The Economics of Kenneth J. Arrow: A Selective Review

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Abstract:

This essay reviews Kenneth Arrow's seminal work in economics, giving special emphasis to his contributions to social choice theory and general equilibrium theory.

Keywords: Kenneth Arrow, social choice, general equilibrium

The Economics of Kenneth J. Arrow: A Selective Review[*]

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Kenneth Arrow is a giant among economists. In the latter half of the twentieth century, only Paul Samuelson had a comparable effect on the economics profession.[1] Arrow created modern social choice theory, established most of the major results in general equilibrium theory, pioneered conceptual tools for studying asymmetric information and risk, and laid foundations for endogenous growth theory, among many other contributions to economics.

His papers are frequently abstract and technically difficult. But the abstractions enable readers to see the essentials of a complicated issue. Indeed, his work, though highly theoretical, has had significant repercussions for much applied research (e.g., computable general equilibrium and health-care economics) and for many fields outside economics, including political science, philosophy, mathematics, operations research, and ecology.

Arrow's academic output was enormous (on the order of 300 research papers and 22 books), and his work has been exposited many times before (see, for example, Shoven 2009). So, I will be highly selective in my choice of articles and books to discuss in this review; indeed, I will concentrate primarily on the work for which he was awarded the Nobel Memorial Prize: social choice and general equilibrium. By presenting the major results in some detail, I hope to make up for with depth what the essay lacks in breadth.

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[1] Milton Friedman was better known to the public than either Arrow or Samuelson, but his scholarly work did not rival theirs for influence.

I begin in section 1 with a short biographical sketch. Section 2 then treats social choice and section 3 general equilibrium. I briefly mention Arrow's most influential other work in section 4. I conclude in section 5 with a discussion of his contributions beyond research. In sections 2 and 3, I first discuss the material non-technically and then, in most cases, offer a more formal presentation in the "starred" version of that section.[2] Readers uninterested in technicalities, however, can safely skip over the starred sections.

1. *Biographical Sketch*

Kenneth Joseph Arrow was born on August 23, 1921 in New York City. His mother and father were Jewish emigrants from Romania, who, though poor, prized education and learning. According to Ken's sister Anita[3,4], his parents willingly cut back on meat to afford the 10-cent daily subway fare when he was admitted to Townsend Harris High School, a magnet school in Queens. Growing up during the Great Depression was a deeply formative experience for Ken. It fostered his interest in social welfare and even led him to give socialism careful consideration (see, for example, Arrow 1978).

Arrow got his bachelor's degree at the City College of New York (then the "poor man's Ivy League") in 1940. The degree was in social science with a major in mathematics - - foreshadowing his later preoccupations. His original goal had been to become a high school math teacher, but the queue for jobs was so long that he decided instead to go to graduate school at Columbia University in statistics (housed in the mathematics department); he was thinking of a

[2] Here I am emulating the expositional device of Amartya Sen in his classic monograph *Collective Choice and Welfare* (Sen 1970, expanded edition 2017).
[3] in a talk given at the Arrow Memorial Symposium, Stanford University, October 9, 2017
[4] Economics seems to have been in the family DNA. Anita became an economics professor herself and married economist Robert Summers, a brother of Paul Samuelson. One of Anita and Bob's sons is economist Lawrence Summers.

career as a life-insurance actuary. He received his M.A. in 1941, and planned to work under Harold Hotelling on his Ph.D. research. Needing a fellowship, he asked Hotelling for a letter of recommendation. But Hotelling had little influence in the math department and persuaded Arrow to switch to economics (Hotelling's primary affiliation), where arranging for financial support wouldn't be difficult. As Ken liked to say, he went into economics because he was bought.

