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DECISION-MAKING UNDER IGNORANCE WITH IMPLICATIONS FOR SOCIAL CHOICE

ABSTRACT. A new investigation is launched into the problem of decision-making in the face of 'complete ignorance', and linked to the problem of social choice. In the first section the author introduces a set of properties which might characterize a criterion for decision-making under complete ignorance. Two of these properties are novel: 'independence of non-discriminating states', and 'weak pessimism'. The second section provides a new characterization of the so-called principle of insufficient reason. In the third part, lexicographic maximin and maximax criteria are characterized. Finally, the author's results are linked to the problem of social choice.

Several authors, [2], [3], [7], and [9] have dealt with the problem of an individual who must choose from a set of alternatives when he cannot associate a probability distribution with the possible outcomes of each alternative. The problem has been called that of decision-making under complete ignorance; presumably 'complete ignorance' captures the notion that the axioms of subjective probability cannot be fulfilled. This paper is a further investigation in that tradition. In the first section we suggest a set of properties which might characterize a criterion for decision-making under ignorance. Most of these properties are familiar, but, in particular, 'independence of non-discriminating states' and 'weak pessimism' are new in this context. The second section recapitulates some of the important decision criteria in the literature and suggests a new characterization of the so-called principle of insufficient reason. In the third part, we drop the assumption invariably made by previous authors that preferences for consequences satisfy the von Neumann-Morgenstern axioms, and characterize the so-called lexicographic maximin and maximax criteria. We also provide a new axiomatization of the ordinary maximin principle. Finally, we show that several of our results translate quite easily into the theory of social choice.

THE PROPERTIES

Let C be a consequence or 'outcome' space. C contains a subset C^* of 'sure' or 'certain' outcome as well as all finite lotteries[1] with outcomes in C^*. We

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assume that the decision-maker has a preference ordering $\succsim$ on C which satisfies the von Neumann-Morganstern axioms for decision-making under uncertainty. Let $\mathscr{U}^*$ be the family of von Neumann-Morgenstern utility representations of $\succsim$. Define a *decision* d as a map $d:\Omega \to C$ where Ω is an exhaustive list of possible states of nature. Obviously there are many ways in which nature can be described, and therefore many conceivable Ω's could apply to the same world. Following Arrow-Hurwicz [2] we shall define a *decision problem* P as a set of decisions which share a common domain $\Omega(P)$. The decision-maker solves a non-empty decision problem P by choosing a non-empty subset $\hat{P} \subseteq P$. $\hat{P}$ is interpreted as the 'choice' or 'optimal' subset of P. Let $\mathscr{P}$ be the class of all non-empty decision problems P such that P and $\Omega(P)$ are finite.[2] A *decision criterion* f is a mapping f: $\mathscr{P} \to \mathscr{P}$ such that $\forall P \in \mathscr{P}$, $f(P) \subseteq P$, $f(P) \neq \emptyset$ and such that $d(w) \sim d'(w)$[3], for all $w \in \Omega$, implies that $d \in f(P)$ if and only if $d' \in f(P)$. The following are conditions that have, at various times, been deemed reasonable properties for a decision criterion f to satisfy.

PROPERTY (1). $\forall P_1, P_2 \in \mathscr{P}, d \in P_1 \subseteq P_2 \Rightarrow [d \in f(P_2) \Rightarrow d \in f(P_1)]$

Property (1) is Sen's Property α of rationality [10].

PROPERTY (2). $\forall P_1, P_2 \in \mathscr{P}\ [d, d^1 \in f(P_1)$ and $P_1 \subseteq P_2] \Rightarrow [d \in f(P_2) \Leftrightarrow d^1 \in f(P_2)]$.

Property (2) is Sen's Property β and Milnor's 'row adjunction' [9].

Together (1) and (2) constitute the Arrow-Hurwicz Property A, and, as Herzberger [6] has shown, imply that, for every Ω, f induces an ordering $\succsim^*_\Omega$ on $D_\Omega = \{d \mid \text{domain of } d = \Omega\}$ such that for any P with $\Omega(P) = \Omega$, $d^* \in f(P) \Leftrightarrow d^* \in P$ and $d^* \succsim^*_\Omega d$ for all $d \in P$.

PROPERTY (3). $\forall P_1, P_2 \in \mathscr{P}, d \in P_1 \subseteq P_2 \Rightarrow [d \in f(P_1), d \notin f(P_2) \Rightarrow f(P_2) \backslash P_1 \neq \emptyset]$.

Properties (1) and (3) together are quivalent to Luce's and Raiffa's Axiom 7′ and Chernoff's [3] Postulate 4. (1) and (3) combined are somewhat weaker than the combination of (1) and (2).

PROPERTY (4). $\forall P \in \mathscr{P} d$, $d^1 \in P$, if $d \in f(P)$ and $d^1(w) \succsim d(w)$ for all $w \in \Omega(P)$, then $d^1 \in f(P)$.

Property (4) is the weakest form of the domination principle. It is Arrow-Hurwicz Property D.

PROPERTY (5). $\forall P \in \mathscr{P}\ \forall d^1 \in P\ \ \forall d \in f(P)$, if $d(w) \succ d^1(w)$ for all $w \in \Omega(P)$, then $d^1 \notin f(P)$.

Property (5) is another rather weak version of domination. It is Milnor's 'Strong Domination' property.

PROPERTY (6). $\forall P \in \mathscr{P}\ \forall d,\ d^1 \in P$ if $\forall w \in \Omega(P)$, $d(w) \succsim d^1(w)$ and $\exists w_0 \in \Omega(P)$ such that $d(w_0) \succ d^1(w_0)$, then $d^1 \notin f(P)$.

Property (6) is the usual admissibility condition. It is obviously stronger than property (5). Combined with continuity (see below), it is also stronger than property (4).

PROPERTY (7). $\forall P_1,\ P_2 \in \mathscr{P}$ such that $\Omega(P_1) = \Omega(P_2) = \Omega$, if, for some $u \in \mathscr{U}^*$, there exist $k \in \mathscr{P}$, $w_0 \in \Omega$, and bijection $h: P_1 \to P_2$ such that

$$\forall d \in P_1, u(h(d)(w)) = \begin{cases} u(d(w)) + k, & w = w_0, \\ u(d(w)), & w \neq w_0. \end{cases}$$

then, $d \in f(P_1)$ if and only if $h(d) \in f(P_2)$.

Property (7) is Milnor's column linearity condition. It amounts to demanding that if two decision problems are isomorphic except that in one, the utility derived from any decision if a certain state of nature w_0 prevails is uniformly higher than the utility from the corresponding decision in the other problem when w_0 arises, then if a given decision is optimal in the other.

