THE
QUARTERLY JOURNAL
OF ECONOMICS

Vol. CXI November 1996 Issue 4

A WALRASIAN THEORY OF MONEY AND BARTER*

ABHIJIT V. BANERJEE AND ERIC S. MASKIN

We study a barter economy in which each good is produced in two qualities and no trader can distinguish between the qualities of those goods he neither consumes nor produces. We show that in competitive equilibrium there exists a (unique) good—the one for which the discrepancy between qualities is smallest—that serves as the *medium of exchange:* this good mediates every trade. Equilibrium is inefficient because production of the medium would be lower if it were not for its mediating role. Introducing fiat money enhances welfare by eliminating this distortion. However, high inflation drives traders back to the commodity medium.

I. INTRODUCTION

Money has always been something of an embarrassment to economic theory. Everyone agrees that it is important; indeed, much of macroeconomic policy discussion makes no sense without reference to money. Yet, for the most part theory fails to provide a good account for it. Indeed, in the best developed model of a competitive economy—the Arrow-Debreu [1954] framework—there is no role for money at all. Rather than there being a medium of exchange, prices are quoted in terms of a fictitious unit of account, agents trade at those prices, and that is the end of the story.

One important exception to the rule that money plays no essential part in theory is the overlapping generations consumption-loan model [Samuelson 1958]. In that model, on which there

*The basic model in this paper was formulated in January 1988, while the second author was visiting St. John's College, Cambridge, United Kingdom. We thank St. John's College and the National Science Foundation for research support. We are grateful to Robert Barro, Theodore Bergstrom, Martin Hellwig, Andreu Mas-Colell, Thomas Sargent, and two referees for helpful comments.

is a considerable literature, the introduction of fiat money permits borrowing and lending across generations and so can dramatically alter the nature of equilibrium. But even there money is indispensable only because, by assumption, there are no other durable assets. Once we admit other assets (e.g., land) that survive over time (and so can serve as a store of value), fiat money loses its central purpose.[1]

Of course, we know from everyday experience that money acts not only as an intergenerational store of value but as a *medium of exchange*. As Jevons [1875] pointed out, it eliminates the need for a "double coincidence of wants." If I have apples but want bananas, then, in a barter economy, I must wait until I can find someone willing to give up bananas for apples in order to trade, and this delay may be costly. By contrast, in a monetized economy the trader who buys my apples need not be the same as the one who sells me bananas, and this decoupling of identities relaxes the constraints on trade.

But there are at least two reasons why this contrast does not completely clinch the case against barter. First, just as in a well-organized economy there is a place where one can go to sell apples and a place where one can buy bananas, we can, in principle, imagine a place where one could go to exchange apples for bananas. If such a place were to exist, there would be no delay in finding trading partners, and so one important argument against barter would collapse. Of course, if similar provisions were made for all pairs of goods, our economy would require $n(n-1)/2$ markets (assuming n goods), instead of the usual n, and one might object that this proliferation of markets would itself be costly. But, second, even if there were only n markets, we could still ask why, if I want bananas, I cannot simply go to the banana market and pay for bananas with apples. After all, even if the banana-seller does not want the apples herself, she can always sell them again. In other words, barter seems no worse than monetary exchange if apples can serve as media of exchange.[2]

1. In an economy with uncertainty it may still have value in helping to diversify portfolios, but in this respect it is not distinguished from many nonmonetary assets.

2. This position was challenged by Ostroy and Starr [1974], who argued that one cost of barter is the long chain of exchange transactions that must occur before an economy clears. However, their perspective conceives of a barter economy as one without well-organized markets, and regards the cost of transaction as proportional to the length of the chain. If, as we have tried to do, one instead assumes that markets *are* well organized, and measures cost as proportional to the length of an *individual* trader's transaction chain, the distinction between barter and

The point of view we take in this paper is that the reason the banana-seller may not accept apples is that she cannot properly evaluate them. That is, if she does not know much about apples, she may not be able to discern the value of the apples she is presented with.[3] There are at least two ways in which we can interpret this failure of discernment. The orange-seller may simply be unable to appreciate the apples' *physical* characteristics, e.g., perhaps she cannot distinguish between ripe and rotten fruit. Alternatively, she may be ignorant of their *market* characteristics. That is, she may not know how much she can resell them for later on. (The difficulties that ignorance of market characteristics pose for barter have been emphasized by Friedman [1960]). The banana-seller's ignorance about apples might not matter much if she could be sure that I am as ignorant as she. If we were both approximately risk-neutral, we could set the banana/apple exchange rate in terms of the *expected* price of apples. But if she suspects that I know more about apples than she does, she may worry that I will take advantage of her. That is (if the reader will excuse the mélange of fruit), she may fear that I will try to stick her with a "lemon" [Akerlof 1971]. By postulating such an asymmetry, we are trying to capture an elementary fact of economic life. As traders we are reasonably familiar with the physical properties and prices of the goods we buy and sell on a regular basis. But for each of us there is a vast array of goods with which we have little experience. Moreover, for different traders the sets of unfamiliar goods are quite different, so that if someone tries to sell us something that we do not know much about, we become wary of being exploited.

From this perspective, the role of money becomes clear. Money is simply a good whose physical characteristics can be reasonably well discerned by *every* trader, and whose current and future market prices are known to the trader in terms of the goods that he buys and sells frequently. Thus, it is a device for overcoming the adverse selection problem that arises in barter. In other words, in Jevons' phrase, money is *identifiable*. It is this attribute of money, namely the ability to overcome asymmetric

monetary exchange, we argue, largely vanishes (see the discussion following Proposition 5).

3. The problem confronting the banana-seller would be underscored more dramatically if instead of attempting to use apples as payment I offered, say, an Impressionist painting, for which the difficulty of evaluation would presumably be particularly acute. Hence, an early title for this paper was "A Monet Theory of Money."

information problems, which, according to Alchian [1977], is the principal advantage of monetary exchange over barter.

The model we develop below is an attempt to bring out this point formally. We stay as close to a standard Walrasian model as possible. Thus, we assume that (i) trade is anonymous (so that traders cannot trust one another to keep promises about quality or delivery, nor can such promises be enforced by third parties);[4] (ii) markets are well organized but decentralized (for each good there is a known location where that good can be bought and sold, but there is no central clearinghouse); and (iii) markets are competitive, i.e., each trader takes all prices as given. The major point of departure is that we suppose that each good comes in two qualities—say, high and low—and that, although producers and consumers of a given good can distinguish between its qualities, other traders cannot. Thanks to the lemons problem, these uninformed traders, when presented with this good in a transaction, will naturally presume that it is of *low* quality. They will, therefore, be willing to pay the low-quality but not the high-quality price for the good. We deduce that a version of *Gresham's law* pertains to our model: only low-quality versions of goods are candidates to be media of exchange, since only low-quality goods can be properly priced. This already makes a significant contrast with the Arrow-Debreu setting, where any good can serve as a medium of exchange.

We will show, however, that not all low-quality goods can function as media of exchange. In fact, generically, the medium is *unique*. Moreover, it corresponds to (the low-quality version of) the good for which the *discrepancy* (in a well-defined sense) *between high and low qualities is smallest.*

This finding seems to accord with the evolution of gold and certain other metals—particularly when used for coins—as widespread media of exchange. Historically, two innovations were important to these metals' success: Archimedes' specific gravity test and the serrated edge. Both inventions, in effect, reduced variation in unobservable quality: the specific gravity test by making it hard to pass off base metal as gold or silver, and the serrated edge by defeating the practice of "coin-clipping." Thus, our theoretical finding can be thought of as a formal explanation for the historical prevalence of gold as a medium of exchange.

4. The ban on contracts (third party enforcement of promises) is a stronger assumption than invoked by many Walrasian models. However, it corresponds well to many everyday trades in which money (as opposed to credit) is used.

Another implication of our model is a relation quite similar to the fundamental equation of the classical quantity theory of money:

$$PQ = MV, \tag{*}$$

where P is the price level, Q is output, M is the money supply, and V is the velocity of money. It is notable that *velocity* turns out to be a well-defined and meaningful concept in our framework. As formula (*) suggests, moreover, we can show that if velocity is somehow exogenously increased the equilibrium quantity of money (the good that functions as the medium of exchange) falls. However, in contrast to the standard quantity theory, these shifts are not allocation-neutral in our model. This is because the need for money creates a production distortion. Specifically, in our model a low-quality good is produced and consumed *only* because it can serve as money. In the absence of an informational asymmetry, it is too poor a substitute for the high-quality good to persist in positive quantities. Thus, an increase in velocity (which means that the same quantity of money circulates more frequently, thereby mediating more exchanges) enables the economy to get by with a lower quantity of the low-quality good, a desirable thing. To return to our historical analogy, gold is hardly the most intrinsically useful commodity. Although it serves an ornamental function and has certain industrial uses, most of its value throughout history has derived from its central monetary role. Thus, one can plausibly argue that, relative to an economy with no informational constraints, too much of it is produced, where the distortionary costs are the resources devoted to prospecting, mining, and refining it.

From this standpoint, the great virtue of *fiat money* is its capacity to function as money while being essentially costless to produce. That is, unlike gold, fiat money creates no real distortion in the economy. This perspective helps explain why inflation is costly: a major cost is the risk of "demonetizing" the economy—of driving traders back to using gold. We examine these ideas more carefully below.

Our approach to studying the limitations of barter and the value of money is not the only possible one. Following the seminal paper by Kiyotaki and Wright [1989], a sizable literature has developed that uses a search framework[5] to address some of the

5. We are using the term "search" to denote a class of decentralized trading models, including some in which there is no active search. The use of such models to study money goes back to Diamond [1984].

questions discussed here.[6] One significant difference between the two approaches is that in a search model such as theirs, where there are significant strategic complementarities, one need not appeal to informational asymmetry for there to be costs of barter. The mere fact that a segment of the population is unwilling to accept good A in exchange for other goods will make every other trader also unwilling to accept good A (because he will have a hard time getting rid of it). As a result, certain types of barter may simply be impossible in the particular equilibrium being played. This kind of coordination failure cannot occur in our framework because of the price-taking assumptions we make. Similarly, the strategic complementarities of the search equilibrium framework mean that there are normally multiple equilibria: a variety of goods can serve as the medium of exchange. In contrast, our Walrasian setup permits a sharper prediction about which good will act as money. Indeed, as we have noted, one of our primary theoretical results (Proposition 1) establishes the (generic) uniqueness of equilibrium (and hence of the medium of exchange).

Besides its weaker predictions the search framework has several other drawbacks from our standpoint. First, although search is indeed important in certain markets (such as the labor market), the markets for many other goods are, and historically have been, relatively well integrated and seem to fit the Walrasian framework better.[7] Second, many of the broad macro/policy implications of these models are quite different from those of a Walrasian model. Therefore, one may run into trouble when basing the theory of money on search frictions while relying on the Walrasian model for one's other macroeconomic intuitions. Finally, partly because of the inevitable complexities of search models, the early Kiyotaki and Wright papers did not provide a theory of how the money supply affects the price level. Given that this is a central question of monetary policy, the omission seems important. More recent papers in this tradition—Hayashi and Matsui [1996], Shi [1994, 1995], Trejos and Wright [1995], and Green and Zhou [1995], for example—have gone further toward developing a theory of the price level, but they all either rule out barter or rely in important ways on indivisibilities. By contrast, a Walrasian framework permits us to avoid such constraints.

6. See, for example, Kiyotaki and Wright [1989, 1991, 1993]; Aiyagari and Wallace [1992]; and Marimon, McGrattan, and Sargent [1990].

7. For example, in India everyone knows where to go to buy or sell fish.

There is a broader literature on the role of money in settings with imperfect information. Examples of such papers include Jones [1976], King and Plosser [1986], Bernhardt and Engineer [1987], Smith [1986], Townsend [1989], and Williamson [1990]. All these papers concentrate on issues quite different from the ones we treat. Williamson and Wright [1994], which introduces informational asymmetries about the goods traded into a Kiyotaki-Wright model, is the paper that comes closest to our work (see also Cuadras-Morato [1994]). However, like the Kiyotaki and Wright papers, that article focuses on the indeterminacy of the medium of exchange and the possibility of multiple Pareto-ranked equilibria in this kind of environment. While we agree that this indeterminacy is of interest, we also feel that these results are best viewed against a Walrasian benchmark, such as the one provided by our model. In this sense, we view our efforts as complementary.

In Section II we lay out the basic, finite-period model. In Section III we characterize the (essentially) unique equilibrium of this model and show that it entails the existence of a unique medium of exchange. The connection with the quantity theory of money is also drawn. We consider two straightforward extensions in Section IV: a positive discount rate and differential durability of goods. Then, in Section V we embed our finite-period model in an infinite-horizon framework in order to discuss steady states. One virtue of an infinite horizon is that it enables us to explore the role of fiat money. Specifically, in Section IV we consider the welfare effects of money and inflation. We conclude in Section VII with some comments on future directions for research.

II. The Model

II.1. An Informal Description

It may be helpful to begin with an informal description of the physical setting we have in mind. This will enable us to motivate the formal assumptions that follow in subsections II.2 and II.3.

Imagine an economy with no fiat money in which there are many goods and where the markets for different goods are geographically dispersed. Thus, there is an apple region, a banana region, and so on. There is no central clearinghouse where one can go to buy or sell all these goods.

Each trader in the economy produces a single kind of good—apples, bananas, or whatever—and there are many traders producing each kind of good. Every apple-producer divides his production between high- and low-quality apples (where the marginal benefit and cost of the former are higher). Just as he is single-minded about production, every trader consumes only a single kind of good, but not the same kind as he produces; i.e., traders are not self-sufficient.

Because a trader has dual roles as producer and consumer, it is convenient to think of his having two identities. If he is an apple-producer, one incarnation stays at home in the apple region and sells apples in his own shop. (Thus, there will be many small shops in each region, one for each trader.) The other goes out to buy the good the trader consumes (say, bananas) or, in principle, other goods to be resold (for bananas) at a later date (as we will see, however, this latter sort of arbitrage does not actually occur in equilibrium).

Trade is *bilateral*. That is, to buy bananas, a buyer goes to a banana shop and transacts directly with the banana-seller there. Just as there is no central clearinghouse for the whole economy, neither are there clearinghouses within individual regions.

Trade takes place in a finite number of discrete *trading periods*. The geographic dispersion of markets means that a trader can visit only one region per period. However, once in a region he can visit whichever shop he wishes; this keeps the shops within the region competitive. Moreover, the fact that he is free to choose which region to visit in any given period serves to eliminate arbitrage opportunities across regions. Hence, although there is no Walrasian auctioneer, prices will nevertheless be Walrasian.

