

## OPTIMAL AUCTIONS WITH RISK AVERSE BUYERS

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We characterize a seller's optimal scheme for the sale of an indivisible good to one of  $n$  risk averse buyers. We also compare certain commonly used schemes, such as the high bid and second bid auctions, under the hypothesis of risk aversion.

THIS PAPER STUDIES AUCTIONS designed to maximize the expected revenue of a seller facing risk averse bidders with unknown preferences.<sup>2</sup> Although we concentrate on auctions where a seller sells a single indivisible item, the principles that emerge apply to a much wider class of "principal-agent" problems, as we argue in Maskin and Riley [12].

The properties of auctions that are optimal for the seller when buyers are risk neutral and their preferences independently distributed have been intensely studied (see, for example, Myerson [21]; Maskin and Riley [10]; Harris and Raviv [4, 5]; and Riley and Samuelson [22]). One conclusion that emerges from this work is that, for many distributions of preferences (the exceptions are discussed in Remark 8.1), the standard "high bid" and "English" auctions, modified to allow for a seller's reserve price, are equivalent (i.e., they generate the same expected revenue for the seller) and optimal. These classical auctions, however, are not equivalent from the seller's viewpoint when buyers are risk averse (see Theorem 4 below). Moreover, neither is optimal. This is for two essentially conflicting reasons: the desirability of insuring these buyers against risk, and the desirability of exploiting their risk-bearing in order to screen them.

The classical auctions ordinarily confront buyers with risk—i.e., the marginal utility of income if a buyer wins is typically not the same as that if he loses. Clearly, the seller can extract a payment for removing this risk while keeping the buyer at the same utility. Thus, holding utilities fixed, introducing insurance will enhance the seller's revenue. The qualification "holding utilities fixed," however, is crucial since the insurance will usually induce buyers to alter their bidding strategies. Indeed, as we shall see below, the introduction of perfect insurance may so alter buyers' behavior that the seller is better off offering no insurance at all.

Thus, the seller will generally find it optimal not to offer perfect insurance (see, however, the discussion following Theorem 11). The degree of risk-bearing he imposes on buyers is determined by *screening* considerations. To see the role of risk in screening, imagine that each buyer either has a high reservation price

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<sup>2</sup> Matthews [15] studies much the same problem when buyers' preferences belong to our Case 1 and satisfy constant absolute risk aversion. Moore [19] establishes many of our results without invoking the density condition (45) but under somewhat stronger conditions on preferences.

(eager buyer) or a low reservation price (reluctant buyer) for the item being auctioned. The seller's problem in designing an auction is how to prevent the eager buyers from bidding too low. Suppose that the seller can devise an auction so that buyers who bid low face risk, but that eager buyers who bid low face *greater* risk than the reluctant ones. Then the seller will derive less revenue from the reluctant buyers than if he offered them complete insurance, but this will be more than made up for by inducing the eager buyers to bid higher than they otherwise would. In the language of the incentives literature, relaxation of the incentive constraint overcompensates for the loss due to risk-bearing. (The incentive constraint is simply the guarantee that an eager buyer should derive at least the expected payoff from bidding high rather than low. For a more comprehensive discussion of the use of risk for screening, see Maskin [9].)

This two-class example suggests that whereas it is desirable to confront low bidders with risk (to induce the eager buyers to bid high), nothing is gained from a high bidder's bearing much risk. Indeed, these principles are quite general (see Theorems 11 and 12 below). It also suggests that, although buyers bear risk in the high bid and English auctions, the nature of this risk is not optimal, since the eager buyers bear the most risk.

In Theorems 8 and 9 below we show that designing an optimal auction can often be reduced to solving a standard control problem. We then use this fact to derive a number of general properties of optimal auctions, such as the principles above. First, as long as a buyer's marginal utility of income decreases with his eagerness to buy, then the probability of winning the auction (getting the item) and the amount paid if the auction is won increase with a buyer's eagerness (Theorem 10). As for the nature of risk bearing, there are, in principle, two ways in which a seller can confront a buyer with risk: (i) the buyer's marginal utility of income can be made to differ depending on whether he wins or loses, and (ii) contingent on winning or losing, his payment can be a random variable. We shall see that under the hypothesis just mentioned and given that aversion to income risk either decreases or does not increase too fast with eagerness (which, for many utility functions, simply means that absolute risk aversion does not increase too fast with income), method (ii) is not desirable (Theorem 9). However, method (i) *is*. Indeed we show (Theorem 11) that it is desirable for the marginal utility of income in the losing state to exceed that in the winning state for all buyers except the most eager. For the utility functions we consider, this means that those buyers are better off winning than losing (Corollary to Theorem 11). This result suggests that sufficiently reluctant buyers might even be made to pay a *penalty* if they lose (Theorem 13), although when risk aversion is nonincreasing they will pay more if they win than if they lose (Theorem 14). Moreover, sufficiently low bids will be refused by the seller (Theorem 16). On the other hand, very eager buyers will receive a subsidy if they lose (Theorem 13), and the most eager will be perfectly insured (Theorem 12). Notwithstanding this insurance, more eager buyers pay more on average (Theorem 15).

In Section 1 we begin by presenting a general model of auctions when buyers are risk averse. In Section 2 we consider the standard high bid and English

auctions. We establish existence and uniqueness of symmetric equilibrium in these auctions (Theorems 2 and 3) and show quite generally (Theorem 4) that when buyers are risk averse, the high bid auction generates greater expected revenue for the seller than the English auction.<sup>3</sup> We argue, moreover, that the seller's preference for the high bid auction is intensified if he is risk averse (Theorem 5). We also consider the "perfect insurance auction," in which buyer's marginal utilities of income are the same whether they win or lose. We show (Theorem 6) that, for an important class of cases, the English and perfect insurance auctions generate the same expected revenue for the seller. In Section 3 we take up optimal auctions and show that the seller's optimization reduces to a straightforward control problem (Theorems 8 and 9). In Section 4, we discuss the properties of optimal auctions mentioned above and also one-buyer auctions (Theorem 17). Finally, Section 5 comprises a few concluding remarks. An Appendix contains the proof of the technically complex Theorem 7.

#### 1. THE MODEL

We consider the problem of a seller who wishes to maximize his expected revenue from the sale of a single item. This formulation assumes that the seller is risk neutral toward revenue. We discuss the reasons for this assumption in Section 5. The formulation also implicitly supposes that the seller himself attaches no value to the item. But the analysis would require only slight modification to accommodate a positive seller's value. The seller chooses a selling procedure, or *auction*, which is a game among the potential buyers,  $n$  in number ( $n \geq 1$ ). Each buyer  $i$  has a strategy space  $S_i$ . On the basis of the  $n$ -tuple of strategies  $(s_1, \dots, s_n)$ , the auction assigns buyer  $i$  a probability of winning  $H_i(s_1, \dots, s_n)$  and requires him to make payment  $\tilde{\beta}_i(s_1, \dots, s_n)$  if he wins and payment  $\tilde{\alpha}_i(s_1, \dots, s_n)$  if he loses, where the tildas reflect the possibility that  $\tilde{\beta}_i$  and  $\tilde{\alpha}_i$  are *random* functions. Feasibility requires that

$$(1) \quad \sum_{i=1}^n H_i(s_1, \dots, s_n) \leq 1$$

for all  $(s_1, \dots, s_n)$ . To prevent the seller from extracting unlimited payments, we must allow each buyer the option of not participating in the auction. Formally, this option can be expressed by including in each strategy space a null strategy, which ensures the buyer a zero probability of winning and a zero payment independently of what other buyers do.

We shall suppose that a buyer's preferences can be parameterized by the scalar  $\theta \in [0, 1]$ . We will suppose that the  $\theta$ 's of different buyers are independently and identically distributed<sup>4</sup> according to the c.d.f.  $F$ . We assume that  $F'(\theta) > 0$  for

<sup>3</sup> This last result has been established in special cases by various others, including Butters [1], Holt [8], Matthews [14], and Riley and Samuelson [22].

<sup>4</sup> The assumption of *identical* distributions is inessential; however, the independence assumption is crucial to the methods and results of this paper (see Section 5).

all  $\theta \in [0, 1]$ . Let  $u(-t, \theta)$  be the utility of a buyer of type  $\theta$  who wins the auction and pays the amount  $t$ . Let  $w(-t)$  be the utility of a buyer who loses and pays the amount  $t$ .<sup>5</sup> Thus, given strategies  $(s_1, \dots, s_n)$ , the expected utility of buyer  $i$  with parameter  $\theta$  is

$$E_{\tilde{\beta}_i, \tilde{\alpha}_i} [H_i(s_1, \dots, s_n) u(-\tilde{\beta}_i(s_1, \dots, s_n), \theta) + (1 - H_i(s_1, \dots, s_n)) w(-\tilde{\alpha}_i(s_1, \dots, s_n))]$$

where “ $E$ ” denotes the expectation operator.

We shall suppose that  $u(x, \theta)$  and  $w(x)$  satisfy the following rather innocuous restrictions.

ASSUMPTION A:

- A1.  $u(x, \theta)$  and  $w(x)$  are thrice continuously differentiable.
- A2.  $u_1 > 0, w_1 > 0$ .
- A3.  $w(0) = 0$ .
- A4.  $u_{11} < 0, w_{11} < 0$ .
- A5.  $u_2 > 0$ .

Subscripts denote the argument with respect to which a partial derivative is taken.

It is natural to assume that utility is increasing in income; hence A2. Assumption A3 is simply a convenient normalization of preferences. Because we are interested in risk averse buyers, we assume that both  $u$  and  $w$  are concave functions of income (A4). Finally, in A5, we parameterize preferences so that increasing  $\theta$  implies greater utility (greater “eagerness” in the terminology of the introduction).

For some of the results of this paper, we shall require the following more substantive assumptions.

ASSUMPTION B:

- B1.  $u_{12} < 0$ .
- B2.  $u_{22} < 0$ .
- B3.  $u_1(-t_1, \theta) < w_1(-t_2)$  implies  $u(-t_1, \theta) > w(-t_2)$ .
- B4.  $u_{122} \geq 0$ .
- B5.  $u_{112} \geq 0$ .

If we equate  $\theta$  with “wealth,” then B1 simply requires that marginal utility of income decline with wealth, whereas B2 stipulates that the gains from increasing wealth should be diminishing. Assumption B3 requires that if a buyer is better off losing than winning an auction, his marginal utility of income must be higher in the winning state. Assumption B4 does not have such an obvious economic

<sup>5</sup> Our formulation in terms of  $u$  and  $w$  assumes that buyers’ preferences over money are identical in the event they lose. This assumption is inessential for the results, but it somewhat simplifies the analysis. The formulation also implies that buyers’ appraisals of the item are not influenced by what other buyers think. This simplification is also not crucial. For a treatment that allows for interdependence of tastes, see the predecessor of this paper, Maskin and Riley [10].

interpretation but is nonetheless satisfied by several important models, as we shall see below.

It is commonly thought to be empirically true that people become less risk averse as their well-being increases. One formulation of this “law” asserts that absolute risk aversion declines (or at least does not increase) with income. That is,

$$(2) \quad \frac{\partial}{\partial x} \left( \frac{-u_{11}(x, \theta)}{u_1(x, \theta)} \right) = \frac{-u_1 u_{111} + u_{11}^2}{u_1^2} \leq 0.$$

If we interpret  $\theta$  as a measure of well-being, then an alternative formalization is

$$(3) \quad \frac{\partial}{\partial \theta} \left( \frac{-u_{11}}{u_1} \right) = \frac{-u_1 u_{112} + u_{11} u_{12}}{u_1^2} \leq 0.$$

Given Assumptions A and B2, (3) implies that  $u_{112} \geq 0$ . Thus Assumption B5 simply requires that absolute aversion to risk not increase too fast with  $\theta$ .

This way of modelling preferences is sufficiently general to incorporate many cases of interest. The case most studied in the existing literature is where the only uncertainty facing a buyer is the outcome of the auction; i.e., where the quality of the item itself is known and has equivalent monetary value. Here it is natural to let  $\theta$  represent this monetary value.

*CASE 1—Certain Quality, Equivalent Monetary Value:*

$$u(-t, \theta) = U(\theta - t),$$

$$w(-t) = U(-t),$$

where  $U$  is a concave increasing von Neumann–Morgenstern utility function.

Case 1 is just an example of the more general case in which the item is of certain quality and contributes additively to the utility of money but may not have an equivalent monetary value:<sup>6</sup>

*CASE 2:*

$$u(-t, \theta) = U(\theta + \Psi(-t)),$$

$$w(-t) = U(\Psi(-t)),$$

where  $U$  and  $\Psi$  are concave and increasing and  $U(0) = \Psi(0) = 0$ .

Another possibility of interest is where the buyer is unsure of the (monetary) value of the object. Assume that the possible values are represented by the random

<sup>6</sup> Let  $z=0$  represent no purchase and  $z=1$  represent purchase of the object. Let  $x$  represent income. Then the CES utility function

$$(\theta z^\alpha + x^\alpha)^{1/\beta}, \quad 0 < \alpha < \beta,$$

is a simple example of Case 2 preferences.

variable  $v$  with c.d.f.  $K(v|\theta)$ , where for all  $\theta$   $K(\underline{v}|\theta) = 0$ ,  $K(\bar{v}|\theta) = 1$ ,  $K_1(\underline{v}|\theta) = K_1(\bar{v}|\theta) = 0$ . To capture the idea that higher values of  $\theta$  represent more favorable distributions over  $v$ , we assume that the distribution for a higher  $\theta$  exhibits first order stochastic dominance over that for a lower  $\theta$ . That is,

$$K_2(v|\theta) \equiv \frac{\partial}{\partial \theta} K(v|\theta) \leq 0, \quad v \in [\underline{v}, \bar{v}],$$

with strict inequality over a subset of non-zero measure. In this case preferences take the following form:

CASE 3—*Uncertain Quality*:

$$u(-t, \theta) = \int U(v-t)K_1(v|\theta) dv,$$

$$w(-t) = U(-t).$$

As a final illustration, suppose that the item is of certain quality and has equivalent monetary value but also has an intensifying effect—so that higher values of  $\theta$  represent a greater ability to derive pleasure, crudely translated into a higher marginal utility of income. A simple example of such an effect is as follows.

CASE 4—*Intensification*:

$$u(-t, \theta) = (\theta + 1)U(\theta - t),$$

$$w(-t) = U(-t).$$

We now derive conditions under which Assumptions A and B are satisfied in each of these four cases. The reader may wish to turn directly to Theorem 1 which summarizes these conditions.

Notice first that the five parts of Assumption A are satisfied by each case as long as  $U$  is concave, thrice differentiable, and normalized so that  $U(0) = 0$ . As for Assumption B1, in Cases 1–3,  $u_{12} < 0$ .<sup>7</sup> The preferences of Cases 1 and 2 satisfy B2 and, if  $U''' \geq 0$ , also B4. (From (2) it can be seen that  $U''' \geq 0$ , given

<sup>7</sup> In Case 3

$$u_{12}(-t, \theta) = \int U'(v-t)K_{12}(v|\theta) dv$$

$$= U'K_2|_{\underline{v}} - \int U''K_2 dv.$$

But  $K(\underline{v}|\theta) = 0$  for all  $\theta$  and  $K(\bar{v}|\theta) = 1$  for all  $\theta$ . Therefore the first term is zero. Also  $K_2 \leq 0$  by assumption. Thus  $u_{12} \leq 0$ .

nonincreasing absolute risk aversion.) The preferences of Case 3 satisfy B2 if

$$\int_v^{\bar{v}} U(v-t)K_{122}(v|\theta) dv \leq 0.$$

Integrating by parts twice and defining

$$T(v|\theta) = \int_v^v K_{22}(x|\theta) dx,$$

we have

$$\int_v^{\bar{v}} U(v-t)K_{122}(v|\theta) dv = \int_v^{\bar{v}} U''(v-t)T(v|\theta) dv - U'(\bar{v}-t)T(\bar{v}|\theta).$$

Therefore a sufficient condition for Case 3 to satisfy B2 is that  $T(v|\theta)$  be everywhere nonnegative. Note that for the special case  $K(v|\theta) = H(v-\theta)$  we have  $T(v|\theta) = H'(v-\theta) \geq 0$ . This suggests that the restriction  $T(v|\theta) \geq 0$  is relatively mild.

Cases 1 and 2 satisfy B3 automatically. To establish that Case 3 does also, if absolute risk aversion is nonincreasing in income, we make use of the following Lemma.