PROPERTY (8). $\forall P_1, P_2 \in \mathscr{P}$ such that $\Omega(P_1) = \Omega(P_2)$ and $|P_1| = |P_2|$, if for some $u \in \mathscr{U}^*$, there exists a bijection $g: P_1 \to P_2$ such that for some $a > 0$, $b \in \mathscr{P}$, $u(g(d)(w)) = au(d(w)) + b$ for all $d \in P_1$ and $w \in \Omega$, then $d \in f(P_1) \Leftrightarrow g(d) \in f(P_2)$.

Property (8) is Milnor's linearity condition.

PROPERTY (9). $\forall P \mathscr{P}\ \forall d_1, d_2, d \in P$, if $\exists u \in \mathscr{U}^*$ such that $u \circ d = \frac{1}{2} u \circ d_1 + \frac{1}{2} u \circ d_2$, then $d_1, d_2 \in f(P) \Rightarrow d \in f(P)$.

Property (9) is Milnor's convexity condition.

PROPERTY (10). Consider a sequence $\{P_i\} \subseteq \mathscr{P}$ and $P \in \mathscr{P}$. Suppose that for all i, $\Omega(P_i) = \Omega(P)$ and $|P_i| = |P| = n$. Write $P = \{d_1, \cdots, d_n\}$, $P^i = \{d_1^i, \cdots, d_n^i\}$. Then, if $\exists u \in \mathscr{U}^*$ such that $\forall j \forall w \in \Omega(P)$ $\lim_{i \to \infty} u(d_j^i(w)) = u(d_j(w))$,

$d_j^i(P^i) \in f(P^i)$ for all i implies $d_j\, f(P)$.

Property (10) is Milnors's continuity axion.

Up to this point, all of the stated properties are arguably reasonable, but none embodies the idea of ignorance. Properties 11–13 are an attempt to capture this notion.

PROPTERTY (11). Suppose there exists a bijection $h: \Omega^1 \to \Omega$. For P with $\Omega(P) = \Omega$, define P^1 with $\Omega(P^1) = \Omega^1$ as

$$P^1 = \{d^1 \mid d^1 = d \circ h \text{ for } d \in P\}.$$

Then, $d \in f(P)$ if and only if $d \circ h \in f(P^1)$.

Property (11) is the Arrow-Hurwicz Property B and the Milnor Symmetry condition. It insists that the labelling of states and decisions be irrelevant for the decision criterion.

Consider $P_1, P_2 \in \mathscr{P}$. Following Arrow-Hurwicz, P_2 is said to be derived from P_1 by deletion of repetitious states ($P_1 \to P_2$) if $\Omega(P_2) \subseteq \Omega(P_1)$ and if there exists a bijection $h: P_1 \to P_2$ such that $\forall w \in \Omega(P_2)\ h(d)(w) = d(w)$ and such that $\forall w \in \Omega(P_1)/\Omega(P_2)$, $\exists w^1 \in \Omega(P_2)$ with $d(w) = d(w^1)$ for all $d \in P_1$.

PROPERTY (12). $\forall P_1, P_2 \in \mathscr{P}$, if $P_1 \to P_2$ via bijection h, then $h(d) \in f(P_2) \Leftrightarrow d \in f(P_1)$.

Property (12) is the Arrow-Hurwicz Property C and the Milnor 'Deletion of Repetitious States'. More than any other property, it captures the idea of complete ignorance, for, in effect, it asserts that dividing a state into several substates should have no effect on the chosen decision. The next condition is just a weakened version of Property (12).

PROPERTY (13). $\forall P_1, P_2 \in \mathscr{P}$, if $P_1 \to P_2$ via bijection h and if for all d_1, $d_2 \in P_1$ with $d_1 \neq d_2$, $\forall w, w^1 \in \Omega(P_1)$, not $d_1(w) \sim d_2(w^1)$, then $h(d) \in f(P_2) \Leftrightarrow d \in f((P_1)$.

PROPERTY (14). Consider $P_1, P_2 \in \mathscr{P}$ with $\Omega(P_2) \supseteq \Omega(P_1)$ and a surjection $g: P_1 \to P_2$ such that $\forall d \in P_1\ \forall w \in \Omega(P_1)\ g(d)(w) = d(w)$. Then, if for d, $d^1 \in f(P_1)$, $g(d)(w) \sim g(d^1)(w)$ for all $w \in \Omega(P_2)/\Omega(P_1)$, $g(d) \in f(P_2) \Leftrightarrow g(d^1) \in f(P_2)$.

This last property requires that adding additional states for which all decisions are equivalent does not affect the choice of optimal decisions. In effect this requirement is the strong separability axiom of Debreu [5]. Although it has not previously appeared in discussions of decision-making under ignorance, entirely analogous properties have been used recently in the social choice literature under the names of 'elimination of indifferent individuals' [4] and 'unanimity' [10].

II. THE DECISION CRITERIA

We may now state the results for the case where preferences obey the von Neumann-Morgenstern axioms.

THEOREM 1. (Arrow-Hurwicz): A decision criterion f satisfies properties (1), (2), (4), (11), (12) if and only if for each $u \in \mathscr{U}^*$ there exists a weak ordering $\succsim_u^*$ in the space of real ordered pairs (M, m) with $m \leqslant M$ such that

(a) $M_1 \geqslant M_2$ and $m_1 \geqslant m_2$ implies that $(M_1, m_1) \succsim_u (M_2, m_2)$,

(b) $\forall P \ \mathscr{P}$,

$$f(P) = \{d \in P | (\max u(d(w)), \min u(d(w)) \succsim_u^* (\max u(d'(w)), \min u(d'(w))) \quad \text{for all} \quad d' \in P\}.$$

DEFINITION. A criterion f is the Hurwicz α-criterion for $\alpha \in [0, 1]$ if $\forall P \in \mathscr{P}\, \forall u \in \mathscr{U}$, $d^* \in f(P)$ if and only if $\alpha \max_w u(d^*(w)) + (1 - \alpha) \min_w u(d^*(w)) \geqslant \max_w u(d(w)) + (1 - \alpha) \min_w u(d(w))$ for all $d \in P$.

THEOREM 2. A decision criterion f satisfies properties (1), (2), (4), (5), (8), (11), (12) if and only if $\exists \alpha \in [0, 1]$ such that $\forall P \in \mathscr{P}, f(P) \subseteq f^{\alpha}(P)$ where f^{α} is the Hurwicz α-criterion.

Remark. It should be noted that this theorem does not require continuity (property (10)). If, however, continuity is also stipulated, we obtain Theorem 3 (see below).

