

Arrow's Theorem, May's Axioms, and Borda's Rule

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Abstract

We argue that Arrow's (1951) independence of irrelevant alternatives condition (IIA) is unjustifiably stringent. Although, in elections, it has the desirable effect of ruling out spoilers (Candidate A spoils the election for B if B beats C when all voters rank A low, but C beats B when some voters rank A high - - A splits off support from B), it is stronger than necessary for this purpose. Worse, it makes a voting rule insensitive to voters' preference intensities. Accordingly, we propose a modified version of IIA to address these problems. Rather than obtaining an impossibility result, we show that a voting rule satisfies modified IIA, Arrow's other conditions, and May's (1952) axioms for majority rule if and only if it is the Borda count (Borda 1781), i.e., rank-order voting.

1. Arrow, May, and Borda

A. Arrow's IIA Condition

In his monograph *Social Choice and Individual Values* (Arrow 1951), Kenneth Arrow introduced the concept of a *social welfare function* (SWF) – a mapping from profiles of individuals' preferences to social preferences.¹ The centerpiece of his analysis was the celebrated

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¹ Formal definitions are provided in section 2.

Impossibility Theorem, which establishes that, with three or more social alternatives, there exists no SWF satisfying four attractive conditions: *unrestricted domain* (U), the *Pareto Principle* (P), *non-dictatorship* (ND), and *independence of irrelevant alternatives* (IIA).

Condition U requires merely that a social welfare function be defined for all possible profiles of individual preferences (since ruling out preferences in advance could be difficult). P is the reasonable requirement that if all individuals (strictly) prefer alternative x to y , then x should be (strictly) preferred to y socially as well. ND is the weak assumption that there should not exist a single individual (a “dictator”) whose strict preference always determines social preference.

These first three conditions are all so undemanding that virtually any SWF studied in theory or used in practice satisfies them all. For example, consider *plurality rule* (or “first-past-the-post”), in which x is preferred to y socially if the number of individuals ranking x first is bigger than the number ranking y first.² Plurality rule satisfies U because it is well-defined regardless of individuals’ preferences. It satisfies P because if all individuals strictly prefer x to y , then x must be ranked first by more individuals than y .³ Finally, it satisfies ND because if everyone else ranks x first, then even if the last individual strictly prefers y to x , y will not be ranked above x socially.

By contrast, IIA – which requires that social preferences between x and y should depend only on individuals’ preferences between x and y , and not on preferences concerning some third

² As used in elections, plurality rule is, strictly speaking, a *voting rule*, not a SWF: it merely determines the *winner* (the candidate who is ranked first by a plurality of voters). By contrast, a SWF requires that *all* candidates be ranked socially (Arrow 1951 sees this as a contingency plan: if the top choice turns out not to be feasible, society can move to the second choice, etc.). See Section 4 for further discussion of voting rules.

³ This isn’t quite accurate, because it is possible that x is *never* ranked first. But we will ignore this small qualification.

alternative – is satisfied by few SWFs.⁴ Even so, it has a compelling justification: to prevent *spoilers* and vote-splitting in elections.⁵

To understand the issue, consider Scenario 1 (modified from Maskin and Sen 2016). There are three candidates – Donald Trump, Marco Rubio, and John Kasich (the example is inspired by the 2016 Republican primary elections) – and three groups of voters. One group (40%) ranks Trump above Kasich above Rubio; the second (25%) places Rubio over Kasich over Trump; and the third (35%) ranks Kasich above Trump above Rubio (see Figure A).

<u>40%</u>	<u>25%</u>	<u>35%</u>
Trump	Rubio	Kasich
Kasich	Kasich	Trump
Rubio	Trump	Rubio

Figure A: Scenario 1

Many Republican primaries in 2016 used plurality rule; so the winner was the candidate ranked first by more voters than anyone else.⁶ As applied to Scenario 1, Trump is the winner with 40% of the first-place rankings. But, in fact, a large majority of voters (60%, i.e., the second and third groups) prefer Kasich to Trump. The only reason why Trump wins in Scenario 1 is that

⁴ One SWF that does satisfy IIA is *majority rule*, in which alternative x is socially preferred to y if a majority of individuals prefer x to y . However, unless individuals’ preferences are restricted, social preferences with majority rule may cycle (i.e., x may be preferred to y , y preferred to z , and yet z preferred to x), as Condorcet (1785) discovered. In that case, majority rule is not actually a SWF (since its social preferences are intransitive). That is, majority rule violates U.

⁵ Eliminating spoilers has frequently been cited the voting literature as a rationale for IIA. See, for example the Wikipedia article on vote splitting https://en.wikipedia.org/wiki/Vote_splitting, especially the section on “Mathematical definitions.”

⁶ In actual plurality rule elections, citizens simply vote for a single candidate rather than rank candidates. But this leads to the same winner as long as citizens vote for their most preferred candidate.

Rubio *spoils* the election for Kasich by splitting off some of his support;⁷ Rubio and Kasich split the first-place votes that don't go to Trump.

An SWF that satisfies IIA avoids spoilers and vote-splitting. To see this, consider Scenario 2, which is the same as Scenario 1 except that voters in the middle group now prefer Kasich to Trump to Rubio (see Figure B).

<u>40%</u>	<u>25%</u>	<u>35%</u>
Trump	Kasich	Kasich
Kasich	Trump	Trump
Rubio	Rubio	Rubio

Figure B: Scenario 2

Pretty much any non-pathological SWF will lead to Kasich being ranked above Trump in Scenario 2 (Kasich is not only top-ranked by 60% of voters, but is ranked second by 40%; by contrast, Trump reverses these numbers: he is ranked first by 40% and second by 60%).

However, if the SWF satisfies IIA, it must also rank Kasich over Trump in Scenario 1, since each of the three groups has the same preferences between the two candidates in both scenarios.

Hence, unlike plurality rule, a SWF satisfying IIA circumvents spoilers and vote-splitting:

Kasich will win in Scenario 1.

⁷ In everyday language, candidate A spoils the election for B if (i) B wins when A doesn't run, and (ii) C wins when A does run (because some voters support A strongly, and this support would otherwise have gone to B). In Arrow's (1951) framework (which we adopt here), however, there is a *fixed* set of candidates, and so we interpret "not running" as being ranked at the bottom by all voters. Similarly, we interpret "some voters supporting A strongly" as their ranking A high (i.e., above B and C). Thus, formally, A is a spoiler for B if B beats A when all voters rank A at the bottom, but C beats B when some voters switch to ranking A above B and C (with no other changes to the preference profile).

But imposing IIA is too demanding: It is stronger than necessary to prevent spoilers (as we will see), and makes sensitivity to preference intensities impossible.⁸ To understand this latter point, consider Scenario 3, in which there are three candidates x , y , and z and two groups of voters, one (45% of the electorate) who prefer x to z to y ; and the other (55%), who prefer y to x to z (see Figure C).

<u>45%</u>	<u>55%</u>	Under the Borda count
x	y	x gets $3 \times 45 + 2 \times 55 = 245$ points
z	x	y gets $3 \times 55 + 1 \times 45 = 210$ points
y	z	z gets $2 \times 45 + 1 \times 55 = 145$ points
		x
		so the social ranking is y
		z

Figure C: Scenario 3

For this scenario, let's apply the *Borda count* (rank-order voting), in which, if there are m candidates, a candidate gets m points for every voter who ranks her first, $m-1$ points for a second-place ranking, and so on. Candidates are then ranked according to their vote totals. The calculations in Figure C show that in Scenario 3, x is socially preferred to y and y is socially preferred to z . But now consider Scenario 4, where the first group's preferences are replaced by x over y over z (see Figure D).

⁸ Arrow (1951) assumes that a SWF is a function only of individuals' *ordinal* preferences, which means that preference intensities cannot directly be expressed in his framework. However, this does not rule out the possibility of *inferring* intensities from ordinal data, as we argue in footnote 9.

<u>45%</u>	<u>55%</u>	Under the Borda count, the
x	y	
y	x	social ranking is now
z	z	y x , a z
		violation of IIA as applied to x and y

Figure D: Scenario 4

As calculated in Figure D, the Borda social ranking becomes y over x over z . This violates IIA: in going from Scenario 3 to 4, no individual’s ranking of x and y changes, yet the social ranking switches from x above y to y above x .

However, the anti-spoiler/anti-vote-splitting rationale for IIA doesn’t apply to Scenarios 3 and 4. Notice that candidate z doesn’t split first-place votes with y in Scenario 3; indeed, she is *never* ranked first. Moreover, her position in group 1 voters’ preferences in Scenarios 3 and 4 provides potentially useful information about the intensity of those voters’ preferences between x and y . In Scenario 3, z lies between x and y – suggesting that the preference gap between x and y may be substantial. In the second case, z lies below both x and y , implying that the difference between x and y is not as big. Thus, although z may not be a strong candidate herself (i.e., she is, in some sense, an “irrelevant alternative”), how individuals rank her vis à vis x and y is arguably pertinent to social preferences^{9, 10} i.e., IIA should not apply to these scenarios.