**LEMMA 1:** *Suppose utility  $u = \phi(x, z)$  is an increasing function of income  $x$  and that absolute risk aversion,  $-\phi_{11}/\phi_1$ , is nonincreasing in  $z$ . Then*

$$E\{\phi(\tilde{x}, z)\} = \phi(y, z) \rightarrow E\{\phi_2(\tilde{x}, z)\} \geq \phi_2(y, z).$$

**PROOF:** Since  $\phi_1 > 0$  we may also define the inverse function  $x = \phi^{-1}(u, z)$  for fixed  $z$ . We shall first show that the function

$$g(u, z) = \phi_2(\phi^{-1}(u, z), z)$$

is a convex function of  $u$ . Since  $u = \phi(x, z)$  we have

$$g(\phi(x, z), z) = \phi_2(x, z).$$

Differentiating with respect to  $x$  and rearranging we have

$$g_1(\phi(x, z), z) = \phi_{21}/\phi_1 = \frac{\partial}{\partial z} \log \phi_1.$$

Then

$$g_{11}(\phi(x, z), z)\phi_1 = \frac{\partial^2}{\partial x \partial z} \log \phi_1 = -\frac{\partial}{\partial z} (-\phi_{11}/\phi_1) \geq 0,$$

establishing convexity. Thus

$$\begin{aligned}
 E\{\phi_2(\tilde{x}, z)\} &= E\{g(\tilde{u}, z)\}, && \text{from the definitions of } u \text{ and } g, \\
 &\geq g(E\{\tilde{u}\}, z), && \text{by Jensen's Inequality,} \\
 &= g(\phi(y, z), z), && \text{by hypothesis,} \\
 &= \phi_2(y, z), && \text{from the definition of } g. \qquad \qquad \qquad Q.E.D.
 \end{aligned}$$

We now establish the contrapositive of B3 for Case 3 preferences; that is,

$$(4) \quad u(-t_1, \theta) \leq w(-t_2) \rightarrow u_1(-t_1, \theta) \geq w_1(-t_2).$$

For Case 3  $u(-t_1, \theta) = \int U(v - t_1) dK(v|\theta)$ . Then, since  $U$  is an increasing concave function, it is sufficient to show that (4) is true when the left-hand side is an equality. That is,

$$(5) \quad \int U(v - t_1) dK(v|\theta) = U(-t_2) \rightarrow \int U'(v - t_1) dK(v|\theta) \geq U'(-t_2).$$

Define  $\phi(v, -t) \equiv U(v - t)$ .

Then  $\phi(v, -t)$  is increasing in  $v$  and

$$-\frac{\partial}{\partial t} \left( \frac{-\phi_{11}}{\phi_1} \right) = -\frac{\partial}{\partial t} \left( \frac{-U''}{U'} \right) = \frac{\partial}{\partial v} \left( \frac{-U''}{U'} \right) \leq 0,$$

if absolute risk aversion is nonincreasing in income. Thus  $\phi(v, -t)$  satisfies the assumptions of Lemma 1. Moreover  $\phi_2(v, -t) = U'(v - t)$ . Hence Lemma 1 implies (5) so that B3 is indeed satisfied for Case 3.

The preferences of Case 3 satisfy B4 if

$$\int_v^{\bar{v}} U'(v - t) K_{122}(v|\theta) dv \geq 0.$$

Integrating by parts twice we have

$$\int_v^{\bar{v}} U'(v - t) K_{122}(v|\theta) dv = \int_v^{\bar{v}} U'''(v - t) T(v|\theta) dv - U''(\bar{v} - t) T(\bar{v}|\theta).$$

Thus, assuming  $U''' \geq 0, T(v|\theta)$  everywhere nonnegative is again a sufficient condition.

Turning to Assumption B5 it is readily verified that for Cases 1-3 a sufficient condition is  $U''' \geq 0$ .<sup>8</sup>

In Case 4 rather more stringent conditions are required to satisfy Assumption B. For this case  $u_{12} = (1 + \theta)U'' + U'$  while  $u_{22} = (1 + \theta)U'' + 2U'$ . Then, since  $\theta$

<sup>8</sup> In Case 3

$$u_{112}(-t, \theta) = \int U''' K_2 dv > 0$$

from the same kind of integration by parts as in footnote 7.



is nonnegative, B1 and B2 are satisfied if the degree of absolute risk aversion,  $-U''/U'$ , exceeds 2. Condition B3 is satisfied automatically.

Condition B4 is satisfied if  $u_{122} = (1 + \theta)U''' + 2U'' \geq 0$ . But

$$\begin{aligned} U''' + 2U'' &= U' \left[ \frac{U'''}{U'} - 2 \left( \frac{-U''}{U'} \right) \left( \frac{U''}{U'} \right)^2 \right] \\ &> U' \left[ \frac{U'''}{U'} - \left( \frac{U''}{U'} \right)^2 \right] \quad \text{if } -U''/U' > 2 \\ &= U' \frac{d}{d(-t)} \left( \frac{U''}{U'} \right). \end{aligned}$$

Thus a sufficient condition for B4 is that absolute risk aversion everywhere exceeds 2 and is nonincreasing with income.

Finally  $u_{112} = (1 + \theta)U''' + U'' > u_{122}$ . Therefore if B4 is satisfied so is Assumption B5.

Summarizing we have:

**THEOREM 1:** *Assumptions A and B are satisfied by Cases 1 and 2 if  $U''' \geq 0$ ; Case 3 if absolute risk aversion is nonincreasing,*

$$\frac{d}{dx} \left( \frac{-U''(x)}{U'(x)} \right) \leq 0, \quad \text{and} \quad T(v|\theta) = \int_0^v K_{22}(x|\theta) dx \geq 0;$$

*Case 4 if absolute risk aversion is nonincreasing and everywhere exceeds 2.*

To predict the outcome of an auction, we must specify a solution concept. We shall assume that the functional forms  $u(\cdot, \cdot)$  and  $w(\cdot)$  and the distribution  $F$  are common knowledge among buyers and seller but that only buyer  $i$  knows the value of the parameter  $\theta_i$ . In this case, the Bayesian equilibrium of Harsanyi [7] is appropriate. For this solution concept, the “revelation principle”<sup>9</sup> (see Dasgupta, Hammond, and Maskin [3]; Harris and Townsend [6]; and Myerson [20]) tells us that we can confine our attention, without loss of generality, to auctions where the strategy space coincides with the set of possible parameters, i.e.,  $[0, 1]$ , and where there exists an equilibrium in which each buyer plays his *true* parameter as his strategy. That is,

(6a)  $\max E_{\theta_{-i}} \pi(x|\theta_{-i}) = E_{\theta_{-i}} \pi(\theta_i|\theta_{-i}),$

(6b)  $\pi(x|\theta_{-i}) = H_i(x, \theta_{-i})u(-\tilde{\beta}(x, \theta_{-i}), \theta_i)$   
 $+ (1 - H_i(x, \theta_{-i}))w(-\tilde{\alpha}(x, \theta_{-i})), \quad \text{and}$   
 $(x, \theta_{-i}) = (\theta_1, \dots, x, \dots, \theta_n).$

Such auctions can be called *direct revelation*<sup>10</sup> auctions.

<sup>9</sup> This term is due to Myerson [20]. For an informal discussion of the application of this principle to auction design see Riley and Samuelson [22].

<sup>10</sup> This term is borrowed from Dasgupta, Hammond, and Maskin [3].

Because the buyers are *ex ante* identical, we may confine our attention to *symmetric* auctions, i.e., those where families of  $H_i, \tilde{\beta}_i,$  and  $\tilde{\alpha}_i$  functions are permutation symmetric.<sup>11</sup> Define

$$(7) \quad G(\theta_i) = \int_{\theta_{-i}} H_i(\theta_i, \theta_{-i}) \prod_{j \neq i} dF(\theta_j).$$

Note that, from symmetry,  $G$  requires no subscript. Then (6) becomes

$$(8) \quad \begin{aligned} \max_x E[G(x)u(-\tilde{b}(x), \theta_i) + (1 - G(x))w(-\tilde{a}(x))] \\ = E[G(\theta_i)u(-\tilde{b}(\theta_i), \theta_i) + (1 - G(\theta_i))w(-\tilde{a}(\theta_i))], \end{aligned}$$

where  $\tilde{b}$  and  $\tilde{a}$  reflect the randomness both from  $\tilde{\beta}_i$  and  $\tilde{\alpha}_i$  and from  $\theta_{-i}$ . They too need no subscripts. Because buyers have the option of not participating,

$$E[G(\theta_i)u(-\tilde{b}(\theta_i), \theta_i) + (1 - G(\theta_i))w(-\tilde{a}(\theta_i))] \geq 0.$$

Since  $u_2 > 0$ , we need stipulate the nonparticipation constraint only for  $\theta_i = 0$ . Thus,

$$(9) \quad E[G(0)u(-\tilde{b}(0), 0) + (1 - G(0))w(-\tilde{a}(0))] \geq 0.$$

Notice that the only characteristics of an auction in which either the seller or the buyers are interested are the functions  $G, \tilde{b}$ , and  $\tilde{a}$ , as buyer  $i$ 's payoff is

$$E[G(\theta_i)u(-\tilde{b}(\theta_i), \theta_i) + (1 - G(\theta_i))w(-\tilde{a}(\theta_i))],$$

and the seller's payoff is

$$(10) \quad nE \int [G(\theta)\tilde{b}(\theta) + (1 - G(\theta))\tilde{a}(\theta)] dF(\theta).$$

For this reason, we shall often represent an auction by the triple  $\langle G, \tilde{b}, \tilde{a} \rangle$ .

## 2. STANDARD AUCTIONS

Before turning to the optimal choice of  $\langle G, \tilde{b}, \tilde{a} \rangle$ , we first consider the standard auctions, the high bid, and English auctions. In the high bid auction, sealed bids are simultaneously submitted by the buyers. The high bidder wins (a tie is broken by a coin flip) and pays his bid. Losers pay (and receive) nothing. In the English auction, bids are submitted successively and openly; each bid must be greater

<sup>11</sup> A family of functions  $\{f_1, \dots, f_n\}$ , where each function  $f_i$  has  $n$  arguments, is *permutation symmetric* if, for all  $i$  and  $j$  and all vectors  $x$  and  $y$  in the domains of  $f_i$  and  $f_j$ ,

$$f_i(x) = f_j(y)$$

if  $x_i = y_j, x_j = y_i,$  and  $\forall k \neq i, j, x_k = y_k$ .

To see that it suffices to consider permutation symmetric families of  $H_i$ 's,  $\alpha_i$ 's and  $\beta_i$ 's, consider the case of two buyers. Suppose an asymmetric auction  $A_1$  were optimal. By symmetry, the auction  $A_2$  obtained from  $A_1$  by reversing the roles of the buyers is also optimal. But then the symmetric auction  $A_{12}$  obtained by flipping a coin to decide which of the two auctions,  $A_1$  or  $A_2$ , to play is also optimal. The argument generalizes to more than two buyers.

than the preceding one. The winner is the last buyer to bid (again, ties are broken by a randomizing device), and he pays his bid, while losers, again, pay nothing.<sup>12</sup> As Vickrey [25] argued,<sup>13</sup> the English auction is equivalent, if the  $\theta_i$ 's are independently distributed, to a sealed bid auction in which the higher bidder wins but pays only the second highest bid, i.e., to a "second bid" auction (here we assume the continuous price formulation of footnote 12). Because it is easier to work with, we shall study the second bid formulation. Our ability to do this, however, depends crucially on independence; the two auctions are not equivalent otherwise. Indeed, as Milgrom and Weber [17] show, buyers who are either risk neutral or exhibit constant absolute risk aversion bid higher, on average, in an English auction than in a second bid auction if, roughly speaking, the  $\theta_i$ 's are positively correlated.

We begin by showing that under some of the conditions discussed in Section 1, equilibria in high and second bid auctions exist, are unique and have the property that bids are increasing as functions of  $\theta_i$ . Because they are sometimes optimal when buyers are risk neutral, we shall consider high and second bid auctions with *seller reserve prices*, i.e., minimum permissible bids. In the second bid auction with reserve price,  $b^0$ , a winning buyer pays  $b^0$  if no bid other than his own is greater than  $b^0$ . Because the buyers are *ex ante* identical, it is natural to focus attention on symmetric equilibria. In Remarks 2.3 and 3.2, we discuss the possibility of asymmetric equilibria.

**THEOREM 2:** *In a high bid auction with  $n \geq 2$  buyers suppose the seller announces a minimum price,  $b^0$ , such that at least one buyer type is indifferent between buying and not buying at this price, that is,*

$$(11) \quad u(-b^0, \theta^0) = 0, \quad \text{for some } \theta^0 \in [0, 1].$$

*Then if preferences satisfy Assumption A and  $\partial(u_1/u)/\partial\theta < 0$ , those for whom  $\theta < \theta^0$  will not submit an acceptable bid, and there exists a unique symmetric equilibrium bid function  $b(\theta)$ ,  $\theta \geq \theta^0$ . Moreover,  $b(\theta)$  is increasing and differentiable.*

**PROOF:** If an equilibrium with  $b(\theta)$  increasing exists, then, for each  $\theta$ , the probability that a buyer with parameter  $\theta$  has of winning is  $G(\theta)$ , where

$$(12) \quad G(\theta) = \begin{cases} 0, & \theta < \theta^0, \\ F^{n-1}(\theta), & \theta \geq \theta^0. \end{cases}$$

<sup>12</sup> This description of the English auction is ill-specified because rational buyers may wish to raise their bids by infinitesimals. This problem can be avoided either by postulating a minimum quantity by which bids must be raised, or by adopting the following continuous price formulation. Suppose that the seller quotes a price that rises continuously over time. At any instant, a buyer can choose either to stay in or to drop out (forever). The winner is the last buyer to remain (again, ties are broken by a randomizing device), and he pays the price prevailing at the time the penultimate buyer drops out. Losers, as usual, pay nothing.

<sup>13</sup> Actually, Vickrey did not treat equivalence for the case where Bayesian equilibrium is the solution concept (he worked instead with dominant strategies). For such a treatment see Remark 3.2 following Theorem 3.

Furthermore, if  $b(\theta)$  is an equilibrium bid function, then, for each  $\theta \geq \theta^0$ ,  $x = \theta$  maximizes

$$(13) \quad E\{u\} = G(x)u(-b(x), \theta).$$

Assuming that  $b(\theta)$  is differentiable and differentiating, we have

$$\begin{aligned} \frac{\partial}{\partial x} E\{u\} &= G'(x)u(-b(x), \theta) - b'(x)G(x)u_1(-b(x), \theta) \\ &= G(x)u_1(-b(x), \theta) \left[ \frac{u(-b(x), \theta)}{u_1(-b(x), \theta)} \frac{G'(x)}{G(x)} - b'(x) \right]. \end{aligned}$$

With  $\partial(u_1/u)/\partial\theta < 0$  the first term in the bracketed expression is strictly increasing in  $\theta$ . Thus

$$\frac{\partial}{\partial x} E\{u\} \cong G(x)u_1(-b(x), \theta) \left[ \frac{u(-b(x), x)}{u_1(-b(x), x)} \frac{G'(x)}{G(x)} - b'(x) \right] \quad \text{as } x \cong \theta.$$

Therefore if  $b(\theta)$  is defined by the differential equation and boundary conditions

$$(14) \quad \begin{cases} b'(\theta) = \frac{G'(\theta)u(-b(\theta), \theta)}{G(\theta)u_1(-b(\theta), \theta)}, \\ b(\theta^0) = b^0, \quad \text{where } u(-b^0, \theta^0) = 0, \end{cases}$$

we have

$$\frac{\partial}{\partial x} E\{u(-b(x), \theta)\} \cong 0 \quad \text{as } x \cong \theta; \quad x, \theta \in [\theta, 1].$$

So  $x = \theta$  yields the global maximum of  $E\{u(-b(x), \theta)\}$ . This establishes existence.

To prove uniqueness we first show that an equilibrium random bid function  $\tilde{b}(\theta)$  (in principle, an equilibrium could involve mixed strategies) must be increasing for  $\theta \geq \theta^0$ . More precisely, we show that if  $\bar{b}(\cdot)$  is a deterministic selection from  $\tilde{b}(\cdot)$  (i.e.,  $\bar{b}(\theta)$  is a number in the support of  $\tilde{b}(\theta)$  for all  $\theta$ ), then  $\bar{b}$  must be increasing for  $\theta \geq \theta^0$ . Suppose that  $\bar{b}(\cdot)$  is decreasing over some interval. Then there exist  $\theta^1$  and  $\theta^2$  with  $\theta^1 < \theta^2$  such that  $b^1 \equiv \bar{b}(\theta^1) > b^2 \equiv \bar{b}(\theta^2)$ . Hence,  $\bar{G}_1 > \bar{G}_2$ , where  $\bar{G}_i$  is the probability that a buyer who bids  $\bar{b}(\theta^i)$  wins; otherwise a buyer with parameter  $\theta^2$  would be better off bidding  $\bar{b}(\theta^2)$ . By definition of equilibrium

$$(15) \quad \bar{G}_2 u(-b^2, \theta^2) \geq \bar{G}_1 u(-b^1, \theta^2)$$

and

$$(16) \quad \bar{G}_2 u(-b^2, \theta^1) \leq \bar{G}_1 u(-b^1, \theta^1).$$

Combining (15) and (16), we obtain

$$(17) \quad \frac{u(-b^2, \theta^2)}{u(-b^2, \theta^1)} \geq \frac{u(-b^1, \theta^2)}{u(-b^1, \theta^1)}.$$

Therefore,

$$\begin{aligned} 0 &\leq \log \frac{u(-b^2, \theta^2)}{u(-b^2, \theta^1)} - \log \frac{u(-b^1, \theta^2)}{u(-b^1, \theta^1)} \\ &= \int_{b^1}^{b^2} \int_{\theta^1}^{\theta^2} \frac{\partial^2}{\partial \theta \partial b} \log u \, d\theta \, db \\ &= - \int_{b^1}^{b^2} \int_{\theta^1}^{\theta^2} \frac{\partial}{\partial \theta} (u_1/u) \, d\theta \, db. \end{aligned}$$

By assumption the integrand is negative. Then (18) holds if and only if  $b^2$  exceeds  $b^1$ . Hence (17) holds if and only if  $\bar{b}(\theta^2) \geq \bar{b}(\theta^1)$  contradicting the hypothesis. Thus  $\bar{b}(\cdot)$  is nondecreasing. Now suppose that  $\bar{b}(\theta) = \hat{b}$  on the interval  $[\theta^1, \theta^2]$ . Assume that  $\bar{b}(\theta) < \hat{b}$  for  $\theta < \theta^1$  and  $\bar{b}(\theta) > \hat{b}$  for  $\theta > \theta^2$ . Then a parameter  $\theta^2$  buyer who bids  $\bar{b}(\theta^2)$  has probability of winning

$$H = \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} (F(\theta^2) - F(\theta^1))^k F(\theta^1)^{n-1-k}.$$

But if the buyer bids  $\bar{b}(\theta^2) + \epsilon$ , for  $\epsilon > 0$ , his probability of winning is greater than  $F(\theta^2)^{n-1}$ , which in turn is greater than  $H$ . Thus by an infinitesimal increase  $\epsilon$  in his bid, the buyer can gain a discrete increase of, at least,  $F(\theta^2)^{n-1} - H$  in his probability of winning, and so  $\bar{b}(\theta^2)$  cannot be an equilibrium bid.

