\) for portion \(\nu\) of good \(Z'\) in period \(t\). Then the complementary transaction \(\tau'\), in which some other trader trades \(\nu\) for \(\mu\) in period \(t\), is also in \(\mathcal{I}(\alpha)\) (If \(\tau \in \mathcal{I}(\alpha)\), then \(\mu\) and \(\nu\) are related to \(\alpha\). From the definition of equilibrium the complementary transaction \(\tau'\) is also made, and so \(\tau' \in \mathcal{I}(\alpha)\).) Now, we can associate transaction \(\tau\) with price \(p\(\tau(Z,Z')\)), in which case the complementary trade \(\tau'\) is associated with ratio \(p\(\tau(Z',Z)\)). Hence, the product of all the prices associated with trades in \(\mathcal{I}(\alpha)\) is \(1\). Now, we can partition the different transactions in \(\mathcal{I}(\alpha)\) into different complete trader-sequences (recall that each transaction belongs to a unique complete trader-sequence), and so from (A.1) we can write the product of prices as

\[
\left(\frac{(k^A)^{\pi_A}}{2^n} \frac{(k^B)^{\pi_B}}{k^A} \frac{(k^C)^{\pi_C}}{k^B} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{\rho_A} \right. \\
\left. \times \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{\rho_B} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{\rho_C} \right) = 1,
\]

where, for \(X = A, B, C\), \(\rho_X\) is the number of complete trader-sequences that end in the consumption of good \(X_1\) or \(X_2\), \(\pi_X\) is the number of complete trader-sequences that end in the consumption of good \(X_2\), and \(\eta\) is the number of complete trader-sequences that begin with trading \(A_2, B_2,\) or \(C_2\). We next claim that

\[
\pi_A + \pi_B + \pi_C = \eta;
\]

i.e., that the number of complete trader-sequences (with trades belonging to \(\mathcal{I}(\alpha)\)) that begin with a low-quality good equals the number of sequences that end with a low-quality good. Of course, any low-quality good that is produced must eventually be consumed, and in that sense, formula (A.3) is plausible. However, it is not immediately obvious; we must first rule out, for example,
the possibility that there are more initial transactions than terminal transactions because the latter entail larger quantities of low-quality goods.

Notice first that any complete trader-sequence can be thought of as a sequence of ordered pairs of 1's and 2's, e.g., \{(1,2),(2,2),(2,1)\}, where the first number in each pair refers to the quality of the good being sold by the trader in question and the second to that of the good being bought. Every such sequence of ordered pairs has the properties that (i) at least one 2 appears in each pair (since a low-quality good must be involved in every transaction), and (ii) if the second number in some pair is \(m\) \((m = 1,2)\), then the first number in the next pair is also \(m\) (since whatever a trader buys that he does not consume, he must instead sell). Because \(T \in \mathcal{T}(\alpha)\) implies that the complementary transaction \(\tau\) belongs to \(\mathcal{T}(\alpha)\), we also have property (iii): among the sequences made up of elements of \(\mathcal{T}(\alpha)\), there is a one-to-one correspondence between pairs of the form \((m_1,m_2)\) and pairs of the form \((m_2,m_1)\), where \((m_1,m_2)\) appears in a different complete trader-sequence from \((m_2,m_1)\).

To establish (A.3), we assert that any finite set \(\mathcal{J}\) of sequences of ordered pairs that satisfy (i), (ii), and (iii) has the property that (iv) the number of sequences beginning with a 2 is the same as the number ending with a 2. (Note that if \(\mathcal{J}\) satisfies (iii) it contains an even number of ordered pairs.) This is true by inspection if the number of ordered pairs in \(\mathcal{J}\) is two: in that case, (i)–(iii) imply that

\[
\hat{\mathcal{J}} = \begin{cases} 
\left\{ \{(1,2)\}, \{(2,1)\} \right\} \\
\text{or} \\
\left\{ \{(2,2)\}, \{(2,2)\} \right\} \\
\text{or} \\
\left\{ \{(2,1)\}, \{(1,2)\} \right\}, 
\end{cases}
\]

and in all three cases, (iv) holds.

To complete the induction, we must show that if (iv) holds when \(\mathcal{J}\) contains \(2n - 2\) pairs, it holds when \(\mathcal{J}\) contains \(2n\) pairs. Suppose, therefore, that \(\mathcal{J}\) contains \(2n\) ordered pairs.