⁹ Here is one setting in which we can make the argument formal: Imagine that, from the perspective of an outside spectator (society), each of a voter’s utilities $u(x)$, $u(y)$, and $u(z)$ (where u captures preference intensity) is drawn randomly and independently from some distribution. For reasons given in footnote 10, however, the spectator cannot directly observe these utilities; she can observe only the voter’s ranking of alternatives. Nevertheless, the expected difference $u(x) - u(y)$ conditional on z being between x and y in the voter’s preference ordering is greater than the difference conditional on z *not* being between x and y . Thus, the spectator can infer cardinal information from the ordinal ranking.

¹⁰ One might wonder why, instead of depending only on individuals’ ordinal rankings, a SWF is not allowed to depend directly on their *cardinal* utilities, as in Benthamite utilitarianism (Bentham, 1789) or majority judgment

Accordingly, we propose a relaxation of IIA.¹¹ Under *modified independence of irrelevant alternatives* (MIIA), if given two alternatives x and y and two profiles of individuals' preferences, (i) each individual ranks x and y the same way in the first profile as in the second, and (ii) each individual ranks the same number of alternatives *between* x and y in the first profile as in the second, then the social ranking of x and y must be the same for both profiles.

If we imposed only requirement (i), then MIIA would be identical to IIA. Requirement (ii) is the one that permits preference intensities to figure in social rankings. Specifically, notice that, since z lies between x and y in group 1's preferences in Scenario 3 but not in Scenario 4, MIIA does *not* require the social rankings of x and y to be the same in the two scenarios. That is, accounting for preference intensities is permissible under MIIA.

Even so, MIIA is strong enough to rule out spoilers and vote-splitting (i.e., a SWF satisfying MIIA cannot exhibit the phenomenon of footnote 7). In particular, it rules out plurality rule: in neither Scenario 1 nor Scenario 2 do group 2 voters rank Rubio between Kasich and Trump. Therefore, MIIA implies that the social ranking of Kasich and Trump must be the *same* in the two scenarios, contradicting plurality rule.

(Balinski and Laraki, 2010). But it is not at all clear how to ascertain these utilities, even leaving aside the question of deliberate misrepresentation by individuals. Indeed, for that reason, Lionel Robbins (1932) rejected the idea of cardinal utility altogether, and Arrow (1951) followed in that tradition. Notice that in the case of ordinal preferences, there is an experiment we can perform to verify an individual's ranking: if he says he prefers x to y , we can offer him the choice and see which he selects. But there is no corresponding experiment for cardinal utility - - except in the case of risk preferences, where we can offer lotteries (in the von Neumann-Morgenstern 1944 procedure for constructing a utility function, utilities are cardinal in the sense that they can be interpreted as probabilities in a lottery). Yet, risk preferences are not the same thing as preference intensities. And introducing risk preferences in social choice situations entailing no uncertainty seems of dubious relevance (for that reason, Harsanyi's 1955 derivation of utilitarianism based on risk preferences is often criticized). Finally, even if there were an experiment for eliciting utilities, misrepresentation might interfere with it. Admittedly, there are circumstances when individuals have the incentive to misrepresent their rankings with the Borda count. But a cardinal SWF is subject to much greater misrepresentation because individuals have the incentive to distort even when there are only two alternatives (see Dasgupta and Maskin 2020). Thus, we are left only with the possibility of inferring preference intensities from *ordinal* preferences, as in footnote 9.

¹¹ Other authors who have considered variants of IIA include Brandl and Brandt (2020), Dhillon and Mertens (1999), Eden (2020), Fleurbaey, Suzumura, and Tadenuma (2005) and (2005a), Osborne (1976), Roberts (2009), Saari (1998), Young (1988), and Young and Levenglick (1978).

Runoff voting is also ruled out by MIIA. Under that voting rule, a candidate wins immediately if he is ranked first by a majority of voters.¹² But failing that, the two top vote-getters go to a runoff. Notice, that if we change Scenario 1 so that the middle group constitutes 35% of the electorate and the third group constitutes 25%, then Trump (with 40% of the votes) and Rubio (with 35%) go to the runoff (and Kasich, with only 25%, is left out). Trump then wins in the runoff, because a majority of voters prefer him to Rubio. If we change Scenario 2 correspondingly (so that the 25% and 35% groups are interchanged), then Kasich wins in the first round with an outright majority. Thus, runoff voting violates MIIA for the same reason that plurality rule does.

B. May's Axioms for Majority Rule

When there are just two alternatives, majority rule is far and away the most widely used democratic method for choosing between them. Indeed, almost all other commonly used voting rules – e.g., plurality rule, runoff voting, and the Borda count – reduce to majority rule in this case.

May (1952) crystallized why majority rule is so compelling in the two-alternative case by showing that it is the only voting rule satisfying *anonymity* (A), *neutrality* (N), and *positive responsiveness* (PR). Axiom A is the requirement that all individuals be treated equally, i.e., that if they exchange preferences with one another (so that individual j gets i 's preferences, individual k get j 's, and so on), social preferences remain the same. N demands that all alternatives be treated equally, i.e., that if the alternatives are permuted and individuals'

¹² Like plurality rule, runoff voting in *practice* is usually administered so that a voter just picks one candidate rather than ranking them all (see footnote 6).

preferences are changed accordingly, then social preferences are changed in the same way.¹³ And PR requires that if alternative x rises relative to y in some individuals' preference orderings, then (i) x doesn't fall relative to y in the social ordering, and (ii) if x and y were previously tied socially, x is now strictly above y .

C. Borda's Rule and Condorcet Cycles: A Central Special Case

The main result of this paper establishes that a SWF satisfies U, MIIA, A, N, and PR (the other Arrow conditions – P and ND – are redundant) if and only if it is the Borda count.¹⁴

Checking that the Borda count satisfies the five axioms is straightforward.¹⁵

To illustrate the main idea of the proof in the other direction, let us focus on the case of three alternatives x , y , and z and suppose that F is a SWF satisfying the five axioms. We will show that

when F is restricted to the domain of preferences $\left\{ \begin{array}{ccc} x & y & z \\ y, z, x \\ z & x & y \end{array} \right\}$ (i.e., when we consider only

profiles with preferences drawn from this domain¹⁶), it must coincide with the Borda count.

¹³ May (1952) expressed the A, N, and PR axioms only for the case of two alternatives. In section 2 we give formal extensions for three or more alternatives (See also Dasgupta and Maskin 2020).

¹⁴ Young (1974) provides a well-known axiomatization of the Borda count, but his axioms are quite different from ours. Saari (2000) and (2000a) provide a vigorous defense of the Borda count based on its geometric properties.

¹⁵ To see that the Borda count satisfies MIIA, note that if two profiles satisfy the hypotheses of the condition, then the difference between the number of points a given voter contributes to x and the number she contributes to y must be the *same* for the two profiles (because the number of alternatives ranked between x and y is the same). Thus, the difference between the total Borda scores of x and y – and hence their social rankings – are the same.

¹⁶ From U, F is defined for every such profile.

Consider, first, the profile in which $\frac{1}{3}$ of individuals have ranking $\begin{matrix} x \\ y \\ z \end{matrix}$; $\frac{1}{3}$ have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$;

and $\frac{1}{3}$ have ranking $\begin{matrix} z \\ x \\ y \end{matrix}$.¹⁷ We claim that the social ranking of x and y that F assigns to this profile

is *social indifference*:

$$(1) \quad \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} x \sim y$$

If (1) doesn't hold, then either

$$(2) \quad \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \end{array}$$

or

$$(3) \quad \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}$$

If (2) holds, then apply permutation σ – with $\sigma(x) = y$, $\sigma(y) = z$, and $\sigma(z) = x$ – to (2).