Therefore, for  $\theta \geq \theta^0$ ,  $\bar{b}(\theta)$  must be strictly increasing, and so  $G$  is given by (12). If  $\bar{b}(\theta)$  is not continuous, then there exists  $\theta^* \geq \theta^0$  with

$$\limsup_{\theta < \theta^*} \bar{b}(\theta) < \liminf_{\theta > \theta^*} \bar{b}(\theta).$$

But, for  $\epsilon > 0$  sufficiently small, a buyer bidding  $\limsup \bar{b}(\theta) - \epsilon$  has a probability of winning that is arbitrarily close to that of one bidding  $\liminf \bar{b}(\theta) + \epsilon$ . This is impossible, however, since  $\limsup \bar{b}(\theta)$  is strictly less than  $\liminf \bar{b}(\theta)$ , so no one would ever bid  $\liminf \bar{b}(\theta) + \epsilon$ . Hence  $\bar{b}(\theta)$  is continuous.

To see that  $\bar{b}(\theta)$  is differentiable for  $\theta \geq \theta^0$ , note that for any  $\Delta\theta$

$$G(\theta)u(-\bar{b}(\theta), \theta) \geq G(\theta + \Delta\theta)u(-\bar{b}(\theta + \Delta\theta), \theta)$$

and

$$G(\theta + \Delta\theta)u(-\bar{b}(\theta + \Delta\theta), \theta + \Delta\theta) \geq G(\theta)u(-\bar{b}(\theta), \theta + \Delta\theta).$$

Hence, invoking the mean value theorem, we obtain

$$\begin{aligned} (G(\theta) - G(\theta + \Delta\theta))u(-\bar{b}(\theta), \theta) + G(\theta + \Delta\theta)u_1(-b^*, \theta) \\ \times (\bar{b}(\theta + \Delta\theta) - \bar{b}(\theta)) \geq 0 \end{aligned}$$

and

$$\begin{aligned} (G(\theta + \Delta\theta) - G(\theta))(u(-\bar{b}(\theta + \Delta\theta), \theta + \Delta\theta)) + G(\theta)u_1(-b^{**}, \theta + \Delta\theta) \\ \times (\bar{b}(\theta) - \bar{b}(\theta + \Delta\theta)) \geq 0, \end{aligned}$$

where both  $b^*$  and  $b^{**}$  are between  $\bar{b}(\theta)$  and  $\bar{b}(\theta + \Delta\theta)$ . Combining these last two inequalities, we have

$$\begin{aligned} \frac{(G(\theta + \Delta\theta) - G(\theta))u(-\bar{b}(\theta + \Delta\theta), \theta + \Delta\theta)}{G(\theta)u_1(-b^{**}, \theta + \Delta\theta)\Delta\theta} &\geq \frac{\bar{b}(\theta + \Delta\theta) - \bar{b}(\theta)}{\Delta\theta} \\ &\geq \frac{(G(\theta + \Delta\theta) - G(\theta))u(-\bar{b}(\theta), \theta)}{G(\theta + \Delta\theta)u_1(-b^*, \theta)\Delta\theta}. \end{aligned}$$

But because  $\bar{b}$  is continuous, the left and right-most terms of this double inequality tend to

$$\frac{G'(\theta)u(-\bar{b}(\theta), \theta)}{G(\theta)u_1(\bar{b}(\theta), \theta)}$$

as  $\Delta\theta \rightarrow 0$ . Hence  $\bar{b}(\theta)$  is differentiable for  $\theta > \theta^0$  and thus satisfies (14) everywhere. Since  $\bar{b}(\theta)$  was an arbitrary selection,  $\tilde{b}(\theta)$  must satisfy (14) too. If  $\tilde{b}(\theta^0) < b^0$ , then for  $\alpha > 0$  sufficiently small  $b(\theta^0 + \alpha) < b^0$ . Thus the payoff of a buyer with parameter  $\theta^0 + \alpha$  is zero. But if he bids  $b^0$ , his expected payoff is

$$F^{n-1}(\theta^0)u(-b^0, \theta^0 + \alpha) > 0,$$

a contradiction. If  $\tilde{b}(\theta^0) > b^0$ , then  $u(-\tilde{b}(\theta^0), \theta^0) < 0$ , also an impossibility. Thus  $\tilde{b}(\theta^0) = b^0$ , and so  $\tilde{b}(\theta) = b(\theta)$  for  $\theta \geq \theta^0$ . Q.E.D.

REMARK 2.1: If the seller gets a minimum price so low that even the least eager buyers ( $\theta = 0$ ) would strictly prefer to buy at that price, there is still a unique equilibrium  $b(\theta)$  with boundary condition  $u(-b(0), 0) = 0$ .

REMARK 2.2: Theorem 2 is stated for preferences such that  $u_{11} < 0$  and  $w_{11} < 0$ , but it is clear from the proof that it is true as well if these inequalities hold weakly.

REMARK 2.3: This theorem does not discuss the possibility of asymmetric equilibria. Under the hypotheses of the theorem, however, one can show (see Maskin and Riley [13]) that the only equilibrium is the symmetric equilibrium. A crucial hypothesis in this uniqueness result is that the distribution  $F$  has bounded support. When the support of  $F$  has no upper bound, there can be a continuum of asymmetric equilibria.

REMARK 2.4: We can incorporate the possibility that buyers care about others' parameter values by writing the utility of buyer  $i$  as  $u(-b_i, \theta_i, \theta_{-i})$ . Then if the equilibrium bid function  $b(\theta_i)$  is increasing and buyer  $i$  is the winner, his expected utility is

$$\bar{u}(-b, \theta_i) = E\{u(-b, \theta_i, \theta_{-i}) | \theta_j \leq \theta_i, j \neq i\}.$$

Appealing to the results of Milgrom and Weber [17] it is readily confirmed that if  $u(x, \theta_i, \theta_{-i})$  is a strictly increasing and concave function of  $x$ , if  $\partial u(x, \theta_i, \theta_{-i})/\partial x$  is a nonincreasing function, and if parameter values are "affiliated" (roughly speaking, positively correlated),<sup>14</sup> then  $\bar{u}$  satisfies Assumption A and  $\partial(\bar{u}_1/\bar{u})/\partial\theta < 0$ . We may therefore apply the argument of Theorem 2 to establish existence and uniqueness with affiliated parameter values. This generalizes Milgrom and Weber's existence proof for the case of risk neutral buyers.

REMARK 2.5: To see that the assumption  $\partial(u_1/u)/\partial\theta < 0$  is weak suppose a buyer of type  $\theta$  is indifferent between paying  $b$  for the item and paying  $b - \delta(\theta)$  for a lottery in which the item is not awarded with some small probability  $p$ . Since  $w(0) = 0$ , the buyer's indifference can be expressed as

$$(1-p)u(-b + \delta(\theta), \theta) = u(-b, \theta).$$

Taking a Taylor's expansion of the left-hand side and rearranging we obtain

$$\delta(\theta) \approx \left( \frac{p}{1-p} \right) \frac{u(-b, \theta)}{u_1(-b, \theta)}.$$

We are therefore assuming only that an individual who places a higher value on the item would require greater compensation in order to accept such a risk.

Turning now to the second bid auction, we define  $\sigma(\theta)$  to be the reservation price of a buyer with parameter  $\theta$ . That is

$$(19) \quad u(-\sigma(\theta), \theta) = 0.$$

THEOREM 3: *In a second bid auction with  $n \geq 2$  buyers suppose the seller announces a minimum price,  $b^0$ , such that at least one buyer is indifferent between buying and not buying at this price, that is*

$$u(-b^0, \theta^0) = 0, \quad \text{for some } \theta^0 \in [0, 1].$$

*Then if preferences satisfy Assumption A there is an equilibrium in which each buyer for whom  $\theta \geq \theta^0$  bids his reservation price while the remainder do not submit acceptable bids. Moreover, this is the unique symmetric equilibrium.*

PROOF: Given  $u(-b^0, \theta^0) = 0$  all those for whom  $\theta < \theta^0$  are better off not bidding. Suppose that all buyers use the same bid function  $h(\theta)$  in equilibrium and that  $h(\theta)$  is increasing. Since  $u(-b^0, \theta) > 0$  for all  $\theta > \theta^0$  we must have  $h(\theta) \geq b^0$  for all  $\theta \geq \theta^0$ . Then if buyers bid according to  $h(\theta)$ ,  $G(\theta)$  satisfies (12).

<sup>14</sup> For joint density functions  $f(\theta_1, \dots, \theta_n)$  which are twice differentiable and non-zero for all  $\theta_i \in [0, 1]$ ,  $i = 1, \dots, n$ , the  $n$  variables are affiliated if

$$\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log f(\theta_1, \dots, \theta_n) \geq 0, \quad \text{for all } i, j = 1, \dots, n.$$

Thus for all  $\theta \geq \theta^0$ ,  $x = \theta$  maximizes

$$(20) \quad E\{u\} = u(-b^0, \theta)F^{n-1}(\theta^0) + \int_{\theta^0}^x u(-h(t), \theta) dF^{n-1}(t).$$

Differentiating (20) by  $x$  we obtain

$$\frac{\partial}{\partial x} E\{u\} = u(-h(x), \theta) dF^{n-1}(x)/dx.$$

By assumption  $h(\theta)$  is increasing and by Assumption A1  $u_1 > 0$ . Thus

$$\frac{\partial}{\partial x} E\{u\} \leq u(-h(\theta), \theta) dF^{n-1}(x)/dx, \quad x \geq \theta.$$

But from the definition of  $\sigma(\theta)$ ,  $u(-\sigma(\theta), \theta) = 0$ . Moreover, since  $u_2 > 0$ ,  $\sigma(\theta)$  is an increasing function. Then  $h(\theta) = \sigma(\theta)$ ,  $\theta > \theta^0$ , is an equilibrium of the second bid auction.

To prove uniqueness, suppose  $\tilde{h}(\theta)$  is the (possibly random) bid function in a symmetric equilibrium. Let  $\bar{h}(\theta)$  be a deterministic selection.

Suppose first that  $\Pr\{\theta \geq \theta^0, \bar{h}(\theta) > \sigma(\theta)\} > 0$ . Then there exists  $\theta^* \geq \theta^0$  such that  $\bar{h}(\theta^*) > \sigma(\theta^*)$  and for all  $\varepsilon > 0$

$$\Pr\{\bar{h}(\theta) \in [\bar{h}(\theta^*) - \varepsilon, \bar{h}(\theta^*)]\} > 0.$$

But then a buyer with parameter  $\theta^*$  is better off bidding less than  $\bar{h}(\theta^*)$ , since otherwise there is a positive probability he will pay more than his reservation price.

Next suppose that  $\Pr\{\theta \geq \theta^0, \bar{h}(\theta) < \sigma(\theta)\} > 0$ . Then there exists  $\theta^{**} \geq \theta^0$  such that  $\bar{h}(\theta^{**}) < \sigma(\theta^{**})$  and for all  $\varepsilon > 0$

$$\Pr\{\bar{h}(\theta) \in [\bar{h}(\theta^{**}), \bar{h}(\theta^{**}) + \varepsilon]\} > 0.$$

But then a buyer with parameter  $\theta^{**}$  is better off bidding more than  $\bar{h}(\theta^{**})$  since otherwise there is a positive probability that he will lose to a bid less than his reservation price.

Hence,  $\Pr\{\bar{h}(\theta) = \sigma(\theta) | \theta \geq \theta^0\} = 1$ , and so  $\bar{h}(\theta) = \sigma(\theta)$  for all  $\theta \in [\theta^0, 1)$ . Therefore  $\tilde{h}(\theta) = \sigma(\theta)$  for all  $\theta \in [\theta^0, 1)$ . Q.E.D.

REMARK 3.1: Examining the last line of the proof, we see that equilibrium is not quite unique. Although bids must coincide with  $\sigma(\theta)$  for all  $\theta \in [\theta^0, 1)$ , they need not for  $\theta = 1$ . All that is necessary is that  $\tilde{h}(1) \geq \sigma(1)$ .

REMARK 3.2: One can also show that there are no asymmetric equilibria with three or more buyers in the second bid auction with a minimum price greater than  $\sigma(0)$  (see Maskin and Riley [13]). With two buyers, however, there is a vast family of asymmetric equilibria. For example, consider the continuum of pairs of bid functions in which one buyer announces that if his reservation price  $\sigma(\theta)$  exceeds  $b^* > b^0$  he will bid  $\sigma(1)$ , otherwise he will bid his reservation value, and the other buyer bids  $\min\{\sigma(\theta), b^*\}$ . It is readily confirmed that all these pairs are equilibria.



Even in the case  $n = 2$ , however, there is good reason to single out the symmetric equilibrium. Besides its appeal from its very symmetry, the symmetric equilibrium is also the unique *dominant strategy* equilibrium. Closely related to this point is the fact that it corresponds to the unique (subgame) perfect equilibrium of the English auction (recall that our motivation for examining second bid auctions was their equivalence to English auctions). There is a one-to-one correspondence between the asymmetric equilibria of the second bid and English auctions. However, those in the latter auction fail to be subgame perfect, whereas those in the former are not trembling-hand perfect (see Selten [24]). For greater elaboration of these points see Maskin and Riley [13].

REMARK 3.3: If the seller sets a minimum price so low that even a buyer with parameter  $\theta = 0$  has a reservation price  $\sigma(0) > b^0$  the unique symmetric equilibrium is for all buyers to bid their reservation values.

REMARK 3.4: Like Theorem 2, Theorem 3 holds for risk neutral buyers, that is,  $u_{11} = 0, w_{11} = 0$ .

We are now ready to show that, under weak assumptions, high bid auctions are superior to second bid auctions from the seller's viewpoint. In general, to compare two auctions entails specifying which equilibria in each are to be examined. We shall in fact compare the (unique) *symmetric* equilibria. However, in view of Remarks 2.3 and 3.2, we need not have made this qualification, since, at least in the case  $n \geq 3$ , equilibrium in both the high bid and second bid auctions is unique.<sup>15</sup>

THEOREM 4: *Under the assumptions of Theorem 2 the symmetric equilibrium of the high bid auction with reserve price  $b^0$  generates greater expected revenue for the seller than the symmetric equilibrium of the second bid auction with the same reserve price.*

REMARK 4.1: For the preferences of Case 1 this result has already been established in those papers mentioned in Footnote 3.

PROOF: Let  $B(\theta)$  be the expected payment by a winner with parameter  $\theta$  in the second bid auction. We shall establish that  $b(\theta) \geq B(\theta)$  for all  $\theta$ , with strict inequality for  $\theta \geq \theta^0$  where  $b(\theta)$  satisfies (14). From Theorem 3 if a buyer with parameter value  $\theta$  is the winner in the second bid auction his payment is a random variable

$$(21) \quad \tilde{B} = \max \{b^0, \sigma(t)\} \text{ where } t \text{ is the highest of the other } n-1 \text{ buyer's parameter values}.$$

<sup>15</sup> Even in the case  $n = 2$ , it is not necessary to single out any particular equilibrium in the second bid auction since, as one can easily confirm, all asymmetric equilibria are dominated, from the seller's perspective, by the symmetric equilibrium.

Therefore the expected payment by the winner,  $B(\theta)$ , satisfies

$$(22) \quad B(\theta) = E\{\tilde{B}\} = \left[ b^0 F^{n-1}(\theta^0) + \int_{\theta^0}^{\theta} \sigma(t) dF^{n-1}(t) \right] / F^{n-1}(\theta).$$

Differentiating (22) with respect to  $\theta$  we obtain

$$(23) \quad \begin{cases} \frac{dB(\theta)}{d\theta} = \frac{G'(\theta)}{G(\theta)}(\sigma(\theta) - B(\theta)), & \theta \geq \theta^0, \\ B(\theta^0) = b^0, & \text{where } u(-b^0, \theta^0) = 0, \end{cases}$$

where  $G(\theta)$  satisfies (12); that is,  $G(\theta) = F^{n-1}(\theta)$ , for  $\theta \geq \theta^0$ .

