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Optimal Nonlinear Pricing with Two-Dimensional Characteristics

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Most work on nonlinear pricing has modeled consumer differences by a single parameter. In this chapter we begin the much harder problem of multidimensional differences. Using a simple model of linear demand, we determine the optimal (i.e., revenue-maximizing) price schedule, assuming that the distributions of slopes and intercepts are independent and uniform.

1. Introduction

Recently a large literature has developed that treats the pricing problem of a monopolist who has imperfect information about buyers (see Maskin and Riley 1984; Mussa and Rosen 1978; Roberts 1979; Spence 1977; among others). This problem is a particular instance of a large class of incentive problems, the general analysis of which was pioneered by Hurwicz (see, for example, Hurwicz 1960, 1972). Other special cases that are formally similar include optimal tax theory and the design of public decision mechanisms.

The monopolist's problem is to select a (nonlinear) pricing scheme to maximize, say, its profit subject to the constraint that, given the scheme, buyers will act to maximize their own utility. The problem is thus a double maximization. One common way of solving for an optimal pricing scheme in the case where all functions are differentiable is to represent buyers' maximization by their first-order conditions. The problem then can be reduced to a conventional constrained optimization program.

There are two serious limitations to this method. First, because of nonconvexities, the first-order conditions may not be sufficient for a maximum (see Mirrlees 1975 or Guesnerie and Laffont 1978). Second, even with convexity, the first-order conditions may be difficult to incorporate as constraints if the imperfect information about buyers requires two or more parameters for its representation. Consequently, no work of which we are aware has considered more than a single unknown parameter.

In this chapter, by contrast, we solve explicitly for the optimal nonlinear price schedule when a monopolist is uncertain about both the slopes and the intercepts of the individual demand curves it faces. We find that when these parameters are independently and uniformly distributed, optimal revenue is piecewise quadratic (with two pieces) in quantity sold and has a decreasing first derivative.

The analysis is fairly involved. The key element, however, is a simple change of variables technique that may be applicable to many other multidimensional optimization problems that have heretofore remained intractable.

In Section 2 we lay out the model. We then determine the optimal nonlinear schedule in Section 3. Finally, in Section 4, we briefly discuss the change of variables technique.

2. The Model

We consider a monopolist who produces a single good and confronts a continuum of buyers. Each buyer is represented by a pair of parameters (a,b), uniformly distributed on a compact, connected subsete of R^2, taken to be $[0,1]^2$ without loss of generality. The utility function of the buyer is parameterized as

$$U(a,b,x,t) = ax - \frac{1}{2}(b + 1)x^2 - t, \tag{1}$$

where x is consumption of the good and t is a monetary transfer.

A nonlinear price schedule is a function $t(x)$ relating the monetary transfer to the amount of good selected by the buyer.[1] Let $x(a,b)$ represent the consumption choice made by a buyer of characteristics (a,b). Assuming, for simplicity, that production costs are zero and that production capacity is infinite, the monopolist's objective function is

$$\int_0^1 \int_0^1 t(x(a,b))dadb.$$

ASSUMPTION 1. $t: R_+ \to R_+$ is a continuously differentiable nondecreasing function such that $t(0) = 0$ and

[1]This formulation rules out the possibility of random transfers. However, it is not difficult to show that, in our model, such transfers would never be desirable for the monopolist (see Maskin and Riley 1984).

$$t'(x) + x \text{ is strictly increasing in } x. \tag{2}$$

The behavior of a buyer (a,b) is determined by the program

$$\max_{x \geq 0} \left\{ ax - \frac{1}{2}(b+1)x^2 - t(x) \right\}. \tag{3}$$

PROPOSITION 1. Under Assumption 1, the program (3) has a unique solution $x = x(a,b)$ for all (a,b) in $[0,1]^2$ where

$$\begin{cases} x = 0 & \text{if } a \leq t'(0) \\ a - bx = x + t'(x) & \text{if } a > t'(0). \end{cases} \tag{4}$$

PROOF. Straightforward.

Defining ϕ: $R_+ \to R_+$ by

$$\phi(u) = \begin{cases} 0, \text{ for all } u \in [0,t'(0)] \\ x, \text{ for } u = x + t'(x) \end{cases} \tag{5}$$

we see that (4) is equivalent to

$$x = \phi(a - bx). \tag{6}$$

By Assumption 1, the function ϕ defined by (5) is continuous and nondecreasing, and $\phi(0) = 0$. In Proposition 3 we conveniently re-express the monopolist's program using this auxiliary function. Let $\Phi = \{\phi: R_+ \to R_+ | \phi$ is continuous and nondecreasing, and $\phi(0) = 0\}$.