From N, we obtain

$$(4) \quad \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ y & x & z \\ z & y & x \\ x & z & y \end{array} \xrightarrow{F} \begin{array}{c} y \\ z \end{array}$$

Applying σ to (4) and invoking N, we obtain

¹⁷ From A, we don't need to worry about which individuals have which preferences.

$$(5) \quad \begin{array}{ccc} \frac{1/3}{z} & \frac{1/3}{y} & \frac{1/3}{x} \\ x & z & y \\ y & x & z \end{array} \xrightarrow{F} \begin{array}{c} z \\ x \end{array}$$

But the profiles in (2), (4), and (5) are the same except for permutations of individuals' preferences, and so, from A, give rise to the same social ranking under F , which in view of (2), (4), and 5 must be

$$\begin{array}{c} x \\ y \\ z \\ x \end{array},$$

violating transitivity. The analogous contradiction arises if (3) holds. Hence, (1) must hold after all. From MIIA and (1), we have

$$(6) \quad \begin{array}{ccc} \frac{a}{x} & \frac{b}{z} & \frac{1/3}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} x \sim y, \text{ for all } a \geq 0 \text{ and } b \geq 0 \text{ such that } a + b = 2/3$$

From PR and (6), we have

$$(7) \quad \begin{array}{ccc} \frac{a}{x} & \frac{b}{z} & \frac{1-a-b}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \end{array}, \text{ where } a + b > 2/3, \text{ and } a, b, 1-a-b \geq 0,$$

and

$$(8) \quad \begin{array}{ccc} \frac{a}{x} & \frac{b}{z} & \frac{1-a-b}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}, \text{ where } a + b < 2/3, \text{ and } a, b, 1-a-b \geq 0.$$

But (6), (7), and (8) collectively imply that x is socially preferred to y if and only if x 's Borda score exceeds y 's Borda score,¹⁸ i.e., F is the Borda count. Q.E.D

¹⁸ For example, in (7), x 's Borda score is $3a + 2b + 1 - a - b$ and y 's Borda score is $3(1 - a - b) + 2a + b$. Hence x is Borda-ranked above y if and only if

$$3a + 2b + 1 - a - b > 3(1 - a - b) + 2a + b,$$

The domain $\left\{ \begin{array}{ccc} x & z & y \\ y, & x, & z \\ z & y & x \end{array} \right\}$ is called a Condorcet cycle because, as Condorcet (1785)

showed, majority rule may cycle for profiles on this domain (indeed, it cycles for the profile in (1)). This domain is the focus of much of the social choice literature, e.g., Arrow (1951) makes crucial use of Condorcet cycles in the proof of the Impossibility Theorem; Barbie et al (2006) show that it is essentially the unique domain (for three alternatives) on which the Borda count is strategy-proof; and Dasgupta and Maskin (2008) show that no voting rule can satisfy all of P, A, N, and IIA on this domain. One implication of our result in this section is that there is a sense in which the Borda count comes closer than any other voting rule to satisfying these four axioms on a Condorcet cycle domain - - it satisfies P, A, and N and captures (through MIIA) the “essence” of IIA.

D. Roadmap

In Section 2, we lay out the model and provide most of the formal definitions. In Section 3, we first prove the characterization result assuming that a SWF’s indifference curves for pairs of alternatives¹⁹ are continuous (Theorem 1). We then show that continuity is, in fact, implied by the axioms (Theorem 2). Finally, Section 4 discusses a few open questions.

which reduces to $a + b > 2/3$.

¹⁹ Given a SWF F , the *indifference curve* for alternatives x and y consists of the set of profiles for which there is social indifference between x and y .

2. Formal Model and Definitions

Consider a society consisting of a continuum of individuals²⁰ (indexed by $i \in [0,1]$) and a finite set of social alternatives X , with $|X| = m$.²¹ For each individual i , let \mathfrak{R}_i be a set of possible *strict* rankings²² of X for individual i and let \succsim_i be a typical element of \mathfrak{R}_i ($x \succsim_i y$ means that individual i prefers alternative x to y). Then, a *social welfare function* (SWF) F is a mapping

$$F : \prod_{i \in [0,1]} \mathfrak{R}_i \rightarrow \mathfrak{R} ,$$

where \mathfrak{R} is the set of all possible social rankings (here we *do* allow for indifference and the typical element is \succsim).

With a continuum of individuals, we can't literally count the number of individuals with a particular preference; we have to work with proportions instead. For that purpose, let μ be Lebesgue measure on $[0,1]$. Given profile \succsim , interpret $\mu(\{i | x \succsim_i y\})$ as the proportion of individuals who prefer x to y .²³

²⁰ In assuming a continuum, we are following Dasgupta and Maskin (2008) and (2020). Those earlier papers invoked this assumption primarily to ensure that ties are nongeneric. The assumption plays that role in this paper too, but more importantly, it guarantees together with Positive Responsiveness that ties *occur*. Indeed, our proof technique relies critically on analyzing a SWF's indifference curves, i.e., the sets of profiles for which there are ties.

²¹ $|X|$ is the number of alternatives in X .

²² Thus, we rule out the possibility that an individual can be indifferent between two alternatives. However, we conjecture that our results extend to the case where she can be indifferent (see Section 4).

²³ To be accurate, we must restrict attention to profiles \succsim for which $\{i | x \succsim_i y\}$ is a measurable set.

The Arrow conditions for a SWF F are:

Unrestricted Domain (U): The SWF must determine social preferences for all possible preferences that individuals might have. Formally, for all $i \in [0,1]$, \mathfrak{R}_i consists of *all* strict orderings of X .

Pareto Property (P): If all individuals (strictly) prefer x to y , then x must be strictly socially preferred. Formally, for all profiles $\succsim_i \in \times \mathfrak{R}_i$ and all $x, y \in X$, if $x \succ_i y$ for all i , then $x \succ_F y$, where $\succsim_F = F(\succsim_i)$.

Nondictatorship (ND): There exists no individual who always gets his way in the sense that if he prefers x to y , then x must be socially preferred to y , regardless of others' preferences. Formally, there does *not* exist i^* such that for all $\succsim_i \in \times \mathfrak{R}_i$ and all $x, y \in X$, if $x \succ_{i^*} y$, then $x \succ_F y$, where $\succsim_F = F(\succsim_i)$.

Independence of Irrelevant Alternatives (IIA): Social preferences between x and y should depend only on individuals' preferences between x and y , and not on their preferences concerning some third alternative. Formally, for all $\succsim_i, \succsim'_i \in \times \mathfrak{R}_i$ and all $x, y \in X$, if, for all i , $x \succ_i y \Leftrightarrow x \succ'_i y$, then \succsim_F ranks x and y the same way that \succsim'_F does, where $\succsim_F = F(\succsim_i)$ and $\succsim'_F = F(\succsim'_i)$.

Because we have argued that IIA is too strong, we are interested in the following relaxation:

Modified IIA: If, given two profiles and two alternatives, each individual (i) ranks the two alternatives the same way in both profiles and (ii) ranks the same number of other alternatives *between* the two alternatives in both profiles, then the social preference between x and y should be the same for both profiles. Formally, for all $\succsim_i, \succsim'_i \in \times \mathfrak{R}_i$ and all $x, y \in X$, if, for all i ,

$|\{z|x \succ_i z \succ_i y\}| = |\{z|x \succ'_i z \succ'_i y\}|$, $|\{z|y \succ_i z \succ_i x\}| = |\{z|y \succ'_i z \succ'_i x\}|$, and $x \succ_i y \Leftrightarrow x \succ'_i y$, then \succ_F and \succ'_F rank x and y the same way, where $\succ_F = F(\succ)$ and $\succ'_F = F(\succ')$.²⁴

May (1952) characterizes majority rule axiomatically in the case $|X| = 2$. We will consider natural extensions of his axioms to three or more alternatives:

Anonymity (A): If we permute a preference profile so that individual j gets i 's preferences, k gets j 's preferences, etc., then the social ranking remains the same. Formally, fix any (measure-preserving)²⁵ permutation of society $\pi: [0,1] \rightarrow [0,1]$. For any profile $\succ \in \times \mathfrak{R}_i$, let \succ^π be the profile such that, for all i , $\succ_i^\pi = \succ_{\pi(i)}$. Then $F(\succ^\pi) = F(\succ)$.

Neutrality (N): Suppose that we permute the alternatives so that x becomes y , y becomes z , etc., and we change individuals' preferences in the corresponding way. Then, if x was socially ranked above y originally, now y is socially ranked above z . Formally, for any permutation $\rho: X \rightarrow X$ and any profile $\succ \in \times \mathfrak{R}_i$, let \succ^ρ be the profile such that, for all $x, y \in X$ and all $i \in [0,1]$, $x \succ_i y \Leftrightarrow \rho(x) \succ_i^\rho \rho(y)$. Then, for all $x, y \in X$, $x \succ_F y \Leftrightarrow \rho(x) \succ_F^\rho \rho(y)$, where $\succ_F = F(\succ)$ and $\succ_F^\rho = F(\succ^\rho)$.

Positive Responsiveness (PR)²⁶: If we change individuals' preferences so that alternative x moves up and y moves down relative to each other and to other alternatives (and no other changes are made), then x moves up socially relative to y (i.e., if x and y were previously socially

²⁴ There is a similar but weaker condition developed in Maskin (2020).

²⁵ Because we are working with a continuum of individuals we must explicitly assume that $\mu(\{i|x \succ_i y\}) = \mu(\{\pi(i)|x \succ_{\pi(i)} y\})$, which holds automatically with a finite number of individuals.