Comparing (14) and (23) first note that  $b(\theta^0) = B(\theta^0)$ . Thus if we can show that

$$(24) \quad \frac{u(-b(\theta), \theta)}{u_1(-b(\theta), \theta)} > \sigma(\theta) - b(\theta) \quad \text{whenever } b = B,$$

then (14) and (23) imply that  $b(\theta) > B(\theta)$  for  $\theta > \theta^0$ . Consider the left- and right-hand sides of (24) as functions of  $b$ . For  $b = \sigma(\theta)$ , both sides vanish. The derivative of the left-hand side is  $-1 + uu_{11}/u_1^2$  whereas the derivative of the right-hand side is  $-1$ . Therefore, because  $b(\theta) < \sigma(\theta)$  we conclude that (24) holds.

*Q.E.D.*

The proof of Theorem 4 actually establishes a bit more than the theorem asserts. The proof indicates that, for each  $\theta > \theta^0$ ,  $b(\theta) > B(\theta)$ ; that is, the high bid auction generates greater expected revenue for each possible value of  $\theta$ . From this observation we can draw strong conclusions if the seller is himself risk averse—with strictly concave utility function,  $v(\cdot)$ . In the second bid auction the seller's expected utility is

$$\begin{aligned} E \int_{\theta^0}^1 v(\tilde{B}(\theta)) dF^n(\theta) &< \int_{\theta^0}^1 v(B(\theta)) dF^n(\theta), \\ &\text{by Jensen's Inequality, since } B(\theta) = E\tilde{B}(\theta), \\ &< \int_{\theta^0}^1 v(b(\theta)) dF^n(\theta), \quad \text{since } B(\theta) < b(\theta), \\ &\text{from Theorem 4.} \end{aligned}$$

This last expression is just the seller's expected utility in the high bid auction. We have therefore proved the following theorem.

**THEOREM 5:** *Under the assumptions of Theorem 2 a risk averse seller strictly prefers the high bid to the second bid auction.*

**REMARK 5.1:** This result was established by Vickrey [25] for a uniform distribution of risk neutral buyers and by Matthews [14] for Case 1 preferences.

If buyers are risk averse, one might expect an auction where they are insured against losing to generate more revenue for the seller than the high bid auction.

After all, the seller could extract a premium for the insurance in such an auction. As we shall see, however, this is not normally the case because the insurance interferes with the seller's ability to screen. By a perfect insurance auction with reserve price  $b^0$  (see Riley and Samuelson [22] for a discussion of such auctions), we shall mean a triple  $(G, b, a)$ , where  $G$  satisfies (12),

$$b(\theta^0) = b^0,$$

$$(25) \quad x = \theta \text{ maximizes } G(x)u(-b(x), \theta) + (1 - G(x))w(-a(x)),$$

$$(26) \quad u_1(-b(\theta), \theta) = w_1(-a(\theta)), \quad \text{and}$$

$$(27) \quad G(\theta^0)u(-b(\theta^0), \theta^0) + (1 - G(\theta^0))w(-a(\theta^0)) = 0.$$

Here we have defined perfect insurance to entail equalization of *marginal* utilities across states, since that is what a rational risk bearer will attempt to do. For many utility functions, e.g., those of Case 1, equalizing marginal utilities is the same as equalizing the utilities themselves.

**THEOREM 6:** *For the preferences of Case 1, a perfect insurance auction with reserve price  $b^0$  generates the same expected revenue for the seller as the second bid auction with the same reserve price.*

**PROOF:** For Case 1 preferences, (26) implies

$$\theta - b(\theta) = -a(\theta).$$

Thus, the first order condition for the maximization, (25), is

$$-GU'(\theta - b)b' + (1 - G)U'(\theta - b)(1 - b') = 0.$$

Hence,

$$(28) \quad b' = (1 - G).$$

Also, for Case 1 preferences,

$$(29) \quad \sigma(\theta) = \theta.$$

From (28) and (29), revenue from both the second bid and perfect insurance auctions is independent of the utility function  $U$ . In particular, we may assume  $U(x) = x$ . But for such risk neutral preferences, we know (see Myerson [21, 22], Maskin and Riley [10]) that two auctions generate the same expected revenue if they share the same  $G$  function and if the most reluctant buyers ( $\theta = 0$ ) obtain a zero payoff. *Q.E.D.*

Theorem 6 implies in particular that, for the preferences of Case 1, the high bid auction generates strictly more revenue than the perfect insurance auction. That this proposition does not hold for general preferences, however, will become clear in Section 4 (see the discussion following Theorem 11).

3. CHARACTERIZATION OF THE SELLER'S OPTIMIZATION PROBLEM

We now consider all possible auction schemes and show that the choice of an optimal auction can be characterized as the solution to a control problem. This is summarized in Theorem 9 at the end of the section. Inferences about the properties of optimal auctions are then drawn in Section 4.

3a. *The Buyer's Problem*

Let us restrict attention for the time being to deterministic auctions, that is, auctions where the payment by a buyer with parameter  $\theta$  is the deterministic function  $b(\theta)$  if he wins and  $a(\theta)$  if he loses.

If truth-telling constitutes an equilibrium (which, as we have noted, we can assume without loss of generality), then we can express maximized expected utility as

$$(30) \quad V(\theta, \theta) = \max_x V(x, \theta)$$

where

$$(31) \quad V(x, \theta) \equiv G(x)u(-b(x), \theta) + (1 - G(x))w(-a(x)).$$

From (31)

$$(32) \quad V(x, \theta) - V(x, x) = G(x) \int_x^\theta u_2(-b(x), \alpha) d\alpha.$$

Because  $V(x, \theta) > V(x, x)$  (if  $\theta > x$ ) and (from (30))  $V(\theta, \theta) \geq V(x, \theta)$ ,

$$V(\theta, \theta) - V(x, x) > 0, \quad \theta > x.$$

Also, from (32)

$$\begin{aligned} V(\theta, \theta) - V(\theta, x) &= G(\theta) \int_x^\theta u_2(-b(\theta), \alpha) d\alpha \\ &\leq G(\theta)u_2(-b(\theta), x)(\theta - x), \end{aligned}$$

since  $u_2 > 0$  and  $u_{22} < 0$ . Then, since  $V(x, x) \geq V(\theta, x)$  and  $V(x, x)$  is increasing,

$$(33) \quad 0 \leq V(\theta, \theta) - V(x, x) \leq G(\theta)u_2(-b(\theta), x)(\theta - x), \quad \theta > x.$$

Therefore  $V(x, x)$  is also continuous.

From (30), for all  $x, \theta$

$$(34) \quad \theta \in \arg \min_x [V(x, x) - V(\theta, x)].$$

From (31)  $V(\theta, x)$  is a differentiable function of  $x$ . Moreover we have just argued that  $V(x, x)$  is continuous and increasing, hence differentiable almost everywhere. Then, almost everywhere we can write the first order condition for

(34) as

$$\frac{dV}{dx}(x, x) - V_2(\theta, x) = 0, \quad \text{at} \quad x = \theta.$$

From (31)  $V_2(\theta, \theta) = G(\theta)u_2(-b(\theta), \theta)$ . Then a necessary condition for truth-telling is

$$(35) \quad \frac{d}{d\theta} V(\theta, \theta) = G(\theta)u_2(-b(\theta), \theta) \quad \text{a.e.}$$

We next show that the first order condition (35) may be sufficient for truth-telling.

LEMMA 2: *Suppose that preferences satisfy Assumption A and  $u_{12} \leq 0$ . If  $G$  and  $b$  are nondecreasing and  $V$ , defined by (31), is continuous, then (35) implies that*

$$V(\theta, \theta) - V(x, \theta) \geq 0 \quad \text{for all } x \text{ and } \theta,$$

*i.e., truth-telling constitutes an equilibrium.*

PROOF: From (35) and the continuity of  $V$

$$(36) \quad V(\theta_2, \theta_2) - V(\theta_1, \theta_1) = \int_{\theta_1}^{\theta_2} G(\theta)u_2(-b(\theta), \theta) d\theta.$$

Notice that  $G(x)u_2(-b(x), \theta)$  is nondecreasing in  $x$  since  $G$  and  $b$  are both nondecreasing and  $u_{12} < 0$ . Thus,

$$(37) \quad \begin{aligned} V(\theta_2, \theta_2) - V(\theta_1, \theta_1) &\geq \int_{\theta_1}^{\theta_2} G(\theta_1)u_2(-b(\theta_1), \theta) d\theta \\ &= V(\theta_1, \theta_2) - V(\theta_1, \theta_1). \end{aligned}$$

Hence

$$V(\theta_2, \theta_2) \geq V(\theta_1, \theta_2). \quad \text{Q.E.D.}$$

### 3b. The Seller's Problem

Turning to the seller's problem, we see that the seller's expected revenue from each buyer is given by

$$(38) \quad \int [G(\theta)b(\theta) + (1 - G(\theta))a(\theta)] dF(\theta).$$

Thus the seller chooses  $G, b, a$  to maximize (38) subject to the buyers' incentive constraints (30), the nonparticipation option,

$$(39) \quad V(\theta, \theta) \geq 0,$$

and the constraint that  $G$  be derived via (7) from the symmetric probability functions  $H_1, \dots, H_n$  satisfying (1). Obviously, because  $G$  is itself the probability of winning it must satisfy  $0 \leq G \leq 1$ . For technical reasons we impose the tighter constraint

$$(40) \quad 0 \leq G \leq 1 - Z(\theta),$$

where  $Z(\theta) = \varepsilon(1 - \theta)$  and  $\varepsilon > 0$  is small. (We show below, however, that  $G \leq 1 - Z(\theta)$  is not binding at the optimum.) But (40) is not enough. The following theorem characterizes when symmetric  $H_i$ 's can be found, at least when  $G$  is nondecreasing.

**THEOREM 7:** *Suppose that  $G(s)$ , the probability of winning with parameter equal to  $s$ , is piecewise differentiable. If  $G(s)$  can be generated by a direct revelation auction, then, conditional on having a parameter value of at least  $y$ , the expected probability of winning never exceeds the expected probability that  $y$  is the highest parameter value. That is, a necessary condition for there to exist a permutation symmetric family (see footnote 11) of probability functions  $H_j(x)$ ,  $j = 1, \dots, n$ , satisfying  $\sum_j H_j \leq 1$ , such that*

$$(7) \quad G(\theta_i) = \int_{\theta_{-i}} H_i(\theta_i, \theta_{-i}) \prod_{j \neq i} dF(\theta_j) \quad \text{for all } \theta_i$$

is

$$(41) \quad \int_y^1 G(s) dF(s) \leq \int_y^1 F^{n-1}(s) dF(s), \quad 0 \leq y \leq 1.$$

Moreover, if  $G(s)$  is nondecreasing, (41) is sufficient for  $H_j$ 's to exist.

**PROOF:** The demonstration of sufficiency is complex and so the proof of the theorem is relegated to the Appendix.

**REMARK 7.1:** In the original version of this paper we were able to establish only that (41) is sufficient for (7) when  $G$  is a nondecreasing step function. The current proof dispenses with the step function requirement by appealing to a limiting argument due to Steven Matthews [16].

To solve the seller's problem we first consider the control problem of choosing  $G$ ,  $b$ , and  $a$ , and hence  $V$ , where

$$(42) \quad V = Gu + (1 - G)w,$$

to maximize (38) subject to (35), (39), (40), and (41).

To convert (41) to standard form we define

$$Y = \int_{\theta}^1 [G(x) - F^{n-1}(x)] dF(x).$$

Then (41) becomes the constraint

$$(43) \quad Y \leq 0,$$

where

$$(44) \quad \frac{dY}{d\theta} = (F^{n-1}(\theta) - G)F'(\theta).$$

After applying the maximum principle to obtain necessary conditions, we show that there is a solution  $\langle G^*, b^*, a^* \rangle$  that is continuous. Using this result we obtain a condition ensuring that  $G^*$  and  $b^*$  are everywhere nondecreasing. Thus, under this condition, Lemma 2 and Theorem 7 together imply that  $\langle G^*, b^*, a^* \rangle$  is a solution to the seller's (deterministic) problem. To be precise, we establish the following theorem.

**THEOREM 8:** *Suppose  $u$  and  $w$  satisfy Assumptions A and B and that for all  $y$  and  $\theta$  there exists  $x$  such that  $u_1(x, \theta) < w_1(y)$ . Then if the solution to the control problem of maximizing (38) subject to (35) and (39)–(42) satisfies*

$$(45) \quad 0 \leq j(\theta) \equiv 2 + \frac{F''}{(F')^2} \int_{\theta}^1 \frac{w_1(-a(\theta))}{w_1(-a(x))} F'(x) dx$$

*this solution corresponds to an optimum for the seller among all deterministic auctions.*

**REMARK 8.1:** Condition (45) requires that the density function,  $F'$ , not decline too rapidly with  $\theta$ . Indeed, observe that (45) is automatically satisfied if  $F'' \geq 0$ . Thus in particular, it is satisfied by the uniform distribution. In the limiting case of risk neutrality the condition becomes

$$0 \leq 2 + \frac{F''}{(F')^2} (1 - F) = \frac{d}{d\theta} \left( \theta - \frac{1 - F}{F'} \right).$$

From Myerson [21] and Maskin and Riley [10] we know that with risk neutrality (45) guarantees that a high bid or English auction with an appropriately chosen seller's reserve price is optimal. However, it will pay to take  $G' = 0$  income intervals when this inequality is violated. With risk aversion, one can show that a violation of (45) may cause the optimal  $G'$  to be negative in places. Since our methods rely heavily on establishing that  $G' \geq 0$  (see Lemma 2 and the Proof of Theorem 8) condition (45) is indispensable.

**REMARK 8.2:** In an earlier version of this paper we considered finite approximations of the control problem of Theorem 8. In such an approximation, the buyer's first order condition (35) becomes the pair of "adjacent" incentive constraints

$$(*) \quad G_i u(-b_i, \theta_i) + (1 - G_i) w(-a_i) \geq G_{i-1} u(-b_{i-1}, \theta_i) + (1 - G_{i-1}) w(-a_{i-1})$$

and

$$(**) \quad G_i u(-b_i, \theta_i) + (1 - G_i) w(-a_i) \geq G_{i+1} u(-b_{i+1}, \theta_i) + (1 - G_{i+1}) w(-a_{i+1})$$

where  $i$  indexes the possible values of  $\theta$ , and  $i > j$  if and only if  $\theta_i > \theta_j$ . However, we imposed only the “downward” constraints (\*) explicitly. By establishing the finite counterpart of Lemma 2, we showed that if (\*) holds with equality for all  $i$ , then, if  $G_i$  and  $b_i$  are nondecreasing in  $i$ , all incentive constraints are satisfied, i.e.,

$$G_i u(-b_i, \theta_i) + (1 - G_i) w(-a_i) \geq G_j u(-b_j, \theta_j) + (1 - G_j) w(-a_j)$$

for all  $i$  and  $j$ . Thus, we can legitimately consider (35) to be the analogue of the adjacent downward incentive constraints.

PROOF: The Hamiltonian for the control problem is

$$L = (Gb + (1 - G)a)F' + \lambda Gu_2(-b, \theta) + \mu(F^{n-1} - G)F' + \nu(Gu + (1 - G)w - V) + \alpha G + \beta(1 - Z - G) - \gamma Y + \phi V$$

where  $\lambda$  and  $\mu$  are the costate variables for (35) and (44),  $\nu$  is the Lagrange multiplier for (42) and  $\phi, \alpha, \beta, \gamma$  are the Lagrange multipliers for the inequality constraints (39), (40), and (43). Define

$$z = (G, a, b), \quad \kappa = (\lambda, \mu, \nu, \alpha, \beta, \gamma, \phi).$$

Then from the maximum principle, for all feasible  $z$

$$L(z; V^*(\theta), Y^*(\theta); \kappa^*(\theta), \theta) \leq L(z^*; V^*(\theta), Y^*(\theta); \kappa^*(\theta), \theta),$$

where the starred variables denote a solution to the control problem.