Proof. If f satisfies the stipulated properties, we may apply Theorem 1 and, for choice of $u \in \mathscr{U}^*$, define an ordering $\succsim_u^*$ as above. Following Milnor's argument, let α_u be the supremum of all $\alpha^1 \in \mathbb{R}$ such that $(1, 0) \succsim_u^* (\alpha', \alpha')$. By property (5), $0 \leqslant \alpha_u \leqslant 1$. Clearly $(1, 0) \succ_u^* (\alpha', \alpha')$ if $\alpha' < \alpha_u$, and

$(\alpha', \alpha') \succ_u^* (1, 0)$ if $\alpha' > \alpha_u$. By property (8) and the fact that all representations of $\succsim$ differ by positive linear transformations, α_u does not depend on u, and we may consequently delete its subscript. By property (8), $(M, m) \succsim_u^* (\alpha' M + (1-\alpha')m,\ \alpha' M + (1-\alpha')m)$ if $0 \leqslant \alpha' < \alpha$, and $(\alpha' M + (1-\alpha')m,\ \alpha' M + (1-\alpha')m) \succ_u^* (M, m)$ if $\alpha < \alpha' \leqslant 1$. Suppose that for some $P \in \mathscr{P}\ \exists d^* \in f(P)$ and $\exists d \in P$ such that $\alpha M^* + (1-\alpha)m^* < \alpha M^0 + (1-\alpha)m^0$, where $M^* = \max_w (u(d^*(w)),\ m^* = \min_w u(d^*(w)),\ M^0 = \max_w u(d(w)),\ m^0 = \min_w u(d(w))$. Choose sequence of real numbers $\{\epsilon_i\}$ and $\{\delta_i\}$ such that (a) $\lim_{i \to \infty} \epsilon_i = 0$, (b) $\forall i\ \epsilon_i > 0$ if $\alpha < 1$ and $\epsilon_i = 0$ if $\alpha = 1$, (c) $\lim_{i \to \infty} \delta_i = 0$, and (d) $\forall i\ \delta_i > 0$ if $\alpha > 0$ and $\delta_i = 0$ if $\alpha = 0$.

For sufficiently large i, $((\alpha + \epsilon_i)M^* + (1 - \alpha - \epsilon_i)m^*, (\alpha + \epsilon_i)M^* + (1 - \alpha - \epsilon)m^*) \succsim_u^* (M^*, m^*) \succ_u^* (M^0, m^0) \succsim_u^* ((\alpha - \delta_i)M^0 + (1 - \alpha + \delta_i)m^0, (\alpha - \delta_i)M^0 + (1 - \alpha + \delta_i)m^0)$. By definition of $\succsim_u^*$, $(\alpha + \epsilon_i)M^* + (1 - \alpha - \epsilon_i)m^*\ \ (\alpha - \delta_i)M^0 + (1 - \alpha + \delta_i)m^0$. Therefore, $\alpha M^* + (1-\alpha)m^* \geqslant \alpha M^0 + (1-\alpha)m^0$, a contradiction. The other direction of implication is trivial.

Q.E.D.

THEOREM 3 (Milnor). A criterion f satisfies properties (1), (2), (5), (8), (10), (11), (12) if and only if $\exists \alpha \in [0, 1]$ such that f is the Hurwicz α-criterion.

DEFINITION. Criterion f is the maximin criterion if and only if, $\forall P \in \mathscr{P}$ $\forall u \in \mathscr{U}^*$, $d^* \in f(P)$ if and only if $\min_w u(d^*(w)) \geqslant \min_w u(d(w))$ for all $d \in P$.

THEOREM 4 (Milnor). A criterion f satisfies properties (1), (2), (5), (9), (10), (11), (12) if and only if f is the maximin criterion.

THEOREM 5. A criterion f satisfies properties (1), (2), (4), (5), (9), (11), (12) if and only if $\forall P \in \mathscr{P}$, $f(P) \subseteq f^*(P)$ where f^* is the maximin criterion.

Proof. The proof is identical to Milnor's proof of Theorem 4, except for a minor alteration forced by lack of continuity.

DEFINITION. f is the principle of insufficient reason (the Laplace criterion) if $\forall P \in \mathscr{P}\ \forall u \in \mathscr{U}^*$, $d^* \in f(P)$ if and only if $\Sigma_{w \in \Omega(P)}\, u(d^*(w)) \geqslant \Sigma_{w \in \Omega P}\, u(d(w))$ for all $d \in P$.

THEOREM 6 (Chernoff). Criterion f satisfies properties (1), (3), (4), (6), (7), (9), (11) if and only if f is the principle of insufficient reason.

THEOREM 7 (Milnor). Criterion f satisfies properties (1), (2), (5), (7), (11) if and only if f is the principle of insufficient reason.

THEOREM 8. A criterion f satisfies properties (1), (2), (6), (10), (11), (14) if and only if f is the principle of insufficient reason.

Proof. Choose $\Omega = \{w_1, \ldots, w_n\}$ and $u \in \mathscr{U}^*$. Let $D_\Omega = \{d \mid \text{domain of } d = \Omega\}$. (1) and (2) imply that there exists an ordering $\succsim^*_\Omega$ on D_Ω such that $\forall P \in \mathscr{P}$ with $\Omega(P) = \Omega$, $d^* \in f(P) \Leftrightarrow d^* \succsim^*_\Omega d$ for all $d \in P$. By definition of a decision criterion, if $u(d(w)) = u(d'(w))$ for all $w \in \Omega$ and $d, d' \in D_\Omega$ then $d \sim^*_\Omega d'$. Therefore, if we write $\Omega = (w_1, \ldots, w_n)$, $\succsim^*_\Omega$ induces an ordering $\succsim_\Omega$ of $(u(C))^n$ such that $x \succsim_\Omega y$ if and only if $d_x \succsim^*_\Omega d_y$ for d_x, $d_y \in D_\Omega$ such that $x = (u(d_x(w_1)), \ldots, u(d_x(w_n))$, $y = (u(d_y(w_1)), \ldots, u(d_y(w_n)))$. Since C contains all finite lotteries of outcomes in C^*, $u(C)$ is connected. Consider $A_{x_0} = \{x \in (u(C))^n | x_0 \succsim_\Omega x\}$ for some $x_0 \in (u(C))^n$. Choose a convergent sequence $\{x_i\} \subseteq A_{x_0}$ such that $\forall_i, x_0 \succsim_\Omega x_i$. Take $x_\infty = \lim_{i \to \infty} x_i$. Choose a sequence $\{P_i\} \subseteq \mathscr{P}$ and $P_\infty \in \mathscr{P}$ with $\Omega(P_i) = \Omega(P_\infty) = \Omega$ for all i, such that $P_i = \{d_i, d_0\}$ $P_\infty = \{d_\infty, d_0\}$, $(u(d_i(w_1)), \ldots, u(d_i(w_n))) = x_i$, $(u(d_0(w_1)), \ldots, u(d_0(w_n))) = x_0$, $(u(d_\infty(w_1)), \ldots, u(d_\infty(w_n))) = x_\infty$. Since $x_0 \succsim_\Omega x_i$, $d_0 \in f(P_i)$ for all i. By (10), $d_0 \in f(P_\infty)$. Therefore $x_0 \succsim_\Omega x_\infty$. So, A_{x_0} is closed. Similarly $B_{x_0} = \{x \in (u(C))^n | x \succsim^*_\Omega x_0\}$ is closed for any $x_0 \in (u(C))^n$. By (14), one may easily show that the ordering induced by $\succsim_\Omega$ on $\mathbb{R}^{n-m}$ by fixing m components of the vectors in $(u(C))^n$ is independent of the values at which they are fixed. Therefore, the hypotheses of Debreu's Theorem [5] are satisfied, and we conclude that there exist continuous functions $g_1, g_2, \ldots, g_n$: $u(C) \to \mathbb{R}$ such that $\forall d, d^1 \in D_\Omega$

$$(1) \qquad d \sim^*_\Omega d^1 \quad \text{if and only if} \quad \sum_{i=1}^n g_i(u(d(w_i)) \geqslant \sum_{i=1}^n g_i(u(d^1(w)).$$

By (11) all the g_i's are equal to some continuous g: $(u(C)) \to \mathbb{R}$. By (6), g is strictly increasing. By (8) we may use the argument of Maskin [8] to conclude that $g(u)$ is a positive linear transformation of u. This establishes the theorem.