²⁶ For a different generalization of PR to more than two alternatives, see Horan, Osborne, and Sanver (2019).

indifferent, x is now strictly preferred; if x was previously socially preferred to y , it remains so; if y was socially preferred to x for the first profile and x is socially preferred for the second, then there exists an intermediate profile for which x and y are socially indifferent²⁷). Formally, suppose \succsim_i and \succsim'_i are two profiles such that, for some $x, y \in X$ and for all $i \in [0, 1]$,

$$(*) \quad x \succsim_i z \Rightarrow x \succsim'_i z, w \succsim_i y \Rightarrow w \succsim'_i y, \text{ and } r \succsim_i s \Leftrightarrow r \succsim'_i s \text{ for all } z \neq x, w \neq y \text{ and } r, s \in X - \{x, y\}.$$

Then, if $\mu(\{i \mid y \succsim_i x \text{ and } x \succsim'_i y\}) > 0$, we have $x \succsim_{\succsim_F} y \Rightarrow x \succsim'_F y$, where $\succsim_F = F(\succsim_i)$ and

$\succsim'_F = F(\succsim'_i)$. Furthermore, if $y \succ_{\succsim_F} x$ and $x \succ_{\succsim'_F} y$, then there exists profile \succsim''_i satisfying (*) (with \succsim''_i replacing \succsim'_i) such that $x \sim''_F y$.

We can now define the Borda count formally:

Borda Count: Alternative x is socially (weakly) preferred to y if and only if x 's Borda score (where x gets m points every time an individual ranks it first, $m-1$ points every time an individual ranks it second, etc.) is (weakly) bigger than y 's Borda score. Formally, for all $x, y \in X$ and all profiles $\succsim_i \in \times \mathfrak{R}_i$,

$$x \succsim_{\succsim_{Bor}} y \Leftrightarrow \int r_{\succsim_i}(x) d\mu(i) \geq \int r_{\succsim_i}(y) d\mu(i),$$

where $r_{\succsim_i}(x) = |\{y \in X \mid x \succsim_i y\}| + 1$ and \succsim_{Bor} is the Borda ranking corresponding to \succsim_i .

The proof of our characterization result makes heavy use of a SWF F 's indifference curves. To define the concept of an indifference curve, let us focus on the case $X = \{x, y, z\}$ (the extension to more than three alternatives is immediate) and fix a profile \succsim_i . Let $a_{xyz}(\succsim_i)$ be the

fraction of individuals who have ranking $\begin{matrix} x \\ y \\ z \end{matrix}$. Then, if F satisfies A, the 6-tuple

²⁷ In the two-alternative case, the existence of an intermediate profile with social indifference does not need to be assumed; it follows from A, N, and the other provisions of PR.

$$(9) \alpha = (\alpha_{xyz}, \alpha_{zxy}, \alpha_{yxz}, \alpha_{zyx}, \alpha_{xzy}, \alpha_{yzx})$$

$$= (a_{xyz}(\succ), a_{zxy}(\succ), a_{yxz}(\succ), a_{xzy}(\succ), a_{yzx}(\succ), a_{zyx}(\succ))$$

is a sufficient statistic for \succ in determining social preferences \succ_F . We define the *indifference*

curve for x and y , I_F^{xy} , to be the set of 6-tuples for which society is indifferent between x and y :

$$I_F^{xy} = \{\alpha \in \Delta^5 \mid x \sim_F y \text{ for } \alpha \text{ satisfying (9)}\}.$$
²⁸

It will be convenient to represent I_F^{xy} as a function J_F^{xy} . Let us assume that, in addition to

A, F satisfies U, MIIA, N, and PR. Let

$$(10) D^* = \{(\alpha_{xzy}, \alpha_{yzx}) \mid \text{there exists } (\alpha_{xyz}^*, \alpha_{yxz}^*, \alpha_{zxy}^*, \alpha_{zyx}^*) \text{ for which}$$

$$(\alpha_{xyz}^*, \alpha_{zxy}^*, \alpha_{yxz}^*, \alpha_{xzy}^*, \alpha_{yzx}^*, \alpha_{zyx}^*) \in I_F^{xy}\}$$

For any $(\alpha_{xzy}, \alpha_{yzx}) \in D^*$, let

$$(11) J_F^{xy}(\alpha_{xzy}, \alpha_{yzx}) = \alpha_{xzy}^* + \alpha_{yzx}^*,$$

²⁸ For some important SWFs in the literature, I_F^{xy} is *linear*, i.e., it is described by a linear equation. For example, for majority rule, we have

$$I_{maj}^{xy} = \{\alpha \in \Delta^5 \mid \alpha_{xyz} + \alpha_{zxy} + \alpha_{xzy} = \alpha_{yzx} + \alpha_{yxz} + \alpha_{zyx}\},$$

i.e., the indifference curve consists of points where the proportion of individuals who rank x and above y equals that who rank y above x . Similarly, for plurality rule, we have

$$I_{plur}^{xy} = \{\alpha \mid \alpha_{xyz} + \alpha_{xzy} = \alpha_{yzx} + \alpha_{zyx}\},$$

i.e., the proportion of individuals who rank x first equals the proportion who rank y first. Finally, the Borda indifference curve takes the form

$$(**) I_{Bor}^{xy} = \{\alpha \mid \alpha_{xyz} + \alpha_{zxy} + 2\alpha_{xzy} = \alpha_{yzx} + \alpha_{zyx} + 2\alpha_{yxz}\},$$

where the coefficient of α_{xzy} in (**) is 2 because x lies two places above y in $\frac{x}{z}$, and analogously for the

coefficient of α_{yzx} .

where α_{xyz}^* and α_{zxy}^* , are given by (10). $J_F^{xy}(\alpha_{xzy}, \alpha_{yzx})$ is well-defined because, from PR and MIIA, if

$$(\alpha_{xyz}^{**}, \alpha_{zxy}^{**}, \alpha_{yzx}, \alpha_{xzy}, \alpha_{yxz}^{**}, \alpha_{zyx}^{**}) \in I_F^{xy},$$

then

$$\alpha_{xyz}^{**} + \alpha_{zxy}^{**} = \alpha_{xyz}^* + \alpha_{zxy}^*$$

and

$$\alpha_{yxz}^{**} + \alpha_{zyx}^{**} = \alpha_{yxz}^* + \alpha_{zyx}^*.$$

Moreover, from MIIA, if $\alpha_{xyz}, \alpha_{zxy}, \alpha_{yxz}, \alpha_{zyx}$ are proportions such that

$$\alpha_{xyz} + \alpha_{zxy} = \alpha_{xyz}^* + \alpha_{zxy}^*$$

and

$$\alpha_{yxz} + \alpha_{zyx} = \alpha_{yxz}^* + \alpha_{zyx}^*,$$

then

$$(\alpha_{xzy}, \alpha_{yxz}, \alpha_{zyx}, \alpha_{xyz}, \alpha_{zxy}, \alpha_{yzx}) \in I_F^{xy}.$$

Notice that the kind of symmetry argument we used in Section 1C implies

$$(12) (1/6, 1/6, 1/6, 1/6, 1/6, 1/6) \in I_F^{xy},$$

and so

$$(13) J_F^{xy}(1/6, 1/6) = 1/3 \text{ and } (1/6, 1/6) \in D^*$$

The extension to more than 3 alternatives is straightforward and is discussed in the proof of Theorem 1.

3. The Characterization Result

We first establish our characterization result under the assumption that indifference curves are continuous:

Theorem 1: Given SWF F , suppose that, for each pair of alternatives x and y , the indifference curve J_F^{xy} is a continuous function. Then, F satisfies U, MIIA, A, N, and PR if and only if F is the Borda count.²⁹

Proof:

For $|X|=2$, the result follows from May (1952) (since the Borda count coincides with majority rule for the case of two alternatives).

Let us turn to the case $|X|=3$. Our approach will be to show that, if F satisfies the axioms, then J_F^{xy} coincides with the Borda indifference curve

$$(14) J_{Bor}^{xy}(\alpha_{xzy}, \alpha_{yzx}) = 1/2 - 3\alpha_{xzy}/2 + \alpha_{yzx}/2 \quad ^{30}$$

We can then invoke PR, as we did in Section 1C, to argue that F coincides with the Borda count everywhere.

Step 1: We will first show that, just on the basis of neutrality and transitivity, we can already say quite a lot about the xy -indifference curve

It will be useful to transform J_F^{xy} to function $H_F^{xy} : D \rightarrow \mathbb{R}$, where D is the translation of D^* and

$$(15) H_F^{xy}(s, t) = J_F^{xy}(1/6 + s, 1/6 + t) - 1/3 \text{ for all } (s, t) \in D.$$

²⁹ That the Borda count satisfies MIIA was established in footnote 15. That it satisfies the other axioms is obvious.