Exchanges between the buyer and seller in the banana shop are *unmonitorable* by third parties. This has two important implications. First, it means that the buyer must pay for the bananas on the spot. Any kind of credit or deferred payment would be infeasible because, ultimately, such arrangements rely on a court or some other authority being able to ascertain whether or not a particular transaction took place (otherwise, a buyer who gave a seller an IOU for some bananas could later claim that he never received them or that the IOU was not his).[8] Futures contracts and short sales are ruled out for exactly the same reason. Second, because there is no fiat money, it means that the buyer must pay for his bananas with *physical goods* (either the good

8. If the buyer and seller had an on-going relationship, then conceivably the unmonitorability of their transactions by others might not be essential. Each could "punish" the other for any violations of their implicit agreement, as in the repeated games literature. However, such a "self-enforcing" arrangement normally requires an indefinite time horizon (otherwise, the scheme would unravel

that he produces himself, or other goods that he has acquired in previous transactions). The fact that traders are constrained in this way to *barter transactions* is central to the major task of this paper: to show how the institution of "money" emerges endogenously in a barter economy.

We come to the only departure from an otherwise fairly standard competitive framework: the restrictions on traders' information. A trader who produces apples and consumes bananas can distinguish between high- and low-quality apples and between high- and low-quality bananas (he is *informed* with respect to apples and bananas) but not between the qualities of any other good. The most important consequence of this shortcoming, in combination with the bilateral trade and unmonitorability assumption, is that he cannot execute any trade involving the high-quality variety of any good other than apples and bananas. This implication is crucial to our results below, and so it is worthwhile to examine why it is so.

Consider a sequence of trades by which the trader comes into possession of high-quality coconuts, which he cannot distinguish from their low-quality counterpart (he is an *uninformed* trader with respect to coconuts). This can be traced back to an exchange in which some (possibly different) uninformed trader buys high-quality coconuts from an *informed* trader (since only informed traders produce coconuts). But the latter would be foolish to supply anything but low-quality coconuts if he could get away with it. And the former's ignorance, together with the absence of monitoring, ensures that the informed trader *can* get away with it.[9] This is just the "lemons" problem referred to in the introduction.

We conclude therefore that, except for exchanges in which a double coincidence of wants obtains,[10] at least one side of any exchange must involve a *low-quality* good. Note that the restric-

from the end), whereas our basic model has only finitely many periods. Moreover, both the buyer and seller have many alternative trading partners, and so any punishment would be severely constrained by these outside options.

9. We have not discussed whether or not uninformed traders are ex ante identifiable as such. But even if they are not, an informed seller should quickly be able to discover a buyer's ignorance by first offering him a package including some X_2 and seeing how he reacts. Such a scheme, of course, presumes that the seller *has* X_2 in his possession, and yet one of our principal conclusions below is that equilibrium production of X_2 is zero for all but one type of good X. However, this inconsistency can be overcome by considering a slightly richer model in which small quantities of all low-quality goods are unavoidably produced.

10. A double coincidence of wants will presumably be rare in our setting. When an apple-producer who consumes bananas goes to a banana shop, the probability the banana-seller will turn out to be an apple-consumer is low if preferences for all our many goods are equally likely. Indeed, we rule out double coincidences by *assumption* in the formal model.

tion that trade be bilateral (rather than multilateral) figures in this conclusion. If there were a central meeting point where traders of all kinds could come, then in principle it might be possible to organize a multilateral exchange of purely high-quality goods among traders without any double coincidences at all. To see how this could be done even in the absence of outside monitoring, see footnote 20.

II.2. Production, Preferences, and Trade

Let us lay out the model more precisely. Although we have in mind an economy with many goods, for the purpose of formal analysis, we shall suppose that there are just three types of goods, A, B, and C. Each type $X \in (A,B,C)$ comes in two qualities: X_1 (high quality) and X_2 (low quality). Goods are perfectly divisible.

There are also three types of traders, again labeled A, B, and C, according to the type of good they *produce*. Hence, X-traders produce only goods of type X. More specifically, an X-trader is endowed with one unit of labor, which can be applied to a linear production technology: good X_1 requires two units of labor per unit of output, whereas X_2 requires one unit of labor per unit.[11] An X-trader can allocate his labor endowment in any way he chooses between X_1- and X_2-production. We suppose that there are large but finite (and, for symmetry, equal) numbers of each of A-, B-, and C-traders, so that assuming that traders take prices as given makes sense.

Just as production is linear, we suppose that preferences are linear. A-traders consume only goods of type B, B-traders only type C, and C-traders only type A. Notice that this assumption means that, whenever two traders exchange the goods they produce, there cannot be a double coincidence of wants; one of the traders must accept goods that he does not consume. An A-trader's preferences can be represented by the utility function,

$$k^B b_1 + b_2,$$

where, for $i = 1,2$, b_i is consumption of good B_i, and k^B is a scalar coefficient. Analogously, B- and C-traders' preferences are also linear, with coefficients k^C and k^A, respectively.

11. Except for our discussion of welfare (see subsection III.4), disutility of labor plays no role in the model. To avoid keeping track of it, therefore, let us suppose that it is small enough so that a trader always supplies his entire endowment of labor at equilibrium prices.

We are particularly interested in the production distortions induced by our informational constraints. To highlight these distortions, we shall assume that

$$(**) \qquad k^X > 2 \quad \text{for all } X = A, B, C.$$

That is, the marginal rate of substitution between high- and low-quality goods exceeds the corresponding marginal rate of transformation. Condition (**) means that, in the absence of any informational imperfection, only high-quality goods are efficient to produce. In particular, no low-quality goods will be produced in the perfect-information Walrasian equilibrium. Contrapositively, any low-quality production that occurs once the informational constraints are imposed can be attributed directly to those constraints.

As discussed in subsection II.1, trade is restricted to be *bilateral:* each exchange involves just two parties. Because exchanges are also *unmonitorable* by third parties, any sort of *credit, short sales,* or *futures trading* is ruled out. We are left only with barters—direct swaps of physical goods—which require no contractual agreement. Let us assume that all barters consist of one single good being exchanged for another.[12]

We suppose that an A-trader can distinguish between A_1 and A_2 (since he produces goods of type A) and between B_1 and B_2 (since he consumes type B goods), but that he cannot distinguish between C_1 and C_2 (which he neither produces nor consumes). Similarly, B- and C-traders cannot distinguish between qualities of type A and type B goods, respectively. As mentioned earlier, this *informational restriction* together with our other assumptions implies that an A-trader cannot execute any trade involving C_1, a B-trader any trade involving A_1, and a C-trader any trade involving B_1. Let us suppose that traders of the same type never trade with one another.[13] Then our bilateral trade and informa-

12. In principle, we could imagine a trader exchanging one good for *two* others, so that *three* goods all told would be involved in the trade. However, to simplify matters, we assume for now that each trade entails only two goods. In subsection II.4 we will show that this assumption can be invoked without loss of generality.

13. We can impose this prohibition without loss of generality. To see this, suppose that, say, one A-trader exchanges A_2 for C_2 with another A-trader in equilibrium. Because the two traders are ex ante identical, each would be as well off if this transaction were *not* made, as long as the former carried out all subsequent transactions that the latter would have made with the A_2, and the latter carried out all subsequent transactions that the former would have carried out with the C_2. Moreover, no other trader would be affected by the fact that the identity of his trading partner may thereby switch from one A-trader to the other. Hence, in the

tional restrictions also imply that any trade must involve at least one low-quality good (For example, the only high-quality that both an A- and a B-trader can distinguish is B_1, and so no other high-quality good can be involved in a trade between them.)

Even without the informational restrictions, our assumptions imply that if there is to be any equilibrium trade, the model must have *multiple trading periods.* In a one-period model an A-trader will wish to exchange the type A goods he produces for type B goods. Given that only B-traders produce B goods and that all trade is bilateral, the exchange must be with a B-trader. But the B-trader will not be happy about receiving type A goods, which he can neither consume nor—in the absence of a subsequent trading period—resell.[14] Thus, the value of an intertemporal trading framework is that it permits reselling.[15]

In our basic model we assume that there are a finite number T of discrete trading periods indexed by $t = 1, \ldots, T$. We take T to be exogenous (but in subsection III.4 explore the implications of different values of T). As discussed in subsection II.1, it is conceptually helpful to think of a trader as comprising two individuals: a buyer and a seller. From this perspective, a trader executes (at most) two transactions each period, one in each capacity. In view of the absence of credit, he cannot sell a quantity of any good that was not in his possession at the beginning of the period. For simplicity, we assume that all production occurs before trade begins. More importantly, we suppose that all goods not yet consumed disappear after period T.

Until Section IV we will suppose that traders do not discount the future at all. That is, as long as consumption of a given good occurs sometimes within the T trading periods, a trader is indifferent about exactly when it occurs. In Section IV we show that our qualitative findings extend to discounting, provided that the discount rate is not too high.

Although there are no futures markets, we assume that each good is tradable for other contemporaneous goods. As noted

aggregate the transaction between the A-traders has no effect at all. And since our characterization and uniqueness results are only about aggregates (see subsection III.4), none of these is affected by our ruling out trade between A-traders.

14. If short-sales were possible, the B-trader could sell good A short for good C while simultaneously trading good B for A.

15. In the absence of the informational constraints that we introduce, two periods would suffice for a fully efficient equilibrium: B-traders could buy A_1 from A-traders (using B_1) in period 1, and resell it to C-traders for C_1 in period 2. As we will see, however, no finite number of periods is adequate for efficiency once there is imperfect information.

above, traders are price-takers. For any two goods, X_i and Y_j, let $p_t(X_i,Y_j)$ be the relative price in period t of X_i in terms of Y_j; i.e., how much a Y_j trader must sell in order to buy one unit of X_i. (Hence $p_t(X_i,Y_j) = 1/p_t(Y_j,X_i)$.) Notice that by expressing prices in this way, we implicitly assume that they are independent of the quantities traded (i.e., that they are "linear"). In fact, this linearity follows immediately from arbitrage.[16]

II.3. Individual Choice

We now formulate a Y-trader's decision problem, for $Y \in \{A, B, C\}$. In each period t he selects a set $\mathcal{T}_t$ of (informationally feasible) bilateral *transactions* that are executed simultaneously. If we strictly followed the story in subsection II.1, $\mathcal{T}_t$ would consist of (at most) two transactions: one corresponding to the trader's role as buyer, and one to that as seller. Nothing in the formal analysis, however, depends on $\mathcal{T}_t$ being limited to only two transactions. Each transaction $\tau \in \mathcal{T}_t$ specifies the pair of goods $\mathscr{X}(t) = \{X_i,X'_j\}$ that the Y-trader exchanges and the quantities exchanged, $q^\tau(X_i)$ and $q^\tau(X'_j)$ (where a positive quantity denotes a purchase and a negative quantity a sale). The constraint that τ be *informationally feasible* can be expressed as

$$(1)\qquad \begin{cases} q^r(A_1) = 0 \ \text{ and } q^\tau(B_1)q^\tau(C_1) = 0, & \text{if } Y = B \\ q^r(B_1) = 0 \ \text{ and } q^\tau(A_1)q^\tau(C_1) = 0, & \text{if } Y = C \\ q^r(C_1) = 0 \ \text{ and } q^\tau(A_1)q^\tau(B_1) = 0, & \text{if } Y = A. \end{cases}$$

To understand (1), note that a B-trader cannot trade A_1; hence $q^\tau(A_1) = 0$. Moreover, as noted in subsection II.2, he cannot make a trade involving more than one high-quality good; hence $q^\tau(B_1)q^\tau(C_1) = 0$. The other two lines follow similarly.

Because there is no credit, the net value of each transaction executed must be zero; i.e., for all $\tau \in \mathcal{T}_t$, if $\mathscr{X}(t) = \{X_i,X'_j\}$, then

16. Suppose, to the contrary, that there were two different relative prices $p_t(X_i,Y_j)$ and $p'_t(X_i,Y_j)$ in equilibrium, according to whether a trader bought, say, two or three units of X_i, respectively. Let $p''_t(X_i,Y_j)$ be the equilibrium price of buying one unit of X_i in terms of Y_j. Then either (i) $p''_t(X_i,Y_j) < p_t(X_i,Y_j)$, or (ii) $p''_t(X_i,Y_j) < p'_t(X_i,Y_j)$. But in the former case the trader would be better off buying one unit of X_i in two separate transactions than two units in a single transaction, and in the latter the trader would be better off buying one unit of X_i in three separate transactions than three units in a single transaction. Hence, at least one of the prices $p_t(X_i,Y_j)$ and $p'_t(X_i,Y_j)$ cannot prevail in equilibrium. (We are perhaps belaboring this point because as footnote 18 will illustrate there are some (standard) arbitrage arguments that do *not* go though in our model.)

$$(2) \qquad p_t\left(X_i, X_j'\right)q^\tau\left(X_i\right) + q^\tau\left(X_j'\right) = 0.$$

Let $e_t(X_i)$ be the amount of good X_i that the Y-trader consumes (eats) in period t (of course, if, say, $Y = A$, then $e_t(X_i)$ will be positive only if X_i is B_1 or B_2), and let $z_t(X_i)$ be the quantity of good X_i that he has at the end of period t (Thus, $z_0(X_i)$ can be interpreted as the quantity of good X_i that the trader produces.) The trader's holdings of good i at the end of period t equal what he began with plus what he acquired (a sale counts as a negative acquisition) minus what he ate. That is, for all $t = 1, \ldots, T$,

$$(3) \qquad z_t\left(X_i\right) = z_{t-1}\left(X_i\right) + \sum_{\tau \in \mathcal{T}_t} q^\tau\left(X_i\right) - e_t\left(X_i\right).$$

The constraint that the trader can sell only goods in his possession is formalized by the requirement,

$$(4) \qquad z_{t-1}\left(X_i\right) + \sum_{\tau \in \mathcal{T}_t\,(X_i)} q^\tau\left(X_i\right) - e_t\left(X_i\right) \geq 0 \quad \text{for all } X_i \text{ and } t,$$

where $\mathcal{T}_t(X_i) = \{\tau \in \mathcal{T}_t \mid q^\tau(X_i) < 0\}$. Hence, given prices $\{p_t(\cdot,\cdot)\}_{t=1}^{T}$, a Y-trader chooses a production/trade/consumption plan $\alpha \equiv \{\mathcal{T}_t, e_t(\cdot), z_0(\cdot)\}_{t=1}^{T}$ to maximize

$$(**) \qquad \sum_{t=1}^{T}\left(k^{Y+1}e_t\left((Y+1)_1\right) + e_t\left((Y+1)_2\right)\right),$$

(where $Y + 1 = B$ if $Y = A$, etc.) such that, for all t, constraints (1)–(4) hold,

$$(5) \qquad e_t(\cdot) \text{ and } z_t(\cdot) \text{ are nonnegative for all } t,$$

and productive feasibility is satisfied:

$$(6) \qquad 2z_0\left(Y_1\right) + z_0\left(Y_2\right) \leq 1, \text{ and } z_0\left(X_i\right) = 0 \quad \text{if } X_i \notin \left\{Y_1, Y_2\right\}.$$

II.4. Transactions with More Than Two Goods

We asserted earlier (see footnote 12) that we could restrict ourselves to two-good transactions without loss of generality. Here is a demonstration. Suppose that we allow three-good transactions and that in equilibrium there exists a three-good transaction in which some trader exchanges good X_i for goods X_j' and X_k'' in period t. Choose the units of the goods so that quantity traded of each in this transaction is one unit. We claim that there exists $\lambda \in (0,1)$ such that

$$(\mathrm{i}) \qquad \lambda p_t\left(X_i, X_j'\right) = 1$$

and

(ii) $$(1-\lambda)p_t(X_i, X_k'') = 1.$$

Note first that $p_t(X_i, X_j') > 1$. Otherwise, our trader's partner would be better off selling $p_t(X_i, X_j')$ units of X_j' (instead of one unit each of X_j' and X_k'' for one unit of X_i), a contradiction of equilibrium. Hence, we can find $\lambda \in (0,1)$ satisfying (i). Now if, say, $(1 - \lambda)p_t(X_i, X_k'') > 1$, our trader would be better off than in equilibrium by selling λ units of X_i for 1 unit of X_j' and $1 - \lambda$ units of X_i for $(1 - \lambda)p_t(X_i, X_k'')$ units of X_k''. Similarly, the trader's partner could find a pair of better-than-equilibrium trades if $(1 - \lambda)p_t(X_i, X_k'') < 1$. Hence, (ii) must hold after all. We conclude that the transaction in which one unit of X_i is exchanged for one unit each of X_j' and X_k'' can be thought of as two exchanges: one in which λ units of X_i are exchanged for 1 unit of X_j', and the other in which $1 - \lambda$ units of X_i are exchanged for 1 of X_k''.