III. DECISION-MAKING WITHOUT THE VON NEUMANN-MORGENSTERN AXIOMS

It is perhaps a bit odd to insist that individual preferences obey a set of probabilistic axioms in order to develop a theory which rejects the use of probabilities. In this section, we drop the assumption that $\succsim$ satisfies the von Neumann-Morgenstern axioms and that C need contain all finite lotteries of sure outcomes. For convenience we shall assume that $\succsim$ is representable by a class $\mathscr{U}$ of real-valued utility functions on C.[4] Obviously, for $u \in \mathscr{U}$, any monotone increasing transformation of u is also in $\mathscr{U}$. Properties (1)–(6), (11), (12), (13), (14) remain the same in this framework as before. Properties (7)–(10) can be modified by substituting $\mathscr{U}$ for $\mathscr{U}^*$. Properties (7), (8), and (9), of course, now make no intuitive sense. However, (9) can be modified appropriately in the following obvious way.

PROPERTY (9′). $\forall P \in \mathscr{P} \forall d_1, d_2, d \in P$, if, for each $w \in \Omega(P)$, $d_1(w) \succsim d(w) \succ d_2(w)$, or $d_2(w) \succsim d(w) \succ d_1(w)$, or $d_1(w) \sim d_2(w) \sim d(w)$, then $d_1, d_2 \in f(P)$ implies that $d \in f(P)$.

Among the results of the previous section, Theorems 2, 3, and 6–8 will not hold in this new context because they depend on all representations of $\succsim$ being linear transformations of one another. Theorem 1 will carry over, but Theorems 4 and 5, as indeed several other theorems to follow, depend crucially on the number and distribution of indifference classes of $\mathscr{U}$. When the von Neumann-Morgenstern axioms are assumed, this is no problem because whenever there are at least two distinct indifference classes, there is a continuum of them. Without these axioms, however, we may run into trouble. Let us, for example, examine Milnor's proof of Theorem 4. For utility function u, he first observes that because of Theorem 1, a decision $d: \Omega \to C$ may, for the purposes of ranking the decisions in D, be identified with the pair (m, M), where $m = \min u(d(w))$ and $M = \max u(d(w))$. He then considers the matrix

	w_1	w_2	w_3
$u \circ d_1$	m	$\frac{1}{2}(m+M)$	$\frac{1}{2}(m+M)$
$u \circ d_2$	m	m	M
$u \circ d_3$	m	M	m

By using this matrix, he implicitly assumes that there exists a consequence x in C such that $u(x) = \frac{1}{2}(m + M)$, an assumption validated by the von Neumann-Morgenstern axioms. In our present context, however, we must make the following supposition.

DENSENESS ASSUMPTION. For all $x, y \in C$, such that $x \succ y$, $\exists z \in C$ for which $x \succ z \succ y$.

THEOREM 9. If $\gtrsim$ satisfies the Denseness Assumption, then a criterion f satisfies properties (1), (2), (5), (9′), (10), (11), and (12) if and only if f is the maximin criterion.

Proof. The proof is an adaption of Milnor's proof of Theorem 4. Choose $u \in \mathscr{U}$. Suppose $\gtrsim$ satisfies the Denseness Assumption and f satisfies the hypothesized properties. For any Ω, let $\gtrsim_\Omega^*$ be the ordering induced by f on D_Ω. Consider $d \in D_\Omega$ such that $M > m$ where $M = \max u(d(w))$ and $m = \min u(d(w))$. Take $k_0 = \inf\{k \mid k \in u(C),\ k > m\}$. The following argument will demonstrate that we may assume that $k_0 = m$. Suppose instead that $k_0 > m$. If there exists $x_0 \in C$ such that $u(x_0) = k_0$, then, by the Denseness Assumption, $\exists y_0 \in C$ such that $k_0 > u(y_0) > m$, a contradiction of k_0. Therefore, there does not exist $x_0 \in C$ such that $u(x_0) = k_0$.

Take
$$u_0(x) = \begin{cases} u(x), u(x) \leqslant m\ , \\ u(x) - k_0 + m, u(x) > m\ . \end{cases}$$

Clearly $u_0 \in \mathscr{U}$ and $\inf\{k \mid k \in u_0(C), k > m\} = m$.

So we may replace u with u_0, thereby allowing us to take $k_0 = m$. Since $k_0 = m$, we may choose a sequence $\{k_n\} \subseteq u(C) \cap (m, M)$ such that $\lim_{n \to \infty} k_n = m$. Choose $W = \{w_1, w_2, w_3\} \subseteq \Omega^5$ and decisions $\bar{d}, \bar{\bar{d}}, d_0, d_n$: $\Omega \to C$ such that

$$u(\bar{d}(w_1)) = m, u(\bar{d}(w_2)) = m, u(\bar{d}(w_3)) = M, u(d(w)) = m$$
$$\text{for } w \notin W$$
$$u(\bar{\bar{d}}(w_1)) = m, u(\bar{\bar{d}}(w_2)) = M, u(\bar{\bar{d}}(w_3)) = m, u(\bar{\bar{d}}(w)) = m$$
$$\text{for } w \notin W$$
$$u(d_0(w)) = m \text{ for all } w \in \Omega,$$

and for all n, $u(d_n(w_1)) = m$, $u(d_n(w_2)) = k_n$, $u(d_n(w_3)) = k_n$, $u(d_n(w)) = m$ for $w \notin W$.

By Property 11, $\bar{d} \sim^*_\Omega \bar{\bar{d}}$. By convexity ((9′)), for all n, $\bar{d} \sim^*_\Omega \bar{\bar{d}}$. Strong domination ((5)) and continuity ((10)) together imply that f satisfies property (4). Therefore, $\bar{d} \sim^*_\Omega \bar{\bar{d}} \sim^*_\Omega d$. By convexity ((9′)), $d_n \sim^*_\Omega \bar{d}$ for all n. Hence, by continuity ((10)), $d_0 \sim^*_\Omega \bar{d} \sim^*_\Omega d_0$. f is clearly the maximum criterion.