³⁰ To see that J_{Bor}^{xy} satisfies (14), observe that the difference between the Borda scores for x and y is

$$(\#) 2(\alpha_{xzy} - \alpha_{yzx}) + J_{Bor}^{xy} - (1 - \alpha_{xzy} - \alpha_{yzx} - J_{Bor}^{xy})$$

Along the Borda indifference curve for x and y , (#) equals 0, giving us (14)

To simplify notation, we will suppress the superscript “ xy ” of H_F^{xy} , since from N, $H_F^{xy}(s,t)$ doesn’t depend on x and y .

Note first that from N,

$$(16) J_F^{xy}(1/6+s, 1/6+s) = 1 - (1/3 + 2s) - J_F^{xy}(1/6+s, 1/6+s), \text{ for all } s \in [-1/6, 1/3].$$

Hence,

$$(17) J_F^{xy}(1/6+s, 1/6+s) = 1/3 - s,$$

and so

$$(18) H(s, s) = -s \text{ for all } s \in [-1/6, 1/3]$$

Next, consider profile

$$(19) \gamma^* = \begin{matrix} \frac{1/6-s}{x} & \frac{1/6-s}{z} & \frac{1/6-s}{y} & \frac{1/6+s}{z} & \frac{1/6+s}{y} & \frac{1/6+s}{x} \\ \frac{1/6-s}{y} & \frac{1/6-s}{x} & \frac{1/6-s}{z} & \frac{1/6+s}{y} & \frac{1/6+s}{x} & \frac{1/6+s}{z} \\ \frac{1/6-s}{z} & \frac{1/6-s}{y} & \frac{1/6-s}{x} & \frac{1/6+s}{x} & \frac{1/6+s}{z} & \frac{1/6+s}{y} \end{matrix}, \text{ where } |s| \leq 1/6$$

Arguing exactly as we did for the Condorcet profile in Section 1C, we can show that

$$(20) x \sim_F^* y, \text{ where } \succ_F^* = F(\succ_F^*)$$

From (19) and (20) $J_F^{xy}(1/6+s, 1/6-s) = 1/3 - 2s$ for all s with $|s| < 1/6$. Hence

$$(21) H_F(s, -s) = -2s \text{ for all } |s| \leq 1/6$$

By adding transitivity of the social ordering into the analytic mix, we now derive an additional strong restriction on H_F . Given $(s, t) \in D$, consider profile

$$\alpha = (\alpha_{xyz}, \alpha_{zyx}, \alpha_{yxz}, \alpha_{xzy}, \alpha_{yzx}, \alpha_{zxy}) \text{ such that}$$

$$(22) \begin{cases} \alpha_{xzy} = \alpha_{zyx} = 1/6 + s \\ \alpha_{yzx} = \alpha_{xyz} = 1/6 + t \\ \alpha_{zxy} = J_F^{xy}(\alpha_{xzy}, \alpha_{yzx}) - (1/6 + t) \\ \alpha_{yxz} = J_F^{yx}(\alpha_{yzx}, \alpha_{xzy}) - (1/6 + s) \end{cases}$$

From (22),

$$(23) \alpha_{xyz} + \alpha_{zxy} = J_F^{xy}(\alpha_{xzy}, \alpha_{yzx})$$

Hence,

$$(24) \alpha \in I_F^{xy}$$

From (22) and (23)

$$(25) \alpha_{yzx} + \alpha_{zxy} = J_F^{xy}(\alpha_{xzy}, \alpha_{yzx})$$

Now, from N, $J_F^{xy}(\cdot, \cdot) = J_F^{zx}(\cdot, \cdot)$ and so from (22)

$$(26) J_F^{xy}(\alpha_{xzy}, \alpha_{yzx}) = J_F^{xy}(1/6 + s, 1/6 + t) = J_F^{zx}(1/6 + s, 1/6 + t) = J_F^{zx}(\alpha_{zyx}, \alpha_{xyz})$$

Hence, from (25) and (26)

$$(27) \alpha_{yzx} + \alpha_{zxy} = J_F^{zx}(\alpha_{zyx}, \alpha_{xyz}),$$

which implies that

$$(28) \alpha \in I_F^{zx}$$

Thus, (24), (28) and the transitivity of social preferences imply that

$$(29) \alpha \in I_F^{zy}$$

From (22) and (29)

$$(30) \begin{aligned} 1/3 + 2t &= \alpha_{yzx} + \alpha_{zxy} = J_F^{yz}(\alpha_{yxz}, \alpha_{zxy}) \\ &= J_F^{yz}(J_F^{yx}(1/6 + t, 1/6 + s) - (1/6 + s), J_F^{xy}(1/6 + s, 1/6 + t) - (1/6 + t)) \end{aligned}$$

From (15), we can rewrite (30) as

$$(31) \quad 2t = H_F(H_F(t, s) - s, H_F(s, t) - t) \text{ for all } (s, t) \in D$$

Step 2: From N, $H_F(s, t)$ is defined at all points (s, s) and $(s, -s)$ for which $|s| \leq 1/6$. We next show that $H_F(s, t)$ is defined at all (s, t) such that $|s| \leq 1/6$ and $|t| \leq 1/6$. That is, $(s, t) \in D$ for all such (s, t) .

Suppose $s > t > 0$. If, given profile \succ_{\cdot} , $1/6 + s$ is the proportion of the electorate with

ranking $\begin{matrix} x & & y \\ z & & z \\ y & & x \end{matrix}$ and $1/6 + t$ is the proportion with $\begin{matrix} & & y \\ & & z \\ & & x \end{matrix}$, then, from MIA, we can summarize \succ_{\cdot} by the 4-

tuple $(1/6 + s, 1/6 + t, \alpha_{xy}, \alpha_{yx})$, where

$$\alpha_{xy} = a_{xyz}(\succ_{\cdot}) + a_{zxy}(\succ_{\cdot})$$

and

$$\alpha_{yx} = a_{yxz}(\succ_{\cdot}) + a_{zyx}(\succ_{\cdot})$$

If there exist profiles $\succ_{\cdot}^* = (1/6 + s, 1/6 + t, \alpha_{xy}^*, \alpha_{yx}^*)$ and $\succ_{\cdot}^{**} = (1/6 + s, 1/6 + t, \alpha_{xy}^{**}, \alpha_{yx}^{**})$ with

$$x \succ_F^* y, \text{ where } \succ_F^* = F(\succ_{\cdot}^*) \text{ and } y \succ_F^{**} x, \text{ where } \succ_F^{**} = F(\succ_{\cdot}^{**}),$$

then, from PR, there exists profile \succ_{\cdot}^{***} such that $\alpha_{xzy}^{***} = 1/6 + s$ $\alpha_{yzx}^{***} = 1/6 + t$, and

$$x \sim_F^{***} y, \text{ where } \succ_F^{***} = F(\succ_{\cdot}^{***}).$$

Hence, if $(s, t) \notin D$, we must have either

$$(32) \quad x \succ_F y \text{ for all profiles } \succ_{\cdot} = (1/6 + s, 1/6 + t, \alpha_{xy}, \alpha_{yx}) \text{ where } \succ_F = F(\succ_{\cdot})$$

or

$$(32a) \quad y \succ_F x \text{ for all profiles } \succ_{\cdot} = (1/6 + s, 1/6 + t, \alpha_{xy}, \alpha_{yx}), \text{ where } \succ_F = F(\succ_{\cdot})$$

From (21),

$$(33) \quad x \sim_F^\circ y \text{ for some } \gamma_\cdot^\circ = (1/6 + s, 1/6 - s, \alpha_{xy}^\circ, \alpha_{yx}^\circ) = (1/6 + s, 1/6 - s, 2/3 - \alpha_{yx}^\circ, \alpha_{yx}^\circ)$$

From (18),

$$(34) \quad x \sim_F^{\circ\circ} y \text{ for some } \gamma_\cdot^{\circ\circ} = (1/6 + s, 1/6 + s, \alpha_{xy}^{\circ\circ}, \alpha_{yx}^{\circ\circ}) = (1/6 + s, 1/6 + s, 2/3 - \alpha_{yx}^{\circ\circ} - 2s, \alpha_{yx}^{\circ\circ})$$

Now, if (32) holds then, in particular,

$$(35) \quad x \succ_F^{\circ\circ\circ} y \text{ for } \gamma_\cdot^{\circ\circ\circ} = (1/6 + s, 1/6 + t, 2/3 - s - t - \alpha_{yx}^\circ, \alpha_{yx}^\circ) \text{ with } \succ_F^{\circ\circ\circ} = F(\gamma_\cdot^{\circ\circ\circ})$$

But $t > -s$ and $2/3 - \alpha_{yx}^\circ > 2/3 - s - t - \alpha_{yx}^\circ$, so in going from the profile in (33) to that in (35), y

risers relative to x and z , and so $x \succ_F^{\circ\circ\circ} y$ contradicts PR.