III. Equilibrium

III.1. Definition of Equilibrium

In the Walrasian tradition an *equilibrium* for this model consists of prices $\{\hat{p}_t(X_i, Y_j)\}$ for all periods t and all pairs of goods (X_i, Y_j); and production/trade/consumption plans $\{\hat{\alpha}^h(\cdot)\}$, one for each trader h, such that (i) each trader is optimizing, i.e., if trader h is of type Y, then

(7) $$\hat{\alpha}^h \text{ maximizes } (**) \text{ subject to } (1)-(6);$$

and (ii) all markets clear, i.e., each trader h can find a trading partner for each of his transactions τ:

(8) for all t, there is a one-to-one correspondence between the set of period t transactions $\hat{\mathcal{T}}_t \left(\equiv \bigcup_h \hat{\mathcal{T}}_t^h\right)$ and itself in which each transaction $\tau \in \hat{\mathcal{T}}_t$ is paired with its complement τ^c.[17]

17. The complement τ^c of transaction τ is the same transaction with the trading partners' roles reversed. Hence, τ^c is defined so that

$$q^{\tau}\left(X_i\right) = -q^{\tau^c}\left(X_i\right)$$

for all $X_i \in \mathcal{X}(\tau)$.

Because of the no-credit and informational constraints on the feasible set, our model does not exactly fit the conventional Walrasian framework. Indeed, these constraints preclude the existence of equilibrium in the case $T = 1$. To see this, note that in equilibrium all prices must be strictly positive. Otherwise, excess demand will be unbounded. This means that in a one-period model a B-trader, say, will be unwilling to buy a positive quantity of A_2; he cannot resell the A_2; and so he is better off using his B-goods to buy C-goods. Hence, excess demand by A-traders for B-goods is necessarily strictly positive (since in a one-period model A-traders must obtain B-goods from B-traders using A_2), a violation of equilibrium.

Nevertheless, our model is conventional enough so that for $T \geq 2$ equilibrium *does* exist. We next exhibit an equilibrium for the case $T = 5$ (see Proposition 6 for a demonstration by construction that equilibrium exists for general $T \geq 5$). We construct this equilibrium so that, in every equilibrium transaction, good A_2 is exchanged for some other good X_i. Hence, in exhibiting the equilibrium in Table I, we report only the prices $\hat{p}(X_i,A_2)$. For each $X_j' \neq A_2$, we take $\hat{p}_t(X_i,X_j') = \hat{p}(X_i,A_2)/\hat{p}_t(X_j',A_2)$.[18]

III.2. An Example

Assume that $k^C > k^B > k^A$. We shall exhibit an equilibrium for the case $T = 5$. This equilibrium is *symmetric* in the sense that all traders of a given type behave identically. Thus, we may speak of a "representative X-trader" for $X = A, B, C$.

Define a_i to be the average production of good A_i by an A-trader in equilibrium. Similarly, let b_i and c_i be the equilibrium per capita production levels of goods B and C, respectively. Take $a_1 = [(k^A/2)^{1/2} - 1]/(k^A - 2)$, $a_2 = 1 - 2a_1$, $b_1 = c_1 = ½$, and $b_2 = c_2 = 0$. (The logic behind these choices will be given in subsection III.4.) Equilibrium transactions and prices are described by Table I. In each period, the *entire* produced quantity of A_2 (i.e., a_2 per capita) is traded. Hence, the table completely describes aggregate

18. In a more standard Walrasian framework this equation would follow automatically from arbitrage. For example, if the left-hand side were greater than the right, a trader interested in selling X_j' to buy X_i would be better off selling X_j' to buy A_2 and using the A_2 to buy X_i than in exchanging X_j' for X_i directly, and so $\hat{p}_t(X_i,X_j')$ could not be an equilibrium price. However, in our model the A_2 obtained from selling X_j' could not be used to purchase X_i until the next period, at which point relative prices might have changed. Therefore, the equation need not hold.

TABLE I

Periods	Traders			Prices (p_t (X_i, A_2))					
	A	B	C	A_1	A_2	B_1	B_2	C_1	C_2
1	$-A_2$, $+B_1$	$-B_1$, $+A_2$		2	1	2	q	2	q
2		$-A_2$, $+C_1$	$-C_1$, $+A_2$	$2/q$	1	2	q	2	q
3	$-A_1$, $+A_2$		$-A_2$, $+A_1$	$2/q$	1	$2/q$	q	2	q
4	$-A_2$, $+B_1$	$-B_1$, $+A_2$		$2/q$	1	$2/q$	q	$2/q$	q
5		$-A_2$, $+C_1$	$-C_1$, $+A_2$	$2/q^2$	1	$2/q$	q	$2/q$	q

production and trade in equilibrium. That this in fact constitutes an equilibrium can be verified mechanically.

The feature to emphasize first about this example is that there is a unique medium of exchange, namely, good A_2. That is, as already discussed, A_2 is on one side of every transaction. We will show below (Propositions 1–3) that equilibrium in this example is (essentially) unique, so that A_2 in fact *necessarily* functions as the medium of exchange in the economy. Notice that in the course of five periods good A_2 makes almost two complete cycles through the economy. It begins with the representative A-trader (who produces it), then moves successively to the B-, C-, and A-, and B-traders. and finally to the C-trader (who consumes it). We will see below (Proposition 3) that as the number of trading periods T increases, the number of cycles that A_2 makes rises correspondingly. Thus, T can be viewed as a measure of the velocity of money.

Another property to note is that the prices of A_1, B_1, C_1 (in terms of A_2) increase over time (since $q < 1$). The value of A_2 derives from its dual roles as consumption good and medium of exchange. But as the last period (period 5) approaches, the mediating function becomes less and less important because there are fewer future trading opportunities left. Hence a decline in the relative price of A_2 (i.e., an increase in the relative prices of A_1, B_1, and C_1) is to be expected. This property will also be shown to generalize (Proposition 2).

The final thing to observe is that, relative to the first-best (an economy without informational constraints), equilibrium in this example is inefficient: the fact that a positive quantity of A_2 is produced entails a loss of welfare.

III.3. Two Implications of Arbitrage

As our comments about the above example suggest, equilibrium in the model of Section II is (generically) unique, not in the sense of individuals' trading patterns (where there is considerable indeterminacy[19]) but in the *aggregate* quantities produced, traded, and consumed. It turns out that these aggregate equilibrium quantities are completely determined by two arbitrage relations. Let T^* be the greatest integer less than or equal to $(T + 1)/3$, and for $i = 1,2$, define a_i, b_i, and c_i to be per capita production of A_i, B_i, and C_i, respectively, as in the example of subsection III.2.

LEMMA 1. The inequality,

$$\frac{k^X}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{T^*}\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{T^*}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{T^*} \geq 1, \tag{10}$$

holds for $X = A, B, C$, in equilibrium.

Proof. The argument relies entirely on arbitrage. Consider an A-trader. One option he has is to produce one unit of A_2 and sell it to a B-trader in period 1 in exchange for B_1. By doing so, he obtains $p_1(A_2,B_1)$ units of B_1. Thus, we infer that

$$p_1(A_2,B_1) \leq (k^B b_1 + b_2)/k^B. \tag{11}$$

(The expression $k^B b_1 + b_2$ is the A-trader's equilibrium utility. Thus, if the left-hand side of (11) were greater than the right-hand side, he could obtain higher than equilibrium utility, a contradiction.)

Next consider a B-trader. One strategy he could follow would be to produce 1/2 unit of B_1, sell it to an A-trader in period 1 for (1/2) $p_1(B_1,A_2)$ units of A_2, and then sell the A_2 to a C-trader in period 2 for (1/2) $p_1(B_1,A_2)p_2(A_2,C_1)$ units of good C_1. This would result in utility:

$$(k^C/2)p_1(B_1,A_2)p_2(A_2,C_1).$$

Following the above logic, we conclude that

$$p_1(B_1,A_2)p_2(A_2,C_1) \leq 2(k^c c_1 + c_2)/k^C. \tag{12}$$

19. For instance, in the example of subsection III.2, a B-trader is indifferent about whether he exchanges B_1 for A_2 in period 1 or 4, and so any shift on his part between those two periods can be made consistent with equilibrium by correspondingly adjusting the trades of other B-traders and their partners.

Similarly, a C-trader who sells 1/2 unit of C_1 for A_2 in period 2 and then uses this to buy A_1 in period 3 obtains utility:

$$(k^A/2)p_2(C_1,A_2)p_3(A_2,A_1),$$

and so

$$p_2(C_1,A_2)p_3(A_2,A_1) \le 2(k^A a_1 + a_2)/k^A. \tag{13}$$

Continuing in the same way, we obtain

$$p_3(A_1,A_2)p_4(A_2,B_1) \le \frac{2(k^B b_1 + b_2)}{k^B}. \tag{14}$$

$$\vdots \qquad\qquad \vdots$$

$$p_{T-1}(B_1,A_2)p_T(A_2,C_1) \le \frac{2(k^C c_1 + c_2)}{k^C}.$$

Finally, in period T, a C-trader could sell 1/2 unit of C_1 to a B-trader for (1/2) $p_t(C_1,A_2)$ units of A_2. Hence,

$$p_T(C_1,A_2) \le 2(k^A a_1 + a_2). \tag{15}$$

Multiplying the expressions in (11)–(15) together, we find that all prices cancel, and we are left with

$$1 \le \left[\frac{k^B b_1 + b_2}{k^B}\right]\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{T^*}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{T^*-1} \left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{T^*-1}\left[2(k^A a_1 + a_2)\right]. \tag{16}$$

Rewriting (16), we obtain (10) when $X = A$. The argument is entirely symmetric when $X = B$ and $X = C$.

QED

The above argument invoking (11)–(16) places an upper bound on the utilities attainable, given equilibrium prices, from feasible (but not necessarily optimal) transactions involving A_2, namely, the *equilibrium* utilities. Now, of course, it may not even be optimal for an A-trader to produce A_2 in equilibrium, in which case we would expect the inequality in (10) to be strict. However, suppose that producing A_2 *were* optimal; i.e., that the level of A_2 were positive in equilibrium. Intuitively, one would anticipate the upper bound to be attained for equilibrium sequences involving

A_2. In other words, (10) should hold with equality when $X = A_2$. Formally, we have

LEMMA 2. If, for given $X \in \{A, B, C\}$, the equilibrium production of X_2 is positive, then (10) holds with equality for that type of good X.

Proof. We present the argument for the case $T = 2$ (the general proof is relegated to the Appendix). Suppose that the equilibrium production of A_2 is positive. Then there must be an exchange in which a C-trader acquires some A_2 for consumption. Given the informational constraints, the C-trader must sell A_1, B_2, C_2, or C_1 to obtain this A_2.

Now, the case in which the C-trader sells A_1 can be ruled out easily. Note that A_1 could be bought only by an A- or a C-trader. The latter possibility is eliminated because we have assumed (without loss of generality) that two C-traders never trade. But the former possibility is also ruled out since the exchange must occur in the second period (how else could the C-trader have acquired the A_1 he is selling?), and so the A-trader must consume the A_1 he buys, an impossibility.

Next suppose that the C-trader sells B_2. This must occur in the second period (since the B_2 must have been acquired in the first). Hence, the B_2 must be sold to an A-trader (since only A-traders consume B_2). The A_2 that the A-trader sells must have been produced by him. He could not have acquired it by selling A_1 in the first period (since that would have entailed trade between two A-traders). Hence, following the logic of the proof of Lemma 1,

$$p_2(A_2,B_2) = k^B b_1 + b_2. \tag{17}$$

Now, the C-trader must sell C_1 or C_2 in the first period to acquire his B_2. Assume that he sells C_1 (the argument is very similar in the C_2 case). Then,

$$(1/2)p_1(C_1,B_2)p_2(B_2,A_2) = k^A a_1 + a_2. \tag{18}$$

The C_1 sold by the C-trader must be purchased by a B-trader (since an A-trader cannot buy C_1). Moreover, the B-trader cannot not resell it for C_2 in period 2, since that would entail trade between two B-traders (only a B-trader would buy C_1 in period 2). Hence,

$$(1/2)k^C p_1(B_2,C_1) = k^C c_1 + c_2. \tag{19}$$

Multiplying (17)–(19) together, we obtain

$$1 = \frac{k^B k^A}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]\left[\frac{2(k^C c_1 + c_2)}{k^C}\right],$$

and so

$$1 > \frac{k^A}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]\left[\frac{2(k^C c_1 + c_2)}{k^C}\right], \tag{20}$$

which contradicts Lemma 1. A very similar contradiction follows if the C-trader sells C_2 to obtain A_2.