Q.E.D.

Our next series of results does not require the Denseness Assumption. The theorems do, however, necessitate the following weaker hypothesis.

COUNTABILITY ASSUMPTION. Either $\gtrsim$ has only a single indifference class, or it has at least countably infinitely many.

We shall need to add to our list two additional properties. The first is a strengthening of property (8). The second merely states that the decision-maker does not always choose as if the best possible outcome will occur.

PROPERTY (15). For $P, P' \in \mathscr{P}$ such that $\Omega(P) = \Omega(P')$ and $|P| = |P'|$, suppose there exists a $\gtrsim$-preserving bijection[6] $\gamma: P \to P'$. Then $d \in f(P)$ if and only if $\gamma(d) \in f(P')$.

PROPERTY (16). (Weak Pessimism): There exist $P \in \mathscr{P}$, $d \in f(P)$, $d' \in P \backslash f(P)$ and $u \in \mathscr{U}$ such that $\max_{w \in \Omega(P)} u(d(w)) < \max_{w \in \Omega(P)} u(d'(w))$.

For any d with domain Ω and $w_0 \in \Omega$, let $D^d(w_0) = \{w \in \Omega | d(w_0) \gtrsim d(w)\}$.

DEFINITION. A criterion f is the lexicographic maximin if and only if for any $P \in \mathscr{P}$, $d \in f(P)$ implies that there do not exist $d^* \in P$, $w_0 \in \Omega$ and permutation g: $\Omega \to \Omega$, such that $d^*(w_0) \succ d(g(w_0))$ and $d^*(w) \sim d(g(w))$ for all $w \in D^{d^*}(w_0)$.

The lexicographic maximax criterion is defined in the obvious analogous way.

THEOREM 10. If $\gtrsim$ satisfies the Countability Assumption, then a criterion f which satisfies properties (1), (2), (6), (11), (14), (15) must be either the lexicographic maximin or lexicographic maximax.

THEOREM 11. If $\succsim$ satisfies the Countability Assumption, then a criterion f which satisfies properties (1), (2), (6), (11), (14), (15) and (16) must be the lexicographic maximin.

Observe that neither Theorem 10 or 11 invokes property (13). The theorems are proved by demonstrating that their statements can be translated into the language of social choice and by then applying a result due to d'Aspremont and Gevers [4]. We must first develop the necessary social choice terminology. Following Maskin [8], let $N = \{1, \ldots, n\}$ be a set of individuals who constitute society and let X be a set of social alternatives. Let $\mathscr{R}$ be the set of all orderings of X and $\mathscr{V}$ the set of all bounded $\mathbb{R}$-valued functions on $X \times N$. For $v \in \mathscr{V}$, $v(x, i)$ is the utility that the ith individual derives from alternative x. A social welfare functional (SWFL) g is a mapping $g: \mathscr{V} \to \mathscr{R}$. The following are possible properties which a SWFL g may possess.

INDEPENDENCE. $\forall v_1, v_2 \in \mathscr{V}$ $\forall B \subseteq X$, if $v_1(x,.) = v_2(x,.)$ for all $x \in B$, then $g(v_1)$ and $g(v_2)$ coincide on B.

STRONG PARETO PROPERTY. $\forall x, y \in X\, \forall v \in \mathscr{V}$, if $\forall i \in N$, $v(x, i) \geqslant v(y, i)$ and $\exists j \in N$ such that $v(x, j) > v(y, j)$, then $x\,P\,y$ where P is the strong ordering corresponding to $g(v)$.

ANONYMITY. For any permutation σ of N, if for v_1, $v_2 \in \mathscr{V}\, \forall i \in N$, $\forall x \in X\; v_1(x, i) = v_2(x, \sigma(i))$, then $g(v_1) = g(v_2)$.

ELIMINATION OF INDIFFERENT INDIVIDUALS. $\forall v_1, v_2 \in \mathscr{V}$, if $\exists M \subseteq N$ such that $\forall i \in M$, $v_1(\cdot, i) = v_2(\cdot, i)$ while $\forall j \in N \backslash M$, $\forall x, y \in X$, $v_1(x, j) = v_1(y, j)$ and $v_2(x, j) = v_2(y, j)$, then $g(v_1) = g(v_2)$.

COORDINALITY. Consider v_1, $v_2 \in \mathscr{V}$ such that $v_1 = \phi(v_2)$ where ϕ is a strictly monotone increasing function. Then a SWFL g satisfies coordinality if $g(v_1) = g(v_2)$.

Above, we defined $\mathscr{V}$ as containing *all* bounded $\mathbb{R}$-valued functions on $X \times N$. Because coordinality is assumed, however, one can modify arguments due to d'Aspremont and Gevers to obtain

THEOREM 12. For $N = \{1, 2, \ldots, n\}$ and a set of social alternatives X containing at least three elements, let $\mathscr{V}$ be a set of bounded $\mathbb{R}$-valued functions of $X \times N$ which is sufficiently large to induce all orderings of $X \times N$.[7] Then a SWFL $g\colon \mathscr{V} \to \mathscr{R}$ which satisfies independence, the strong Pareto property, anonymity, coordinality, and elimination of indifferent individuals is either the lexicographic maximin or maximix principle.

Proofs of Theorems 10 and 11. Suppose f satisfies the hypothesized properties. Choose $u \in \mathscr{U}$. For a given nature space $\Omega = \{w_1, \ldots, w_n\}$, let each state $w_i \in \Omega$ be interpreted as an individual i. Take $N = \{1, 2, \ldots, n\}$. Choose a positive integer m, and select a decision problem $P_0 \in \mathscr{P}$ such that $|P_0| = m$ and $\Omega(P_0) = \Omega$. Let $\mathscr{P}^m_\Omega = \{P \in \mathscr{P} \mid \Omega(P) = \Omega,\ |P| = m\}$. For each $P \in \mathscr{P}^m_\Omega$, select a bijection $h_P: P_0 \to P$. Write $P_0 = \{d_1, \ldots, d_m\}$. Associate with each d_j a social alternative x^j. Take $X^m_\Omega = \{x^1, \ldots, x^m\}$. A decision problem $P \in \mathscr{P}^m_\Omega$ induces an interpersonal utility function $v_P: X^m_\Omega \times N_\Omega \to \mathbb{R}$ where

$$v_P(x^j, i) = u(h_P(d_j)(w_i)) \quad \text{for all} \quad i, j$$

$v_P(x^j, i)$ is interpreted as the utility that individual i derives from alternative x^j. For given $\Omega, \mathscr{P}^m_\Omega, P_0$, and set of bijections $\{h_P \mid h_P: P_0 \to P,\ P \in \mathscr{P}^m_\Omega\}$, the decision criterion f induces a SWFL $g^m_\Omega: \mathscr{V}^m_\Omega \to \mathscr{R}^m_\Omega$, where $\mathscr{V}^m_\Omega = \{v_P \mid P \in \mathscr{P}^m_\Omega\}$ and $\mathscr{R}^m_\Omega$ is the set of all orderings of X^m, via the following relation