If (32a) holds instead, then, in particular,

$$(36) \quad y \succ_F^{\circ\circ\circ} x \text{ for } \gamma_\cdot^{\circ\circ\circ} = (1/6 + s, 1/6 + t, 2/3 - s - t - \alpha_{yx}^{\circ\circ}, \alpha_{yx}^{\circ\circ}), \text{ where } \succ_F^{\circ\circ\circ} = F(\gamma_\cdot^{\circ\circ\circ})$$

But $s > t$ and $-s - t - \alpha_{yx}^{\circ\circ} > -2s - \alpha_{yx}^{\circ\circ}$, and so in going from the profile in (36) to that in (34), y

risers relative to x and z , and so $x \sim_F^{\circ\circ} y$ contradicts PR. We conclude that, for both (32) and (32a)

$(s, t) \in D$. We have been supposing that $s > t > 0$, but the other cases are completely symmetric.

Step 3: Now, because H_F is assumed to be continuous and (31) holds, the Weierstrass

approximation theorem implies that, for all $\varepsilon > 0$, there exist an integer q and a polynomial

$$H_F^\varepsilon(s, t) = \sum_{i+j=1}^q A_{ij} s^i t^j \text{ of degree } q \text{ such that}$$

$$(37) \quad \|H_F^\varepsilon - H_F\| < \varepsilon$$

and

$$(38) \quad \|H_F^\varepsilon(H^\varepsilon(t, s) - s, H^\varepsilon(s, t) - t) - 2t\| < \varepsilon,$$

where $\|f(s, t)\| = \max_{(s, t) \in D} |f(s, t)|$ for any continuous function f .

The remainder of the argument consists of showing given that, given (37) and (38),

$H_F^\varepsilon(s, t)$ is also a good approximation of

$$(39) \quad H_{Bor}(s, t) = -3s/2 + t/2,$$

implying (from PR) that $H_F = H_{Bor}$.

To see this, let us suppose first that H_F^ε is of degree 1, i.e., linear:

$$(40) \quad H_F^\varepsilon(s, t) = A_{10}s + A_{01}t.$$

Let $r = 1/6$. From (18) and (21)

$$A_{10}r + A_{01}r = -r + \varepsilon_1, \text{ where } |\varepsilon_1| < \varepsilon$$

and

$$A_{10}r - A_{01}r = -2r + \varepsilon_2, \text{ where } |\varepsilon_2| < \varepsilon,$$

so

$$(41) \quad A_{10} = -3/2 + \frac{\varepsilon_1 + \varepsilon_2}{2r}$$

and

$$(42) \quad A_{01} = 1/2 + \frac{\varepsilon_1 - \varepsilon_2}{2r}.$$

From (41) and (42)

$$(43) \quad \begin{aligned} & \|A_{10}s + A_{01}t - (-3s/2 + t/2)\| \\ &= \max_{|s|, |t| \leq r} \left| \frac{(\varepsilon_1 + \varepsilon_2)}{2r} s + \frac{(\varepsilon_1 - \varepsilon_2)}{2r} t \right| < \varepsilon \end{aligned}$$

Next, suppose that H_F^ε is quadratic, i.e.,

$$(44) \quad H_F^\varepsilon(s, t) = B_{10}s + B_{01}s + B_{20}s^2 + B_{11}st + B_{02}t^2.$$

From (18) and (21), we have, for all $v \in [-r, r]$

$$B_{10}v + B_{01}v + (B_{20} + B_{02})v^2 + B_{11}v^2 = -v + \alpha_1, \text{ where } |\alpha_1| < \varepsilon$$

and

$$B_{10}v - B_{01}v + (B_{20} + B_{02})v^2 - B_{11}v^2 = -2v + \alpha_2, \text{ where } |\alpha_2| < \varepsilon$$

Hence,

$$(45) \quad B_{10}v + (B_{20} + B_{02})v^2 = -3/2v + \frac{\alpha_1 + \alpha_2}{2}$$

$$(46) \quad B_{01}v + B_{11}v^2 = 1/2v + \frac{\alpha_1 + \alpha_2}{2}$$

Case I: $B_{10} - (-3/2) > 0$ and $B_{20} + B_{02} > 0$

Then, setting $v = r$ in (45), we obtain

$$(48) \quad B_{10} - (-3/2) = \lambda \left(\frac{\alpha_1 + \alpha_2}{2r} \right), \text{ where } 0 < \lambda < 1$$

$$(49) \quad B_{20} + B_{02} = (1 - \lambda) \left(\frac{\alpha_1 + \alpha_2}{2r^2} \right)$$

Case II: $B_{10} - (-3/2) < 0$ $B_{20} + B_{02} > 0$

Then, by setting $v = -r$ in (45)

$$(50) \quad B_{10} - (-3/2) = -\lambda \left(\frac{\alpha_1 + \alpha_2}{2r} \right)$$

$$(51) \quad B_{20} + B_{02} = (1 - \lambda) \left(\frac{\alpha_1 + \alpha_2}{2r^2} \right)$$

From similar calculations for the other two cases and for B_{01} and B_{11} , we obtain

$$(52) \quad B_{10} = -3/2 + \varepsilon_{10}, \text{ where } |\varepsilon_{10}| < \varepsilon$$

$$(53) \quad B_{01} = 1/2 + \varepsilon_{01}, \quad \text{where } |\varepsilon_{01}| < \varepsilon$$

$$(54) \quad B_{20} + B_{02} = \varepsilon_2, \quad \text{where } |\varepsilon_2| < \varepsilon$$

$$(55) \quad B_{11} = \varepsilon_{11}, \quad \text{where } |\varepsilon_{11}| < \varepsilon$$

Now, from (38) and (44), we obtain

$$(56) \quad \begin{aligned} & B_{10} \left(B_{10}t + (B_{01} - 1)s + B_{20}t^2 + B_{11}ts + B_{02}s^2 \right) \\ & + B_{01} \left(B_{10}s + (B_{01} - 1)t + B_{20}s^2 + B_{11}st + B_{02}t^2 \right) \\ & + B_{20} \left(B_{10}t + (B_{01} - 1)s + B_{20}t^2 + B_{11}ts + B_{02}s^2 \right)^2 \\ & + B_{02} \left(B_{10}s + (B_{01} - 1)t + B_{20}s^2 + B_{11}st + B_{02}t^2 \right)^2 \\ & + B_{11} \left(B_{10}t + (B_{01} - 1)s + B_{20}t^2 + B_{11}st + B_{02}s^2 \right) \left(B_{10}s + (B_{01} - 1)t + B_{20}s^2 + B_{11}st + B_{02}t^2 \right) \\ & = 2t + \gamma, \quad \text{where } |\gamma| < \varepsilon \end{aligned}$$

Substituting for B_{10}, B_{01}, B_{02} , and B_{11} in (56) using (52) - (55) and setting $S = r$ and $t = 0$, we

obtain

$$\begin{aligned} & (-4r^3 + \text{terms in } \varepsilon \text{ or smaller}) B_{20}^2 + ((2\varepsilon_{10} - 3\varepsilon_2)r^2 + (\varepsilon_{11} - 4\varepsilon_2)r^3) B_{20} \\ & + 3/4 \varepsilon_{11} r^2 - 3\varepsilon_{11} r - 3\varepsilon_2 r^2 - \gamma = 0 \end{aligned}$$

From the quadratic formula,

$$(57) \quad \begin{aligned} B_{20} = & -((2\varepsilon_{10} - 3\varepsilon_2)r^2 + (\varepsilon_{11} - 4\varepsilon_2)r^3)/8r^3 \\ & \pm \frac{\sqrt{[(2\varepsilon_{10} - 3\varepsilon_2)r^2 + (\varepsilon_{11} - 4\varepsilon_2)r^3]^2 + 16r^3(3/4 \varepsilon_{11} r^2 - 3\varepsilon_{11} r - 3\varepsilon_2 r^2 - \gamma)}}{8r^3} \end{aligned}$$

Now, for ε small enough, $\frac{\sqrt{\varepsilon}}{\varepsilon} \gg 1$, and so (57) implies

$$(58) \quad |B_{20}| < \varepsilon^{1/2}/r^{3/2}.$$

Hence, from (52) - (55) and (57) - (58)

$$\left\| B_{10}s + B_{20}t + B_{20}s^2 + B_{02}t^2 + B_{11}st - (-3s/2 + t/2) \right\|$$

$$\begin{aligned}
(59) \quad & < \left\| \frac{\varepsilon_{10}}{r} s + \frac{\varepsilon_{01}}{r} t + \frac{\varepsilon^{1/2}}{r^{3/2}} s^2 + \frac{\varepsilon^{1/2}}{r^{3/2}} t^2 + \frac{\varepsilon_{11}}{r^2} st \right\| \\
& < \frac{3\varepsilon^{1/2} r^2}{r^{3/2}}
\end{aligned}$$