World War II interrupted Arrow's doctoral studies. From 1942-6 he was a weather officer in the Army Air Corps, which led to his first published paper (Arrow 1949). Afterwards, he returned to Columbia for a year. But, unable to generate a thesis topic he was happy with, he moved in 1947 to a research position at the Cowles Commission, a research institute at the University of Chicago devoted to mathematical economics and econometrics. Finally, in the summer of 1949 (spent at the RAND Corporation) he found the big question he had hoped for - - and quickly developed his Impossibility Theorem in social choice theory. This work ultimately became his Columbia dissertation in 1951 (although the faculty there at first doubted that the subject matter was truly part of economics).

Arrow found Cowles and Chicago to be a highly stimulating intellectual environment. And, on the personal side, he met and married Selma Schweitzer there, a marriage that was to last 67 years until her death in 2015. But partly because of Milton Friedman's arrival in Chicago (Friedman was quite hostile toward Cowles) and partly because of Stanford University's attractions, Ken and Selma moved to Palo Alto in 1949. They stayed until 1968, when Ken accepted a professorship at Harvard. Yet, the Arrow family (by that time including sons David and Andy) left their hearts in California, and returned to Stanford every summer; they moved back for good in 1979. Ken formally retired in 1991, but remained active in research, teaching, and public service to the end of his life. He died at the age of 95 on February 21, 2017.

Arrow's work did not lack for recognition. To mention just a few of his honors: in 1957, he received the John Bates Clark Medal, awarded to an outstanding American economist under 40 (at the medal ceremony, George Stigler urged him to begin his acceptance speech by saying "Symbols fail me"). In 1972 he shared the Nobel Memorial Prize in Economics with John Hicks for their (separate) work in general equilibrium and welfare theory (Arrow was then 51 and remains the youngest-ever recipient of that prize). He was awarded the National Medal of Science in 2004.

2. *Social Choice*[5]

As my introduction mentions, Kenneth Arrow created the modern field of social choice theory, the study of how society should make collective decisions on the basis of individuals' preferences. There had been scattered contributions to social choice before Arrow, going back (at least) to Jean-Charles Borda (1781) and the Marquis de Condorcet (1785). But most earlier writers had focused on elections and voting exclusively. Indeed, they usually examined the properties of *particular* voting rules (I am ignoring here the large literature on utilitarianism- following Jeremy Bentham (Bentham1789) – which I touch on below). Arrow's approach, by contrast, encompassed not only *all possible* voting rules (with some qualifications discussed below), but also the issue of aggregating individuals' preferences or welfares more generally.

Arrow's first paper in this field was "A Difficulty in the Concept of Social Welfare" (Arrow 1950), which he then expanded into the celebrated monograph *Social Choice and Individual Values* (Arrow 1951). His formulation starts with two things: (1) a *society*, which is a group of individuals and (2) a set of *social alternatives* from which society must choose.

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[5] This section draws heavily on my essay for the Econometric Society on Arrow's contributions to social choice (Maskin 2017) and my foreword to the third edition of *Social Choice and Individual Values* (Maskin 2012)

The interpretation of this set-up depends on the context. For example, imagine a town that is considering whether or not to build a bridge across the local river. Here, "society" comprises the citizens of the town, and the "social alternatives" are just the options "build the bridge" and "don't build it." Or, we can think of a situation involving pure distribution. Suppose there is a jug of milk and a plate of cookies to be divided among a group of children. Then, the children are society and the different ways the milk and cookies could be allocated among them are the alternatives. As a third example, think of a committee that must elect a chairperson. In this case, society is the committee and the social alternatives are the various candidates for chair.

Those are just a few interpretations of the Arrow set-up, and there is clearly an unlimited number of other possibilities. An important feature of the formulation is its generality.

Now, presumably, each member of society has *preferences* over the social alternatives. That means that the individual can rank the alternatives from best to worst. Thus, in the bridge example, a citizen might prefer building the bridge to not building it. A *social welfare function* (SWF), according to Arrow, is a rule for going from the citizens' rankings to social preferences (i.e., a social ranking). Thus, social preferences are a function of citizens' preferences[6]. In the bridge setting, one possible SWF is *majority rule*, meaning that if a majority of citizens prefer building the bridge to not building it, then building is socially preferred - - the town should build the bridge.