$$\forall v_P \in \mathscr{V}^m_\Omega\ \forall x^k, x^l \in X^m_\Omega,\ x^k g^m_\Omega(v_P) x^l \text{ iff } h_P(d_k) \succsim^*_\Omega h_P(d_l),$$

where $\succsim^*_\Omega$ is the ordering induced by f on D_Ω. ($\succsim^*_\Omega$ exists since f satisfies properties (1) and (2)). Because f satisfies properties (1) and (2), g^m_Ω is clearly well-defined and satisfies independence. Since f satisfies property (6), g^m_Ω satisfies the Pareto property. Because f satisfies property (11), g^m_Ω is obviously anonymous. From property (14) g^m_Ω satisfies the elimination of indifferent individuals property. Property (15) clearly translates into coordinality. It should be observed that the set $\mathscr{V}^m_\Omega$ may not include all bounded $\mathbb{R}$-valued functions on $X^m_\Omega \times N_\Omega$. However, by the Countability Assumption $\succsim$ has more than nm indifference classes. Thus $\mathscr{V}^m_\Omega$ will induce all possible orderings of $X^m_\Omega \times N_\Omega$. Thus $\forall \Omega$, $\forall m$, g^m_Ω satisfies all the hypotheses of Theorem 12 and must therefore be either the lexixographic maximin or maximax. Translating back into the language of decision-making under ignorance, f must be either the lexicographic maximin or maximax. Thus Theorem 10 is established. To establish Theorem 11, we note that property (16) translates into the statement that for some $\bar{\Omega}, \bar{m}$,

$$(*) \qquad \begin{cases} \exists v \in \mathscr{V}_{\bar{\Omega}}^{\bar{m}}\ \exists x, y \in X_{\bar{m}} \text{ such that} \\ \max_i v(x, i) < \max_i v(y, i) \text{ but } x g_{\bar{\Omega}}^{\bar{m}}(v) y. \end{cases}$$

Now $g_{\bar{\Omega}}^{\bar{m}}$ is, by the above arguments, either the lexicographic maximin or maximax. But by (*), it cannot be the lexicographic maximax. Therefore, $g_{\bar{\Omega}}^{\bar{m}}$ is the lexicographic maximin. But since f must be either the lexicographic maximin or maximax, this in turn implies that f is actually the lexicographic maximin.

Q.E.D.

It is well known that when a domain of individual preferences is restricted, the set of social welfare functions which satisfy a given list of properties is enlarged. For example, restricting the domain of preferences often enables one to define social welfare functions satisfying all of Arrow's properties [1], although no such SWF exists for the unrestricted domain. As we have seen, positing a minimum number of indifference classes of $\succsim$ is essential to show that the corresponding SWF has unrestricted domain. Therefore, limiting the number of indifference classes is equivalent to restricting the corresponding domain of preferences. One would expect, then, that the properties of Theorems 10 and 11 might not be sufficient to uniquely characterize the lexicographic maximin if a minimum number of indifference classes is lacking. This conjecture is validated by the following example.

Suppose that $\succsim$ has only three indifference classes and that for $x, y, z \in C$, $z \succ y \succ x$. Let f^* be the criterion such that, for any $\Omega, d, d' \in D_\Omega$, $d \succsim_\Omega^* d'$, if and only if $d \succsim_\Omega^{Lm} d'$, where $\succsim_\Omega^*$ and $\succsim_\Omega^{Lm}$ are the orderings induced on D by, respectively, f^* and the lexicographic maximin, unless $\exists w_1, w_2, w_3 \in \Omega$ and $\exists$ some permutation σ of Ω such that $d'(w_1) \sim x$, $d'(w_2) \sim z$, $d'(w_3) \sim z$, $d(\sigma(w_1)) \sim y$, $d(\sigma(w_2)) \sim y$, $d(\sigma(w_3)) \sim y$, and $d'(w) \sim d(\sigma(w))$ for all $w \in \Omega \backslash \{w_1, w_2, w_3\}$ in which case $d' \succ_\Omega^* d$. To check that f^* indeed induces a transitive ordering of D_Ω, consider $\Omega = \{w_1, w_2, w_3\}$ and $d, d' \in D_\Omega$ such that

	w_1	w_2	w_3
d'	x	z	z
d	y	y	y

By hypothesis, $d' \succ^*_\Omega d$. If there are any intransitivities in $\succsim^*_\Omega$, it must be because $\exists \bar{d}, \bar{\bar{d}} \in D_\Omega$ such that $\bar{d} \succsim^*_\Omega d'$ and $d \succsim^*_\Omega \bar{\bar{d}}$ but $\bar{\bar{d}} \succsim^*_\Omega \bar{d}$. Now if $\bar{d} \succsim^*_\Omega d'$, either there exists a permutation of $\{1, 2, 3\}$ such that $\bar{d}(w_{\sigma(i)}) \sim d'(w_i)$ for all $i \in \{1, 2, 3\}$, or, for all i, $\bar{d}(w_i) \succsim y$. If $d \succsim^*_\Omega \bar{\bar{d}}$, then either $\bar{\bar{d}}(w_i) \sim y$ for all $i \in \{1, 2, 3\}$ or there exists $i \in \{1, 2, 3\}$ such that $\bar{\bar{d}}(w_i) \sim x$. It is straightforward to check that in all cases $\bar{d} \succ^*_\Omega \bar{\bar{d}}$ as desired. Thus f^* satisfies properties (1), (2), (6), (11), (14), (15), and (16), but is obviously not the lexicographic maximin. Clearly, additional properties must be hypothesized to obtain a result like Theorem 11 without stipulating the Countability Assumption. It turns out that actually only one additional property must be added to the list: property (13), the weakened form of (12). We shall assume from now on that $\succsim$ has at least 6 indifference classes. If this assumption is not met, the proofs of Theorems 13–15 are even simpler.

THEOREM 13. A criterion f which satisfies properties (1), (2), (6), (11), (13), (14), (15) is either the lexicographic maximin or lexicographic maximax.