Analogously, if H^ε is a polynomial of degree $q(\varepsilon)$, i.e., $H^\varepsilon(s, t) = \sum_{i+j=1}^{q(\varepsilon)} C_{ij} s^i t^j$, we obtain

$$\begin{aligned}
C_{10} &= -3/2 + \frac{\varepsilon_{10}}{r}, \text{ where } |\varepsilon_{10}| < \varepsilon \\
C_{01} &= 1/2 + \frac{\varepsilon_{01}}{r}, \text{ where } |\varepsilon_{01}| < \varepsilon \\
(60) \quad C_{ij} &< \frac{\varepsilon_{ij}}{r} \text{ where } |\varepsilon_{ij}| < \varepsilon \text{ for } 1 < i+j < q
\end{aligned}$$

and

$$|C_{ij}| < \varepsilon^{1/q} / r^{q+1/q}, \text{ where } i+j = q$$

From (60) and for ε small enough,

$$\begin{aligned}
(61) \quad & \left\| \sum_{i+j=1}^{q(\varepsilon)} C_{ij} s^i t^j - (-3s/2 + t/2) \right\| \\
& < \frac{(q(\varepsilon)+1)\varepsilon^{1/q(\varepsilon)} r^{q(\varepsilon)}}{r^{(q(\varepsilon)+1)/q(\varepsilon)}}
\end{aligned}$$

To complete the argument, we need to show that the right-hand side of (61) goes to 0 as $\varepsilon \rightarrow 0$. Taking the logarithm of the right-hand side of (61), we obtain

$$(62) \quad \log(q(\varepsilon)+1) + \frac{1}{q(\varepsilon)} \log \varepsilon + q(\varepsilon) \log r - \frac{q(\varepsilon)+1}{q(\varepsilon)} \log r$$

Suppose first that $q(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Then, in (62),

$$(63) \quad \log(q(\varepsilon)+1) \rightarrow \infty$$

$$(64) \frac{1}{q(\varepsilon)} \log \varepsilon \rightarrow -\infty \text{ or a finite number}$$

$$(65) q(\varepsilon) \log r \rightarrow -\infty$$

$$(66) \frac{q(\varepsilon)+1}{q(\varepsilon)} \log r \rightarrow \log r$$

Now, the convergence in (65) is faster than that in (63). Furthermore neither (64) nor (65) go to $+\infty$. Hence (62) goes to $-\infty$, and so the right-hand side of (61) goes to zero, as required.

Finally, suppose that $\lim_{\varepsilon \rightarrow 0} q(\varepsilon) = q^*$, a finite number. Then

$$(67) \log(q(\varepsilon)+1) \rightarrow \log(q^*+1), \quad q(\varepsilon) \log r \rightarrow q^* \log r, \quad \frac{q(\varepsilon)+1}{q(\varepsilon)} \log r \rightarrow \log r$$

and

$$(68) \frac{1}{q(\varepsilon)} \log \varepsilon \rightarrow -\infty$$

Thus again (62) goes to $-\infty$, and so the right-hand side of (61) goes to zero, as we required.

We next examine the case of four alternatives: $X = \{x, y, z, w\}$. Fix profile \succ_i . Let

$$a_{x..y}(\succ_i) = \mu(\{i | x \succ_i z \succ_i w \succ_i y \text{ or } x \succ_i w \succ_i z \succ_i y\})$$

$$a_{x.y}(\succ_i) = \mu(\{i | x \succ_i w \succ_i y \succ_i z \text{ or } x \succ_i z \succ_i y \succ_i w \text{ or } w \succ_i x \succ_i z \succ_i y \text{ or } z \succ_i x \succ_i w \succ_i y\})$$

and

$$a_{.xy}(\succ_i) = \mu(\{i | x \succ_i y \succ_i z \succ_i w \text{ or } x \succ_i y \succ_i w \succ_i z \text{ or } z \succ_i x \succ_i y \succ_i w$$

$$\text{or } w \succ_i x \succ_i y \succ_i z \text{ or } z \succ_i w \succ_i x \succ_i y \text{ or } w \succ_i z \succ_i x \succ_i y\})$$

Define $a_{y..x}(\succ_i), a_{y.x}(\succ_i)$, and $a_{yx}(\succ_i)$ analogously.

By analogy with the $|X| = 3$ case, let

$D^* = \{(\alpha_{x..y}, \alpha_{y..x}, \alpha_{x.y}, \alpha_{y.x}) \mid \text{there exist } \alpha_{xy} \text{ and } \alpha_{yx} \text{ with}$

$$(\alpha_{x..y}, \alpha_{y..x}, \alpha_{x.y}, \alpha_{y.x}) \in I_F^{xy} \}$$

For any $(\alpha_{x..y}, \alpha_{y..x}, \alpha_{x.y}, \alpha_{y.x}) \in D^*$, define

(69) $J_F^{xy}(\alpha_{x..y}, \alpha_{y..x}, \alpha_{x.y}, \alpha_{y.x}) = \alpha_{xy}$ such that there exists α_{yx} with

$$(\alpha_{x..y}, \alpha_{y..x}, \alpha_{x.y}, \alpha_{y.x}, \alpha_{xy}, \alpha_{yx}) \in I_F^{xy}$$

From MIIA, N, and PR, J_F^{xy} is well-defined. From N,

$$(70) \quad J_F^{xy}(s, s, t, t) = 1/2 - s - t$$

Consider the profile \succ_{\circ}^{**} such that

$$(71) \quad \begin{array}{cccc} \frac{1/24 + c_1}{x} & \frac{1/24 + c_1}{y} & \frac{1/24 + c_1}{z} & \frac{1/24 + c_1}{w} \\ y & z & w & x \\ z & w & x & y \\ w & x & y & z \end{array}$$

$$(72) \quad \begin{array}{cccc} \frac{1/24 + c_2}{x} & \frac{1/24 + c_2}{y} & \frac{1/24 + c_2}{w} & \frac{1/24 + c_2}{z} \\ y & w & z & x \\ w & z & x & y \\ z & x & y & w \end{array}$$

$$(73) \quad \begin{array}{cccc} \frac{1/24 + c_3}{x} & \frac{1/24 + c_3}{w} & \frac{1/24 + c_3}{y} & \frac{1/24 + c_3}{z} \\ w & y & z & x \\ y & z & x & w \\ z & x & w & y \end{array}$$

$$(74) \quad \begin{array}{cccc} \frac{1}{24+c_4} & \frac{1}{24+c_4} & \frac{1}{24+c_4} & \frac{1}{24+c_4} \\ x & z & y & w \\ z & y & w & x \\ y & w & x & z \\ w & x & z & y \end{array}$$

$$(75) \quad \begin{array}{cccc} \frac{1}{24+c_5} & \frac{1}{24+c_5} & \frac{1}{24+c_5} & \frac{1}{24+c_5} \\ x & z & w & y \\ z & w & y & x \\ w & y & x & z \\ y & x & z & w \end{array}$$

$$(76) \quad \begin{array}{cccc} \frac{1}{24+c_6} & \frac{1}{24+c_6} & \frac{1}{24+c_6} & \frac{1}{24+c_6} \\ x & w & z & y \\ w & z & y & x \\ z & y & x & w \\ y & x & w & z \end{array}$$

From the same argument we used to establish that $x \sim_F^* y$ in (16) above, we can show that

$$(77) \quad x \sim_F^{**} y, \text{ where } \zeta^{**} = F(\zeta^{**})$$

From (70) - (76),

$$(78) \quad J_F^{xy}(1/12+c_5+c_6, 1/12+c_1+c_2, 1/6+2c_3+2c_4, 1/6+2c_3+2c_4) = 1/4+3c_1+3c_2$$

Now, define

$$(79) \quad H_F^{xy}(s, t, u, v) = J_F^{xy}(1/12+s, 1/12+t, 1/6+u, 1/6+v) - 1/4$$

Then (70), (78), and (79) imply that

$$(80) \quad H_F^{xy}(s, s, t, t) = -s - t$$

and

$$(81) \quad H_F^{xy}(s, t, -2s-2t, -2s-2t) = 3t$$

The rest of the argument for $|X|=4$ is completely analogous to that for $|X|=3$, and the same is true for $|X|>4$. Q.E.D.

Finally, we show that continuity of indifference curves is implied by our axioms:

Theorem 2: If SWF F satisfies U, MIIA, A, N, and PR, then for all pairs x and y , J_F^{xy} is continuous.

Proof: As in the proof of Theorem 1, we will work with H_F rather than J_F^{xy} . We will confine attention to the case $|X|=3$, since the argument for $|X|>3$ is essentially the same.