If the pair \((2,2)\) appears in some sequence in \(\mathcal{J}\), then from (iii), \((2,2)\) appears in some other sequence as well. Suppose that we delete both appearances of \((2,2)\). Because \(\mathcal{J}\) satisfies (i)–(iii), the resulting set \(\mathcal{J}'\) (which contains \(2n - 2\) pairs) also does. By
inductive hypothesis, (iv) holds for \( \mathcal{F}' \). But from (i) and (ii), \( \mathcal{F} \) and \( \mathcal{F}' \) have the same number of sequences that begin with 2 and also have the same number of sequences that end with 2 (and so \( \mathcal{F} \) satisfies (iv) too), unless at least one of the deleted pairs (2,2) constitutes an entire sequence by itself in \( \mathcal{F} \). If this last condition holds, then, for each deleted pair that constitutes an entire sequence by itself, \( \mathcal{F} \) has one more sequence that begins with 2 and one more sequence that ends with 2 than does \( \mathcal{F}' \). And so again by inductive hypothesis (iv) holds for \( \mathcal{F} \).

Therefore, to finish the inductive step, we may assume that \( \mathcal{F} \) contains only pairs (1,2) and (2,1).

Suppose first that \( \mathcal{F} \) has a sequence \( \sigma \) that ends with the pair (1,2). If \( \mathcal{F} \) also has a sequence \( \sigma' \) that ends with the pair (2,1), then delete both these pairs to obtain the set \( \mathcal{F}'' \). \( \mathcal{F}'' \) satisfies (i)–(iii) and so by inductive hypothesis satisfies (iv). If (2,1) is the only pair in \( \sigma' \), then from (ii), \( \mathcal{F}'' \) has one fewer sequence that begins with 2 and one fewer sequence that ends with 2 than \( \mathcal{F} \) does. If (2,1) is not the only pair in \( \sigma' \), then from (ii), \( \mathcal{F}'' \) has the same number of sequences that begin with 2 and the same number of sequences that end with 2 as does \( \mathcal{F} \). In either case since \( \mathcal{F}'' \) satisfies (iv), so does \( \mathcal{F} \). Therefore, assume that all sequences in \( \mathcal{F} \) end with the pair (1,2). If there exists a sequence \( \sigma'' \in \mathcal{F} \) (possibly \( \sigma \) itself) that begins with the pair (2,1), then delete the final (1,2) in \( \sigma \) and the first (2,1) in \( \sigma'' \) to obtain the set \( \mathcal{F}^* \), which satisfies (i)–(iii). Because \( \mathcal{F}^* \) satisfies (iv) by inductive hypothesis, and contains one fewer sequence that begins with 2 and one fewer sequence that ends with 2 than does \( \mathcal{F} \), we conclude once again that \( \mathcal{F} \) satisfies (iv). Thus, we are left only with the case in which \( \mathcal{F} \) consists entirely of sequences that both begin and end with the pair (1,2). Now from (ii), pairs alternate between (1,2) and (2,1) along a sequence. Hence, in this final case each sequence in \( \mathcal{F} \) contains one more occurrence of (1,2) than of (2,1). But this violates property (iii), so this final case is impossible.

Now, we have been assuming that there exists a sequence \( \sigma \) that ends with (1,2). But a similar argument applies if all sequences end with (2,1). We conclude that (A.3) holds after all.

For convenience (so that we do not have to deal with fractions), suppose that \( T + 1 \) is divisible by 3. We next claim that

\[
(A.4) \quad 3\rho_x \leq \eta(T + 1), \quad \text{for all } X = A, B, C.
\]

To see that (A.4) holds, we will use a different accounting method for transactions. Earlier we partitioned the transactions in \( \mathcal{T}(\alpha) \)
into complete trader-sequences, each of which traces out the succession of transactions by a particular trader that his initial sale of a physical portion of some good gives rise to. We now introduce the concept of a complete $X_2$-sequence ($X_2 = A_2, B_2, C_2$), which traces out the trading history of a particular physical portion of $X_2$. Consider, for example, a complete trader-sequence (with transactions in $\mathcal{T}(\alpha)$) for an $A$-trader that begins with him trading a physical portion $\alpha'$ of $A_2$. One obtains a complete $A_2$-sequence by taking the full history of equilibrium exchanges that involve $\alpha'$. For example, a possible $A_2$-sequence might be as follows: the $A$-trader trades $\alpha'$ for a physical portion $\beta'$ of $B_1$ (which he consumes), where his trading partner is a $B$-trader who trades $\beta'$ for $\alpha'$; then (in some later period) the $B$-trader trades $\alpha'$ for a physical portion $\gamma'$ of $C_1$ (which he consumes), where the trading partner is a $C$-trader who trades $\gamma'$ for $\alpha'$ (which he consumes). In this case, the complete $A_2$-sequence consists of two exchanges (each comprising two complementary transactions): the first involving $\alpha'$ and $\beta'$, and the second $\alpha'$ and $\gamma'$. Complete $B_2$- and $C_2$-sequences are defined analogously. Notice that an exchange involving two low-quality goods will belong to two complete $X_2$-sequences: one for each good.