Proof. Consider a decision criterion f which satisfies the above axioms. Let $P^0 = \{d_1^0, d_2^0\}$ and $\Omega(P^0) = \Omega^0 = \{w_1, w_2\}$ for which $d_1^0(w_1) \succ d_2^0(w_1) \succ d_2^0(w_2) \succ d_1^0(w_2)$. There are three possible cases

(I) $d_1 \notin f(P^0), d_2 \in f(P^0)$

(II) $d_1 \in f(P^0), d_2 \notin f(P^0)$

(III) $d_1, d_2 \in f(P^0)$

By axioms (11) and (15), if (I) holds, then for any $\tilde{\Omega} = \{w_1, w_2\}$, $\tilde{P} = \{d_1, d_2\}$, $\Omega(\tilde{P}) = \tilde{\Omega}$ such that $d_1(w_1) \succ d_2(w_1) \succ d_2(w_2) \succ d_1(w_2)$, we have $d_1 \notin f(P)$ and $d_2 \in f(P)$. Analogously for (II). Suppose that (III) obtains. Consider $\Omega' = \{w_1, w_2\}$ and $P' = \{d_1, d_2, d_3\}$ with $\Omega(P') = \Omega'$. Suppose that $d_1(w_1) \succ d_3(w_1) \succ d_3(w_2) \succ d_2(w_1) \succ d_2(w_2) \succ d_1(w_2)$.[9] By properties (1) and (2), d_1, d_2, $d_3 \in f(P')$. By d_2 dominates d_1. By property (6), therefore, $d_1 \notin f(P')$, a contradiction. Thus case (III) is impossible. We claim that if (I) holds, f is the lexicographic maximin, and if (II), the lexicographic maximax. We shall assume for the duration of the proof that (I) holds. The argument is entirely analogous for (II). Choose $u \in \mathscr{U}$. Consider $\Omega = \{w_1, \ldots, w_n\}$ and $D_\Omega = \{d \mid \text{domain } d = \Omega\}$. Let $\succsim^*$ be the order on D_Ω induced by f. Choose d_1, $d_2 \in D$. Let σ_1, σ_2 be a permutations of Ω for which $\exists m (\leqslant n)$, such that

$$\forall i > m \quad u(d_1(w_{\sigma_1(i)})) = u(d_2(w_{\sigma_2(i)})),$$

and

$$\forall i, j \leqslant m \quad u(d_1(w_{\sigma_1(i)})) \neq u(d_2(w_{\sigma_2(j)})).$$

If $m = 0, d_1 \sim {}^*d_2$ by property (11). Since this case is trivial, we shall assume that $m \geqslant 1$. By property (14), $d_1 \succsim {}^*d_2$ implies that $d_1^1 \succsim {}_1^*d_2^1$ where d_1^1, $d_2^1 \in D_{\Omega_1}$ $\Omega_1 = \{w_1, \ldots, w_m\}$, $d_k^1(w_i) = d_k(w_{\sigma_3(i)})$ for all $i \in \{1, \ldots, m\}$ and $k \in \{1, 2\}$ and $\succsim {}_1^*$ is the ordering on D_{Ω_1} induced by f. For $k \in \{1, 2\}$, let $M_k = \max_{w \in \Omega_1} u(d_k^1(w))$ and $m_k = \min_{w \in \Omega_1} u(d_k^1(w))$. Choose $\bar{w}, \bar{\bar{w}} \in \Omega_1$ such that $u(d_1^1(\bar{w})) = M_1$ and $u(d_2^1(\bar{\bar{w}})) = m_2$.

$$\text{Define } d_1^2(w) = \begin{cases} d_1^1(\bar{w}), \text{if } u(d_1^1(w)) \neq m_1 \\ d_1^1(w), \text{if } u(d_1^1(w)) = m_1 . \end{cases}$$

$$d_2^2(w) = \begin{cases} d_2^1(\bar{\bar{w}}), \text{if } u(d_2^1(w)) \neq M_2 \\ d_2^1(w), \text{if } u(d_2^1(w)) = M_2 . \end{cases}$$

By property (6), $d_1^1 \succsim {}_1^*d_2^1$ implies that $d_1^2 \succsim {}_1^*d_2^2$.

Let $H = \{(M_1, M_2), (M_2, m_2), (m_1, M_2), (m_1, m_2)\}$. We shall assume that the four elements of H are distinct (the other cases can be argued almost identically).

Let $\Omega_3 = \{w_1, w_2, w_3, w_4\}$ and define $d_1^3, d_2^3 \in D_{\Omega_3}$

$$\text{where} \quad u(d_1^3(w_1)) = u(d_1^3(w_2)) = M_1 ,$$

$$u(d_2^3(w_1)) = u(d_2^3(w_3)) = M_2 ,$$

$$u(d_1^3(w_3)) = u(d_1^3(w_4)) = m_1 ,$$

$$u(d_2^3(w_2)) = u(d_2^3(w_4)) = m_2 .$$

Let $\succsim {}_3^*$ be the ordering on D_{Ω_3} induced by f. By properties (6) and (13), $d_1^2 \succsim {}_1^*d_2^2$ implies that $d_1^3 \succsim {}_3^*d_2^3$. Choose $d_3^3 \in D_{\Omega_3}$ such that $u(d_3^3(w_1)) = u(d_3^3(w_2)) = M_2$, $u(d_3^3(w_3)) = u(d_3^3(w_4) = m_2$. By property (11) $d_2^3 \sim {}_3^*d_3^3$. Therefore $d_1^2 \succsim {}_1^*d_2^2$ implies that $d_1^3 \succsim {}_3^*d_3^3$. Take $\Omega_4 = \{w_1, w_2\}$ and d_1^4, $d_2^4 \in D_{\Omega_4}$ with $u(d_k^4(w_1)) = M_k$ and $u(d_k^4(w_2)) = m_k$ for $k \in \{1, 2\}$. Let $\succsim {}_4^*$ be the ordering of D_{Ω_4} induced by f. Then $d_1^3 \succsim {}_3^*d_3^3$ implies that $d_1^4 \succsim {}_4^*d_2^4$. Collapsing the chain of implications, we obtain

$$d_1 \succsim {}^*d_2 \text{ implies } d_1^4 \succsim {}^4d_2^4 .$$

By construction $M_1 \neq M_2, M_1 \neq m_2, m_1 \neq M_2, m_1 \neq m_2$.

From (I) and property (6), $d_1^4 \succsim {}_4^* d_2^4$ implies that one of the following relations hold

(1) $M_1 > m_1 > M_2 > m_2$.

(2) $M_1 > m_1 > M_2 = m_2$.

(3) $M_1 = m_1 > M_2 > m_2$.

(4) $M_1 = m_1 > M_2 = m_2$.

(5) $M_1 > M_2 > m_1 > m_2$.

(6) $M_1 > M_2 = m_2 > m_1$.

(7) $M_2 > M_1 > m_1 > m_2$.

(8) $M_2 > M_1 = m_1 > m_2$.

Suppose that both $d_1^4 \succsim {}_4^* d_2^4$ and (6) hold. Choose M_2' and $m_2' \in \mathbb{R}$ and $d_3^4 \in D_{\Omega_4}$ such that $m_2 > M_2' > m_2' > m_1$, $u(d_3^4(w_1)) = M_2'$, and $u(d_3^4(w_2)) = m_2'$.[10]

By property (6), $d_2^4 \succ {}_4^* d_3^4$. Therefore $d_1^4 \succ {}_4^* d_3^4$. But, by (I), $d_3^4 \succsim {}_4^* d_1^4$; a contradiction. Therefore, if $d_1^4 \succsim {}_4^* d_2^4$, (6) is impossible. Let $\succsim {}_L^*$ be the ordering induced on D_Ω by the lexicographic maximin criterion.