Writing down the first order conditions we obtain

$$(46) \quad \frac{\partial L}{\partial G} = (b^* - a^*)F' + \lambda^* u_2 - \mu^* F' + \nu^*(u - w) + \alpha^* - \beta^* = 0,$$

$$(47) \quad \frac{\partial L}{\partial a} = (1 - G^*)[F' - \nu^* w_1] = 0,$$

$$(48) \quad \frac{\partial L}{\partial b} = G^*[F' - \lambda^* u_{12} - \nu^* u_1] = 0.$$

From the maximum principle we know that the costate variable  $\lambda^*$  is continuous and piecewise differentiable and satisfies

$$(49) \quad (\lambda^*)' = \frac{-\partial L}{\partial V} = \nu^* - \phi^*.$$

From (35)  $V^*$  is nondecreasing everywhere and is increasing whenever  $G^* > 0$ . Moreover, with  $G^* = 0$   $V^*$  is nonnegative only if  $a^*$  is nonpositive. Therefore, to raise any revenue at all,  $G^*$  must be positive over some subinterval of  $[0, 1]$ . Because  $V^*$  is continuous, it follows that there exists  $\theta_0, 0 \leq \theta_0 < 1$  such that

$$(50) \quad \begin{array}{lll} V^* = 0 & \text{only if} & \theta \leq \theta_0, \\ G^* = 0 & \text{if} & \theta < \theta_0. \end{array}$$



From constraint (40)  $G^* < 1$  for all  $\theta < 1$ . Hence, from (47),  $\nu^* > 0$ , and so (49) and (50) imply that  $(\lambda^*)' > 0$  for  $\theta > \theta_0$ . From the maximum principle,

$$(51) \quad \lambda^*(1) = 0.$$

Therefore,

$$(52) \quad \lambda^*(\theta) < 0 \quad \text{for} \quad \theta < 1.$$

From (40) and (47), the bracketed expression in (47) is zero. Hence

$$(53) \quad \nu^* = \frac{F'}{w_1} > 0.$$

If  $G^* > 0$ , then the bracketed expression in (48) vanishes. Hence

$$(54) \quad \lambda^* = -\frac{F'(u_1 - w_1)}{u_{12}w_1}.$$

We will show that there exists a solution to the control problem that satisfies (46), (53), and (54) everywhere. First, choose  $b^{**}$  and  $\lambda^{**}$  to satisfy

$$F' - \lambda^{**}u_{12} - \nu^*u_1 = 0.^{16}$$

Then  $b^{**} = b^*$  whenever  $G^* > 0$ , and, since  $\langle G^*, b^*, a^* \rangle$  is a solution, so is  $\langle G^*, b^{**}, a^* \rangle$ . Define

$$H^{**}(\theta) \equiv (b^{**} - a^*)F' + \lambda^{**}u_2(-b^{**}, \theta) - \mu^*F' \\ + \nu^*(u(-b^{**}, \theta) - w(-a^*)) - \beta^*.$$

Also define the nonnegative function

$$\alpha^{**}(\theta) = \begin{cases} -H^{**}(\theta), & H^{**}(\theta) \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, with  $\alpha^*$  replaced by  $\alpha^{**}$  and  $b^*$  by  $b^{**}$ , (46)–(48) become

$$(46') \quad \frac{\partial L}{\partial G} \geq 0,$$

$$(47') \quad \frac{\partial L}{\partial a} = 0,$$

$$(48') \quad \frac{\partial L}{\partial b} = 0.$$

<sup>16</sup> To see that such a  $b^{**}$  and  $\lambda^{**}$  exist, notice that because  $u_1 > 0$  and  $u_{11} < 0$ , for any  $\theta$  we can choose  $b^-(\theta)$  sufficiently large that  $u(-b^-(\theta), \theta) < w(-a^*(\theta))$ . Then, Assumption B3 implies that  $u_1(-b^-(\theta), \theta) \geq w_1(-a^*(\theta))$  and so, from Assumption B1, (52), and (53)

$$F'(\theta) - \lambda^*u_{12}(-b^-(\theta), \theta) - \nu^*u_1(-b^-(\theta), \theta) < 0.$$

Choose  $b^+(\theta)$  big enough so that  $u_1(b^+(\theta), \theta) < w_1(-a^*(\theta))$  and  $\lambda^{**}$  close enough to zero so that

$$F'(\theta) - \lambda^*u_{12}(-b^+(\theta), \theta) - \nu^*u_1(-b^+(\theta), \theta) > 0.$$

By continuity  $b^{**}(\theta)$  exists.

By construction the altered program satisfies all the constraints and the corresponding complementary slackness conditions and generates the same expected revenue as  $\langle G^*, b^*, a^* \rangle$ . But if (46') holds strictly on some interval, (46')–(48') together imply that  $\langle G^*, b^{**}, a^* \rangle$  is not optimum. Then  $\langle G^*, b^*, a^* \rangle$  is also not a solution to the control problem, a contradiction. Therefore (46') must hold with equality everywhere, establishing our claim. We shall henceforth assume that  $\langle G^*, b^*, a^* \rangle$  satisfies (53) and (54) everywhere. We shall also drop the superscript  $*$ .

From (51), (52), and (54) it follows that

$$(55) \quad \begin{cases} u_1 - w_1 < 0, & \theta < 1, \\ u_1 - w_1 = 0, & \theta = 1. \end{cases}$$

Then, by Assumption B3 we may conclude that

$$(56) \quad u - w > 0, \quad \theta < 1.$$

From the maximum principle the costate variable  $\mu(\theta)$  is piecewise differentiable and satisfies

$$\mu'(\theta) = -\frac{\partial L}{\partial Y} = \gamma,$$

plus the endpoint condition  $\mu(0) \geq 0$ . Furthermore, one can show that  $\mu$  is continuous.<sup>17</sup> Since the Lagrange multiplier,  $\gamma$ , is everywhere nonnegative, we therefore deduce

$$(57) \quad \mu(\theta) \geq 0, \quad \text{for all } \theta.$$

Next, substituting (53) in (46) we obtain

$$(58) \quad b - a + \frac{\lambda}{F'} u_2 + \frac{u - w}{w_1} + \frac{\alpha}{F'} - \left( \frac{\beta}{F'} + \mu \right) = 0.$$

We now show that  $G(\theta)$ ,  $a(\theta)$ , and  $b(\theta)$  are continuous. Suppose to the contrary that  $a(\theta)$  has an upward discontinuity from the right at  $\theta^*$ . (The argument is virtually the same if the upward discontinuity is from the left.)

Then

$$a^* \equiv \lim_{\Delta \downarrow 0} a^*(\theta^* + \Delta) > a(\theta^*).$$

Similarly, define

$$b^* \equiv \lim_{\Delta \downarrow 0} b(\theta^* + \Delta) \quad \text{and} \quad G^* \equiv \lim_{\Delta \downarrow 0} G(\theta^* + \Delta).$$

<sup>17</sup> We are indebted to Steven Matthews for pointing out to us that the continuity of  $\mu$  does not follow directly from the maximum principle, since  $Y$  is a bounded state variable. However, it *does* follow from the word-for-word translation of an argument that he provides in Matthews [15, p. A11].

From (54),

$$(59) \quad -\frac{\lambda(\theta^*)}{F'(\theta^*)} = \frac{u_1(-b(\theta^*), \theta^*) - w_1(-a(\theta^*))}{u_{12}(-b(\theta^*), \theta^*) w_1(-a(\theta^*))} = \frac{u_1(-b^*, \theta^*) - w_1(-a^*)}{u_{12}(-b^*, \theta^*) w_1(-a^*)}.$$

For each  $x \in [a(\theta^*), a^*]$ , choose  $\hat{b}(x)$  so that  $\hat{b}(a(\theta^*)) = b(\theta^*)$ ,  $\hat{b}(a^*) = b^*$  and

$$(60) \quad -\frac{\lambda(\theta^*)}{F'(\theta^*)} = \frac{u_1(-\hat{b}(x), \theta^*) - w_1(-x)}{u_{12}(-\hat{b}(x), \theta^*) w_1(-x)}.$$

Such a choice is possible from (59) and Assumption B. Consider the expression

$$(61) \quad \begin{aligned} \Omega(x) = & \hat{b}(x) - x + \frac{\lambda(\theta^*)}{F'(\theta^*)} u_2(-\hat{b}(x), \theta^*) \\ & + \frac{1}{w_1(-x)} (u(-\hat{b}(x), \theta^*) - w(-x)). \end{aligned}$$

Differentiating by  $x$  we obtain

$$(62) \quad \begin{aligned} \Omega'(x) = & \hat{b}'(x) - 1 - \frac{\lambda(\theta^*)}{F'(\theta^*)} u_{12}(-\hat{b}(x), \theta^*) \hat{b}'(x) \\ & + \frac{1}{w_1(-x)} \left( -u_1(-\hat{b}(x), \theta^*) \hat{b}'(x) + w_1(-x) \right) \\ & + \frac{(u(-\hat{b}(x), \theta^*) - w(-x)) w_{11}(-x)}{w_1(-x)^2} \\ = & \frac{(u(-\hat{b}(x), \theta^*) - w(-x)) w_{11}(-x)}{w_1(-x)^2} \\ & + \hat{b}'(x) \left[ 1 - \frac{u_1(-\hat{b}(x), \theta^*)}{w_1(-x)} - \frac{\lambda(\theta^*)}{F'(\theta^*)} u_{12}(-\hat{b}(x), \theta^*) \right]. \end{aligned}$$

From (60) the term in brackets is zero. From (56) and the concavity of  $w$  the other term is negative.

Thus  $\Omega(x)$  is decreasing in  $x$  and so (61) is larger when  $x = a(\theta^*)$  than when  $x = a^*$ . From (58), at  $x = a(\theta^*)$  and  $x = a^*$

$$b - a + \frac{\lambda}{F'} u_2 + \frac{u - w}{w_1} = \mu + \frac{\beta - \alpha}{F'}.$$

Since we have just proved that the left-hand side is larger at  $x = a(\theta^*)$  than at  $x = a^* = \lim_{\theta \downarrow \theta^*} a(\theta)$ , and since  $F'$  and  $\mu$  are continuous it therefore follows that  $\beta(\theta^*) - \alpha(\theta^*) > \beta^* - \alpha^*$ . The Lagrange multipliers,  $\alpha$  and  $\beta$  are everywhere nonnegative. Hence either  $\beta(\theta^*) > 0$  implying  $G(\theta^*) = 1$  or  $\alpha^* > 0$  implying  $G^* = 0$ . Since  $0 \leq G \leq 1$  we therefore obtain

$$(63) \quad G(\theta^*) \geq G^* = 0.$$

From (59)

$$(64) \quad -\lambda(\theta^*)(u_{12}(-b^*, \theta^*) - u_{12}(-b(\theta^*), \theta^*)) \\ = F'(\theta^*) \left( \frac{u_1(-b^*, \theta^*)}{w_1(-a^*)} - \frac{u_1(-b(\theta^*), \theta^*)}{w_1(-a(\theta^*))} \right).$$

If  $b^* \leq b(\theta^*)$ , then the left-hand side of (64) is nonnegative by Assumption B5 and  $\lambda < 0$ . But the right-hand side is negative by Assumptions A2 and A4. Hence  $b^* > b(\theta^*)$ .

Since  $V(\theta)$  is continuous, from (42) we have

$$G^*u(-b^*, \theta^*) + (1 - G^*)w(-a^*) \\ = G(\theta^*)u(-b(\theta^*), \theta^*) + (1 - G(\theta^*))w(-a(\theta^*)).$$

Appealing to the mean value theorem we then obtain

$$(65) \quad [u(-b(\theta^*), \theta^*) - w(-a(\theta^*))](G^* - G(\theta^*)) - G^*u_1(-\hat{b}, \theta^*) \\ \times (b^* - b(\theta^*)) - (1 - G^*)w_1(-\hat{a})(a^* - a(\theta^*)) = 0$$

where  $\hat{b} \in [b(\theta^*), b^*]$  and  $\hat{a} \in [a(\theta^*), a^*]$ . Because  $b^* - b(\theta^*)$  and  $a^* - a(\theta^*)$  are positive, (56) and (65) imply  $G^* > G(\theta^*)$ . But this contradicts (63). Hence  $a(\theta)$  cannot have an upward discontinuity. From virtually the same argument it can be established that  $a(\theta)$  cannot have a downward discontinuity. From (64), if  $a(\theta)$  is continuous at  $\theta^*$  then  $b(\theta)$  must be continuous there. Finally, from (65)  $G$  is also continuous.

We now show that for all  $\theta$  such that  $0 < G(\theta) < 1$ ,  $G'$  and  $b'$  are positive.<sup>18</sup> Taking the logarithmic derivative of (54) we obtain

$$(66) \quad \frac{\lambda'}{\lambda} - \frac{F''}{F'} = \frac{u_{12}}{u_1 - w_1} - \frac{u_{122}}{u_{12}} - b' \left[ \frac{u_{11}}{u_1 - w_1} - \frac{u_{112}}{u_{12}} \right] + a' \left[ \frac{w_{11}}{u_1 - w_1} + \frac{w_{11}}{w_1} \right].$$

Also, from (49), (53), and (54),

$$\lambda' = \frac{F'}{w_1} \quad \text{and} \quad \frac{\lambda'}{\lambda} = \frac{-u_{12}}{u_1 - w_1}.$$

Making use of these conditions we can rewrite (66) as

$$(67) \quad b' \left[ \frac{u_{11}}{u_1 - w_1} - \frac{u_{112}}{u_{12}} \right] - a' \left[ \frac{w_{11}u_1}{w_1(u_1 - w_1)} \right] \\ = \left[ \frac{u_{12}}{u_1 - w_1} \right] \left\{ 2 - \lambda w_1 \frac{F''}{(F')^2} \right\} + \left[ \frac{-u_{122}}{u_{12}} \right].$$

<sup>18</sup> While the proof assumes differentiability an almost identical argument can be used to show that  $G$  and  $b$  are increasing at points of nondifferentiability as well.

From (49), (51), and (53),

$$(68) \quad \lambda(\theta) = - \int_{\theta}^1 \frac{F'(x)}{w_1(-a(x))} dx, \quad \text{for all } \theta > \theta_0.$$

Substituting (68) in (67) we have finally

$$(69) \quad b' \left[ \frac{u_{11}}{u_1 - w_1} - \frac{u_{112}}{u_{12}} \right] - a' \left[ \frac{w_{11}u_1}{w_1(u_1 - w_1)} \right] = \left[ \frac{u_{12}}{u_1 - w_1} \right] j(\theta) + \left[ \frac{-u_{112}}{u_{12}} \right],$$

where  $j(\theta)$  is defined by (45).

From the hypotheses of the theorem and (56) and (57), each of the bracketed expressions in (69) is positive. Then if  $a' > 0$ ,  $b' > 0$ .

Next, differentiating (42) by  $\theta$  and making use of (35) we obtain

$$(70) \quad G'(u - w) - b'Gu_1 - a'(1 - G)w_1 = 0.$$

Thus if  $a'$  and  $b'$  are positive so is  $G'$ .

We now show that if the integral constraint, (41), is not binding  $a'$  is necessarily positive. With  $0 < G < 1$  we can differentiate (58) by  $\theta$  to obtain

$$\begin{aligned} b' - a' + \left( \frac{\lambda'}{F'} - \frac{\lambda F''}{(F')^2} \right) u_2 + \frac{\lambda}{F'} (-u_{12}b' + u_{22}) + \frac{u_2}{w_1} + (u - w) \frac{w_{11}}{(w_1)^2} a' \\ + \frac{1}{w_1} (-u_1b' + w_1a') = 0. \end{aligned}$$

Collecting terms and making use of (54) we can rewrite this as

$$\frac{(w_1 - u_1)u_{22}}{w_1u_{12}} + (u - w) \frac{w_{11}}{(w_1)^2} a' + \frac{u_2}{w_1} \left( 1 + \frac{\lambda'w_1}{F'} \right) - \frac{\lambda u_2 F''}{(F')^2} = 0.$$

Making use of (49), (53), and (54) and then multiplying by  $w_1/u_2$  we obtain

$$(71) \quad \left[ \frac{(w - u)w_{11}}{u_2w_1} \right] a' - \left[ \frac{(w_1 - u_1)u_{22}}{u_2u_{12}} \right] = \left\{ 2 - \frac{\lambda w_1 F''}{(F')^2} \right\} = j(\theta).$$

From (56), (57), and Assumptions A and B all the bracketed terms in (71) are positive. Then  $a' > 0$ .

It remains to show that if  $a' < 0$  and the integral constraint is binding both  $G'$  and  $b'$  are positive. But if (41) is binding then locally  $G = F^{n-1}$  and so  $G' > 0$ . Then, from (70), if  $a' < 0$ ,  $b' > 0$ .

Recall that in the statement of the control problem of Theorem 8 we imposed the tighter constraint (40). Note, however, that we can choose  $\varepsilon > 0$  so small that if  $G$  is continuous and nondecreasing and satisfies (41), the constraint  $G(\theta) \leq 1 - \varepsilon(1 - \theta)$  is never binding. Therefore, the solution to the control problem is an optimal deterministic auction. Q.E.D.

We mentioned in Remark 8.1 that the density condition (45) is crucial to the conclusion that  $G$  is nondecreasing in the optimal auction. It is also essential to

ensuring that only the local downward constraint (35) is binding among all the incentive constraints. Violations of (45) can lead to other constraints being binding. Moore [19] drops condition (45) (but strengthens Assumption B). He explicitly introduces *all* the downward constraints (not just (35)) into the control problem and shows that a solution to the revised control problem automatically satisfies all the upper constraints. He then derives many of the same qualitative properties of optimal auctions that we do (excluding, of course,  $G' \geq 0$ ).