Finally, suppose that the C-trader sells C_1. We have

$$(1/2)p_2(C_1,A_2) = k^A a_1 + a_2. \tag{21}$$

Moreover, as above, the buyer must be a B-trader. Since the B-trader sells A_2 in this exchange, the trade must occur in period 2. Thus, the B-trader acquires A_2 (from an A-trader) by selling B_1 or B_2 in period 1. Suppose first that he sells B_2. We have

$$p_1(B_2,A_2)p_2(A_2,C_1) = (k^C c_1 + c_2)/k^C. \tag{22}$$

The A-trader who buys B_2 must consume it (if he sold it for B_1 in period 2, the exchange could only be with another A-trader, which we have ruled out). Hence,

$$p_1(A_2,B_1) = k^B b_1 + b_2. \tag{23}$$

Multiplying (21)–(23) together, we again obtain (20), leading to the same contradiction as before. Suppose therefore that the B-trader sells B_1. This implies that

$$(1/2)p_1(B_1,A_2)p_2(A_2,C_1) = (k^C c_1 + c_2)/k^C. \tag{24}$$

Because the A-trader who buys the B_1 consumed it, we have

$$p_1(A_2,B_1) = (k^B b_1 + b_2)/k^B. \tag{25}$$

Multiplying (21), (24), and (25) together, we obtain

$$1 = \frac{k^A}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]\left[\frac{2(k^C c_1 + c_2)}{k^C}\right],$$

as was to be shown.

QED

III.4. Properties of Equilibrium

Armed with Lemmas 1 and 2, we can now readily characterize equilibrium. As the introduction suggests, the informational feasibility constraint (1) by itself implies something akin to *Gresham's law:* any exchange between traders not of the same type must involve a low-quality good. That is, only *low-quality* goods are media of exchange. In turn, this implies that equilibrium production of at least one low-quality good must be positive, since traders do not consume the goods they produce. The next result establishes that, generically, the medium of exchange and, in fact, the equilibrium quantities of all goods are unique.

PROPOSITION 1. Suppose without loss of generality that

$$k^A \le k^B \le k^C. \tag{26}$$

If the first inequality is strict (which occurs for a generic choice of k^X's satisfying (26)), then, in any equilibrium, a_2 $(= 1 - 2a_1) >$ 0, $b_2 = c_2 = 0$, $b_1 = c_1 = 1/2$, and a_1 satisfies

$$\frac{k^A}{2}\left[\frac{2\left(k^A a_1 + 1 - 2a_1\right)}{k^A}\right]^{T^*} = 1. \tag{27}$$

Proof. From Gresham's law we know that at least one of a_2, b_2, and c_2 is positive. If $b_2 > 0$, then from Lemma 2, (10) holds with equality when $X = B$. But then if $k^A < k^B$, (10) is violated for $X = A$, a contradiction of Lemma 1. We conclude that $b_2 = 0$ and, similarly, that $c_2 = 0$. Hence, $a_2 > 0$. Applying Lemma 2 again and the production constraint $2a_1 + a_2 = 1$, we obtain (27).

QED

If $k^A < k^B \le k^C$ as hypothesized by Proposition 1, then the discrepancy between high and low quality (as measured by the deviation of the marginal rate of substitution from one) is smallest for goods of type A. This is the sense in which our model provides a theoretical explanation for the pervasive use of gold as a medium of exchange: gold is a good for which variations in quality that are undetectable to "uninformed" traders are particularly small.

The fact that the equilibrium medium of exchange minimizes distortion, however, does not directly imply that equilibrium is Pareto-efficient in any standard sense. Indeed, as we have already noted, equilibrium is clearly not first-best efficient; positive production of low-quality goods rules that out. The more pertinent question, however, is whether equilibrium is efficient relative to the informational, bilateral, and unmonitorability constraints that we have imposed on trade. Here the answer is not so clear. We believe that it is yes but do not have a formal proof.

The subtlety of the issue has mainly to do with the unmonitorability constraint. For example, consider the following simple scheme, which attains a first-best allocation. All traders produce only high-quality goods; each A-trader gives his A_1 to a C-trader; each C-trader gives his C_1 to a B-trader; and finally each B-trader gives his B_1 to an A-trader. Such behavior obviously satisfies the informational and bilateral constraints. However, it does not pass muster with unmonitorability. The problem is that, if exchanges cannot be monitored, there is nothing to prevent, say, a C-trader from collecting some A-trader's supply of A_1 without bothering to produce any C_1 himself. In this way he could avoid incurring any disutility of labor (this is the only point where we invoke disutility of labor).

We see then that in a world of unmonitorable trade an important virtue of Walrasian trade is to provide a natural way of identifying those who are "entitled" to others' goods. Specifically, a trader is so entitled if he himself has money or goods that he can offer in exchange. Indeed, one reason why we conjecture that Walrasian equilibrium is efficient is simply that it is so difficult to think of alternative identification schemes. (For an identification scheme akin to that provided by fiat money, see footnote 21.)

In the example of subsection III.2, we noted that prices in terms of money rise over time. We attributed this trend to the decline in value of money's mediating role as the last period of exchange nears. It is easy to see that the property of rising prices is a general feature of equilibrium.

PROPOSITION 2. Suppose that the hypotheses of Proposition 1 hold. Then in any equilibrium

$$\text{(28)} \qquad \begin{cases} 2 = q^r p_{1+3r}(B_1,A_2) \\ 2 = q^r p_{2+3r}(C_1,A_2) \\ 2 = q^r p_{3r}(A_1,A_2) \quad \text{for all } r, \end{cases}$$

where $q = (2(k^A a_1 + a_2))/K^A)$. Moreover, for all r, t, and t', with $t < t'$,

$$(29)\quad \begin{cases} p_t(B_1,A_2) \le 2/q^r \le p_{t'}(B_1,A_2), & \text{if } t \le 1+3r \le t' \\ p_t(C_1,A_2) \le 2/q^r \le p_{t'}(C_1,A_2), & \text{if } t \le 2+3r \le t' \\ p_t(A_1,A_2) \le 2/q^r \le p_{t'}(A_1,A_2), & \text{if } t \le 3r \le t'. \end{cases}$$

Proof. From Lemma 2, (10) holds with equality when $X = A$. Hence, (11)–(15) all hold with equality. We infer that (28) holds.

As for (29) note that, because (11)–(15) hold with equality, a C-trader can attain his equilibrium utility level by selling C_1 for A_2 in periods 2, 5, 8, etc., and buying A_1 with A_2 in periods 3, 6, 9, etc. Similarly, an A-trader can attain his equilibrium utility level by selling A_1 for A_2 in periods 3, 6, 9, etc. and buying B_1 with A_2 in periods 1, 4, 7, etc. Suppose, contrary to (29), that

$$(30)\qquad p_{t'}(A_1,A_2) < p_{3r}(A_1,A_2)$$

for some t' and r with $t' > 3r$. But then a C-trader does better to buy A_1 in period t' than in period $3r$, contrary to our observation that the latter behavior attains the C-trader's equilibrium utility. Hence, (30) is impossible. Alternatively, suppose that

$$(31)\qquad p_t(A_1,A_2) > p_{3r}(A_1,A_2)$$

for some t and r with $t < 3r$. Then an A-trader does better to sell A_1 in period t than in period $3r$, again contrary to our preceding analysis. Thus, (31) is also impossible, and we conclude that the third pair of inequalities in (29) holds. That the first and second pairs hold follows similarly.

QED

We saw that, in the example of subsection III.2, money made two cycles through the economy. It is natural to expect that, as T increases, money will circulate correspondingly more times. This can be put more precisely as follows.

Define a *standard B_1-exchange* to be an exchange in which an A-trader buys B_1 from a B-trader for A_2. Call a standard B_1-exchange *regular* if it occurs in some period 1, 4, . . . , or $1 + 3(T^* - 1)$. Similarly, a *standard C_1-exchange* entails a B-trader buying C_1 from a C-trader for A_2 (and is deemed regular if it occurs in one of periods 2, 5, . . . , $2 + 3(T^* - 1)$), and a *standard A_1-exchange* involves a C-trader buying A_1 from an A-trader for A_2 (it is regular if it occurs in period 3, 6, . . . , or $3 + 3(T^* - 2)$). Collectively, the standard A_1-, B_1-, and C_1-exchanges constitute the *standard exchanges*.

PROPOSITION 3. Suppose that the hypotheses of Proposition 1 hold. In any equilibrium all exchanges are regular and standard. Furthermore, in any period the *entire* quantity produced of A_2 is exchanged for high-quality output.

Proof. See the Appendix.

From Proposition 3 we can trace exactly how money moves through the economy in equilibrium. Specifically, in period 1, A-traders buy B_1 from B-traders using A_2. In period 2, B-traders buy C_1 from C-traders using the A_2 they acquired in period 1. In period 3, C-traders buy A_1 from A-traders using the A_2 they acquired in period 2. In period 4, A-traders again buy B_1, and so on until period $3T^* - 1$, when B- and C-traders exchange A_2 for C_1 and each group consumed what it has received.

Thus, given T, money makes T^* cycles through the economy, and so T is a measure of the *velocity* of money. If we think of subdividing a given interval of time more finely so as to increase the number of trading periods, money will circulate correspondingly more times in that interval. In line with the quantity theory of money, moreover, a given quantity of money can mediate more exchanges as velocity increases, and so less money is needed.

PROPOSITION 4. Under the hypotheses of Proposition 1, the equilibrium per capita quantity of money a_2 is a decreasing function of T. Moreover, in the limit as $T \to \infty$, a_2 tends to 0.

Proof. This follows directly from (27).

Although the inverse relation between velocity and quantity is entirely classical, there is an important way in which our model deviates from orthodoxy. Namely, a fall in the quantity of money is not welfare neutral. Indeed, as a_2 falls, welfare rises (more precisely, the welfare of C-traders rises; that of A- and B-traders remains the same), as equilibrium production of A_1 increases.

PROPOSITION 5. Under the hypotheses of Proposition 1, equilibrium utility of C-traders is an increasing function of T (and that of A- and B-traders is independent of T). In the limit of $T \to \infty$, C-trader utility tends to the first-best level.

Proof. This follows immediately from Proposition 1 and formula (27).

The fact that an increase in T leads to welfare improvement underscores the fact that in a model like ours counting the number of transactions needed before the economy clears is an inappropriate measure of the inefficiency of the economy (see footnote 2). Indeed, as $T \to \infty$, Proposition 3 shows that the number of transactions also tends to infinity, and yet, from Proposition 5, welfare converges to the first-best level. However, note that from Proposition 2 there is no reason for an *individual* trader ever to make more than two transactions: selling the good he produces for A_2 and then reselling the A_2 for the good he consumes. In fact, this is the same number of transactions he would execute even if the economy were monetized.

Propositions 1–5 are devoted to characterizing equilibrium. For completeness we will also confirm existence. The example of subsection III.2 exhibits an equilibrium when $T = 5$. We now turn to arbitrary T. We do this by exhibiting a symmetric equilibrium explicitly.

PROPOSITION 6. Under the hypotheses of Proposition 1, there exists a symmetric equilibrium in which, for $i = 1, 2$, a_i, b_i, and c_i satisfy the formulae of Proposition 1; trade is characterized by Proposition 3; for all t and r, prices of high-quality goods satisfy

$$\begin{aligned} p_t(B_1,A_2) &= 2/q^r, \quad \text{if } 3r \le t \le 3r+2, \\ p_t(C_1,A_2) &= 2/q^r, \quad \text{if } 1+3r \le t \le 3r+3, \\ p_t(A_1,A_2) &= 2/q^r, \quad \text{if } 3r-1 \le t \le 3r+1, \end{aligned}$$

and prices of low-quality goods satisfy $p_t(B_2,A_2) = p_t(C_2,A_2) = q$ for all t, where $q = 2(k^A a_1 + 1 - 2a_1)/k^A$, and a_1 satisfies (27).

Proof. Merely a matter of mechanical verification.

III.5. The Bilateral Trade and Unmonitorability Assumptions

Let us return to the bilateral trade and unmonitorability assumptions introduced in subsection II.1. As we noted, the former assumption gets at the idea that trade is *decentralized*, i.e., that markets, although well organized in the sense that traders know where to find the goods they want, are geographically dispersed rather than centrally located. This means that if an A-trader wishes to buy B_1 he can readily find the market where it is sold but, once there, he is unlikely to find a C-trader, who could complete a three-way exchange of A_1 for B_1.[20] Thus, the A-trader will not be able to buy the B_1 using A_2.

This discussion should make clear that what is important about the bilateral trade assumption is not that there literally be just two parties to every trade but simply that trading circles as in footnote 20 be too costly to arrange. If, as in subsection II.1, there were many more than three goods, such circles would typically have to be quite large, and so ruling them out would be a relatively weak assumption.

As for the unmonitorability assumption, it seems very much in the spirit of the anonymity that most everyday cash transactions entail. To understand the role that it plays, consider the model of Section II, but let us now relax the assumption that a good must be physically in a trader's possession in order for him to sell it; i.e., let us drop constraint (4). Then for $T = 1$, we claim that the following is an equilibrium. Let the price of all low-quality goods be 1 and that of all high-quality goods be 2. Suppose that each A-trader exchanges 1 unit of A_2 for 1/2 unit of B_1 with some B-trader; each B-trader exchanges 1 unit of A_2 for 1/2 unit of C_1 with some C-trader; and each C-trader trades 1 unit of A_2 for 1/2 of A_1 with some A-trader.[21]

Note first that such behavior does indeed violate constraint (4) because, in a one-period model, B- and C-traders do not actually have the A_2 that they are trading. However, observe that good A_2 is in zero net demand by all traders, i.e., every trader both buys and sells one unit of it. Thus, no physical production

20. If a C-trader *could* be found, then in principle it would be possible to conduct a three-way exchange even in the absence of outside monitoring. For example, consider the following stylized scheme. Three traders—one each of types A, B, and C—sit in a circle around a rotating table. Each party puts half a unit of the high-quality good he produces on the table directly in front of him, and all parties have the opportunity to inspect one another's goods. Thus, the A-trader can scrutinize the B_1, etc. Once a party is satisfied, he can press a button, and when everyone has done this, the table rotates 120 degrees, so that the A-trader gets the B_1, the B-trader gets the C_1, and the C-trader gets the A_1. Unless everyone presses the button, the table does not rotate.

21. A similar sort of trading arrangement is the following "money-lending" mechanism. The money-lender (who could be either one of the traders or some outside authority) first issues each trader with one unit of "money" and imposes the requirement that it be repaid after the final period. Each A-trader uses his money to buy 1/2 unit of B_1 from a B-trader, each B-trader buys 1/2 unit of C_1 from a C-trader with his money, and each C-trader purchases 1/2 of A_1 from an A-trader. Each trader thus ends up with one unit of money, which he then repays to the money-lender. Because of the requirement of repayment, this scheme also falls afoul of the unmonitorability constraint. But it is very similar to the model of fiat money (which does *not* violate unmonitorability but requires an infinite horizon) presented in Section VI.

of A_2 is actually required, and therefore, the outcome we have described is first-best efficient.