Observe that any of the cases (1)–(5), (7) and (8) are consistent with $d_1 \succsim {}_L^* d_2$. Therefore, $d_1 \succsim^* d_2$ implies that $d_1 \succsim {}_L^* d_2$. By repeating essentially the same argument, we can show that $d_1 \succ^* d_2$ implies that $d_1 \succ {}_L^* d_2$. Therefore f is the lexicographic maximin.

Q.E.D.

The following result is an immediate corollary.

THEOREM 14. A criterion f which satisfies properties (1), (2), (6), (11), (13), (14), (15), (16) is the lexicographic maximin.

Proof. By Theorem 9, f is either the lexicographic maximin or maximax. By (16), f cannot be the latter.

The final result of this part is a new characterization of the ordinary maximin criterion. Its principal advantages over the axiomatizations in Theorems 4 and 9 are (i) it does not require a convexity axiom and (ii) it does not depend on a denseness assumption, nor, indeed, on any other assumption about the indifference classes of $\succsim$.

THEOREM 15. A criterion f which satisfies axioms (1), (2), (5), (10), (11), (12), (15), (16) is the maximin criterion.

Proof. Choose $u \in \mathscr{U}$. Properties (1), (2), (5) and (10) together imply that (4) holds. Thus, the hypotheses of Theorem 1 are satisfied, and there exists a weak ordering $\succsim_u^*$ as in the statement of Theorem 1. Choose $P \in \mathscr{P}$, $d, d' \in P$ as in property (16). Let $m = \min_w u(d(w))$, $m' = \min u(d'(w))$, $M = \max u(d(w))$, $M = \max u(d'(w))$. Then $M < M'$. By property (4), $m > m'$. So,

$$(9) \qquad m' < m \leqslant M < M' .$$

Suppose there exists Ω such that for some $d_1, d_2 \in D_\Omega$, $d_1 \succsim_u^* d_2$ but $m_1 < m_2$ where $m_1 = \min(d_1(w))$ and $m_2 = \min(d_2(w))$. By (5), we have $m_1 < m_2 \leqslant M_2 \leqslant M_1$ where $M_1 = \max u(d_1(w))$, $M_2 = \max u(d_2(w))$. Choose $d_1' \in D_\Omega$ such that $m_1 < m_1' < m_2 \leqslant M_2 \leqslant M_1 < M_1'$.[11] Then, by (5), $d_1' \succ_u^* d_1 \succsim_u^* d_2$. But, by (15) and (9), $d_2 \succ_u^* d_1$, a contradiction. Therefore, $\forall \Omega\, \forall d_1, d_2 \in D_\Omega$, $\min u(d_1(w)) > \min u(d_2(w))$ implies that $d_1 \succ_u^* d_2$. By continuity (property 10), $\min u(d_1(w)) \geqslant \min u(d_2(w))$ implies that $d_1 \succsim_u^* d_2$. Therefore, f is the maximin criterion.

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SUMMARY OF THE PRINCIPAL PROPERTIES

Property	
(1), (2)	The ranking of decisions must be an ordering
(4)	Weak Dominance
(5)	Strict Dominance
(6)	Admissibility
(7)	Column Linearity
(8)	Linearity
(9), (9′)	Convexity
(10)	Continuity
(11)	Symmetry
(12)	Deletion of Repetitious States
(13)	Weakened Form of Deletion of Repetitious States
(14)	Independence of Non-discriminating States
(15)	Ordinality
(16)	Weak Pessimism

SUMMARY OF RESULTS

WITH THE VON NEUMANN-MORGENSTERN AXIOMS

Properties	H	M (Milnor)	IR (Chernoff)	IR (Milnor)	IR (Maskin)
(1)	⊗	⊗	⊗	⊗	⊗
(2)	⊗	⊗	X	⊗	⊗
(3)	X	X	⊗	X	X
(4)	X	X	⊗	X	X
(5)	X	X	X	⊗	X
(6)			⊗	X	X
(7)			⊗	⊗	X
(8)	⊗	X	X	X	⊗
(9)		⊗	⊗	X	X
(10)	⊗	⊗	X	X	⊗
(11)	⊗	⊗	⊗	⊗	⊗
(12)	⊗	⊗			
(13)	X	X			
(14)			X	X	⊗
(15)		X			
(16)	X*	X	X	X	X

Key:
H – Hurwicz criterion
M – Maximin criterion
LM – Lexicographic maximin criterion
IR – Principle of Insufficient Reason
X – Criterion satisfies this property
⊗ – Criterion axiomatized by this propert
* – satisfies this property only if $\alpha \neq 1$.

WITHOUT THE VON NEWMANN-MORGENSTERN AXIOMS

Properties	M (Modified Milnor)	LM	LM	M (Maskin)
(1)	⊗	⊗	⊗	⊗
(2)	⊗	⊗	⊗	⊗
(3)	X	X	X	X
(4)	X	X	X	X
(5)	X	X	X	⊗
(6)		⊗	⊗	
(7)				
(8)	X	X	X	X
(9′)	⊗	X	X	X
(10)	⊗			⊗
(11)	⊗	⊗	⊗	⊗
(12)	⊗			⊗
(13)	X	X	⊗	X
(14)		⊗	⊗	
(15)	X	⊗	⊗	⊗
(16)	X	⊗	⊗	⊗
Denseness Assumption	⊗			
Countability Assumption	X	⊗		

NOTES

[1] A lottery is finite if it has only finitely many branches.
[2] Finiteness is a convenient but unnecessary supposition.
[3] '$x \sim y$' denotes '$x \gtrsim y$ and $y \gtrsim x$'.
[4] This is convenient for notational purposes but substantively unnecessary.
[5] We may assume that Ω contains at least three elements because if not, we can always add extra states via property (12).
[6] A mapping $g:P \to P'$ is $\gtrsim$-preserving iff $\forall d_1, d_2 \in P \, \forall w, w' \in \Omega(P)$, $d_1(w) \gtrsim d_2(w')$ implies that $g(d_1)(w) \gtrsim g(d_2)(w')$ and $d_1(w) \succ d_2(w')$ implies that $g(d_1)(w) \succ g(d_2)(w')$.
[7] The assumption that $\mathscr{V}$ induces all possible orderings of $X \times N$ is actually equivalent to Hammond's Unrestricted Domain Condition [12].
[8] These exist by our assumption about the number of indifference classes of $\gtrsim$.
[9] See Note 8.
[10] M_2' and m_2' will not exist if M_1, M_2, and m_1 represent adjacent utility levels. However, by our assumption that there are at least six indifference classes and by property (15), we may as well assume that there is a gap of at least two indifferences classes between M_2 and m_1, so that M_2' and m_2' will exist.
[11] d_1' may not actually exist for reasons similar to those of Note 10. However, we may assume its existence, without loss of generality, by the same argument as above.

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