Consider a sequence $\{(s_n, t_n)\}$, with $(s_n, t_n) \in D$ for all n , such that $(s^n, t^n) \rightarrow (s^*, t^*)$.

We must show that $\lim_{n \rightarrow \infty} H_F(s_n, t_n) = H_F(s^*, t^*)$

Case I: $\{s_n\}$ is an increasing sequence and $\{t_n\}$ is decreasing.

We first show that

$$(82) \quad H_F(s_n, t_n) > H_F(s_{n+1}, t_{n+1}) \text{ for all } n$$

Suppose instead that

$$(83) \quad H_F(s_n, t_n) \leq H_F(s_{n+1}, t_{n+1}) \text{ for some } n.$$

Assume first that

$$(84) \quad s^n + t^n \leq s^{n+1} + t^{n+1}$$

From MIIA,

$$(85) \quad \alpha^n = (\alpha_{xzy}^n, \alpha_{yzx}^n, \alpha_{xyz}^n, \alpha_{zxy}^n, \alpha_{zyx}^n, \alpha_{yxz}^n) \\ = (s_n, t_n, H_F(s_n, t_n), 0, 0, 1 - s_n - t_n - H_F(s_n, t_n)) \in I_F^{xy}$$

and

$$\alpha^{n+1} = (\alpha_{xzy}^{n+1}, \alpha_{yzx}^{n+1}, \alpha_{xyz}^{n+1}, \alpha_{zxy}^{n+1}, \alpha_{zyx}^{n+1}, \alpha_{yxz}^{n+1}) \\ = (s_{n+1}, t_{n+1}, H_F(s_{n+1}, t_{n+1}), 0, 0, 1 - s_{n+1} - t_{n+1} - H_F(s_{n+1}, t_{n+1})) \in I_F^{xy}$$

But from (83) and (84), x rises relative to both y and z in going from α^n to α^{n+1} , and so the fact that $\alpha^n, \alpha^{n+1} \in I_F^{xy}$ contradicts PR.

Assume next that

$$(86) \quad s_n + t_n > s_{n+1} + t_{n+1}$$

From MIIA,

$$\begin{aligned} \hat{\alpha}^n &= (\hat{\alpha}_{xzy}^n, \hat{\alpha}_{yzx}^n, \hat{\alpha}_{xyx}^n, \hat{\alpha}_{zyx}^n, \hat{\alpha}_{zyx}^n, \hat{\alpha}_{yxz}^n) \\ &= (s_n, t_n, H_F(s_n, t_n), 0, 1 - s_n - t_n - H_F(s_n, t_n), 0) \in I_F^{xy} \end{aligned}$$

and

$$\begin{aligned} \hat{\alpha}^{n+1} &= (\hat{\alpha}_{xzy}^{n+1}, \hat{\alpha}_{yzx}^{n+1}, \hat{\alpha}_{xyx}^{n+1}, \hat{\alpha}_{zyx}^{n+1}, \hat{\alpha}_{zyx}^{n+1}, \hat{\alpha}_{yxz}^{n+1}) \\ &= (s_{n+1}, t_{n+1}, H_F(s_{n+1}, t_{n+1}), 0, 1 - s_n - t_n - H_F(s_n, t_n), c) \in I_F^{xy}, \end{aligned}$$

where $c = (1 - s_{n+1} - t_{n+1} - H_F(s_{n+1}, t_{n+1})) - (1 - s_n - t_n - H_F(s_n, t_n))$.

But, from (83) and (86), x rises relative to both y and z , in going from $\hat{\alpha}^n$ to $\hat{\alpha}^{n+1}$, and so the fact that $\hat{\alpha}^n, \hat{\alpha}^{n+1} \in I_F^{xy}$ contradicts PR.

Hence (82) holds after all, and so $\lim_{n \rightarrow \infty} H_F(s_n, t_n)$ exists and

$$(87) \quad \lim_{n \rightarrow \infty} H_F(s_n, t_n) \leq H_F(s_k, t_k) \text{ for all } k$$

Now, if $H_F(s^*, t^*) > \lim_{n \rightarrow \infty} H_F(s_n, t_n)$, we can derive the same contradiction we did from (83).

Hence, assume

$$(88) \quad H_F(s^*, t^*) < \lim_{n \rightarrow \infty} H_F(s_n, t_n)$$

From MIIA

$$\begin{aligned} \alpha^\circ &= (\alpha_{xzy}^\circ, \alpha_{yzx}^\circ, \alpha_{xyx}^\circ, \alpha_{zyx}^\circ, \alpha_{zyx}^\circ, \alpha_{yxz}^\circ) \\ &= (s^*, t^*, 0, H_F(s^*, t^*), 1 - H_F(s^*, t^*) - s^* - t^*, 0) \in I_F^{xy} \end{aligned}$$

And so from PR and (88), given

$$\begin{aligned}\alpha^{\circ\circ} &= (\alpha_{xzy}^{\circ\circ}, \alpha_{yzx}^{\circ\circ}, \alpha_{zxy}^{\circ\circ}, \alpha_{xyz}^{\circ\circ}, \alpha_{yxz}^{\circ\circ}, \alpha_{yxz}^{\circ\circ}) \\ &= (s^*, t^*, 0, \lim H_F(s_n, t_n), 1 - \lim H_F(s_n, t_n) - s^* - t^*, 0)\end{aligned}$$

$$(89) \quad x \succ_F^{\circ\circ} y, \text{ where } \succ_F^{\circ\circ} = F(\alpha^{\circ\circ}).$$

From (82) and PR, for any n^* , given

$$\begin{aligned}\alpha_n^{\circ\circ} &= (\alpha_{xzy}^{\circ\circ\circ}, \alpha_{yzx}^{\circ\circ\circ}, \alpha_{zxy}^{\circ\circ\circ}, \alpha_{xyz}^{\circ\circ\circ}, \alpha_{yxz}^{\circ\circ\circ}, \alpha_{yxz}^{\circ\circ\circ}) \\ &= (s_{n^*}, t_{n^*}, 0, \lim H_F(s_n, t_n), 1 - \lim H_F(s_n, t_n) - s_{n^*} - t_{n^*}, 0),\end{aligned}$$

$$(90) \quad y \succ_F^{\circ\circ\circ} x, \text{ where } \succ_F^{\circ\circ\circ} = F(\alpha^{\circ\circ\circ}).$$

From PR, (89) and (90) there exists $\lambda \in (0,1)$ such that for

$$(s^{**}, t^{**}) = \lambda(s^*, t^*) + (1-\lambda)(s_{n^*}, t_{n^*}),$$

$$\alpha^{**} = (s^{**}, t^{**}, 0, \lim H_F(s_n, t_n), 1 - \lim H_F(s_n, t_n) - s_{n^*} - t_{n^*}, 0) \in I_F^{xy}$$

But for any k ,

$$\alpha_k = (s_k, t_k, 0, H_F(s_k, t_k), 1 - H_F(s_k, t_k) - s_k - t_k, 0) \in I_F^{xy}$$

and yet, for k big enough, $s_k > s^{**}, t_k < t^{**}, H_F(s_k, t_k) > \lim H_F(s_n, t_n)$, a contradiction of PR, and

so $\lim H_F(s_n, t_n) = H_F(s^*, t^*)$ after all.

The arguments for the other cases are very similar.

Q.E.D.

4. Open Questions

There are at least four questions that seem worth pursuing in follow-up work.

First, we have assumed throughout that, although society can be indifferent between a pair of alternatives, x and y , individuals are never indifferent. We conjecture that if individual indifference were allowed, we would obtain the natural extension of the Borda count, i.e., if an individual is indifferent between x and y , then instead of x getting p points and y getting $p - 1$ (as would be the case if the individual ranked x immediately above y), the alternatives split the point count $p + p - 1 = 2p - 1$ equally.

Second, we have made important use of the continuum of voters in our proof. Specifically, the continuum, together with our axioms, guarantees that there will be profiles for which society is indifferent between x and y . It would be interesting to explore to what extent the characterization result extends to the case of finitely many voters.

Third, this paper studies SWFs, which rank *all* alternatives. By contrast, a voting rule simply selects the *winner* (see footnote 2). In a previous draft of this paper, we proposed a way to modify the axioms to obtain a characterization of the voting-rule version of the Borda count (i.e., the winner is the alternative with the highest Borda score). However, that draft considered only the case in which social indifference curves are linear or polynomial. Whether that characterization holds in the more general setting of the current draft is not yet known.

Finally, although the anonymity and neutrality axioms are quite natural in political elections, they don't apply universally (think, for example, of corporate elections where voters are weighted by their ownership stake and certain alternatives – e.g., the status quo – may be privileged). It is clear that certain variants of the Borda count – e.g., where different people can have different weights or some particular alternatives get extra Borda points – satisfy the remaining axioms when A and N are dropped, but we do not have a full characterization of all SWFs satisfying those axioms.

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