So far in this section, we have confined our attention to deterministic auctions—ones where  $\tilde{b}$  and  $\tilde{a}$  are deterministic. That this restriction is justified, assuming the hypotheses of Theorem 8, is confirmed by the following result:

**THEOREM 9:** *Under the hypotheses of Theorem 8, the optimal auction is deterministic.*

**PROOF:** Suppose that  $\langle G, \tilde{b}, \tilde{a} \rangle$  is an optimal (possibly random) auction. By analogy with the proof of Theorem 8 we define

$$V^*(x, \theta) = E\{G(x)u(-\tilde{b}(x), \theta) + (1 - G(x))w(-\tilde{a}(x))\}.$$

Then  $\langle G, \tilde{b}, \tilde{a} \rangle$  must satisfy the first order condition

$$(72) \quad \frac{dV^*}{d\theta}(\theta, \theta) = G(\theta)Eu_2(-\tilde{b}, \theta).$$

We next show that there exists an alternative *deterministic*  $\langle G, \bar{b}, a_0 \rangle$  that satisfies the local condition (72) (which is the same as (35)) and generates more revenue. From Theorem 8 we know that the deterministic revenue-maximizing  $\langle G^*, b^*, a^* \rangle$  generates at least as much revenue as  $\langle G, \bar{b}, a_0 \rangle$ , since Theorem 8 employs only condition (72) (not the global condition (30)). Therefore  $\langle G^*, b^*, a^* \rangle$  generates greater revenue than  $\langle G, \tilde{b}, \tilde{a} \rangle$ .

Suppose  $\tilde{a}(\theta)$  is random. Then define  $a^{**}(\theta)$  such that  $w(-a^{**}(\theta)) = Ew(-\tilde{a}(\theta))$ . It follows that  $V^*(\theta, \theta)$  continues to satisfy (72) if  $a^{**}$  replaces  $\tilde{a}$ . Also, since  $w(-a)$  is a concave function of  $a$ ,

$$Ew(-\tilde{a}(\theta)) < w(-\bar{a}(\theta)), \quad \text{where} \quad \bar{a}(\theta) = E\{\tilde{a}(\theta)\}.$$

Then, since  $w$  is increasing,

$$-a^{**}(\theta) < -\bar{a}(\theta) \rightarrow \bar{a}(\theta) < a^{**}(\theta).$$

Thus the seller's expected revenue is at least as great if he replaces  $\tilde{a}(\theta)$  by  $a^{**}(\theta)$ .

Next, suppose  $\tilde{b}(\theta)$  is random. Defining  $\bar{b}(\theta) = E\tilde{b}(\theta)$  we note that, since  $u$  is concave in its first argument

$$G(\theta)u(-\bar{b}(\theta), \theta) + (1 - G(\theta))w(-a(\theta)) \geq V^*(\theta, \theta) \quad \text{for all } \theta.$$

Moreover, since  $u_2$  is convex in its first argument,

$$Gu_2(-\bar{b}(\theta), \theta) \leq GEu_2(-\tilde{b}(\theta), \theta) = \frac{dV^*}{d\theta}.$$

The functions  $a_0(\theta)$  and  $V_0(\theta) = G(\theta)u(-\bar{b}(\theta), \theta) + (1 - G(\theta))w(-a_0(\theta))$  are defined implicitly by the differential equation  $dV_0/d\theta = Gu_2(-\bar{b}(\theta), \theta)$  and the boundary condition  $V_0(0) = V^*(0, 0)$ . From the above inequality we therefore have

$$G(\theta)u(-\bar{b}(\theta), \theta) + (1 - G(\theta))w(-a(\theta)) \geq V_0(\theta)$$

and so  $a(\theta) \leq a_0(\theta)$ . Then  $\langle G, \bar{b}, a_0 \rangle$  satisfies (72) and generates at least as much revenue as the random scheme. Q.E.D.

A restriction like  $u_{112} \geq 0$  on the rate at which absolute risk aversion can increase with  $\theta$  is essential for the conclusion that the optimal auction is deterministic. To see that randomization may pay if  $u_{112}$  is negative suppose that  $\theta$  can take on two values  $\theta_1$  and  $\theta_2(\theta_2 > \theta_1)$  where

$$\begin{aligned} u(-t, \theta_1) &= \theta_1 - t, \\ u(-t, \theta_2) &= \log(1 + \theta_2 - t), \end{aligned}$$

and

$$w(-t) = -t.$$

Since  $u$  is risk neutral for  $\theta = \theta_1$  and risk averse for  $\theta = \theta_2$ , risk aversion is increasing with  $\theta$ . Suppose that there is just one buyer. Consider a scheme in which the seller offers to sell the item (with probability one) if the buyer accepts either of the following two payments schedules:

$$\tilde{b}_1 = \begin{cases} 0, & \text{with probability } 1 - \frac{\theta_1}{1 + \theta_2}, \\ 1 + \theta_2, & \text{with probability } \frac{\theta_1}{1 + \theta_2}, \end{cases}$$

$$b_2 = \theta_2.$$

It is readily confirmed that if  $\theta = \theta_1$ , the buyer opts for  $\tilde{b}_1$  and that if  $\theta = \theta_2$  the buyer prefers  $b_2$ . Moreover, given these choices, the scheme extracts all buyer surplus. Thus the scheme is certainly optimal. Furthermore, it is evident that no scheme where  $\tilde{b}_1$  is deterministic can extract all surplus. Hence randomization is essential.

Theorem 9 establishes that the first order conditions (46)–(48) are necessary for a maximum. They need not be sufficient, however, because, although the objective function is concave and the constraints (39)–(41) are convex, the incentive constraint, (35), is nonconvex. Indeed, without that nonconvexity, establishing that the optimal  $\tilde{a}$  and  $\tilde{b}$  are deterministic would be trivial and would not require any assumptions about how risk aversion changes in  $\theta$  (only that the

buyer actually be risk averse); we could simply replace  $\tilde{a}$  and  $\tilde{b}$  by their certainty equivalents.

#### 4. PROPERTIES OF OPTIMAL AUCTIONS

The proof of Theorem 8, in addition to demonstrating that designing an optimal auction reduces to a conceptually simple and standard control problem, establishes and suggests certain interesting properties of optimal auctions. We now present some of these properties explicitly.

**THEOREM 10:** *Under the hypotheses of Theorem 8, the probability of winning and the amount a buyer pays if he wins in an optimal auction are increasing functions of his eagerness to buy, if the probability of his winning is positive. That is,  $b' > 0$  and  $G' > 0$  if the constraint  $G \geq 0$  is not binding.*

**PROOF:** Established in the proof of Theorem 8.

**THEOREM 11:** *Under the hypotheses of Theorem 8, the marginal utility of income in an optimal auction is lower when a buyer wins than when he loses. That is,  $u_1(-b(\theta), \theta) < w_1(-a(\theta))$ , if  $\theta^0 \leq \theta < 1$ , where  $\theta^0 = \inf \{\theta | G(\theta) > 0\}$ .*

**PROOF:** Established in the proof of Theorem 8 (see condition (55)).

Theorem 11 establishes that, under the hypotheses of Theorem 8, it is desirable for the seller to make all buyers, except the most conceivably eager ( $\theta = 1$ ) and those who have no chance of winning ( $G = 0$ ), bear risk in order to exploit this risk for screening. The result that an optimal incentive scheme introduces "inefficiency" for all values of the unknown parameter  $\theta$  but one, is a very general principle in the incentives literature. In the optimal income tax literature (see Mirrlees [18]), for example, it implies that all but the very ablest agent should face a positive marginal tax rate. The main interest of Theorem 11, therefore, is its description of the *nature* of the inefficiency, namely, that  $u_1 < w_1$ . The direction of the inequality  $u_1 < w_1$  is due to the Hypothesis B1, i.e.,  $u_{12} < 0$ . If B1 holds and  $u_1(-b(\theta), \theta) < w_1(-a(\theta))$  for given  $\theta$ , then the difference between  $u_1(-b(\theta), \hat{\theta})$  and  $w_1(-a(\hat{\theta}))$  is greater for  $\hat{\theta} > \theta$  than for  $\hat{\theta} = \theta$ . In other words, by having a  $\theta$ -buyer bear risk, the seller can relax the incentive constraint (35) by making  $(G(\theta), b(\theta), a(\theta))$  appear still riskier for buyers with parameters greater than  $\theta$ .

From this reasoning, it is evident that when  $u_{12} = 0$ , i.e., when preferences take the form

$$u(-t, \theta) = \theta - v(t),$$

$$w(-t) = -v(t),$$



there is no value to buyers bearing risk. It is easy to show that for such preferences, the optimal auction entails full insurance.<sup>19</sup> This is to be contrasted with Theorem 6 which demonstrates that, for Case 1 preferences, a perfect insurance auction is not only suboptimal but inferior to the high bid auction.

**COROLLARY:** *Under the hypotheses of Theorem 8, a buyer is strictly better off in an optimal auction when he wins than when he loses. That is,  $u(-b(\theta), \theta) > w(-a(\theta))$  for  $\theta^0 \leq \theta < 1$ , where  $\theta^0 = \inf \{\theta | G(\theta) > 0\}$ .*

**PROOF:** Follows directly from Theorem 11 and B3.

*Q.E.D.*

In contrast with Theorem 11, the next result shows that the most eager buyer possible ( $\theta = 1$ ) should be perfectly insured.

**THEOREM 12:** *Under the hypotheses of Theorem 8, the most conceivably eager buyer is perfectly insured against losing in an optimal auction. That is,*

$$u_1(-b(1), 1) = w_1(-a(1)).$$

**REMARK 12.1:** For Case 1 preferences with constant absolute risk aversion, this result is established by Matthews [15].

**PROOF:** Established in the proof of Theorem (8) (see condition (55)).

Theorem 12 is in general false when  $n = 1$ , as Matthews [15] illustrates with Case 1 preferences and constant absolute risk aversion. Intuitively, a high bidder in a multi-buyer auction must be insured against losing because there may always be a higher bidder. But in a one-buyer auction, a sufficiently high bidder will have a probability one chance of winning (see Theorem 17).

Next we consider the behavior of  $a$ , the fee a buyer pays if he loses. We observe that for low values of  $\theta$  where  $G$  is positive,  $a$  is positive and increasing, whereas  $a$  is negative for high  $\theta$ 's. Since, from Theorem 10,  $b$ , the buyer's "bid", is increasing in  $\theta$ , we conclude that if a buyer bids low, he is penalized for losing in an optimal auction but is compensated for losing if he bids high.

<sup>19</sup> If a buyer with parameter value  $\theta$  chooses  $x$  his expected utility from the auction  $\langle G, b, a \rangle$  is

$$E(x, \theta) = G(x)[\theta - v(b(x))] - (1 - G(x))v(a(x)).$$

Since  $v$  is strictly convex there exists  $\delta(x) \geq 0$  with strict inequality whenever  $a(x) \neq b(x)$  and  $0 < G(x) < 1$  such that

$$E(x, \theta) = G(x)[\theta - v(c(x) + \delta(x))] - (1 - G(x))v(c(x) + \delta(x))$$

where

$$c(x) = G(x)b(x) + (1 - G(x))a(x).$$

Thus expected revenue can be increased whenever  $b \neq a$ .

**THEOREM 13:** *Let  $(G, b, a)$  be an optimal auction. Let  $\theta^0$  be the infimum of all  $\theta$ 's such that the constraint (39),  $G \geq 0$ , is not binding. Then under the hypotheses of Theorem 8  $a(\theta) > 0$  and  $a'(\theta) > 0$  for  $\theta (> \theta^0)$  sufficiently close to  $\theta^0$ . If, in addition,  $u_1 = w_1$  implies  $u = w$ , then for  $\theta$  sufficiently close to 1,  $a(\theta) < 0$ .*

**REMARK 13.1:** The first but not the second assertion of Theorem 13 holds for one-buyer auctions. The hypothesis “ $u_1 = w_1$  implies  $u = w$ ” clearly holds for Case 1 preferences and, under constant absolute risk aversion, for those of Case 3. (See (5) and the subsequent argument. The inequality in Lemma 1 holds with equality under constant absolute risk aversion.)

**REMARK 13.2:** One simple way of instituting a positive  $a$ —so that losers as well as winners pay—is to introduce a nonrefundable entry fee. For more on the desirability of entry fees, see Maskin and Riley [10].

**PROOF:** We first observe that the nonparticipation constraint (39) must be binding at  $\theta = \theta^0$ ; otherwise, we could increase  $a$  without altering  $G$  and  $b$  and augment the seller's expected revenue.

First suppose that  $G(\theta^0) = 0$ . Then (39) implies that

$$(73) \quad a(\theta^0) = 0$$

since  $w(0) = 0$ . If the integral constraint (41) is not binding at  $\theta^0$ , then, from the argument in the proof of Theorem 8,  $a'(\theta) > 0$  for all  $\theta (> \theta^0)$  sufficiently close to  $\theta^0$ . In view of (73), this implies that  $a(\theta) > 0$  for all  $\theta (> \theta^0)$  sufficiently close to  $\theta^0$ . If (41) is binding at  $\theta^0$ , then locally  $G = F^{n-1}$ , and so  $G'(\theta^0) > 0$ . From (35)

$$\frac{dV}{d\theta} \equiv G'(u - w) - G(u_1 b' - w_1 a') - w_1 a' + Gu_2 = Gu_2.$$

Then, at  $\theta_0$ , with  $G(\theta_0) = 0$ ,

$$(74) \quad a' = \frac{(u - w)G'}{w_1}.$$

Now from the Corollary to Theorem 11,  $u > w$ . Therefore, (73) and (74) imply that  $a(\theta)$  and  $a'(\theta)$  are positive for  $\theta (> \theta^0)$  close to  $\theta^0$ .

Next suppose that  $G(\theta^0) > 0$ . Then  $\theta^0 = 0$ , from the definition of  $\theta^0$ . Because, as already was observed,  $u > w$  at  $\theta = \theta^0$ , the equality of the nonparticipation constraint implies that  $w(-a(\theta^0)) < 0$ , and so  $a(\theta^0) > 0$ . Because (41) cannot be binding at  $\theta^0$ ,  $a'(\theta^0) > 0$ .

From Theorem 12,  $u_1(-b(1), 1) = w_1(-a(1))$ . If  $u_1 = w_1$  implies  $u = w$ , then  $w(-a(1)) > 0$ . Therefore  $a(1) < 0$ , and so  $a(\theta) < 0$  for  $\theta$  near 1. Q.E.D.

**THEOREM 14:** *Suppose that condition (45) is satisfied. If the preferences of Case 1 exhibit nonincreasing absolute risk aversion, a buyer pays at least as much if he*

wins as if he loses in an optimal auction. That is,  $b \geq a$ , and the inequality is strict if risk aversion is strictly decreasing.

PROOF: If we substitute for  $\lambda$  using (54), (58) becomes

$$(75) \quad b - a = \left( \frac{u_1 - w_1}{u_{12} w_1} \right) u_2 - \left( \frac{u - w}{w_1} \right) + \mu + \frac{\beta}{F'}$$

if  $G > 0$ . Because preferences take the Case 1 form, the first two terms on the right-hand side of (75) can be written as

$$(76) \quad \left[ \frac{U'(\theta - b) - U'(-a)}{U''(\theta - b)U'(-a)} \right] U'(\theta - b) - \left[ \frac{U(\theta - b) - U(-a)}{U'(-a)} \right].$$

Rearranged, (76) becomes

$$(77) \quad \frac{U'(\theta - b)(U(\theta - b) - U(-a))}{U''(\theta - b)U'(-a)} \left[ \frac{U'(\theta - b) - U'(-a)}{U(\theta - b) - U(-a)} - \frac{U''(\theta - b)}{U'(\theta - b)} \right].$$

We will show that the bracketed factor in (77) is negative for decreasing absolute risk aversion, implying that (76) is positive. From the Corollary to Theorem 11,  $U(\theta - b) > U(-a)$ , implying that  $\theta - b(\theta) > -a(\theta)$ . Thus we can show that the bracketed factor is negative by establishing that

$$(78) \quad \frac{U'(x_2) - U'(x_1)}{U(x_2) - U(x_1)} - \frac{U''(x_2)}{U'(x_2)} < 0$$

when  $x_2 > x_1$ . Since  $v = U(x)$  is increasing, define  $x = U^{-1}(v)$ ,  $x_1 = U^{-1}(v_1)$ , and  $x_2 = U^{-1}(v_2)$ . Also take

$$g(v) = U'(U^{-1}(v)).$$

Arguing exactly as in the proof of Lemma 1, we know that

$$g'(v) = \frac{U''(x)}{U'(x)}$$

and that  $g''$  is positive if absolute risk aversion is decreasing. But if  $g'' > 0$ , then

$$\frac{g(v_2) - g(v_1)}{v_2 - v_1} < g'(v_2),$$

and so (78) holds. Thus (76) is positive. From (57)  $\mu$  is nonnegative. Moreover the Lagrange multiplier  $\beta$  is nonnegative. Then (75) implies that  $b > a$ . Q.E.D.

COROLLARY: Under the hypotheses of Theorem 14,  $b > 0$  if  $G > 0$ .

PROOF: From Theorem 13,  $a(\theta^0) \geq 0$ , where  $\theta^0$  is the infimum of all  $\theta$  for which  $G(\theta) \geq 0$  is not binding. From Theorem 14,  $b(\theta^0) \geq 0$ . Thus, from Theorem 10,  $b(\theta) > 0$  for all  $\theta > \theta^0$ . Q.E.D.

We next study the expected revenue generated from a given buyer.