As defined, there are $\eta$ complete $X_2$-sequences ($X_2 = A_2, B_2, \text{or } C_2$) with trades in $\mathcal{T}(\alpha)$. Each complete $X_2$-sequence consists of at most $T$ exchanges. Suppose that we add a “null” exchange at the end of each complete $X_2$-sequence to obtain an extended $X_2$-sequence. Then we have an upper bound of $\eta(T + 1)$ exchanges in all. If we can show that, for each exchange (belonging to some extended $X_2$-sequence) in which there is consumption of $B_1$ or $B_2$ (a $B_1$- or $B_2$-consumption exchange), we can associate two other exchanges (in the union of all extended $X_2$-sequences) in which there is no consumption of $B_1$ or $B_2$ and that are not associated with some other $B$-consumption exchange, then we will have established (A.4) in the case $X = B$ (Clearly, the cases $X = A$ and $X = C$ then follow by symmetry.) Actually, as mentioned above, an exchange in which, say, an $A$-trader sells $A_2$ in order to consume $B_2$ appears in both an extended $A_2$- and a $B_2$-sequence. Therefore, to avoid double counting we shall count this as a $B_2$-consumption exchange only in the extended $X_2$-sequence corresponding to the good not being consumed—in this example the $A_2$-sequence. (It does not matter if we double-count non-$B$-consumption exchanges since the upper bound $(T + 1)\eta$ itself includes all such double counts.)
Let $\sigma_A$ be an extended $A_2$-exchange in which a physical portion $\alpha'$ of $A_2$ is traded. We shall work backwards inductively from the last $B$-consumption exchange in $\sigma_A$ to demonstrate that each $B$-consumption exchange $e$ in $\sigma_A$ can be associated with two non-$B$-consumption exchanges $\hat{e}$ not already associated with some other $B$-consumption exchange and each having one of the following properties: (i) $\hat{e}$ is the null exchange in $\sigma_A$ (if $e$ is the last $B$-consumption exchange in $\sigma_A$); or (ii) $\hat{e}$ is the duplicate of $e$ (if $e$ entails consumption of $B_2$); or (iii) $\hat{e}$ is the final exchange in $\sigma_A$ if it does not entail the purchase of $B_2$ by an $A$-trader; or (iv) $\hat{e}$ is an exchange in $\sigma_A$ in which an $A$-trader acquires $\alpha'$ and which follows $e$ and precedes the next $B$-consumption exchange in $\sigma_A$; or (v) $\hat{e}$ is an exchange in $\sigma_A$ (different from that satisfying (iv)) in which a $B$-trader sells $\alpha'$ and which follows $e$ and precedes the next $B$-consumption exchange in $\sigma_A$; or (vi) $\hat{e}$ is the duplicate of an exchange satisfying (iv) (if the $A$-trader's partner is a $B$-trader and $B_2$ or $C_2$ is exchanged for $\alpha'$); or (vii) $\hat{e}$ is an exchange in which an $A$-trader acquires $B_1$ that he then resells in an exchange satisfying (iv).

To accomplish this demonstration, suppose first that $e$ is the last $B$-consumption in $\sigma_A$. Take the null exchange in $\sigma_A$ as one of the non-$B$-consumption exchanges associated with $e$ (property (i)). If $e$ entails consumption of $B_2$, then it also appears in some complete $B_2$-sequence, where by our accounting rules it does not count as a $B$-consumption exchange. Thus, in this case we associate this duplicate of $e$ with $e$ (property (ii)). If $e$ entails consumption of $B_1$, then $\hat{e}$ cannot be the final exchange in $\sigma_A$. (The final exchange involves $\alpha'$ being consumed, and so in that exchange, $\alpha'$ is bought by a $C$-trader. But a $C$-trader cannot sell $B_1$.) In this case we associate $e$ with the final exchange in $\sigma_A$, provided that it does not entail purchase of $B_2$ by an $A$-trader (property (iii)). If the final exchange does entail purchase of $B_2$ by an $A$-trader (and, hence, sale of $\alpha'$ by that trader), then there exists another exchange in $\sigma_A$—before the final one but after $e$—in which this $A$-trader acquires $\alpha'$. In that case let us associate this exchange with $e$ (property (iv)).

Next suppose that $e$ is a $B$-consumption exchange in $\sigma_A$ such that the inductive hypothesis holds for all subsequent $B$-consumption exchanges. We must show that $e$ also satisfies the inductive hypothesis. Let $e'$ be the next $B$-consumption exchange in $\sigma_A$. Now, $e'$ cannot immediately follow $e$, since first $\alpha'$—which is bought by a non-$A$-trader in $e$ (we have ruled out transactions
between two traders of the same type)—must be acquired by an A-trader in some exchange $e'$ (in order then to be sold in $e''$). From property (iv) we can associate $e''$ with $e$. If there exists another exchange in $\sigma_A$ between $e$ and $e'$ in which a $B$-trader sells $B_2$, then from property (v) it can also be associated with $e$. Therefore, assume that no such exchange exists. Now if $e$ involves $B_2$-consumption, then from property (ii) we can associate its duplicate in the corresponding complete $B_2$-sequence. Therefore, assume that $e$ involves $B_1$-consumption. Then, the $\alpha'$-purchaser in $e$ is a $B$-trader. Moreover, in $e''$ an A-trader buys $\alpha'$ from this $B$-trader. (Otherwise, contrary to our above assumption, there would be a non-$B$-consumption exchange between $e$ and $e''$ in which the $B$-trader sold $\alpha'$ to someone else.) Hence $e''$ takes the form