The idea that A_2 can serve as a medium of exchange without actually being produced may seem somewhat paradoxical until it is remembered that it is not really A_2 but only promises to deliver A_2 (IOU's) that are being traded. These promises therefore constitute money in much the same sense that gold- or silver-certificates formerly did. The only difference between certificates and IOU's is that the former are public promises (i.e., promises by government), whereas the latter are private (but presumably enforced by a public court).

IV. Two Extensions: Discounting and Durability

The analysis so far has presumed that traders are indifferent about when they consume within the T trading periods and that all goods survive to period T. We now briefly examine what happens when these presumptions are dropped.

Let us first introduce a discount factor δ common to all traders. If we continue to assume that $k^A < k^B \le k^C$, then, by analogy with the proofs of Lemmas 1 and 2, we can show that

$$\frac{\delta^{3r} p_{3r}(A_1, A_2)}{p_{3r+1}(B_1, A_2)} = \frac{2(k^B b_1 + b_2)}{k^B}, \qquad 1 \le 3r \le T - 1$$

$$\frac{\delta^{3r+1} p_{3r+1}(B_1, A_2)}{p_{3r+2}(C_1, A_2)} = \frac{2\delta(k^C c_1 + c_2)}{k^C}, \qquad 1 \le 3r + 1 \le T - 1$$

and

$$\frac{\delta^{3r+2} p_{3r+2}(C_1, A_2)}{p_{3r+3}(A_1, A_2)} = \frac{2\delta^2(k^A a_1 + a_2)}{k^A}, \qquad 1 \le 3r + 2 \le T - 1,$$

and so if the analogue of Proposition 1 holds,

$$\frac{\delta^{3T^*} k^A}{2}\left[\frac{2(k^A a_1 + a_1)}{k^A}\right]^{T^*} = \delta^{(T+2)(T-1)/2}. \tag{32}$$

But we can find a_1 and a_2 satisfying (32) if and only if

$$\left[\frac{2}{k^A}\right]^{T^*-1} < \delta^{(T^2-T-4)/2}. \tag{33}$$

It can be shown that if (33) holds and $T \ge 5$, equilibrium quantities are unique and satisfy $a_2 > 0$, $b_2 = c_2 = 0$, $b_1 = c_1 = 1/2$.

If δ is small enough, however, Walrasian equilibrium fails to exist (see the Appendix for a formal demonstration). The difficulty is that everybody will try to consume in the first period, and as we argued in subsection II.2, such intentions cannot be mutually consistent. This illustrates the simple point that a trading arrangement based on delayed gratification cannot work if traders are too impatient.

Next, let us revert to the case of no discounting but now consider a model in which, although goods A_1, B_1, B_2, C_1, and C_2 as before endure for T periods, A_2 survives for only $\hat{T}$ periods, where $\hat{T} < T$.

Arguments similar to those that gave us Lemma 1 yield

$$(34) \qquad \frac{k^A}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{\hat{T}^*}\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{\hat{T}^*}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{\hat{T}^*} \geq 1$$

$$(35) \qquad \frac{k^B}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{T^*}\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{T^*}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{T^*} \geq 1$$

and

$$(36) \qquad \frac{k^C}{2}\left[\frac{2(k^C c_1 + c_2)}{k^A}\right]^{T^*}\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{T^*}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{T^*} \geq 1,$$

where $\hat{T}^*$ is the greatest integer not bigger than $(\hat{T} + 1)/3$. Moreover, arguments similar to those establishing Lemma 2 imply that (i) if $a_2 > 0$, then (34) holds with equality; (ii) if $b_2 > 0$, then (35) holds with equality; and (iii) if $k^A < k^B < k^C$, then $c_2 = 0$.

Now, if $a_2 > 0$, then (35) and the fact that (34) holds with equality imply

$$\frac{k^B}{k^A}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{T^*-\hat{T}^*}\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{T^*-\hat{T}^*} \geq 1$$

which, from the equality version of (34), implies that

$$(37) \qquad \frac{k^A}{2}\left[\frac{k^A}{k^B}\right]^{\hat{T}^*/T^*-\hat{T}^*} \leq 1.$$

But if (37) holds strictly, then it is readily shown that (35) holds strictly, and so $b_2 = 0$. Conversely, if (35) fails to hold, then we must have $b_2 > 0$ and $a_2 = 0$. Hence, in this version of the model there is a trade-off between "identifiability" and "durability." On the one hand, the smaller is the ratio k^A/k^B (i.e., the more identifiable type A-goods are relative to B-goods), the more likely it is that A_2 will function as the medium of exchange. On the other hand, the smaller is the ratio $\hat{T}^*/T^* - \hat{T}^*$ (i.e., the greater is the durability of B_2 relative to A_2), the more likely it is that B_2 will be the medium of exchange.

V. Steady States

So far we have considered basically a one-shot economy: production occurs once-and-for-all, trade unfolds over T periods, and then the economy ends. We now show that this economy can be embedded within an infinite-horizon framework for the purpose of examining steady state equilibrium.

Suppose that in each period $t = 1,2, \ldots$, there is an infusion of m new traders of each type A, B, and C. On their arrival new traders produce according to the linear technology described in Section II. They then trade and consume for T periods, at which point they and any unconsumed goods they have produced disappear. Preferences are the same as in Section II; there is no discounting.

Let us assume that $k_A < k_B \leq k_C$. Then we would expect that good A_2 will function as the medium of exchange in steady state equilibrium. At any time t there are, in fact, T different vintages of A_2 available, and so each of these must have a different price. (There are different vintages of the high-quality goods available too, but since these goods are consumed as soon as they get into the right hands, they need not be differentially priced.) For $1, \ldots, T$ let A_2^t be t-period-old A_2 (define B_2^t and C_2^t analogously). Reinterpreting the prices in Proposition 6, we obtain the following:

PROPOSITION 7. In the above infinite-horizon model, the prices

$$p(B_1,A_2^t) = 2/q^r, \qquad \text{if } 3r \leq t \leq 3r+2$$

$$p(C_1,A_2^t) = 2/q^r, \qquad \text{if } 1+3r \leq t \leq 3r+3$$

$$p(A_1,A_2^t) = 2/q^r, \qquad \text{if } 3r-1 \leq t \leq 3r+1,$$

$$\left.\begin{aligned} p(A_2^t,A_2^1) &= 1/q^r \\ p(B_2^t,A_2^1) &= 1/q^{r-1} \\ p(C^t,A^1) &= 1/q^{r-1} \end{aligned}\right\} \qquad \text{if } 3r-1 \leq t \leq 3r+1$$

for all $t = 1, \ldots, T$ and $r = 0, \ldots, T^*$, constitute a steady state equilibrium together with the following exchanges: for all t, (i) each t-year-old A-trader exchanges a_2 units of A_2^t for $a_2/p(B_1,A_2^t)$ units of B_1 with some t-period-old B-trader (provided that $t - 1$ is a multiple of 3); (ii) each t-year-old B-trader exchanges a_2 units of A_2^t for $a_2/p(C_1,A_2^t)$ units of C_1 with some t-period-old C-trader (provided that $t - 2$ is a multiple of 3); (iii) each t-year-old C-trader exchanges a_2 units of A_2^t for $a_2/p(A_1,A_2^t)$ units of A_1 with some t-period-old A-trader (provided that t is a multiple of 3), where $a_2 = 1 - a_1$ and a_1 satisfies (27).

Proof. The proposition is just a reinterpretation of Propositions 3 and 6. The only thing to check is that the possibility of exchanging A_2, B_2, and C_2 of different vintages (which was not possible in the static model) does not create new arbitrage opportunities for any trader. For example, consider an A-trader. If he trades A_2^1 for A_2^t, he obtains $1/q^r$ units of the latter per unit of the former (where $3r - 1 \leq t \leq 3r + 1$). If in the next period he then sells what now is A_2^{T+1} for B_1, he obtains $q^r/2$ or $q^{r+1}/2$ units of the latter for the former. Thus, the A-trader does no better with these trades than with steady state equilibrium transactions. Similar reasoning applies to B- and C-traders.

QED

Proposition 7 exhibits a steady equilibrium in which trade is completely segregated according to cohort: t-period-old traders transact only with other t-period-old traders and the A_2 they exchange is only of vintage t. But intercohort steady states are also possible (although they continue to entail the same aggregate production). For example, suppose, for some t (with $t - 1$ divisible by 3), that some t-period-old A-trader sells A_2^t in period t to a t + 3-period-old B-trader, and correspondingly, some t + 3-period-old A-trader sells A_2^{T+3} to a t-period-old B-trader. Then as long as the older B-trader continues to trade with the t-cohort (which is possible provided that he sells all his B_1 and buys all his C_1 before period $T - 3$) and the younger B-trader continues to trade with the t + 3-cohort, we still have a steady state equilibrium.

VI. Fiat Money

When $T < \infty$ (i.e., velocity is less than infinite), we have seen that equilibrium is inefficient in the sense that it entails production and consumption of a low-quality good. This suggests that there is a potentially valuable role in our model for *fiat money* (which can be thought of as a good that is essentially costless to produce, that confers no utility, and whose quality is discernible by everybody). Namely, if fiat money takes over as the medium of exchange, the production distortion we have discussed is eliminated.

To introduce fiat money, of course, we must deal with the end-point problem: who will hold the money in the last period? We shall, therefore, appeal to an infinite horizon,[22] but using a somewhat different framework from that of the preceding section. Recall the model of Section II: production followed by T periods of trading. Call this sequence of events an *epoch*. We shall suppose that there is an infinite sequence of epochs indexed by $s = 1,2,\ldots$. Traders are infinitely lived and have a discount factor δ across epochs (there is no discounting within an epoch).

Imagine now that into this economy we introduce another good, which is produced at zero cost by a single "producer," whom we call the government. We shall denote this good—fiat money—by M. Suppose that, in the first period of the first epoch, each trader is endowed with a quantity of money $M_0 > 0$. In each period t of each epoch s, the government spends a quantity of money M_{ts} on each of the three high-quality goods, where

$$M_{ts} = \begin{cases} mM_{t-1s}, & \text{if } t > 1 \\ mM_{Ts-1}, & \text{if } t = 1, \end{cases} \tag{38}$$

and $M_{11} = mM_0$. This setup corresponds well to many analyses of inflation in the macroeconomic literature.

Given prices $\{p_{ts}(\cdot)\}$, each Y-trader in this economy chooses $\{\mathcal{T}_{ts}, e_{ts}(\cdot), z_{0s}(\cdot)\}_{t,s}$ (which includes trades of money) to maximize

$$\sum_{s=1}^{\infty} \delta^{s-1} \sum_{t=1}^{T} \left(k^{Y+1} e_{ts}\left((Y+1)_1\right) + e_{ts}\left((Y+1)_2\right)\right), \tag{39}$$

subject to

$$p_{ts}\left(X_i, X_j'\right) q^{\tau}\left(X_i\right) + q^{\tau}\left(X_j'\right) = 0, \quad \text{for all } \tau \in \mathcal{T}_{ts} \text{ and } X_i, X_j' \in \mathcal{X}(\tau); \tag{40}$$

$$z_{ts}\left(X_i\right) = z_{t-1s}\left(X_i\right) + \sum_{\tau \in \mathcal{T}_{ts}} q^{\tau}\left(X_i\right) - e_t\left(X_i\right), \quad \text{for all } t, z, \text{ and } X_i \in \mathcal{X}(\tau); \tag{41}$$

22. A well-known finite-horizon solution to the end-point problem is the device of requiring traders to return the money at the end of the last period (see footnote 21).

$$(42)\quad z_{t-1s}(X_i) + \sum_{\tau \in \mathcal{T}_{ts}(X_i)} q^{\tau}(X_i) - e_t(X_i) \geq 0, \qquad \text{for all } t, s, \text{ and } X_i \in \mathscr{X}(\tau);$$

$$(43)\quad e_{ts}(X_i) \geq 0 \text{ and } z_{ts}(X_i) \geq 0, \quad \text{for all } t, s, \text{ and } X_i \in \mathscr{X}(\tau);$$

and

$$(44)\quad 2z_{0s}(Y_1) + z_{0s}(Y_2) \leq 1, z_{0s}(X_i) = 0, \quad X_i \notin \{Y_1, Y_2\} \text{ and all } s,\ X_i \in \mathscr{X}(\tau) - \{M\} \text{ and } Z_0(M) = M_0.$$

Let $\mathcal{T}^g_{ts}$ be the set of transactions that the government carries out in period t of epoch s. Let $\hat{\mathcal{T}}_{ts}$ be the union of *all* traders' ts-transactions (including $\mathcal{T}^g_{ts}$). Then the market-clearing requirement takes the form,

(45) for all t and s, there is a one-to-one correspondence between the set of period ts-transactions $\hat{\mathcal{T}}_{ts}$ and itself in which each transaction $\tau \in \hat{\mathcal{T}}_{ts}$ is paired with its complement τ^c.

One equilibrium satisfying (38)–(45) is the same as that in the model without money. That is, the equilibrium given by Proposition 6 will simply be replicated in every epoch. Under certain conditions, however, there exists another equilibrium. In this other equilibrium, for all goods X and Y, all periods t, and epochs s, prices are given by

$$(46)\quad p_{ts}(X_1, M) = 2(1+m)^{(s-1)T+t} \cdot M_0 \frac{1-\delta}{\delta^{(t-1)/T}}(1 - \delta^{1/T})$$

$$(47)\quad p_{ts}(X_2, M) = p_{ts}(X_1, M)/2,$$

$$(48)\quad p_{ts}(X_1, Y_2) = 2\left(\frac{1+m}{\delta^{1/T}}\right)^t.$$

As for quantities, if $q^Y_{ts}(X_i)$ is the amount of good X_i bought by a Y-trader in period t of epoch s,

$$(49)\quad q^Y_{ts}(X_i) = \begin{cases} -\dfrac{1}{2}^{\delta(t-1)/T} \dfrac{1-\delta^{1/T}}{1-\delta}, & \text{if } X_i = Y_1 \\ \dfrac{-q^Y_{ts}(Y_1)}{1+m}, & \text{if } X_i = (Y+1)_1 \\ 0, & \text{otherwise.} \end{cases}$$

To understand (46), note that for a Y-trader to be willing to sell good Y_1 in each period $t = 1, \ldots, T$ of epoch s and then to buy good $(Y + 1)_1$ in the following period $t + 1 = 2, \ldots, T$, and period 1 of the next epoch, prices must satisfy

$$(50)\quad \frac{p_{1s}(X_1,M)}{p_{2s}(X_1,M)} = \frac{p_{2s}(X_1,M)}{p_{3s}(X_1,M)} = \cdots = \frac{p_{T-1s}(X_1,M)}{p_{Ts}(X_1,M)} = \frac{\delta p_{Ts}(X_1,M)}{p_{1s+1}(X_1,M)}$$

because, in equilibrium, $p_{ts}(X_1,M) = p_{ts}(Y_1,M) = p_{ts}(Y + 1)_1,M)$. Moreover, to equilibrate supply and demand, we have

$$(51)\quad \sum_{t=1}^{T} \frac{(m+1)^{(s-1)T+t} M_0}{p_{ts}(X_1,M)} = \frac{1}{2}.$$

Solving (50) and (51), we get (46). Now,

$$q^Y(Y_1) = -\frac{(m+1)^{(s-1)T+t} M_0}{p_t(Y_i,M)}, \quad q^Y((Y+1)_1) = \frac{(m+1)^{(s-1)T+t-1} M_0}{p_t(Y_i,M)},$$

which, in view of (46), gives us (49).