**THEOREM 15:** *Under the hypotheses of Theorem 8, the expected revenue from a given buyer is an increasing function of his willingness to buy. That is,  $R(\theta) = G(\theta)b(\theta) + (1 - G(\theta))a(\theta)$  is an increasing function of  $\theta$ .*

**PROOF:** From the proof of Theorem 8,  $\langle G, b, a \rangle$  is continuous in  $\theta$  and hence  $R(\theta)$  is continuous. Then, if the theorem is false there is some interval  $[\theta_1, \theta_2] \subset [\theta_0, 1]$  over which  $R(\theta)$  is nonincreasing. Define

$$\hat{G}(\theta) = \begin{cases} G(\theta_1), & \theta \in [\theta_1, \theta_2], \\ G(\theta), & \text{otherwise,} \end{cases}$$

$$\hat{b}(\theta) = \begin{cases} b(\theta_1), & \theta \in [\theta_1, \theta_2], \\ b(\theta), & \text{otherwise.} \end{cases}$$

Also, we can implicitly define the functions  $\hat{a}(\theta)$  and  $\hat{V}(\theta) = \hat{G}(\theta)U(-\hat{b}(\theta), \theta) + (1 - \hat{G}(\theta))w(-\hat{a}(\theta))$  by the differential equation

$$\frac{d\hat{V}}{d\theta} = \hat{G}(\theta)u_2(-\hat{b}(\theta), \theta)$$

and the boundary condition  $\hat{V}(0) = V(0)$ .

Since  $u_2 > 0$  and  $u_{12} < 0$ ,  $G u_2(-b, \theta)$  is an increasing function of  $G$  and  $b$ . But, from the proof of Theorem 8,  $G(\theta)$  and  $b(\theta)$  are strictly increasing. Then for  $\theta \in (\theta_1, \theta_2]$ ,

$$\frac{dV}{d\theta} = G(\theta)u_2(-b(\theta), \theta) > \hat{G}(\theta)u_2(-\hat{b}(\theta), \theta) = \frac{d\hat{V}}{d\theta}.$$

From the definition of  $\hat{V}$ ,  $\hat{V}(\theta) = V(\theta)$ ,  $\theta \in [0, \theta_1]$ . Therefore, for  $\theta \in (\theta_1, \theta_2]$ ,  $\hat{V}(\theta) < V(\theta)$ .

Also, since

$$\frac{d\hat{V}}{d\theta} = \hat{G}u_2(-\hat{b}, \theta),$$

$$\hat{V}(\theta) = \hat{V}(\theta_1) = G(\theta_1)\{u(-b(\theta_1), \theta) - u(-b(\theta_1), \theta_1)\}, \quad \theta \in [\theta_1, \theta_2].$$

Therefore

$$\hat{V}(\theta) - G(\theta_1)u(-b(\theta_1), \theta) + (1 - G(\theta_1))w(-a(\theta_1)), \quad \theta \in [\theta_1, \theta_2].$$

Hence  $\hat{a}(\theta) = a(\theta_1)$  for  $\theta \in [\theta_1, \theta_2]$  and so

$$\hat{R}(\theta) = R(\theta_1); \quad \theta \in [\theta_1, \theta_2].$$

Thus

$$\hat{R}(\theta) \geq R(\theta), \quad \theta \in [0, \theta_2].$$

For  $\theta > \theta_2$ ,  $dV/d\theta = d\hat{V}/d\theta$ . Then, since  $\hat{V}(\theta) < V(\theta)$  at  $\theta = \theta_2$  this inequality holds for all  $\theta \in (\theta_2, 1]$  as well. Moreover, since  $\hat{G} = G$  and  $\hat{b} = b$  over this interval, we must have  $\hat{a} > a$ .

Since  $\hat{G} \leq G$ , the integral constraint (41) is satisfied and, by construction,  $(\hat{G}, \hat{b}, \hat{a})$  satisfies the local self-selection condition (35). Moreover, since  $\hat{G}$  and  $\hat{b}$  are nondecreasing functions, it follows from Lemma 2 that (35) is also sufficient.

Then  $(\hat{G}, \hat{b}, \hat{a})$  is feasible for the seller. But, by construction  $\hat{R}(\theta) \geq R(\theta)$ ,  $\theta \leq \theta_2$ . Finally, over  $[\theta_2, 1)$ , since  $\hat{G} = G$ ,  $\hat{b} = b$  and  $\hat{a} > a$ ,  $\hat{R}(\theta) > R(\theta)$ . But this contradicts the hypothesis that  $(G, b, a)$  maximizes expected revenue. *Q.E.D.*

Theorem 15 has analogues in many other "monopoly" problems. In the optimal tax literature, for example, its counterpart is the property that taxes should be increasing in individuals' skill (see, e.g., Mirrlees [18]). Theorem 15 is less obvious than many of these counterparts, however, because of the feasibility constraint (41) and because there is a two-dimensional vector of payments  $(b(\theta), a(\theta))$  rather a single function relating  $\theta$  to a payment.

We next demonstrate that, at least for Case 1 preferences exhibiting nonincreasing absolute risk aversion, the seller will find it advantageous to set a positive reserve price—that is, he will refuse to sell to a buyer with a  $\theta$  less than some positive level  $\theta^0$ .

**THEOREM 16:** *Under the hypotheses of Theorem 14, there exists  $\theta^0 > 0$  such that  $G(\theta) = 0$  for  $\theta < \theta^0$ .*

**PROOF:** Suppose the theorem is false, and  $G(\theta) > 0$  for all  $\theta > 0$ . From the Corollary to Theorem 11,

$$\theta - b > -a \quad \text{for } 0 < G < 1.$$

Therefore, letting  $\theta$  tend to zero, we have

$$0 \geq b(0) - a(0).$$

From Theorem 14,

$$b(0) - a(0) \geq 0,$$

and so

$$b(0) - a(0) = 0.$$

By hypothesis  $w_1(-a) = U'(-a)$  and  $u_1(-b, \theta) = U'(\theta - b)$ . Then, since  $b(0) = a(0)$ ,  $u_1 = w_1$  at  $\theta = 0$ .

But, with  $G > 0$  for all  $\theta > 0$ , (54) must hold and so  $\lambda(0) = 0$ . But this contradicts (52) so  $G(\theta)$  cannot be strictly positive for  $\theta > 0$ . *Q.E.D.*

Theorem 16 applies as well to the case of a single buyer. In this case, we can establish a corresponding result for *high* values of  $\theta$ ; namely, that for sufficiently high  $\theta$ , the probability of winning is one.

**THEOREM 17:** *Under the hypotheses of Theorem 14, if  $n = 1$ , then there exists  $\theta^* < 1$  such that, for all  $\theta > \theta^*$ ,  $G(\theta) = 1$  in an optimal auction.*

**PROOF:** From the argument of Theorem 8,

$$(58) \quad (b - a)F' - \lambda u_2 - \lambda'(u - w) + \alpha - \beta - \mu F' = 0.$$

Suppose  $G < 1$ , for all  $\theta < 1$ . Then

$$(79) \quad \mu = \beta = 0.$$

Recall that

$$(51) \quad \lambda(1) = 0.$$

Therefore, since

$$\lambda = \frac{u_1 - w_1}{u_{12} w_1} F' \quad \text{for } 0 < G < 1 \text{ (see (54))},$$

$u_1 \rightarrow w_1$  as  $\theta \rightarrow 1$ , and so

$$(80) \quad u \rightarrow w, \quad \text{as } \theta \rightarrow 1.$$

From (80) we conclude that

$$1 - b(1) = -a(1),$$

and thus

$$(81) \quad b(1) - a(1) > 0.$$

But from (49) and (79)–(81) the left-hand side of (58) is positive for  $\theta = 1$ , an impossibility. *Q.E.D.*

Theorems 16 and 17 and the continuity of  $G$  permit us to conclude that, at least for Case 1 preferences with nonincreasing absolute risk aversion, an optimal auction divides the unit interval into three nondegenerate subintervals: the lowest interval has  $G = 0$ ; the middle interval has  $0 < G < 1$ ; the upper interval has  $G = 1$ . The middle interval is perhaps the most interesting. We have taken  $G(\theta)$  to be the probability of winning. In the one buyer case we could alternatively interpret  $G(\theta)$  as the probability that the item does not “fall apart”, i.e., the “quality” of the item. The nondegeneracy of the middle interval then implies that there are values of  $\theta$  for which the seller will offer less than top quality, even though quality is costless to provide. This result hinges crucially on risk aversion. As Riley and Zeckhauser [23] show, the optimal  $G$  equals either 0 or 1 for all values of  $\theta$  if the buyer is risk neutral.

#### 5. CONCLUDING REMARKS

We have been most concerned in this paper with elucidating the interplay between insurance and screening considerations in models of incomplete informa-

tion with risk averse agents. We have studied auctions in particular, but as they are formally very similar to a variety of other monopoly problems, the principles that emerge all apply elsewhere.

We have discussed the roles of most of our assumptions, but it is worth returning to two of them. First, by assuming that the seller maximizes expected revenue, we implicitly suppose that he is risk neutral. For the case of a single buyer this assumption makes no qualitative difference. Indeed, for this case, we could have presented Theorems 8–11 and 13–17 for a risk averse seller with only slightly modified proofs. The assumption of seller risk neutrality is, however, crucial to our methods for two or more buyers. Risk neutrality means that the seller's payoff depends on the underlying probabilities,  $H_i$ , only through the marginal distribution  $G$ . Thus we can work directly with  $G$  rather than with the analytically more difficult  $H_i$ 's.

For much the same reason, the hypothesis that the  $\theta_i$ 's are distributed independently is highly simplifying. Indeed, without independence, a buyer's marginal probability of winning depends not just on his bidding behavior but on his parameter. Thus, again, we are forced to work with the  $H_i$  functions. It is easy to see that the seller can exploit any correlation among the  $\theta_i$ 's. To take an extreme example, suppose that the value of  $\theta_i$  were the same for all buyers. Even if the seller did not know this value, he could extract all surplus from buyers by operating a second bid auction.

As Myerson [21] suggests, it is possible, even with imperfect (but nonzero) correlation, to construct auctions one of whose equilibria extracts all surplus (at least, if  $\theta$  can take on only discrete values) when buyers are risk neutral. Maskin and Riley [11] show that at least in the case where  $\theta$  assumes only two values, such auctions can be constructed with a unique equilibrium. Crémer and McLean [2] show, for a large class of discrete distributions, that auctions can be devised with a dominant strategy equilibrium when correlation is sufficiently strong. How these results fare for the more general distributions, and what optimal auctions look like with correlation when buyers are risk averse remain conjectural.

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#### APPENDIX

**THEOREM 7:** *Suppose that  $G(s)$ , the probability of winning with parameter equal to  $s$ , is piecewise differentiable and nondecreasing. A necessary condition for there to exist a permutation symmetric family of probability functions  $H_j(x)$ ,  $j = 1, \dots, n$ , satisfying  $\sum_j H_j \leq 1$  such that*

$$G(s) = \int H_i(s, x_{-i}) dx_{-i} \quad \text{for all } s$$

is

$$(41) \quad \int_y^1 G(s) dF(s) \leq \int_y^1 F^{n-1}(s) dF(s), \quad 0 \leq y \leq 1.$$

Moreover, if  $G(s)$  is a step function with finitely many steps, (41) is sufficient.

We prove Theorem 7 in four steps.

We first show that we can eliminate  $F$  from the statement of the problem.

LEMMA A1: Suppose we can establish that, for all nonnegative and nondecreasing functions  $G(y)$  on  $[0, 1]$ , if there exist probability functions  $H_1, \dots, H_n$  satisfying the symmetry condition,

$$(i) \quad H_i(x) = H_j(x'), \quad \text{if } x'_i = x_j, \quad x'_j = x_i, \quad \text{and} \quad x'_k = x_k, \quad k \neq i, j,$$

and the feasibility condition

$$(ii) \quad \sum_{i=1}^n H_i \leq 1,$$

such that

$$(iii) \quad G(s) = \int_{x_{-i}} H_i(s, x_{-i}) dx_{-i},$$

then

$$(iv) \quad \int_{s=y}^1 (G(s) - s^{n-1}) ds \leq 0, \quad \text{for all } y \in [0, 1].$$

Suppose, furthermore, that the converse holds if  $G$  is a finite step function. Then Theorem 7 must hold.

PROOF: Since  $F$  is a continuous strictly increasing function we can define  $\theta_i = F^{-1}(x_i)$ , a strictly increasing function from  $[0, 1]$  to  $[0, 1]$ .

For any  $\tilde{H}_i(\theta)$  we can define

$$H_i(x) = \tilde{H}_i(F^{-1}(x_1), \dots, F^{-1}(x_n)).$$

Then the  $H_i$ 's satisfy conditions (i) and (ii) if and only if the  $\tilde{H}_i$ 's do. Similarly, for any  $\tilde{G}(t)$ , we can define  $G(x_i) = \tilde{G}(F^{-1}(x_i))$ . Then  $G$  is nondecreasing if and only if  $\tilde{G}$  is, and  $G$  satisfies (iii) and (iv) if and only if  $\tilde{G}$  satisfies

$$\int_{\theta}^1 (\tilde{G}(t) - F^{n-1}(t)) dF(t) \leq 0, \quad 0 \leq \theta \leq 1,$$

and

$$\tilde{G}(\theta_i) = \int_{\theta_{-i}} \tilde{H}_i(\theta_i, \theta_{-i}) \prod_{j \neq i} dF(\theta_j). \tag{Q.E.D.}$$

We next establish the theorem in one direction.

LEMMA A2: If the probability functions  $H_i, i = 1, \dots, n$ , satisfy (i) and (ii) and if  $G$  is defined by (iii), then (iv) is satisfied.

PROOF: Let  $X_r = \{x | x_i \in [y, 1], i \leq r; x_i \in [0, y), \text{ otherwise}\}$ . Over  $X_r$ , the symmetry of  $H_i$  implies that

$$\int_{x \in X_r} H_1 dx = \dots = \int_{x \in X_r} H_r dx = \frac{1}{r} \int_{x \in X_r} \sum_{i=1}^r H_i dx.$$

Then, since  $\sum_{i=1}^r H_i \leq 1$ ,

$$\int_{x \in X_r} H_i dx \leq \frac{1}{r} (1-y)^r y^{n-r}, \quad \text{for } i = 1, \dots, r.$$



Moreover there are

$$\binom{n-1}{r-1}$$

ways to choose exactly  $r-1$  components of  $(x_2, \dots, x_n)$  to lie in  $[y, 1]$ . Thus by the symmetry of  $H_1, \dots, H_m$

$$\begin{aligned} \int_{x_1=y}^1 G(x_1) dx_1 &= \int_{x_1 \geq y} H_1(x) dx \leq \sum_{r=1}^n \frac{1}{r} \binom{n-1}{r-1} (1-y)^r y^{n-r} \\ &= \frac{1}{n} \sum_{r=1}^n \binom{n}{r} (1-y)^r y^{n-r} \\ &= \frac{1}{n} (1-y^n) = \int_{x_1=y}^1 x_1^{n-1} dx_1 \end{aligned} \quad \text{Q.E.D.}$$

Next, we establish the theorem in the other direction for the case where (iv) holds with equality at  $y=0$ .

LEMMA A3: For any nondecreasing finite step function  $G(s)$  satisfying the integral constraint (iv) for all  $y$  and with equality at  $y=0$  there exist probability functions  $H_i(x), i=1, \dots, n$ , satisfying the symmetry and feasibility conditions (i) and (ii) such that

$$(iii) \quad G(s) = \int H_1(s, x_{-1}) dx_{-1}.$$

PROOF: First note that if the  $H_i(x)$ 's are symmetric in the sense of (i) and satisfy (ii), then

$$\int_0^1 G(s) ds = \int H_1(x) dx = \frac{1}{n} \int \sum_{j=1}^n H_j(x) dx.$$

Also since  $G(s)$  satisfies the integral constraint (iv) with equality at  $y=0$ ,

$$\int_0^1 G(s) ds = \int_0^1 s^{n-1} ds = \frac{1}{n}.$$

It follows that the probability functions  $H_j(x)$  must satisfy the adding up condition,

$$(v) \quad \sum_{j=1}^n H_j(x) = 1.$$

If  $G$  is a finite step function satisfying the hypotheses of the lemma, we can write

$$(vi) \quad G(s) = G_i, \quad y_i \leq s < y_{i+1} \quad (i=1, \dots, m),$$

where  $y_1=0$  and  $y_{m+1}=1$ . From (vi),

$$G_1 y_2 = \int_0^{y_2} G(s) ds = \int_0^1 s^{n-1} ds - \int_{y_2}^1 G(s) ds,$$

where the last equality is just the requirement that (iv) should hold with equality at  $y=0$ .