$$e'' = \begin{cases} 
(a) \text{A-trader trades } B_1 \text{ for } A_2, \text{ and } B\text{-trader trades } A_2 \text{ for } B_1 \\
(b) \text{A-trader trades } B_2 \text{ for } A_2, \text{ and } B\text{-trader trades } A_2 \text{ for } B_2 \\
(c) \text{A-trader trades } C_2 \text{ for } A_2, \text{ and } B\text{-trader trades } A_2 \text{ for } C_2 .
\end{cases}$$

In case (a) there must be some earlier exchange $\hat{e}$ (possibly in an $X_2$-sequence different from $\sigma_A$) in which the A-trader acquired the $B_1$ that he sells in $e'$. In that case let $\hat{e}$ be the other exchange associated with $e$ (This assignment satisfies the inductive hypothesis (vii) since the A-trader resells $B_1$ before $e'$.) In case (b) the exchange $e''$ also appears in some extended $B_2$-sequence, and so from property (vi) we can let this second occurrence also be associated with $e$. In case (c) $e''$ also appears in some extended $C_2$-sequence, and so again from property (vi) we can also associate the duplicate with $e$.

Next suppose that $e$ belongs to an extended $B_2$-sequence $\sigma_B$, in which physical portion $\beta'$ of $B_2$ is traded. We shall work backwards inductively from the last $B$-consumption in $\sigma_B$ to demonstrate that each $B$-consumption exchange $e$ in $\sigma_B$ can be associated with two non-$B$-consumption exchanges $\hat{e}$ not already assigned to some other $B$-consumption exchange and each having one of the following properties: (i') $\hat{e}$ is the null exchange in $\sigma_B$ (if $e$ is the final exchange of $\sigma_B$); (ii') $\hat{e}$ is an exchange in $\sigma_B$ in which a $C$-trader acquires $\beta'$, which he then resells either in $e$ or in a $\sigma_B$ exchange preceding $e$ and following all $B$-consumption exchanges before $e$ in $\sigma_B$; (iii') $\hat{e}$ is an exchange in which an $A$-trader acquires $B_1$, which he then resells in either in $e$ or in a $\sigma_B$-exchange preceding $e$ and following all $B$-consumption exchanges in $\sigma_B$ before $e$; (iv') $\hat{e}$ is an exchange in $\sigma_B$ in which an $A$-trader acquires
β', which he then resells in e; (v') e is the duplicate of an exchange satisfying (iv').

Suppose first that e is the last exchange of σB. Then from (i') we can associate e with the null exchange of σB. Now, by our accounting rule, e does not entail trade of A or C (otherwise, it would not count as a B-consumption exchange in σB). Hence, e must involve the exchange of β' for A1 or B1 (since an A-trader is involved). In the former case a C-trader is involved, and there must exist a previous exchange in σB in which this trader acquires β'. From property (ii') this exchange can then be associated with e. (Notice that this exchange cannot already be associated with a B-consumption exchange in σ because the only non-σ exchanges involving B2 that such exchanges are associated with entail either trade between an A- and B-trader (property (vi)) or consumption of B2 by an A-trader (property (ii)). In the latter case there must be an earlier exchange ce in which the A-trader acquires the B1 that he sells in e. Hence, from property (iii') we can associate ce with e. ce cannot already be associated with a B-consumption exchange in σ because the A-trader acquires the B1 that the A-trader acquires in ce resold in σB.)