As for (48), observe that we must have

$$(52)\quad \frac{1}{p_{1s}(X_1,Y_2)} \leq \frac{1}{2}\left(\frac{\delta^{1/T}}{1+m}\right).$$

Otherwise, a Y-trader is better off selling Y_2 for $(Y + 1)_1$ in period 1 than selling Y_1 for M in period 1 and then buying $(Y + 1)_1$ in period 2. Moreover, we must have

$$(53)\quad \frac{p_{ts}(X_1,Y_2)}{p_{t+1s}(X_1,Y_2)} \leq \frac{p_{ts}(X_1,M)}{p_{t+1s}(X_1,M)}.$$

Otherwise, a Y-trader is better off selling Y_1 for Y_2 in period t and then reselling the Y_2 for $(Y + 1)_1$ in period $t + 2$ than selling Y_1 for M in period t and then buying $(Y + 1)_1$ in period $t + 1$. Formula (48) then follows when (53) holds with equality.

Notice that, in the equilibrium described by (46)–(49), welfare converges to the first best (the allocation that would prevail with barter if there were no informational imperfections) as δ goes to 1 and m goes to 0. Thus, for δ near 1 and m near enough 0, the introduction of money can definitely promote a welfare improvement. Note too that prices in (46) and (47) are proportional to the money supply $(1 + m)^{(s-1)T+t}M_0$.

Now, for (46)–(49) to constitute an equilibrium, it cannot be the case that a C-trader is better off selling C_1 for A_2 than selling

C_1 for M and then buying A_1. Hence, from (46) and (48) we must have

$$\frac{2(1+m)^{T+2}}{\delta} \leq \frac{K^A \delta^{1/T}}{1+m};$$

i.e.,

$$(1+m)^{T+1} \leq \frac{K^A}{2} \delta^{(T+1)/T}. \tag{54}$$

We conclude that m must be sufficiently small to satisfy (54) in order to ensure the existence of a monetary equilibrium. For too rapid a monetary expansion—i.e., for m bigger than the critical values at which (54) holds with equality—the only equilibrium is that of Proposition 6. The disappearance of the monetary equilibrium corresponds to a serious and well-recognized historical problem with hyperinflation: the risk that the economy will be demonetized; i.e., agents will fall back on barter as the form of exchange.[23]

VII. Conclusion

In this paper we develop a simple Walrasian general equilibrium model in which there is a role for a medium of exchange. Although we feel our assumptions are naturally motivated, the model is quite special. The demands of tractability have limited us to a single example from a larger class of models. The characterization of the properties of this broader class is clearly a substantial task remaining to be done. We therefore conclude with some of the leading open questions.

First, in the paper we focus on the particular case of linear preference, and the more general case of concave preferences needs to be examined. If preferences are concave, the marginal rates of substitution between the two types of each good will typically depend on how much of each type was being consumed. As a result, no single good need always be more identifiable than all other goods. We conjecture that this may sometimes lead to more than one good being used as a medium of exchange at the same time.

23. Looking at data from a variety of hyperinflations, Barro [1972] finds that economies seem to behave as if there is some threshold level of inflation beyond which the inflationary spiral becomes unstable.

Second, our results may depend on the sequencing of trades. The trading periods that are relevant for different people may be very different and allowing for this possibility may significantly alter our results.

Third, as a referee suggested, it would be interesting to investigate the nature of nonsteady state equilibria in the fiat money version of the model. This would help relate our approach to others in the literature.

Finally, we have yet to investigate the welfare implications of our model fully. In particular, we do not know whether the equilibrium in our model is constrained Pareto-optimal in an appropriately defined sense. We suspect that the answer is yes, but confirmation must await future work.

APPENDIX

LEMMA 2. If the equilibrium production of X_2 is positive, then

$$\frac{k^X}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{T^*}\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{T^*}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{T^*} = 1. \tag{10}$$

Proof. Suppose that $a_2 > 0$. We must show that (10) holds when $X = A$.

We first introduce the idea of a *physical portion* of a good. Suppose that we divide a stock of given good in half. We will then have two portions of equal quantity but physically distinct. Much of the following argument relies on tracing a particular physical portion of a good around the economy. Thus, the difference between this concept and that of "quantity" should be borne in mind.

Consider the equilibrium behavior of an $(X - 1)$-trader. Suppose that in some period $t(1)$ he trades a physical portion α of a good $Y(0)$ he has produced (hence, $Y(0)$ is either $(X - 1)_1$ or $(X - 1)_2$) for a physical portion β of some other good $Y(1)$. Now, of course, he may retrade (or consume) different subportions of β at different times. However, if we divide up α into subportions appropriately, we can ensure that each subportion of α corresponds to a subportion of β retraded (or consumed) in its entirety at a single time. For example, suppose that in period 1 the $(X - 1)$-trader sells 1 unit of $(X - 1)_1$ for 3/4 units of X_2'. Suppose that he resells 1/3 unit of this X_2' in period 2, 1/4 unit in period 4, and consumes the rest in period 2. We can divide up the unit of $(X - 1)_1$ into physical portions of size 4/9, 1/3, and 2/9, so that the 4/9-unit portion can be thought of as being exchanged for the portion of X_2' resold in period 2, the 1/3-unit portion is exchanged for the portion of X_2' resold in period 4, and the 2/9-unit portion is exchanged for the portion of X_2' that is consumed.

Clearly, this argument generalizes. Indeed (since in equilibrium only finitely many transactions are made),[24] if α is chosen appropriately (i.e., by suitable subdivision), we can guarantee not only that the entire physical portion of $Y(1)$ which the trader obtains for α is retraded at a single time, but that the same is true of the physical portion $Y(2)$ that *it* is exchanged for, and so on, for $Y(3)$, $Y(4)$ etc., up to the physical portion of $Y(l)$ ($Y(l) = X_1$ or X_2) that the trader ultimately consumes. Define a *complete trader-sequence* of equilibrium transactions for our $(X - 1)$-trader to be such a sequence: i.e., one in which he trades a physical portion of $Y(0)$ for a portion of $Y(1)$ in period $t(1)$; he trades the portion of $Y(1)$ for a physical portion of $Y(2)$ in period $t(2)$; and so on, until finally in period $t(l)$, he trades the portion of $Y(l - 1)$ acquired in period $t(l - 1)$ for a physical portion of $Y(l)$, which he consumes.[25]

Following the logic of the proof of Lemma 1, one can readily establish that

$$p_{t(1)}\big(Y(0), Y(1)\big)p_{t(2)}\big(Y(1), Y(2)\big) \cdots p_{t(l)}\big(Y(l-1), Y(l)\big)$$

$$\text{(A.1)} \quad = \begin{cases} 2\big(k^X x_1 + x_2\big)/k^X, & \text{if } Y(0) = X_1 \text{ and } Y(l) = X_1 \\ \big(k^X x_1 + x_2\big)/k^X, & \text{if } Y(0) = X_2 \text{ and } Y(l) = X_1 \\ 2\big(k^X x_1 + x_2\big), & \text{if } Y(0) = X_1 \text{ and } Y(l) = X_2 \\ k^X x_1 + x_2, & \text{if } Y(0) = X_2 \text{ and } Y(l) = X_2, \end{cases}$$

where x_1 and x_2 are the equilibrium per capita quantities of X_1 and X_2, respectively.

We shall say that a certain physical portion β of good Y' is *first-order related* to a certain physical portion α of good Y if β and α each belong to transactions in the same complete trader-sequence. A physical portion β of good Y' is *second-order related*

24. There are only finitely many periods and finitely many traders, and each trader executes only finitely many transactions in any period.

25. Notice that by appropriately choosing the physical portions as above, we can ensure that the set of equilibrium transactions executed by our $(X - 1)$-trader is *partitioned* by his complete trader sequences. That is, each transaction belongs to a unique complete trader sequence.

to α if there exists a physical portion γ of some good Y'' such that β is first-order related to γ and γ is first-order related to α. Continuing iteratively, for any n we can define what it means for a physical portion of good Y' to be nth-order related to α. We shall say that a physical portion β of good Y' is *related* to α if, for some n, β is nth-order related to α.

Fix a physical portion α of some good Y. Let $\mathcal{T}(\alpha)$ be the set of all equilibrium transactions τ such that τ belongs to a complete trader-sequence in which a physical portion β (of some good Y') related to α is traded. Consider a transaction $\tau \in \mathcal{T}(\alpha)$, in which some trader trades physical portion μ of good Z for portion ν of good Z' in period t. Then the complementary transaction τ^c, in which some other trader trades ν for μ in period t, is also in $\mathcal{T}(\alpha)$ (If $\tau \in \mathcal{T}(\alpha)$, then μ and ν are related to α. From the definition of equilibrium the complementary transaction τ^c is also made, and so $\tau^c \in \mathcal{T}(\alpha)$.) Now, we can associate transaction τ with price $p_t(Z,Z')$, in which case the complementary trade τ^c is associated with ratio $p_t(Z',Z)$. Hence, the product of all the prices associated with trades in $\mathcal{T}(\alpha)$ is 1. Now, we can partition the different transactions in $\mathcal{T}(\alpha)$ into different complete trader-sequences (recall that each transaction belongs to a unique complete trader-sequence), and so from (A.1) we can write the product of prices as

$$\text{(A.2)}\quad \frac{(k^A)^{\pi_A}(k^B)^{\pi_B}(k^C)^{\pi_C}}{2^{\eta}}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{\rho_A} \times \left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{B\rho}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{\rho_C} = 1,$$

where, for $X = A,B,C$, ρ_X is the number of complete trader-sequences that end in the consumption of good X_1 or X_2, π_X is the number of complete trader-sequences that end in the consumption of good X_2, and η is the number of complete trader-sequences that begin with trading A_2, B_2, or C_2. We next claim that

$$\text{(A.3)}\qquad \pi_A + \pi_B + \pi_C = \eta;$$

i.e., that the number of complete trader-sequences (with trades belonging to $\mathcal{T}(\alpha)$) that begin with a low-quality good equals the number of sequences that end with a low-quality good. Of course, any low-quality good that is produced must eventually be consumed, and in that sense, formula (A.3) is plausible. However, it is not immediately obvious; we must first rule out, for example, the possibility that there are more initial transactions than terminal transactions because the latter entail larger quantities of low-quality goods.

Notice first that any complete trader-sequence can be thought of as a sequence of ordered pairs of 1's and 2's, e.g., $\{(1,2),(2,2),(2,1)\}$, where the first number in each pair refers to the quality of the good being sold by the trader in question and the second to that of the good being bought. Every such sequence of ordered pairs has the properties that (i) at least one 2 appears in each pair (since a low-quality good must be involved in every transaction), and (ii) if the second number in some pair is m ($m = 1,2$), then the first number in the next pair is also m (since whatever a trader buys that he does not consume, he must instead sell). Because $\tau \in \mathcal{T}(\alpha)$ implies that the complementary transaction τ^c belongs to $\mathcal{T}(\alpha)$, we also have property (iii): among the sequences made up of elements of $\mathcal{T}(\alpha)$, there is a one-to-one correspondence between pairs of the form (m_1,m_2) and pairs of the form (m_2,m_1), where (m_1,m_2) appears in a different complete trader-sequence from (m_2,m_1).

To establish (A.3), we assert that any finite set $\hat{\mathcal{S}}$ of sequences of ordered pairs that satisfy (i), (ii), and (iii) has the property that (iv) the number of sequences beginning with a 2 is the same as the number ending with a 2. (Note that if $\hat{\mathcal{S}}$ satisfies (iii) it contains an even number of ordered pairs.) This is true by inspection if the number of ordered pairs in $\hat{\mathcal{S}}$ is two: in that case, (i)–(iii) imply that

$$\hat{\mathcal{S}} = \begin{cases} \{\{(1,2)\}, \{(2,1)\}\} \\ \qquad\text{or} \\ \{\{(2,2)\}, \{(2,2)\}\} \\ \qquad\text{or} \\ \{\{(2,1)\}, \{(1,2)\}\}, \end{cases}$$

and in all three cases, (iv) holds.

To complete the induction, we must show that if (iv) holds when $\hat{\mathcal{S}}$ contains $2n - 2$ pairs, it holds when $\hat{\mathcal{S}}$ contains $2n$ pairs. Suppose, therefore, that $\hat{\mathcal{S}}$ contains $2n$ ordered pairs.

If the pair (2,2) appears in some sequence in $\hat{\mathcal{S}}$, then from (iii), (2,2) appears in some other sequence as well. Suppose that we delete both appearances of (2,2). Because $\hat{\mathcal{S}}$ satisfies (i)–(iii), the resulting set $\hat{\mathcal{S}}'$ (which contains $2n - 2$ pairs) also does. By inductive hypothesis, (iv) holds for $\hat{\mathcal{S}}'$. But from (i) and (ii), $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}'$ have the same number of sequences that begin with 2 and also have the same number of sequences that end with 2 (and so $\hat{\mathcal{S}}$ satisfies (iv) too), unless at least one of the deleted pairs (2,2) constitutes an entire sequence by itself in $\hat{\mathcal{S}}$. If this last condition holds, then, for each deleted pair that constitutes an entire sequence by itself, $\hat{\mathcal{S}}$ has one more sequence that begins with 2 and one more sequence that ends with 2 than does $\hat{\mathcal{S}}'$. And so again by inductive hypothesis (iv) holds for $\hat{\mathcal{S}}$.

Therefore, to finish the inductive step, we may assume that $\hat{\mathcal{S}}$ contains only pairs (1,2) and (2,1).