Inequality (iv) also implies that

$$\int_{y_2}^1 G(s) ds \leq \int_{y_2}^1 s^{n-1} ds.$$

Therefore  $G_1$  must satisfy the constraint

$$(vii) \quad G_1 \geq \frac{1}{y_2} \int_0^{y_2} s^{n-1} ds = \frac{1}{n} y_2^{n-1}.$$

Moreover, using (iv) once more

$$G_m(1-y_m) = \int_{y_m}^1 G(s) ds \leq \int_{y_m}^1 s^{n-1} ds = \frac{1}{n} (1-y_m^n).$$

Hence  $G_m$  must satisfy the constraint

$$(viii) \quad G_m \leq \frac{1}{n} \frac{(1 - y_m^n)}{1 - y_m}.$$

Next define  $\bar{H}_1(x)$  to be the probability of winning in a second bid auction modified so that, for all  $i$ , all bids in the interval  $[y_i, y_{i+1})$  are treated as equal. Then,

$$(ix) \quad \bar{H}_1(x) = \begin{cases} 1/1 + c, & \text{if for some } i, y_i \leq x_1 \leq \max_{j=1, \dots, n} \{x_j\} < y_{i+1} \text{ where } c \text{ is the} \\ & \text{number of components of } x_{-1} \text{ which are elements of } [y_i, y_{i+1}), \\ 0, & \text{otherwise.} \end{cases}$$

We shall find it useful to modify this probability function in the following manner. Define

$$(x) \quad X_r = \left\{ x \mid y_{r-1} \leq x_1 \leq \max_{j=1, \dots, n} \{x_j\} < y_{r+1} \text{ and at least one component of } x_{-1} \text{ is an element of } [y_{r-1}, y_{r+1}) \right\}.$$

For  $p = (p_1, \dots, p_{n-1}) \in P = \{p \mid 0 \leq p_i \leq 1/(1+i) \text{ for all } i\}$ , take

$$(xi) \quad \bar{H}'_1(x) = \begin{cases} \bar{H}_1(x), & x \notin X_r, \\ \frac{1}{1+c}, & x_1 \in [y_{r-1}, y_r), c \text{ components of } x_{-1} \text{ are elements} \\ & \text{of } [y_{r-1}, y_r) \text{ and no component of } x_{-1} \text{ is an} \\ & \text{element of } [y_r, y_{r+1}), c \geq 1, \\ \frac{1}{1+c} - p_c, & x_1 \in [y_{r-1}, y_r), c \text{ components of } x_{-1} \text{ are elements of} \\ & [y_{r-1}, y_{r+1}) \text{ and } d \text{ components of } x_{-1} \text{ are} \\ & \text{elements of } [y_r, y_{r+1}), d \geq 1, \\ \frac{1}{1+c} + \left(\frac{c-d}{1+d}\right) p_c, & x_1 \in [y_r, y_{r+1}), c \text{ components of } x_{-1} \text{ are elements} \\ & \text{of } [y_{r-1}, y_{r+1}) \text{ and } d \text{ components of } x_{-1} \text{ are} \\ & \text{elements of } [y_r, y_{r+1}). \end{cases}$$

It is a straightforward although tedious exercise to confirm that if  $\bar{H}'_j(x), j = 2, \dots, n$ , are defined symmetrically, then for any choice of  $p = (p_1, \dots, p_{n-1}) \in P$ , the  $\bar{H}'_j(x)$  are nonnegative functions satisfying both the symmetry condition, (i), and the adding up condition, (v). With these preliminaries completed we now suppose the lemma to be true for any  $(m-1)$ -step function and show that it must then also hold for any  $m$ -step function. Since it holds trivially for any 1-step function (set  $H_j(x) = 1/n$ ) this will establish the lemma.

Consider any nondecreasing  $m$ -step function  $G(s)$  defined by (vi). We can, in effect, delete the  $r$ th step by defining

$$(xii) \quad G^r(s) = \begin{cases} G_i, & y_i \leq s < y_{i+1}, & (i = 1, \dots, r-2), \\ \left(\frac{y_r - y_{r-1}}{y_{r+1} - y_{r-1}}\right) G_{r-1} + \left(\frac{y_{r+1} - y_r}{y_{r+1} - y_{r-1}}\right) G_r, & y_{r-1} \leq s < y_{r+1}, \\ G_i, & y_i \leq s < y_{i+1}, & (i = r+1, \dots, m). \end{cases}$$

Since  $G(s)$  is nondecreasing so is  $G^r(s)$ . By inductive hypothesis there exist probability functions  $H'_j(x)$  such that (i), (iii), and (v) are satisfied. Define

$$(xiii) \quad \hat{H}'_j(x) = \begin{cases} H'_j(x), & x \notin X_r, \\ \bar{H}'_j(x), & x \in X_r, \end{cases}$$

for some  $p = (p_1, \dots, p_{n-1}) \in P$ .

We will show that there exist choices of  $p$  and  $r$  such that

$$(xiv) \quad G_i = \int \hat{H}'_i(y_i, x_{-1}) dx_{-1} \quad (i = 1, \dots, m+1).$$

Notice that for any choice of  $p$  and  $r$ , (xiv) holds for all  $i \neq r-1, r$  by definition of  $\hat{H}'_i(x)$  and  $H'_i$ . Also, since both the  $H'_j(x)$  and  $\bar{H}'_j(x)$  functions satisfy the adding up condition so must the  $\hat{H}'_j(x)$ .

Hence, from symmetry,

$$\int_0^1 G(s) ds = \frac{1}{n} = \int \hat{H}_1^r(x) dx.$$

Since (xiv) holds for all  $i \neq r-1, r$  we have

$$(xv) \quad (y_r - y_{r-1}) \left[ G_{r-1} - \int \hat{H}_1^r(y_{r-1}, x_{-1}) dx_{-1} \right] + (y_{r+1} - y_r) \left[ G_r - \int \hat{H}_1^r(y_r, x_{-1}) dx_{-1} \right] = 0.$$

Thus, if (xiv) holds for  $i = r$  it must also hold for  $i = r-1$ . First set  $p = (0, 0, \dots, 0)$  in (xi) and (xiii). Then, for all  $x_{-1}$ , it follows from the definitions of  $H_1^r(x)$  and  $\bar{H}_1^r(x)$  that

$$(xvi) \quad \hat{H}_1^r(y_{r-1}, x_{-1}) = \hat{H}_1^r(y_r, x_{-1}).$$

Hence, since  $G(s)$  is nondecreasing, (xv) and (xvi) imply that

$$(xvii) \quad G_r \geq \int \hat{H}_1^r(y_r, x_1) dx_1 \quad \text{when} \quad p = (0, 0, \dots, 0).$$

Suppose that for all  $p \in P$ , and all  $r$ ,

$$G_r > \int \hat{H}_1^r(y_r, x_{-1}) dx_{-1}.$$

From (xv) we must therefore have

$$G_{r-1} < \int \hat{H}_1^r(y_{i-1}, x_{-1}) dx_{-1}.$$

Hence

$$(xviii) \quad G_r - G_{r-1} > \int [\hat{H}_1^r(y_r, x_{-1}) - \hat{H}_1^r(y_{r-1}, x_{-1})] dx_{-1}.$$

From the definition of  $\hat{H}_1^r$ ,

$$(xix) \quad \hat{H}_1^r(y_r, x_{-1}) - \hat{H}_1^r(y_{r-1}, x_{-1}) = \bar{H}_1^r(y_r, x_{-1}) - \bar{H}_1^r(y_{r-1}, x_{-1}),$$

if  $(y_r, x_{-1}) \in X_r$ .

Again from definition,

$$(xx) \quad \hat{H}_1^r(y_r, x_{-1}) - \hat{H}_1^r(y_{r-1}, x_{-1}) = \bar{H}_1^r(y_r, x_{-1}) - \bar{H}_1^r(y_{r-1}, x_{-1}) = 0,$$

if  $(y_r, x_{-1}) \notin X_r$ .

If  $p = (\frac{1}{2}, \frac{1}{3}, \dots, 1/n)$ , then for all  $x$

$$\bar{H}_1^r(x) = \bar{H}_1(x).$$

Hence,

$$(xxi) \quad \bar{H}_1^r(y_r, x_{-1}) - \bar{H}_1^r(y_{r-1}, x_{-1}) = \bar{H}_1(y_r, x_{-1}) - \bar{H}_1(y_{r-1}, x_{-1}),$$

for  $p = \left(\frac{1}{2}, \dots, \frac{1}{n}\right)$ .

Therefore

$$(xxii) \quad \bar{H}_1(y_r, x_{-1}) - \bar{H}_1(y_{r-1}, x_{-1}) = 0 \quad \text{if} \quad (y_r, x_{-1}) \notin X_r.$$

Combining (xix)–(xxii) we obtain

$$\hat{H}_1^r(y_r, x_{-1}) - \hat{H}_1^r(y_{r-1}, x_{-1}) = \bar{H}_1(y_r, x_{-1}) - \bar{H}_1(y_{r-1}, x_{-1}) \quad \text{for} \quad p = \left(\frac{1}{2}, \dots, \frac{1}{n}\right).$$

Hence, from (xviii)

$$G_r - G_{r-1} > \int [\bar{H}_1(y_r, x_{-1}) - \bar{H}_1(y_{r-1}, x_{-1})] dx_{-1} \quad \text{for} \quad p = \left(\frac{1}{2}, \dots, \frac{1}{n}\right).$$

Summing over  $r$ , we obtain

$$(xxiii) \quad G_m - G_1 > \int [\bar{H}_1(y_m, x_{-1}) - \bar{H}_1(y_1, x_{-1})] dx_{-1}, \quad \text{for } p = \left(\frac{1}{2}, \dots, \frac{1}{n}\right).$$

But from the definition of  $\bar{H}_1(x)$ ,

$$\int \bar{H}_1(y_1, x_{-1}') dx_{-1} = \frac{1}{n} y_2^{n-1}$$

and

$$\begin{aligned} \int \bar{H}_1(y_m, x_{-1}) dx_{-1} &= \sum_{c=0}^{n-1} \frac{1}{1+c} \binom{n-1}{c} (1-y_m)^c y_m^{n-1-c} \\ &= \frac{1}{n} \frac{1-y_m^n}{1-y_m}. \end{aligned}$$

Thus we may rewrite (xxiii) as

$$(xxiv) \quad G_m - G_1 > \frac{1}{n} \left( \frac{1-y_m^n}{1-y_m} \right) - \frac{1}{n} y_2^{n-1}.$$

But (xxiv) contradicts (vii) and (viii). Thus we conclude that there exist  $p \in P$  and  $r$  such that  $G_r = \int \hat{H}_1^r(y_r, x_{-1}) dx_{-1}$ , to complete the induction. Q.E.D.

The final step is to extend the previous lemma to  $G$ 's for which the constraint (iv) does not hold with equality.

LEMMA A4: *Lemma A3 is also true if the integral constraint (iv) holds with strict inequality at  $y = 0$ .*

PROOF: Suppose that  $G(s)$  is a nondecreasing finite step function satisfying the integral constraint (iv) and with strict inequality at  $y = 0$ . We first claim that there exists a finite step function  $\tilde{G} \geq G$  such that

$$(xxv) \quad \tilde{G} \text{ is nondecreasing}$$

and

$$(xxvi) \quad \int_y^1 (\tilde{G}(z) - z^{n-1}) dz \leq 0 \quad \text{for all } y,$$

where (xxvi) holds with equality for  $y = 0$ .

Let us write  $G(s)$  as in (vi). Define the step function  $G^*(s)$  so that for  $s \in [y_i, y_{i+1})$

$$G^*(s) = \frac{1}{y_{i+1} - y_i} \int_{y_i}^{y_{i+1}} z^{n-1} dz.$$

Clearly  $G^*(s)$  is nondecreasing and satisfies (iv) everywhere and with equality at  $y_i, i = 1, \dots, m$ . Let  $Y_1$  be the set of "crossing points" of  $G(s)$  and  $G^*(s)$ . That is,  $Y_1 = \{y_i | \text{for all } \varepsilon > 0 \text{ sufficiently small } (G(y_i + \varepsilon) - G^*(y_i + \varepsilon))(G(y_i - \varepsilon) - G^*(y_i - \varepsilon)) \leq 0\}$ . If  $Y_1$  is empty, then from (iv),  $G(s) < G^*(s)$  for all  $s$ , and so we can take  $G(s) = G^*(s)$  to establish the claim. Therefore, assume that  $Y_1$  is nonempty. Let

$$(xxvii) \quad Y_2 = \left\{ y_i \in Y_1 \mid \int_0^{y_i} \max \{G^*(s), G(s)\} ds + \int_{y_i}^1 G(z) dz \geq \int_0^1 z^{n-1} dz \right\}.$$

Take

$$y^{**} = \begin{cases} 1, & \text{if } Y_2 \text{ is empty,} \\ \min Y_2, & \text{if } Y_2 \text{ is nonempty,} \end{cases}$$

and

$$y^* = \begin{cases} \max \{y \in Y_1 \mid y < y^{**}\}, & \text{if } \{y \in Y_1 \mid y < y^{**}\} \text{ is nonempty,} \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$(xxix) \quad \tilde{G}(s) = \begin{cases} \max \{G(s), G^*(s)\}, & s < y^*, \\ \lambda G(s) + (1-\lambda) \max \{G(s), G^*(s)\}, & y^* \leq s \leq y^{**}, \\ G(s), & s > y^{**}, \end{cases}$$

for choice of  $\lambda$  between 0 and 1. If  $\lambda = 1$ , then by choice of  $y^*$ ,  $\int_0^1 \tilde{G}(z) dz < \int_0^1 z^{n-1} dz$ . Suppose  $\lambda = 0$ . If  $Y_2$  is empty, then  $y^{**} = 1$ , and so (xxix) implies that  $\tilde{G}(s) \geq G^*(s)$  for all  $s$ . Thus, if  $Y_2$  is empty,  $\int_0^1 \tilde{G}(z) dz \geq \int_0^1 z^{n-1} dz$ . If  $Y_2$  is nonempty then  $y^{**} \in Y_2$ , and (xxvii) implies that, again,  $\int_0^1 \tilde{G}(z) dz \geq \int_0^1 z^{n-1} dz$ . Thus we may choose  $\lambda < 1$  so that

$$(xxx) \quad \int_0^1 \tilde{G}(z) dz = \int_0^1 z^{n-1} dz.$$

Because  $y^* \in Y_1 \cup \{0\}$  and  $y^{**} \in Y_1 \cup \{1\}$ ,  $\tilde{G}(s)$  is nondecreasing. It is obvious from (xxix) that  $\tilde{G}(s) \geq G(s)$ . Thus it remains only to show that (xxvi) holds. Suppose that

$$(xxxi) \quad \int_{\bar{y}}^1 (\tilde{G}(z) - z^{n-1}) dz > 0 \quad \text{for some } \bar{y}.$$

If  $\tilde{G}(\bar{y}) > \bar{y}^{n-1}$ , then  $\int_{\bar{y}}^1 (\tilde{G}(z) - z^{n-1}) dz > 0$ , where  $\bar{y} \in [y_i, y_{i+1})$ . If  $\tilde{G}(\bar{y}) \leq \bar{y}^{n-1}$ , then  $\int_{\bar{y}}^1 (\tilde{G}(z) - z^{n-1}) dz > 0$ , where  $\bar{y} \in [y_{i-1}, y_i)$ . Therefore, we may as well assume that  $\bar{y} = y_i$  for some  $i$ . If  $\bar{y} < y^*$ , then because  $\tilde{G}(s) \geq G^*(s)$  for all  $s < y^*$  and  $\int_0^{\bar{y}} G^*(z) dz = \int_0^{\bar{y}} z^{n-1} dz$  (since (iv) holds with equality for  $G^*$  at  $y = \bar{y}$  and  $y = 0$ ),

$$(xxxii) \quad \int_0^{\bar{y}} \tilde{G}(z) dz \geq \int_0^{\bar{y}} z^{n-1} dz.$$

But (xxxi) and (xxxii) together imply that

$$(xxxiii) \quad \int_0^1 \tilde{G}(z) dz > \int_0^1 z^{n-1} dz,$$

which contradicts (xxx).

Suppose  $y \in [y^*, y^{**})$ . By definition of  $y^*$ , either  $G(s) \leq G^*(s)$  for all  $s \in (y^*, y^{**})$  or  $G(s) > G^*(s)$  for all  $s \in (y^*, y^{**})$ . If the former, then  $\tilde{G}(s) \leq G^*(s)$  for  $s \in (y^*, y^{**})$ , and so

$$(xxxiv) \quad \int_{\bar{y}}^{y^{**}} (\tilde{G}(z) - z^{n-1}) dz \leq 0.$$

From (xxxi) and (xxxiv)

$$\int_{y^{**}}^1 (\tilde{G}(z) - z^{n-1}) dz > 0$$

which contradicts the fact that  $\tilde{G}(s) = G(s)$  for  $s > y^{**}$ . If the latter, then  $\tilde{G}(s) = G(s)$  for  $s > y^*$ , which contradicts (xxxi).

Finally, if  $y \geq y^{**}$ , (xxxi) is impossible, since  $\tilde{G}(s) = G(s)$  for all  $s > y^{**}$ . We conclude that (xxxi) is impossible and that the claim is established.

From Lemma A3 there exist probability functions  $\tilde{H}_j$  satisfying the counterparts of (i)–(iii) for  $\tilde{G}$ . Define  $Q(z) = G(z)/\tilde{G}(z)$  and take

$$H_j(x) = Q(x_j)\tilde{H}_j(x).$$

It is immediate that the  $H_j$ 's satisfy (i)–(iii).

*Q.E.D.*

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