The problem of finding an optimal nonlinear price schedule can be written as

$$\max \int_0^1 \int_0^1 t(x(a,b))dadb, \tag{7}$$

where $t(\cdot)$ satisfies Assumption 1 and x is determined by (5) and (6).

Let $V(a,b)$ be the utility level of agent (a,b) faced with the schedule $t(\cdot)$. Then,

$$V(a,b) = \max_{x \geq 0} \left\{ ax - \frac{1}{2}(b+1)x^2 - t(x) \right\}. \tag{8}$$

PROPOSITION 2. $V(\cdot)$ is convex and continuously differentiable for all (a,b). Furthermore,

$$\frac{\partial V}{\partial a}(a,b) = x(a,b) \tag{9}$$

$$\frac{\partial V}{\partial b}(a,b) = -\frac{1}{2}x^2(a,b). \tag{10}$$

PROOF. A maximum of linear functions is convex, and so is V. Formula (8) implies

$$V(a,b) - V(a',b) \geq (a - a')x(a',b)$$

$$V(a,b) - V(a,b') \geq -\frac{1}{2}(b - b')x^2(a,b').$$

Because V is convex, it is differentiable almost everywhere. Therefore, for almost every (a,b) (Rockafellar 1970)

$$x(a,b) = \frac{\partial V}{\partial a}(a,b)$$

and

$$-\frac{1}{2}x^2(a,b) = \frac{\partial V}{\partial b}(a,b).$$

From Assumption 1, it is easy to see that $x(a,b)$ is continuous and consequently V is continuously differentiable (see Rockafellar 1970). QED

PROPOSITION 3. Program (7) is equivalent to the following program

$$\max \int_0^1 \int_0^1 \left\{(2a - 1)x(a,b) - \frac{1}{2}(b + 1)x^2(a,b)\right\} da\,db, \tag{11}$$

where $x(a,b) = \phi(a - bx(a,b))$ for all (a,b) and $\phi \in \Phi$.

PROOF. From Proposition 2 we have

$$V(a,b) = V(0,b) + \int_0^a x(\alpha,b)d\alpha.$$

By (4), $x(a,b) = 0$ if $a \leq t'(0)$. So, in particular, $x(0,b) = 0$ for all b and

$$V(0,b) = V(0,0) - \frac{1}{2}\int_0^b x^2(0,\beta)d\beta = V(0,0) = 0.$$

Now,

$$t(x(a,b)) = ax(a,b) - \frac{1}{2}(b + 1)x^2(a,b) - V(a,b)$$

$$= ax(a,b) - \frac{1}{2}(b + 1)x^2(a,b) - \int_0^a x(\alpha,b)d\alpha.$$

Hence

$$\int_0^1\int_0^1 t(x(a,b))dadb = \int_0^1\int_0^1 \left[(2a-1)x - \frac{1}{2}(b+1)x^2\right]dadb. \quad \text{QED}^2$$

3. The Solution of the Optimization Program

Choose any ϕ in Φ. We have

LEMMA. For every (a,b) in $[0,1]^2$ there exists a unique $z = z(a,b)$ in $[0,1]$ such that

$$a = z + b\phi(z).$$

Moreover, for all b, $a \to z(a,b)$ is Lipschitzian.

PROOF. Straightforward.

We now change variables from (a,b) to (z,b). By Rademacher's theorem, $a \to z(a,b)$ is almost everywhere differentiable and we have for almost every (a,b)

$$\frac{\partial z}{\partial a}(a,b) = (1 + b\phi'(z(a,b)))^{-1}.$$

We observe that $z(0,b) = 0$ for all b and that $b \to z(1,b)$ is a decreasing function such that $z(1,0) = 1$. We set $z_1 = z(1,1)$. Consequently

$$\int_0^1 \left[(2a-1)x - \frac{1}{2}(b+1)x^2\right]da = \int_0^{z(1,b)} \Psi(\phi,z,b)\left|\frac{\partial z}{\partial \alpha}\right|^{-1} dz,$$

where $\Psi(\phi,z,b) = (2z + 2b\phi(z) - 1)\phi(z) - 1/2\ (b+1)\phi^2(z)$ and $(\partial z/\partial a)^{-1} = 1 + b\phi'(z)$.