Next assume that e is a B-consumption exchange in σB such that properties (i')–(v') hold for all subsequent B-consumptions in σB. If e entailed consumption of β', it would have to be the final exchange in σB. Hence, suppose that it entails consumption of B1. Now there must exist a previous exchange e' in σB in which the A-trader who consumes B1 buys β' in order to trade it for B1. From property (iv') we can let e' be one of the non-B-consumption exchanges associated with e. e' cannot be associated with a B-consumption exchange in σ because the facts that it does not belong to σ, and entails the purchase of β' by an A-trader conflict with (i)–(vii)), but that still leaves another to be found. Now in e' the A-trader exchanges Xi for β', where Xi = B1, C2, A2, or A1. If Xi = B1, then there must be an earlier exchange ce in which the A-trader acquires B1, in which case from (iii') we can let ce be the other associated non-B-consumption exchange. (As in the previous paragraph, ce cannot be associated with a B-consumption exchange in σ because, if it were, property (vii) would require that A-trader resell the B1 in a σ-exchange.) If Xi = C2, then the exchange e' also occurs in an extended C-sequence. In which case, from (v'), we can also associate this second occurrence with e. Similarly, if Xi = A2, then e' also appears in an extended A2-sequence, and because e' entails a purchase of B2 by an A-trader,
(i), (vi), and (vii) imply that it is not already associated with a $B$-consumption exchange in such a sequence. Hence, the duplicate occurrence of $e'$ can also be associated with $e$. Finally, if $X_i = A_1$, then the $A$-trader must trade with a $C$-trader in $e'$. Hence, there must be some earlier exchange $e''$ in which this $C$-trader acquires $\beta'$. And so from (ii') $e''$ can be associated with $e$. However, it cannot be associated with a $B$-consumption exchange in $\sigma_A$ because, if it were, (vi) implies that it would entail trade between an $A$- and a $B$-trader.

It remains to consider the case where $e$ belongs to an extended $C_2$-sequence $\sigma_{C_2}$, in which a physical portion $\gamma'$ of $C_2$ is traded. We shall work backwards inductively from the last $B$-consumption in $\sigma_C$ to demonstrate that each $B$-consumption exchange in $\sigma_C$ can be associated with two non-$B$-consumption exchanges $e'$ not already assigned to some other $B$-consumption exchange and each having one of the following properties: (i'') $\hat{e}$ is the null exchange in $\sigma_C$ (if $e$ is the final $B$-consumption exchange in $\sigma_C$); (ii'') $\hat{e}$ is an exchange in $\sigma_C$ in which an $A$-trader acquires $\gamma'$, which he then resells in $e$; (iii'') $\hat{e}$ is the duplicate of $e$ (if $e$ entails consumption of $B_2$); (iv'') $\hat{e}$ is an $\sigma_{C'}$-exchange in which an $A$-trader acquires $\gamma'$ and which occurs after $e$ and before the next $B$-consumption exchange in $\sigma_C$; (v'') $\hat{e}$ is a $\sigma_{C'}$-exchange in which a $B$-trader sells $\gamma'$ to a $C$-trader and which occurs after $e$ and before the next $B$-consumption exchange in $\sigma_C$; (vi'') $\hat{e}$ is an exchange in which an $A$-trader acquires $B_1$ and later sells it in a $\sigma_{C'}$-exchange between $e$ and the next $B$-consumption exchange in $\sigma_C$.

Suppose that the inductive hypothesis holds for all $B$-consumptions following $e$ in $\sigma_C$. We must show that the same is true of $e$. Now in $e$ an $A$-trader sells $\gamma'$ (to buy $B_1$ or $B_2$). Thus, there must be an earlier exchange $e'$ in which this trader acquires $\gamma'$. From (ii'') $e'$ can be associated with $e$. (The exchange $e'$ could not be associated with a $B$-consumption in an extended $A_2$-sequence because (ii), (vi), and (vii) imply that, since it is not an $A_2$-sequence exchange, it would have to entail $B_2$-consumption, or acquisition of $A_2$ or $B_1$ by an $A$-trader. Similarly, $e'$ cannot be associated with a $B$-consumption exchange in an extended $B_2$-sequence because if it were, (iii') and (v') would imply that it would entail an $A$-trader buying $B_1$ or $B_2$.)