Suppose first that $\hat{\mathcal{S}}$ has a sequence $\mathcal{s}$ that ends with the pair (1,2). If $\hat{\mathcal{S}}$ also has a sequence $\mathcal{s}'$ that ends with the pair (2,1), then delete both these pairs to obtain the set $\hat{\mathcal{S}}''$. $\hat{\mathcal{S}}''$ satisfies (i)–(iii) and so by inductive hypothesis satisfies (iv). If (2,1) is the only pair in $\mathcal{s}'$, then from (ii), $\hat{\mathcal{S}}''$ has one fewer sequence that begins with 2 and one fewer sequence that ends with 2 than $\hat{\mathcal{S}}$ does. If (2,1) is not the only pair in $\mathcal{s}'$, then from (ii), $\hat{\mathcal{S}}''$ has the same number of sequences that begin with 2 and the same number of sequences that end with 2 as does $\hat{\mathcal{S}}$. In either case since $\hat{\mathcal{S}}''$ satisfies (iv), so does $\hat{\mathcal{S}}$. Therefore, assume that all sequences in $\hat{\mathcal{S}}$ end with the pair (1,2). If there exists a sequence $\mathcal{s}'' \in \hat{\mathcal{S}}$ (possibly $\mathcal{s}$ itself) that begins with the pair (2,1), then delete the final (1,2) in $\mathcal{s}$ and the first (2,1) in $\mathcal{s}''$ to obtain the set $\hat{\mathcal{S}}^*$, which satisfies (i)–(iii). Because $\hat{\mathcal{S}}^*$ satisfies (iv) by inductive hypothesis, and contains one fewer sequence that begins with 2 and one fewer sequence that ends with 2 than does $\hat{\mathcal{S}}$, we conclude once again that $\hat{\mathcal{S}}$ satisfies (iv). Thus, we are left only with the case in which $\hat{\mathcal{S}}$ consists entirely of sequences that both begin and end with the pair (1,2). Now from (ii), pairs alternate between (1,2) and (2,1) along a sequence. Hence, in this final case each sequence in $\hat{\mathcal{S}}$ contains one more occurrence of (1,2) than of (2,1). But this violates property (iii), so this final case is impossible.

Now, we have been assuming that there exists a sequence $\mathcal{s}$ that ends with (1,2). But a similar argument applies if all sequences end with (2,1). We conclude that (A.3) holds after all.

For convenience (so that we do not have to deal with fractions), suppose that $T + 1$ is divisible by 3. We next claim that

$$3\rho_X \leq \eta(T+1), \quad \text{for all } X = A, B, C. \tag{A.4}$$

To see that (A.4) holds, we will use a different accounting method for transactions. Earlier we partitioned the transactions in $\mathcal{T}(\alpha)$

into complete trader-sequences, each of which traces out the succession of transactions by a particular *trader* that his initial sale of a physical portion of some good gives rise to. We now introduce the concept of a *complete X_2-sequence* ($X_2 = A_2, B_2, C_2$), which traces out the trading history of a particular *physical portion* of X_2. Consider, for example, a complete trader-sequence (with transactions in $\mathcal{T}(\alpha)$) for an A-trader that begins with him trading a physical portion α' of A_2. One obtains a complete A_2-sequence by taking the full history of equilibrium exchanges that involve α'. For example, a possible A_2-sequence might be as follows: the A-trader trades α' for a physical portion β' of B_1 (which he consumes), where his trading partner is a B-trader who trades β' for α'; then (in some later period) the B-trader trades α' for a physical portion γ' of C_1 (which he consumes), where the trading partner is a C-trader who trades γ' for α' (which he consumes). In this case, the complete A_2-sequence consists of two *exchanges* (each comprising two complementary transactions): the first involving α' and β', and the second α' and γ'. Complete B_2- and C_2-sequences are defined analogously. Notice that an exchange involving two low-quality goods will belong to two complete X_2-sequences: one for each good.

As defined, there are η complete X_2-sequences ($X_2 = A_2$, B_2, or C_2) with trades in $\mathcal{T}(\alpha)$. Each complete X_2-sequence consists of at most T exchanges. Suppose that we add a "null" exchange at the end of each complete X_2-sequence to obtain an *extended X_2-sequence.* Then we have an upper bound of $\eta(T + 1)$ exchanges in all. If we can show that, for each exchange (belonging to some extended X_2-sequence) in which there is consumption of B_1 or B_2 (a B_1- or B_2-*consumption exchange*), we can associate two other exchanges (in the union of all extended X_2-sequences) in which there is no consumption of B_1 or B_2 and that are not associated with some other B-consumption exchange, then we will have established (A.4) in the case $X = B$ (Clearly, the cases $X = A$ and $X = C$ then follow by symmetry.) Actually, as mentioned above, an exchange in which, say, an A-trader sells A_2 in order to consume B_2 appears in both an extended A_2- and a B_2-sequence. Therefore, to avoid double counting we shall count this as a B_2-consumption exchange only in the extended X_2-sequence corresponding to the good *not* being consumed—in this example the A_2-sequence. (It does not matter if we double-count *non-B-consumption* exchanges since the upper bound $(T + 1)\eta$ itself includes all such double counts.)

Let σ_A be an extended A_2-exchange in which a physical portion α' of A_2 is traded. We shall work backwards inductively from the last B-consumption exchange in σ_A to demonstrate that each B-consumption exchange $\mathscr{e}$ in σ_A can be associated with two non-B-consumption exchanges $\hat{\mathscr{e}}$ not already associated with some other B-consumption exchange and each having one of the following properties: (i) $\hat{\mathscr{e}}$ is the null exchange in σ_A (if $\mathscr{e}$ is the last B-consumption exchange in σ_A); or (ii) $\hat{\mathscr{e}}$ is the duplicate of $\mathscr{e}$ (if $\mathscr{e}$ entails consumption of B_2); or (iii) $\hat{\mathscr{e}}$ is the final exchange in σ_A if it does not entail the purchase of B_2 by an A-trader; or (iv) $\hat{\mathscr{e}}$ is an exchange in σ_A in which an A-trader acquires α' and which follows $\mathscr{e}$ and precedes the next B-consumption exchange in σ_A; or (v) $\hat{\mathscr{e}}$ is an exchange in σ_A (different from that satisfying (iv)) in which a B-trader sells α' and which follows $\mathscr{e}$ and precedes the next B-consumption exchange in σ_A; or (vi) $\hat{\mathscr{e}}$ is the duplicate of an exchange satisfying (iv) (if the A-trader's partner is a B-trader and B_2 or C_2 is exchanged for α'); or (vii) $\hat{\mathscr{e}}$ is an exchange in which an A-trader acquires B_1 that he then resells in an exchange satisfying (iv).

To accomplish this demonstration, suppose first that $\mathscr{e}$ is the last B-consumption in σ_A. Take the null exchange in σ_A as one of the non-B-consumption exchanges associated with $\mathscr{e}$ (property (i)). If $\mathscr{e}$ entails consumption of B_2, then it also appears in some complete B_2-sequence, where by our accounting rules it does not count as a B-consumption exchange. Thus, in this case we associate this duplicate of $\mathscr{e}$ with $\mathscr{e}$ (property (ii)). If $\mathscr{e}$ entails consumption of B_1, then $\mathscr{e}$ cannot be the final exchange in σ_A. (The final exchange involves α' being consumed, and so in that exchange, α' is bought by a C-trader. But a C-trader cannot sell B_1.) In this case we associate $\mathscr{e}$ with the final exchange in σ_A, provided that it does not entail purchase of B_2 by an A-trader (property (iii)). If the final exchange *does* entail purchase of B_2 by an A-trader (and, hence, sale of α' by that trader), then there exists another exchange in σ_A—before the final one but after $\mathscr{e}$—in which this A-trader acquires α'. In that case let us associate *this* exchange with $\mathscr{e}$ (property (iv)).

Next suppose that $\mathscr{e}$ is a B-consumption exchange in σ_A such that the inductive hypothesis holds for all subsequent B-consumption exchanges. We must show that $\mathscr{e}$ also satisfies the inductive hypothesis. Let $\mathscr{e}'$ be the next B-consumption exchange in σ_A. Now, $\mathscr{e}'$ cannot *immediately* follow $\mathscr{e}$, since first α'—which is bought by a non-A-trader in $\mathscr{e}$ (we have ruled out transactions between two traders of the same type)—must be acquired by an A-trader in some exchange e'' (in order then to be sold in e'). From property (iv) we can associate e'' with e. If there exists another exchange in σ_A between e and e' in which a B-trader sells B_2, then from property (v) it can also be associated with e. Therefore, assume that no such exchange exists. Now if e involves B_2-consumption, then from property (ii) we can associate its duplicate in the corresponding complete B_2-sequence. Therefore, assume that e involves B_1-consumption. Then, the α'-purchaser in e is a B-trader. Moreover, in e'' an A-trader buys α' from this B-trader. (Otherwise, contrary to our above assumption, there would be a non-B-consumption exchange between e and e'' in which the B-trader sold α' to someone else.) Hence e'' takes the form

$$e'' = \begin{cases} \text{(a) } A\text{-trader trades } B_1 \text{ for } A_2, \text{ and } B\text{-trader trades } A_2 \text{ for } B_1 \\ \text{(b) } A\text{-trader trades } B_2 \text{ for } A_2, \text{ and } B\text{-trader trades } A_2 \text{ for } B_2 \\ \text{(c) } A\text{-trader trades } C_2 \text{ for } A_2, \text{ and } B\text{-trader trades } A_2 \text{ for } C_2. \end{cases}$$

In case (a) there must be some earlier exchange $\hat{e}$ (possibly in an X_2-sequence different from σ_A) in which the A-trader acquired the B_1 that he sells in e''. In that case let $\hat{e}$ be the other exchange associated with e. (This assignment satisfies the inductive hypothesis (vii) since the A-trader resells B_1 before e'.) In case (b) the exchange e'' also appears in some extended B_2-sequence, and so from property (vi) we can let this second occurrence also be associated with e. In case (c) e'' also appears in some extended C_2-sequence, and so again from property (vi) we can also associate the duplicate with e.

Next suppose that e belongs to an extended B_2-sequence σ_B, in which physical portion β' of B_2 is traded. We shall work backwards inductively from the last B-consumption in σ_B to demonstrate that each B-consumption exchange e in σ_B can be associated with two non-B-consumption exchanges $\hat{e}$ not already assigned to some other B-consumption exchange and each having one of the following properties: (i′) $\hat{e}$ is the null exchange in σ_B (if e is the final exchange of σ_B); (ii′) $\hat{e}$ is an exchange in σ_B in which a C-trader acquires β', which he then resells either in e or in a σ_B exchange preceding e and following all B-consumption exchanges before e in σ_B; (iii′) $\hat{e}$ is an exchange in which an A-trader acquires B_1, which he then resells in either in e or in a σ_B-exchange preceding e and following all B-consumption exchanges in σ_B before e; (iv′) $\hat{e}$ is an exchange in σ_B in which an A-trader acquires

β', which he then resells in e; (v') e is the duplicate of an exchange satisfying (iv').

Suppose first that e is the last exchange of σ_B. Then from (i') we can associate e with the null exchange of σ_B. Now, by our accounting rule, e does not entail trade of A_2 or C_2 (otherwise, it would not count as a B-consumption exchange in σ_B). Hence, e must involve the exchange of β' for A_1 or B_1 (since an A-trader is involved). In the former case a C-trader is involved, and there must exist a previous exchange in σ_B in which this trader acquires β'. From property (ii') this exchange can then be associated with e. (Notice that this exchange cannot already be associated with a B-consumption exchange in σ_A because the only non-σ_A exchanges involving B_2 that such exchanges are associated with entail either trade between an A- and B-trader (property (vi)) or consumption of B_2 by an A-trader (property (ii)). In the latter case there must be an earlier exchange $\hat{e}$ in which the A-trader acquires the B_1 that he sells in e. Hence, from property (iii') we can associate $\hat{e}$ with e. $\hat{e}$ cannot already be associated with a B-consumption exchange in σ_A because the B_1 that the A-trader acquires in $\hat{e}$ is resold in σ_B.)

Next assume that e is a B-consumption exchange in σ_B such that properties (i')–(v') hold for all subsequent B-consumptions in σ_B. If e entailed consumption of β', it would have to be the final exchange in σ_B. Hence, suppose that it entails consumption of B_1. Now there must exist a previous exchange e' in σ_B in which the A-trader who consumes B_1 buys β' in order to trade it for B_1. From property (iv') we can let e' be one of the non-B-consumption exchanges associated with e (e' cannot be associated with a B-consumption exchange in σ_A because the facts that it does not belong to σ_A and entails the purchase of β' by an A-trader conflict with (i)–(vii)), but that still leaves another to be found. Now in e' the A-trader exchanges X_i for β', where $X_i = B_1, C_2, A_2$, or A_1. If $X_i = B_1$, then there must be an earlier exchange $\hat{e}$ in which the A-trader acquires B_1, in which case from (iii') we can let $\hat{e}$ be the other associated non-B-consumption exchange. (As in the previous paragraph, $\hat{e}$ cannot be associated with a B-consumption exchange in σ_A because, if it were, property (vii) would require that A-trader resell the B_1 in a σ_A-exchange.) If $X_i = C_2$, then the exchange e' also occurs in an extended C_2-sequence. In which case, from (v'), we can also associate this second occurrence with e. Similarly, if $X_i = A_2$, then e' also appears in an extended A_2-sequence, and because e' entails a purchase of B_2 by an A-trader, (i), (vi), and (vii) imply that it is not already associated with a B-consumption exchange in such a sequence. Hence, the duplicate occurrence of e' can also be associated with e. Finally, if $X_i = A_1$, then the A-trader must trade with a C-trader in e'. Hence, there must be some earlier exchange e'' in which this C-trader acquires β'. And so from (ii′) e'' can be associated with e. However, it cannot be associated with a B-consumption exchange in σ_A because, if it were, (vi) implies that it would entail trade between an A- and a B-trader.

It remains to consider the case where e belongs to an extended C_2-sequence σ_C, in which a physical portion γ' of C_2 is traded. We shall work backwards inductively from the last B-consumption in σ_C to demonstrate that each B-consumption exchange in σ_C can be associated with two non-B-consumption exchanges $\hat{e}$ not already assigned to some other B-consumption exchange and each having one of the following properties: (i″) $\hat{e}$ is the null exchange in σ_C (if e is the final B-consumption exchange in σ_C); (ii″) $\hat{e}$ is an exchange in σ_C in which an A-trader acquires γ', which he then resells in e; (iii″) $\hat{e}$ is the duplicate of e (if e entails consumption of B_2); (iv″) $\hat{e}$ is an σ_C-exchange in which an A-trader acquires γ' and which occurs after e and before the next B-consumption exchange in σ_C; (v″) $\hat{e}$ is a σ_C-exchange in which a B-trader sells γ' to a C-trader and which occurs after e and before the next B-consumption exchange in σ_C; (vi″) $\hat{e}$ is an exchange in which an A-trader acquires B_1 and later sells it in a σ_C-exchange between e and the next B-consumption exchange in σ_C.