Let

$$I = \int_0^1\int_0^1 \left[(2a-1)x - \frac{1}{2}(b+1)x^2\right]dadb.$$

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[2]Alternatively, we can derive Proposition 3 along the following lines. Writing t as a function of a and b and assuming differentiability, we obtain the first-order conditions

$$\frac{\partial t}{\partial a}(a,b) = a\frac{\partial x}{\partial a}(a,b) - (b+1)x(a,b)\frac{\partial x}{\partial a}(a,b)$$

$$\frac{\partial t}{\partial b}(a,b) = a\frac{\partial x}{\partial b}(a,b) - (b+1)x(a,b)\frac{\partial x}{\partial b}(a,b)$$

(see Laffont and Maskin 1980).

The integrability of this system requires $\partial^2 t/\partial a\partial b = \partial^2 t/\partial b\partial a$ or $\partial x/\partial b\ (a,b) + x(a,b)\ \partial x/\partial a\ (a,b) = 0$.

The solution of this partial differential equation can be expressed as

$$x = \phi(a - bx),$$

where ϕ is an arbitrary function. The second-order conditions require that ϕ be nondecreasing. Integrating the first-order conditions, we obtain Proposition 3.

By Fubini's theorem, we know that

$$I = \int_0^1 \int_0^{z(1,b)} \Psi(\phi,z,b)(1 + b\phi'(z))dzdb.$$

And, permuting integrals (Fubini again), we determine that

$$I = \int_0^{z_1} \int_0^1 \Psi(\phi,z,b)(1 + b\phi'(z))dbdz$$
$$+ \int_{z_1}^1 \int_0^{(1-z)/\phi(z)} \Psi(\phi,z,b)(1 + b\phi'(z))dbdz.$$

So I splits into two integrals:

$$I_1 = \int_0^{z_1} \left\{(2z - 1)\phi + \frac{1}{4}\phi^2 + \left(z - \frac{1}{2}\right)\phi\phi' + \frac{1}{4}\phi^2\phi'\right\}dz$$

$$I_2 = -\int_{z_1}^1 \left\{\frac{(5z - 1)(z - 1)}{4} + \frac{1}{4}\frac{d}{dz}((z - 1)^2\phi) + (1 - z)\phi \right.$$
$$\left. - \frac{1}{2}z(1 - z)^2 \frac{d}{dz}(\log \phi(z))\right\}dz.$$

Integrating by parts, when possible, and recalling that $\phi(z_1) = 1 - z_1$, we obtain:

$$I = \int_0^{z_1} \left\{(2z - 1)\phi(z) - \frac{1}{4}\phi^2(z)\right\}dz + \int_{z_1}^1 (z - 1)\left(\phi(z) - \frac{3z - 1}{2}\log \phi\right)dz$$
$$+ \frac{1}{12}(1 - z_1)^2(7z_1 + 2) - \frac{1}{2} z_1(1 - z_1)^2 \log (1 - z_1).$$

For any given z_1, we first solve the two programs

$$P_1 \begin{cases} \sup \displaystyle\int_0^{z_1} \left\{(2z - 1)\phi(z) - \frac{1}{4}\phi^2(z)\right\}dz \\ \phi \in \Phi,\ \phi(z_1) \le 1 - z_1 \end{cases}$$

$$P_2 \begin{cases} \sup \displaystyle\int_{z_1}^1 (z - 1)\left\{\phi(z) - \frac{3z - 1}{2}\log \phi(z)\right\}dz \\ \phi \in \Phi,\ \phi(z_1) \ge 1 - z_1. \end{cases}$$

We then optimize with respect to z_1 and show that, for the optimal choice, $\phi(z_1) = 1 - z_1$ so that ϕ is continuous on $[0,1]$.