Now if $e$ is the final $B$-consumption exchange in $\sigma_C$, we can from (i'') choose the null exchange as the other non-$B$-consumption exchange associated with $e$. Therefore, assume that
\( e \) is followed by a \( B \)-consumption exchange \( e' \) in \( \sigma_c \). If \( e \) entails an \( A \)-trader consuming \( B_2 \), then the duplicate of \( e \) also appears in an extended \( B_2 \)-sequence (where, however, it does not count as a \( B \)-consumption exchange). Because of the \( B_2 \)-consumption, \((i')-(vi')\) imply that this duplicate of \( e \) is not associated with any \( B \)-consumption in the \( B_2 \)-sequence. Therefore, from \((iii'')\) we can choose this duplicate to be the other non-\( B \)-consumption exchange associated with \( e \). Therefore, assume that \( e \) entails an \( A \)-trader buying \( B_1 \) (from a \( B \)-trader). Now, because \( e \) and \( e' \) each entail an \( A \)-trader selling \( \gamma' \), there exists an exchange \( e'' \) between \( e \) and \( e' \) in which an \( A \)-trader acquires \( \gamma' \). From \((iv'')\) \( e'' \) can be associated with \( e \) unless it has already been associated with \( e' \) (\( e'' \) cannot be associated with a \( B \)-consumption exchange in an \( A_2 \)-sequence since, if it were, \((ii), (vi), \) and \((vii)\) would imply that it entails an \( A \)-trader buying \( B_2, A_2, \) or \( B_1 \); it cannot be associated with a \( B \)-consumption exchange in a \( B_2 \)-sequence because, if it were, \((iii'')\) and \((v')\) would imply that it entails an \( A \)-trader buying \( B_1 \) or \( B_2 \)). Now in the latter case we may assume that the \( A \)-trader in \( e'' \) acquires \( \gamma' \) from the \( B \)-trader in \( e \) (otherwise, there would exist an exchange between \( e \) and \( e'' \) in which the \( B \)-trader sells \( \gamma' \) to some \( C \)-trader, and from \((v'')\) \( \gamma' \) that exchange could be associated with \( e \)). Suppose first that the \( A \)-trader sells \( A_2 \) in \( e'' \) to buy \( \gamma' \). In that case there must be an earlier exchange \( \hat{e} \) in which the \( A \)-trader acquires \( B_1 \), and from \((vi')\) \( \hat{e} \) can be associated with \( e \). (From \((vii)\) and \((vii')\) \( \hat{e} \) cannot be associated with a \( B \)-consumption exchange in an \( A_2 \)- or \( B_2 \)-sequence \( \sigma_A \) or \( \sigma_B \).) Next, suppose that the \( A \)-trader sells \( A_2 \) to buy \( \gamma' \) in \( e'' \). Now, the duplicate of \( e'' \) appears in an extended \( A_2 \)-sequence \( \sigma_A \). But this duplicate cannot be the last exchange in \( \sigma_A \) (the last exchange entails consumption of \( A_2 \) by a \( C \)-trader), and so it cannot be associated with the last \( B \)-consumption exchange in \( \sigma_A \). Nor can it be associated with any other \( B \)-consumption exchange in \( \sigma_A \) since from \((i)-(vii)\) no such association entails an \( A \)-trader buying \( C_2 \). Hence, this second occurrence of \( e'' \) can be associated with \( e \). Finally, suppose that the \( A \)-trader sells \( B_2 \) in \( e'' \) to buy \( \gamma' \). Then, although the duplicate of \( e'' \) appears in some extended \( B_2 \)-sequence, \((i')-(v')\) imply that this duplicate is not associated with a \( B \)-consumption exchange in that sequence because \( e'' \) entails an \( A \)-trader buying \( C_2 \) from a \( B \)-trader. Hence, again we can associate the duplicate.

26. That exchange cannot already be associated with a \( B \)-consumption exchange in an \( A_2 \)- or \( B_2 \)-sequence, thanks, as usual, to \((ii), (vi), \) and \((vii)\) in the former case and to \((iii'')\) and \((v')\) in the latter case.
of \( e' \) with \( e \). This concludes the demonstration that each \( B \)-
consumption exchange can be uniquely associated with two non-
\( B \)-consumption exchanges, and hence establishes (A.4).

Because the square-bracketed expressions in (A.2) are each
no greater than one, (A.4) implies that

\[
(A.5) \quad \frac{(k^A)^{n_A} (k^B)^{n_B} (k^C)^{n_C}}{2^n} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{\eta(T+1)/3} \times \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{\eta(T+1)/3} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{\eta(T+1)/3} \leq 1.
\]

Now, because \( a_2 > 0 \), \( \pi_A > 0 \). Hence if \( k^C < k^A \) and \( k^C \leq k^B \), then
(A.3) and (A.5) imply that

\[
\frac{k^C}{2} \left[ \frac{2(k^A a_1 + a_2)}{k^A} \right]^{(T+1)/3} \left[ \frac{2(k^B b_1 + b_2)}{k^B} \right]^{(T+1)/3} \left[ \frac{2(k^C c_1 + c_2)}{k^C} \right]^{(T+1)/3} < 1,
\]

a contradiction of Lemma 1. Similarly, \( k^B < k^A \) and \( k^B \leq k^C \) lead
to a contradiction, and so we have \( k^C \geq k^A \) and \( k^B \geq k^A \). Therefore,
(A.5) implies that the left-hand side of (10) is no greater than 1
when \( X = A \). But this and Lemma 1 together imply that (10) holds
when \( X = A \).

QED

**PROPOSITION 3.** Suppose that the hypotheses of Proposition 1
hold. In any equilibrium all exchanges are regular and stan-
dard. Furthermore, in any period the entire produced quan-
tity of \( A_2 \) is exchanged for high-quality output.