Suppose that the inductive hypothesis holds for all B-consumptions following e in σ_C. We must show that the same is true of e. Now in e an A-trader sells γ' (to buy B_1 or B_2). Thus, there must be an earlier exchange e' in which this trader acquires γ'. From (ii″) e' can be associated with e. (The exchange e' could not be associated with a B-consumption in an extended A_2-sequence because (ii), (vi), and (vii) imply that, since it is not an A_2-sequence exchange, it would have to entail B_2-consumption, or acquisition of A_2 or B_1 by an A-trader. Similarly, e' cannot be associated with a B-consumption exchange in an extended B_2-sequence because if it were, (iii′) and (v′) would imply that it would entail an A-trader buying B_1 or B_2,.)

Now if e is the final B-consumption exchange in σ_C, we can from (i″) choose the null exchange as the other non-B-consumption exchange associated with e. Therefore, assume that

e is followed by a B-consumption exchange e' in σ_C. If e entails an A-trader consuming B_2, then the duplicate of e also appears in an extended B_2-sequence (where, however, it does not count as a B-consumption exchange). Because of the B_2-consumption, (i′)–(vi′) imply that this duplicate of e is not associated with any B-consumption in the B_2-sequence. Therefore, from (iii″) we can choose this duplicate to be the other non-B-consumption exchange associated with e. Therefore, assume that e entails an A-trader buying B_1 (from a B-trader). Now, because e and e' each entail an A-trader selling γ', there exists an exchange e'' between e and e' in which an A-trader acquires γ'. From (iv″) e'' can be associated with e unless it has already been associated with e' (e'' cannot be associated with a B-consumption exchange in an A_2-sequence since, if it were, (ii), (vi), and (vii) would imply that it entails an A-trader buying B_2, A_2, or B_1; it cannot be associated with a B-consumption exchange in a B_2-sequence because, if it were, (iii′) and (v′) would imply that it entails an A-trader buying B_1 or B_2). Now in the latter case we may assume that the A-trader in e'' acquires γ' from the B-trader in e (otherwise, there would exist an exchange between e and e'' in which the B-trader sells γ' to some C-trader, and from (v″) *that* exchange could be associated with e).[26] Suppose first that the A-trader sells B_1 in e'' to buy γ'. In that case there must be an earlier exchange $\hat{e}$ in which the A-trader acquires B_1, and from (vi′) $\hat{e}$ can be associated with e. (From (vii) and (vii′) $\hat{e}$ cannot be associated with a B-consumption exchange in an A_2- or B_2-sequence σ_A or σ_B.) Next, suppose that the A-trader sells A_2 to buy γ' in e''. Now, the duplicate of e'' appears in an extended A_2-sequence σ_A. But this duplicate cannot be the last exchange in σ_A (the last exchange entails consumption of A_2 by a C-trader), and so it cannot be associated with the last B-consumption exchange in σ_A. Nor can it be associated with any other B-consumption exchange in σ_A since from (i)–(vii) no such association entails an A-trader buying C_2. Hence, this second occurrence of e'' can be associated with e. Finally, suppose that the A-trader sells B_2 in e'' to buy γ'. Then, although the duplicate of e'' appears in some extended B_2-sequence, (i′)–(v′) imply that this duplicate is not associated with a B-consumption exchange in that sequence because e'' entails an A-trader buying C_2 from a B-trader. Hence, again we can associate the duplicate

26. That exchange cannot already be associated with a B-consumption exchange in an A_2- or B_2-sequence, thanks, as usual, to (ii), (vi), and (vii) in the former case and to (iii′) and (v′) in the latter case.

of e'' with e. This concludes the demonstration that each B-consumption exchange can be uniquely associated with two non-B-consumption exchanges, and hence establishes (A.4).

Because the square-bracketed expressions in (A.2) are each no greater than one, (A.4) implies that

$$\text{(A.5)}\qquad \frac{(k^A)^{\pi_A}(k^B)^{\pi_B}(k^C)^{\pi_C}}{2^\eta}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{\eta(T+1)/3} \times \left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{\eta(T+1)/3}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{\eta(T+1)/3} \leq 1.$$

Now, because $a_2 > 0$, $\pi_A > 0$. Hence if $k^C < k^A$ and $k^C \leq k^B$, then (A.3) and (A.5) imply that

$$\frac{k^C}{2}\left[\frac{2(k^A a_1 + a_2)}{k^A}\right]^{(T+1)/3}\left[\frac{2(k^B b_1 + b_2)}{k^B}\right]^{(T+1)/3}\left[\frac{2(k^C c_1 + c_2)}{k^C}\right]^{(T+1)/3} < 1,$$

a contradiction of Lemma 1. Similarly, $k^B < k^A$ and $k^B \leq k^C$ lead to a contradiction, and so we have $k^C \geq k^A$ and $k^B \geq k^A$. Therefore, (A.5) implies that the left-hand side of (10) is no greater than 1 when $X = A$. But this and Lemma 1 together imply that (10) holds when $X =$ A.

QED

PROPOSITION 3. Suppose that the hypotheses of Proposition 1 hold. In any equilibrium all exchanges are regular and standard. Furthermore, in any period the *entire* produced quantity of A_2 is exchanged for high-quality output.

Proof. For convenience let us suppose that $T + 1 = 3T^*$ (i.e., $T + 1$ is divisible by 3). From Proposition 1 all purchases of B_1 by A-traders must be mediated by A_2. That is, they must be B_1-standard exchanges. Moreover, if all these exchanges are regular (i.e., occur in periods $1 + 3r$, $r = 0, \ldots, T^* - 1$), then because from Proposition 2 the corresponding prices are 2, $2/q, \ldots, 2/q^{T^*-1}$, we conclude that indeed the entire quantity produced of A_2 must be exchanged for B_1 in each of these periods. This follows because

$$\text{(A.6)}\qquad \frac{a_2}{2}\left(1 + q + q^2 + \cdots + q^{T^*-1}\right) = \frac{1}{2},$$

where the left-hand side of (A.6) corresponds to the quantities of B_1 that can be obtained by selling a_2 in each of periods $1, 4, \ldots, 1 + 3(T^* - 1)$, and the right-hand side corresponds to the per capita production of B_1. In other words, if anything less than the entire produced quantity of A_2 were exchanged for B_1 in each period 1, 4, . . . , $1 + 3(T^* - 1)$, then, at the equilibrium prices prevailing in those periods, it would not be possible for the entire produced quantity of B_1 to be sold.

We obtain analogous results for standard C_1- or A_1-exchanges, as long as all these are regular. That is, in each period where these exchanges take place, the entire produced quantity of A_2 is traded for C_1 or A_1. This means that there is no A_2 left over to mediate any nonstandard exchanges.

Hence, the proposition is established provided that we can show that there are no nonregular standard exchanges in equilibrium. To the contrary, suppose that there exists an equilibrium with such an exchange. Consider the last period $\hat{t}$ in which such an exchange occurs. For concreteness assume that the exchange $\hat{e}$ is a standard B_1-exchange. Then, for some $\hat{\tau} = 0, \ldots, T^* - 2$, $\hat{t} \in \{2 + 3\hat{\tau}, 3(\hat{\tau} + 1)\}$. Now, if the B-trader subsequently retrades all the A_2 he acquired in $\hat{e}$ and obtains exclusively B_1 for this A_2, then $\hat{e}$ is, in effect, "cancelled out,"[27] and we can move back to the last period in which there is a nonregular standard exchange that is not cancelled out. (If all nonregular standard exchanges are cancelled out, then, ultimately, B_1, C_1 are acquired by A-, B-, and C-traders entirely through regular standard exchanges, which means—given that from (A.6) all A_2 is devoted to these exchanges—that there can be no nonregular standard exchanges at all.) Thus, assume that at least some of the A_2 acquired by the B-trader in $\hat{e}$ is not resold for B_1. Therefore, it must be resold for C_1; i.e., it is traded in a standard C_1-exchange. But by hypothesis such an exchange must be regular (since it comes after period $\hat{t}$) and so cannot occur until at least period $2 + 3(\hat{\tau} + 1)$. Hence, the price of C_1 in terms of A_2 is at least $2/q^{\hat{\tau}+1}$. This, in turn, implies that the price of B_1 in terms of A_2 in period $\hat{t}$ is at least $2/q^{\hat{\tau}+1}$ (otherwise a B-trader selling B_1 for A_2 and then reselling the A_2 for C_1 would be better off waiting at least until period $1 + 3(\hat{\tau} + 1)$ to sell the B_1, when it would fetch a price of $2/q^{\hat{\tau}+1}$). Now, if the

27. Note that the quantity of B_1 that the B-trader obtains from retrading the A_2 can be no less than the quantity of B_1 he sells in exchange $\hat{e}$ (otherwise, he would be better off not selling B_1 in $\hat{e}$). However, suppose that it were more. Then assume for convenience that he obtains it in a single exchange $\hat{e}'$ with an A-trader. Because *all* A-traders have equal utility, we can think of this as the *same* A-trader as in exchange $\hat{e}$. But then this A-trader would be better off not undertaking exchanges $\hat{e}$ and $\hat{e}'$, since, by refraining, he would end up with more of B_1 and the same quantity of A_2.

We conclude that $\hat{e}$ is *exactly* cancelled out in the sense that the B-trader is left in the same position as though he had never executed it.

A_2 that the A-trader sells in $\hat{e}$ is not obtained from a previous exchange (i.e., he produces it himself), he would be better off selling it in period $1 + 3\hat{\tau}$, when the price of B_1 is only $2/q^{\hat{\tau}}$, than in period $\hat{t}$. Hence, he *must* obtain it from a previous exchange $\hat{e}'$ in period $\hat{t}$, where $1 + 3\hat{\tau} \leq \hat{t}' < \hat{t}$. Specifically, he must obtain it by either (i) selling B_1 to a B-trader or (ii) selling A_1 to a C-trader.

Consider case (i). Because the A-trader sells B_1 in $\hat{e}'$ and then buys B_1 in $\hat{e}$, $p_{\hat{t}'}(B_1,A_2) \geq p_{\hat{t}}(B_1,A_2)$ (otherwise he would be better off not selling B_1 in $\hat{e}'$). Given that prices are nondecreasing over time, therefore, $\hat{t}' = \hat{t} - 1$, and $p_{\hat{t}-1}(B_1,A_2) = 2/q^{\hat{\tau}+1}$. The B-trader who sells A_2 in exchange $\hat{e}'$, in turn, must have acquired it in a still earlier exchange $\hat{e}''$ in period $\hat{t}'' < \hat{t}'$. Moreover, he must have sold either B_1 or C_1 to obtain it. In the former case, $p_{\hat{t}''}(B_1,A_2) \geq p_{\hat{t}'}(B_1,A_2)$ (otherwise, the B-trader would be better off not selling B_1 in period $\hat{t}''$), and hence $p_{\hat{t}''}(B_1,A_2) = 2/q^{\hat{\tau}+1}$. Thus, $p_{\hat{t}-2}(B_1,A_2) = 2/q^{\hat{\tau}+1}$, a contradiction, since $\hat{t}$ is no more than two periods after $1 + 3\hat{\tau}$, and the price of B_1 in terms of A_2 in period $1 + 3\hat{\tau}$ is $p_{1+3\hat{\tau}}(B_1,A_2) = 2/q^{\hat{\tau}}$. In the latter case, $p_{\hat{t}''}(C_1,A_2) = 2/q^{\hat{\tau}+1}$.[28] We can therefore derive the same contradiction as in the former case.

We are left with case (ii), which means that $\hat{e}'$ is a standard exchange. In this case, $p_{\hat{t}'}(A_1,A_2) \leq 2/q^{\hat{\tau}}$ since $\hat{t}' < 3 + 3\hat{\tau}$. Indeed, $p_{\hat{t}'}(A_1,A_2) = 2/q^{\hat{\tau}}$. Otherwise, the A-trader is better off waiting until period $3 + 3\hat{\tau}$ to sell the A_1 and until period $1 + 3(\hat{\tau} + 1)$ (when the price of B_1 is still only $2/q^{\hat{\tau}+1}$) to buy B_1. We conclude that $\hat{e}'$ is nonregular. Thus, we have shown that if there exists a nonregular standard exchange, then the trader selling A_2 in that exchange must have obtained it from a previous exchange that is also nonregular and standard. Moving backward in this way, we ultimately reach a nonregular standard exchange in which a trader sells A_2 that *cannot* have been obtained from a previous exchange, a contradiction.

QED

Proof That Equilibrium Does Not Exist When δ Is Small

Suppose that $T = 2$ and $k^A < k^B < k^C$. We would expect A_2 to be the medium of exchange if an equilibrium existed. This means

28. Otherwise, the quantity Q_B of B_1 which the B-trader buys in period $\hat{t}'$ is smaller than the quantity Q_C of C_1 which he sells in period $\hat{t}''$. Now, the B-trader sells directly or indirectly the quantity Q_B of B_1 for some quantity Q_C' of C_1, which he consumes. Moreover, $Q_C' \leq Q_B$ since $b_1 = c_1$ in equilibrium (if the inequality went the other way, the B-trader could consume $c_1 > b_1$). Hence, $Q_B < Q_C$ implies that $Q_C' < Q_C$; i.e., the B-trader ends up with a smaller quantity of C_1 than he had in period $\hat{t}''$, a contradiction.

that since A_2 cannot be consumed until period 2, a C-trader's equilibrium utility is δ. (There is not time if $T = 2$ for A_1 to get into the hands of a C-trader.) Because in equilibrium a B-trader must first acquire A_2 in period 1 before buying C_1, his equilibrium utility is $\delta k^C/2$. However, since he has the option of selling B_2 for C_2 in the first period, we must have

$$(*) \qquad p_1(B_2, C_2) \leq \delta k^C/2.$$

Now, because an A-trader consumes B_1 in equilibrium, he must buy this all in period 1 (otherwise, there will not be enough time for the A_2 he sells to get into the hands of C-traders). Hence, his equilibrium utility level is $k^B/2$. Alternatively, because he could buy C_2 in period 1 and resell it for B_2 in period 2, we must have

$$\frac{\delta}{p_1(C_2, A_2)p_2(B_2, C_2)} \leq \frac{k^B}{2},$$

which implies, in view of (*), that

$$(**) \qquad 4/k^C k^B \leq p_1(C_2, A_2).$$

But suppose that a C-trader sells all his C_2 for A_2 in period 1. From (**) his utility would be at least $4/k^C k^B$, which exceeds δ when the latter is small, a contradiction of equilibrium. Consequently, no equilibrium exists.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
HARVARD UNIVERSITY

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