Step 1. Solution to P_1

Let

$E_1 = \Phi([0,z_1],R)$, endowed with the uniform topology,

$C_1 = \{\phi \text{ in } E_1,\ \phi(z_1) \leq 1 - z_1\}$,

$\Psi_1: E_1 \to R$ be defined by

$$\Psi_1(\phi) = \int_0^{z_1} \left\{(2z - 1)\phi(z) - \frac{1}{4}\phi^2(z)\right\} dz.$$

Ψ_1 is a strictly concave differentiable mapping and

$$\Psi_1'(\phi)h = \int_0^{z_1} \left\{(2z - 1) - \frac{1}{2}\phi(z)\right\} h(z) dz.$$

Indeed, if

$$\Delta(h) = \left|\Psi_1(\phi + h) - \Psi_1(\phi) - \int_0^{z_1} \left\{(2z - 1) - \frac{1}{2}\phi(2)\right\} h(z) dz\right|,$$

then

$$\Delta(h) \leq \int_0^{z_1} |h(z)| \left|\frac{1}{4} h(z)\right| dz \leq \frac{1}{4}\left\|h\right\|_\infty^2.$$

This implies in particular for all ϕ, h in E_1:

$$\Psi_1(\phi + h) - \Psi_1(\phi) \leq \Psi_1'(\phi)h. \tag{12}$$

Define ϕ^*_1 by

$$\phi_1^*(z) = \begin{cases} 0 \text{ if } z \leq \frac{1}{2} \\ 4z - 2 \text{ if } \frac{1}{2} \leq z \leq z_1 \end{cases} \qquad \text{whenever } z_1 \leq \frac{3}{5}$$

$$\phi_1^*(z) = \begin{cases} 0, \text{ if } z \leq \frac{1}{2} \\ 4z - 2, \text{ if } \frac{1}{2} \leq z \leq \frac{3}{4} - \frac{1}{4} z_1 \\ 1 - z_1, \text{ if } \frac{3}{4} - \frac{1}{4} z_1 \leq z \leq z_1. \end{cases} \qquad \text{whenever } z_1 \geq \frac{3}{5}$$

Let h in E_1 be such that $(\phi_1^* + h)$ is in C_1. Then, $h(0) = 0$, and h is nondecreasing on $[0,1/2]$. Furthermore, if

$$z_1 \leq \frac{3}{5},\ \Psi_1'(\phi_1^*)h = \int_0^{1/2} (2z - 1)h(z)dz \leq 0$$

and if

$$z_1 > \frac{3}{5},\ \Psi_1'(\phi_1^*)h = \int_0^{1/2} (2z - 1)h(z)dz + \int_{3/4-(1/4)z_1}^{z_1} 2\left(z - \frac{3}{4} + \frac{z_1}{4}\right)h(z)dz.$$

But in that case $\phi_1^*(z_1) = 1 - z_1$ so $h(z_1) \leq 0$ and $h(z) \leq 0$ and on $[3/4 - 1/4\ z_1, z_1]$. Thus in any case $\Psi_1'(\phi_1^*)h \leq 0$, and, by (12), ϕ_1^* is the solution to P_1.

Step 2. Solution to P_2

Let

$E_2 = \Phi([z_1,1),R)$, endowed with the topology of uniform convergence,

$C_2 = \{\phi$ in E_2, ϕ nondecreasing, $\phi(z_1) \geq 1 - z_1\}$,

Ψ_2: $E_2 \to R$ be defined by

$$\Psi_2(\phi) = \int_{z_1}^{1} (z - 1)\left\{\phi(z) - \frac{3z - 1}{2} \log \phi(z)\right\}dz.$$

Recalling (5), we know that $\phi(z_1) = 1 - z_1$ is equivalent to $z_1 = 1 - z_1 + t'(1 - z_1)$. Since t' is positive, this implies $z_1 \geq 1/2$. For all such z_1 (in fact $z_1 > 1/3$ would have been enough), Ψ_2 is strictly concave and differentiable on a neighborhood of C_2, and

$$\Psi_2'(\phi)h = \int_{z_1}^{1} (z - 1)\left\{1 - \frac{3z - 1}{2\phi(z)}\right\}h(z)dz.$$

Define ϕ_2^* by

$$\phi_2^*(z) = \frac{3}{2}z - \frac{1}{2},\ z \geq z_1 \qquad \text{if } z_1 \geq \frac{3}{5}$$

$$\phi_2^*(z) = \begin{cases} \frac{3}{2}z - \frac{1}{2} & z \geq 1 - \frac{2}{3}z_1 \\ 1 - z_1 & z_1 \leq z \leq 1 - \frac{2}{3}z_1. \end{cases} \qquad \text{if } z_1 < \frac{3}{5}$$

In the first case, $\phi_2'(\phi)h = 0$ for all h.

