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A PROGRESS REPORT ON KELLY'S MAJORITY CONJECTURES

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Let $C(m, n)$ be the proportion of n-voter profiles on m alternatives that have a majority winner. Jerry Kelly conjectured that $C(m, n) > C(m+1, n)$ for $m \geqslant 3$ and $n = 3$ and $n \geqslant 5$, and $C(m, n) > C(m, n+2)$ for $m \geqslant 3$ and $n \geqslant 3$. We prove these for special cases.

Kelly (1974) proposed the following conjectures for the proportion $C(m, n)$ of n-tuples of linear orders on a set of m alternatives in which some alternative x is ranked ahead of y in *at least* $n/2$ of the n orders, for each $y \neq x$:

Conjecture 1: $C(m, n) > C(m+1, n)$ for $m \geqslant 3$, $n = 3$ or $n \geqslant 5$,

Conjecture 2: $C(m, n) > C(m, n+2)$ for $m \geqslant 3$, $n = 1$ or $n \geqslant 3$.

Although these appear to be extremely difficult to prove (or refute) in their full generality, Fishburn, Gehrlein and Maskin (1978) have verified them for several important cases. We report our conclusions here. The proofs, which rely on complex combinatorial arguments, are available in the longer paper.

Conjecture 1 is true for the smallest relevant number of voters, i.e., for $n = 3$.

Theorem 1. $C(m, 3) > C(m+1, 3)$ *for all* $m \geqslant 2$.

The recursion relations $C(4, n) = 2C(3, n) - 1$ and $C(6, n) = 3 - 5C(3, n) + 3C(5, n)$ for odd n from Gehrlein and Fishburn (1976) also lead to $C(3, n) > C(4, n)$ and $C(3, n) > C(5, n)$ for all odd $n \geqslant 3$.

Our work on Conjecture 2 is based on the parity of n, since majority ties can occur for even n but not for odd n. The basic result for odd n that we have established applies to three alternatives.

Theorem 2. $C(3,n) > C(3,n+2)$ *for all odd* $n \geqslant 1$.

The relation $C(4,n) = 2C(3,n) - 1$ then shows that $C(4,n) > C(4,n+2)$ for odd $n \geqslant 1$.

Although we do not have an identical theorem for even n, it is known that Conjecture 2 holds for three alternatives when the number of voters is large.

Theorem 3. $C(3,n) > C(3,n+2)$ *for all even n greater than some integer N.*

Two other results for three alternatives and even numbers of voters involve simple majority ties, or the lack of ties. First, let $P(3,n)$ be the proportion of n-voter profiles on three alternatives in which some alternative has a strict simple majority over each of the others.

Theorem 4. $P(3,n) < P(3,n+2)$ *for all even* $n \geqslant 2$.

Note that this reverses the inequality in Theorem 3. The possibility of tied majority winners accounts for the difference.

Second, with $\{1,2,3\}$ the set of alternatives, let $T_A(3,n)$ be the proportion of n-voter profiles in which the alternatives in A tie under simple majority and nothing in $\{1,2,3\} \setminus A$ has a strict majority over anything in A. For convenience, let $T_{\{1\}} = T_1$, and so forth. Then C and T are connected by

$$\begin{aligned} C(3,n) &= [T_1(3,n) + T_2(3,n) + T_3(3,n)] \\ &\quad - [T_{12}(3,n) + T_{13}(3,n) + T_{23}(3,n)] + T_{123}(3,n) \\ &= 3T_1(3,n) - 3T_{12}(3,n) + T_{123}(3,n). \end{aligned}$$

Our second auxiliary result for even n is:

Theorem 5. $T_1(3,n) > T_1(3,n+2)$ *for all even* $n \geqslant 2$.

Thus $T_1(3,n)$ versus $T_1(3,n+2)$ behaves in the manner for C indicated in Theorem 3.

References

Fishburn, P.C., W.V. Gehrlein and E. Maskin, 1978, Condorcet proportions and Kelly's conjectures, Mimeo.

Gehrlein, W.V. and P.C. Fishburn, 1976, The probability of the paradox of voting: A computable solution, Journal of Economic Theory 13, 14–25.

Kelly, J.S., 1974, Voting anomalies, the number of voters, and the number of alternatives, Econometrica 42, 239–251.