**Proof.** For convenience let us suppose that \( T + 1 = 3T^* \) (i.e.,
\( T + 1 \) is divisible by 3). From Proposition 1 all purchases of \( B_1 \) by
\( A \)-traders must be mediated by \( A_2 \). That is, they must be \( B_1 \)-
standard exchanges. Moreover, if all these exchanges are regular
(i.e., occur in periods \( 1 + 3r, r = 0, \ldots, T^* - 1 \)), then because
from Proposition 2 the corresponding prices are \( 2, 2/q, \ldots, 2/
q^{T^*-1} \), we conclude that indeed the entire quantity produced of \( A_2 \)
must be exchanged for \( B_1 \) in each of these periods. This follows
because

\[
(A.6) \quad \frac{a_2}{2} \left( 1 + q + q^2 + \cdots + q^{T^*-1} \right) = \frac{1}{2},
\]

where the left-hand side of (A.6) corresponds to the quantities of
\( B_1 \) that can be obtained by selling \( a_2 \) in each of periods \( 1, 4, \ldots, 1 + 3(T^* - 1) \), and the right-hand side corresponds to the per
capita production of $B_1$. In other words, if anything less than the entire produced quantity of $A_2$ were exchanged for $B_1$ in each period $1, 4, \ldots, 1 + 3(T^* - 1)$, then, at the equilibrium prices prevailing in those periods, it would not be possible for the entire produced quantity of $B_1$ to be sold.

We obtain analogous results for standard $C_1$- or $A_1$-exchanges, as long as all these are regular. That is, in each period where these exchanges take place, the entire produced quantity of $A_2$ is traded for $C_1$ or $A_1$. This means that there is no $A_2$ left over to mediate any nonstandard exchanges.

Hence, the proposition is established provided that we can show that there are no nonregular standard exchanges in equilibrium. To the contrary, suppose that there exists an equilibrium with such an exchange. Consider the last period $\hat{t}$ in which such an exchange occurs. For concreteness assume that the exchange $\hat{c}$ is a standard $B_1$-exchange. Then, for some $\hat{\tau} = 0, \ldots, T^* - 2$, $\hat{t} \in \{2 + 3\hat{\tau}, 3(\hat{\tau} + 1)\}$. Now, if the $B$-trader subsequently retrades all the $A_2$ he acquired in $\hat{c}$ and obtains exclusively $B_1$ for this $A_2$, then $\hat{c}$ is, in effect, “cancelled out,” and we can move back to the last period in which there is a nonregular standard exchange that is not cancelled out. (If all nonregular standard exchanges are cancelled out, then, ultimately, $B_1, C_1$ are acquired by $A$-, $B$-, and $C$-traders entirely through regular standard exchanges, which means—that given that from (A.6) all $A_2$ is devoted to these exchanges—that there can be no nonregular standard exchanges at all.) Thus, assume that at least some of the $A_2$ acquired by the $B$-trader in $\hat{c}$ is not resold for $B_1$. Therefore, it must be resold for $C_1$; i.e., it is traded in a standard $C_1$-exchange. But by hypothesis such an exchange must be regular (since it comes after period $\hat{t}$) and so cannot occur until at least period $2 + 3(\hat{\tau} + 1)$. Hence, the price of $C_1$ in terms of $A_2$ is at least $2/q^{\hat{\tau}+1}$. This, in turn, implies that the price of $B_1$ in terms of $A_2$ in period $\hat{t}$ is at least $2/q^{\hat{\tau}+1}$ (otherwise a $B$-trader selling $B_1$ for $A_2$ and then reselling the $A_2$ for $C_1$ would be better off waiting at least until period $1 + 3(\hat{\tau} + 1)$ to sell the $B_1$, when it would fetch a price of $2/q^{\hat{\tau}+1}$). Now, if the

27. Note that the quantity of $B_1$ that the $B$-trader obtains from retrading the $A_2$ can be no less than the quantity of $B_1$ he sells in exchange $\hat{c}$ (otherwise, he would be better off not selling $B_1$ in $\hat{c}$). However, suppose that it were more. Then assume for convenience that he obtains it in a single exchange $c'$ with an $A$-trader. Because all $A$-traders have equal utility, we can think of this as the same $A$-trader as in exchange $\hat{c}$. But then this $A$-trader would be better off not undertaking exchanges $c'$ and $c''$, since, by refraining, he would end up with more of $B_1$ and the same quantity of $A_2$.

We conclude that $c'$ is exactly cancelled out in the sense that the $B$-trader is left in the same position as though he had never executed it.
A WALTASIAN THEORY

To see that the A-trader sells in \( \hat{c} \) is not obtained from a previous exchange (i.e., he produces it himself), he would be better off selling it in period \( 1 + \hat{3} \hat{t} \), when the price of \( B_1 \) is only \( 2/q^* \), than in period \( \hat{t} \). Hence, he must obtain it from a previous exchange \( \hat{c}' \) in period \( \hat{t} \), where \( 1 + 3\hat{t} \leq \hat{t}' < \hat{t} \). Specifically, he must obtain it by either (i) selling \( B_1 \) to a B-trader or (ii) selling \( A_1 \) to a C-trader.