In the second case, if $\phi_2^* + h$ is in C_2, we conclude that $h(z_1) \geq 0$ and

h is nondecreasing. Furthermore

$$\Psi_2'(\phi)h = \int_{z_1}^{1-(2/3)z_1} (z-1)\left(1 - \frac{3z-1}{2-2z_1}\right)h(z)dz$$

and

$$z \leq 1 - \frac{2}{3}z_1,$$

which implies that $3z - 1 \leq 2 - 2z_1$ and $\phi_2'(\phi)h \leq 0$.

Step 3. Maximization with respect to z_1

We denote the maximized values of P_1 and P_2 as $I_1(z_1)$ and $I_2(z_1)$, respectively. Let $A_1(z_1)$ be the remaining term in I. By tedious but straightforward calculations we get

$$A_1(z_1) = \frac{7}{12}z_1^3 - z_1^2 + \frac{1}{4}z_1 + \frac{1}{6} - \frac{1}{2}z_1(1-z_1)^2 \log(1-z_1)$$

$$I_1(z_1) = \frac{1}{162}[293z_1^3 - 573z_1^2 + 393z_1 - 108] + \frac{1}{54}(3z_1-1)^2(3z_1-4)\log(1-z_1)$$

$$I_2(z_1) = \frac{1}{144}[-279z_1^3 + 591z_1^2 - 365z_1 + 53] + \frac{1}{54}(3z_1-1)^2(3z_1-4)\log\frac{3z_1-1}{2}.$$

Now,

$$(I_1' + A_1')(z_1) = \frac{P_1(z_1)}{324(1-z_1)},$$

where

$$P_1(z_1) = -2325z_1^3 + 5365z_1^2 + 3907z_1 + 891.$$

If $z_1 \leq 3/5$, then $-2325z_1^3 \geq -1395z_1^2$, and so $P_1 \geq 0$ on $[0,3/5]$. Also,

$$(I_2' + A_1')(z_1) = P_2(z_1) - \frac{1}{2}(1-z_1)(3z_1-1)\left[\log\frac{3z_1-1}{2} - \log(1-z_1)\right],$$

where

$$P_2(z_1) = -\frac{1}{432}[1899z_1^2 - (2634)z_1 + 1109]$$

and

$$\Delta' = (1317)^2 - 1899 \times 1109 < 0.$$

Thus

$$(I_2' + A_1') < 0 \text{ on } \left[\frac{3}{5}, 1\right].$$

Consequently the maximum is attained at $z_1 = 3/5$.

The optimal nonlinear price is characterized by a function ϕ defined as follows:

$$0 \le z \le \frac{1}{2} \quad \phi(z) = 0$$

$$\frac{1}{2} \le z \le \frac{3}{5} \quad \phi(z) = 4z - 2$$

$$\frac{3}{5} \le z \le 1 \quad \phi(z) = \frac{3}{2} z - \frac{1}{2}.$$

Easy calculations then yield $x(a,b)$, and $t(x)$:

$$x(a,b) = \begin{cases} 0, & a \le \dfrac{1}{2} \\[2mm] \dfrac{4a - 2}{4b + 1}, & \dfrac{1}{2} \le \dfrac{a + 2b}{4b + 1} \le \dfrac{3}{5} \\[2mm] \dfrac{3a - 1}{2 + 3b}, & \dfrac{3}{5} \le \dfrac{2a + b}{2 + 3b} \le 1 \end{cases}$$

$$t(x) = \begin{cases} \dfrac{1}{2}x - \dfrac{3}{8}x^2, & x \le \dfrac{2}{5} \\[2mm] \dfrac{1}{3}x - \dfrac{x^2}{6} + \dfrac{1}{30}, & \dfrac{2}{5} \le x \le 1. \end{cases}$$

4. The Change of Variables

A standard technique for solving one-dimensional nonlinear pricing problems is to express revenue collected in terms of the quantity sold, x, and then optimize with respect to x. Typically, in such problems, the only constraint on x is a monotonicity condition that can be represented conveniently by an inequality. In a two-dimensional problem, however, x is further constrained by an integrability condition of the form

$$x = \phi(x,a,b), \tag{12}$$

which, in our case, took the form (6). Because x is the control variable, it is convenient to express it as a function of a single parameter:

$$x = f(z). \tag{13}$$

Formulas (12) and (13) imply that we can express z as

$$z = f^{-1}(\phi(f(z),a,b)),$$

which can then be used to eliminate a.

Hence, the method is to define a parameter z such that the control variable is a function of that parameter alone and then substitute z for one of the original parameters. This technique should be useful quite generally.

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