Consider case (i). Because the A-trader sells \( B_1 \) in \( \hat{c}' \) and then buys \( B_1 \) in \( \hat{c} \), \( p_{\hat{t}}(B_1,A_2) \geq p_{\hat{t}}(B_1,A_2) \) (otherwise he would be better off not selling \( B_1 \) in \( \hat{c}' \)). Given that prices are nondecreasing over time, therefore, \( \hat{t}' = \hat{t} - 1 \), and \( p_{\hat{t}-1}(B_1,A_2) = 2/q^{i+1} \). The B-trader who sells \( A_2 \) in exchange \( \hat{c}' \), in turn, must have acquired it in a still earlier exchange \( \hat{c}'' \) in period \( \hat{t}'' < \hat{t}' \). Moreover, he must have sold either \( B_1 \) or \( C_1 \) to obtain it. In the former case, \( p_{\hat{t}}(B_1,A_2) \geq p_{\hat{t}}(B_1,A_2) \) (otherwise, the B-trader would be better off not selling \( B_1 \) in period \( \hat{t}'' \), and hence \( p_{\hat{t}}(B_1,A_2) = 2/q^{i+1} \). Thus, \( p_{\hat{t}-2}(B_1,A_2) = 2/q^{i+1} \), a contradiction, since \( \hat{t} \) is no more than two periods after \( 1 + 3\hat{t} \), and the price of \( B_1 \) in terms of \( A_2 \) in period \( 1 + 3\hat{t} \) is \( p_{1+3\hat{t}}(B_1,A_2) = 2/q^* \). In the latter case, \( p_{\hat{t}}(C_1,A_2) = 2/q^{i+1} \). We can therefore derive the same contradiction as in the former case.

We are left with case (ii), which means that \( \hat{c}' \) is a standard exchange. In this case, \( p_{\hat{t}}(A_1,A_2) = 2/q^* \) since \( \hat{t}' < 3 + 3\hat{t} \). Indeed, \( p_{\hat{t}}(A_1,A_2) = 2/q^* \). Otherwise, the A-trader is better off waiting until period \( 3 + 3\hat{t} \) to sell the \( A \), and until period \( 1 + 3(\hat{t} + 1) \) (when the price of \( B_1 \) is still only \( 2/q^{i+1} \)) to buy \( B_1 \). We conclude that \( \hat{c}' \) is nonregular. Thus, we have shown that if there exists a nonregular standard exchange, then the trader selling \( A_2 \) in that exchange must have obtained it from a previous exchange that is also nonregular and standard. Moving backward in this way, we ultimately reach a nonregular standard exchange in which a trader sells \( A_2 \) that \textit{cannot} have been obtained from a previous exchange, a contradiction.

QED

Proof That Equilibrium Does Not Exist When \( \delta \) Is Small

Suppose that \( T = 2 \) and \( k^A < k^B < k^C \). We would expect \( A_2 \) to be the medium of exchange if an equilibrium existed. This means

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28. Otherwise, the quantity \( Q_{a} \) of \( B \), which the B-trader buys in period \( \hat{t}' \) is smaller than the quantity \( Q_{c} \) of \( C \), which he consumes. Moreover, \( Q_{a} \leq Q_{c} \) since \( b_{1} = c_{1} \) in equilibrium (if the inequality went the other way, the B-trader could consume \( c_{1} > b_{1} \)). Hence, \( Q_{a} < Q_{c} \) implies that \( Q_{a} < Q_{c} \), i.e., the B-trader ends up with a smaller quantity of \( C \) than he had in period \( \hat{t}'' \), a contradiction.
that since $A_2$ cannot be consumed until period 2, a $C$-trader’s equilibrium utility is $\delta$. (There is not time if $T = 2$ for $A_1$ to get into the hands of a $C$-trader.) Because in equilibrium a $B$-trader must first acquire $A_2$ in period 1 before buying $C_1$, his equilibrium utility is $\delta k^C/2$. However, since he has the option of selling $B_2$ for $C_2$ in the first period, we must have

\[(*) \quad p_1(B_2, C_2) \leq \delta k^C / 2.\]

Now, because an $A$-trader consumes $B_1$ in equilibrium, he must buy this all in period 1 (otherwise, there will not be enough time for the $A_2$ he sells to get into the hands of $C$-traders). Hence, his equilibrium utility level is $k^B/2$. Alternatively, because he could buy $C_2$ in period 1 and resell it for $B_2$ in period 2, we must have

\[
\frac{\delta}{p_1(C_2, A_2)p_2(B_2, C_2)} \leq \frac{k^B}{2},
\]

which implies, in view of (*), that

\[(**) \quad 4/k^Ck^B \leq p_1(C_2, A_2).\]

But suppose that a $C$-trader sells all his $C_2$ for $A_2$ in period 1. From (**) his utility would be at least $4/k^Ck^B$, which exceeds $\delta$ when the latter is small, a contradiction of equilibrium. Consequently, no equilibrium exists.

REFERENCES