[6] Bergson (1938) and Samuelson (1947) also used the term "social welfare function" to draw a connection between individual and social preferences. But in the Bergson-Samuelson formulation, social preferences are determined for given *fixed* preferences on the part of individuals. B-S do not consider – as Arrow does – how social preferences might change if individual rankings were different. Because he allows for variability in individuals' preferences, Arrow's social welfare function has sometimes been called a *constitution* (see, for example, Kemp and Asimakopoulos 1952) - - a procedure for arriving at a social ranking no matter what individuals' preferences turn out to be.

Although highly permissive in some respects, this way of formulating a SWF still excludes some important possibilities. First, it rules out making use of intensities of individuals' preferences (or other cardinal information). For example, it disallows a procedure in which each individual assigns a numerical "utility" (or "grade") to every alternative (say, on a scale from 1 to 5), and alternatives are then ordered according to the median of utilities (see Balinski and Laraki 2010 for a recent approach along these lines). Arrow's rationale for excluding cardinality – following Robbins (1932) – is that such information cannot be reliably obtained empirically unless individuals trade off alternatives against some other good like money, in which case the set of alternatives we started with does not fully describe the possibilities (one sort of cardinal information that *can* be obtained empirically is data about individuals' risk preferences; I discuss this possibility in section 2*).

A second (and closely related) omission: the formulation doesn't allow for *interpersonal comparisons*.[7] For example, there is no way of expressing the possibility that individual 1 might gain more in going from alternative *a* to *b* than individual 2 loses. Thus, Arrow's set-up excludes classical utilitarianism à la Bentham, according to which *a* is socially preferred to *b* if the sum of individuals' utilities for *a* is greater than for *b*. The formulation also rules out a comparison such as "individual 1 is worse off with alternative *a* than 2 is with alternative *b*." Hence, Rawls's (1971) maximin criterion (in which *a* is socially preferred to *b* if the worst-off individual in alternative *a* is better off than the worst-off individual in alternative *b*) is also off the table. Arrow avoided interpersonal comparisons because, again, he argued that they lack an empirical

[7] See chapter A3* of Sen (2017) for formulations that *do* permit interpersonal comparisons.

basis (he doubted that there are experiments we could perform to test claims such as "This hurts me more than it hurts you" or "My welfare is lower than yours.")

Finally, the requirement that social preferences constitute a complete ranking[8] may seem to attribute a degree of rationality to society that is questionable (see, in particular, James Buchanan 1954 on this issue). Arrow's reason for positing a social ranking, however, was purely pragmatic: it allows us to say what society ought to choose when the feasibility of the various alternatives isn't known in advance. Specifically, society should choose the alternative *a* at the top of the ranking if *a* is feasible, the next best alternative *b*, if *a* is infeasible, and so on.

Yet requiring a social ranking is a potential problem for the best known way of determining social preferences, majority rule. We noted above that majority rule works fine in the bridge example, where they are only two possible choices. Imagine, however, that there are three alternatives: building a bridge (*B*), building a tunnel (*T*), and doing nothing (*N*). Suppose, for example, that 35% of the citizens in the town prefer *B* to *T* and *T* to *N*; 33% prefer *T* to *N* and *N* to *A*; and 32% prefer *N* to *B* to *T* (these preferences are summarized in Table 1).

35%	33%	32%
B	*T*	*N*
T	*N*	*B*
N	*B*	*T*

Table 1

Then, under majority rule, *N* is socially preferred to *B* because a majority (33%+32%) prefer *N*. Furthermore, *T* is socially preferred to *N*, because a majority (35%+33%) prefer *T*. But *B* is

[8] In particular, Arrow requires that social preferences be *complete* (any pair of alternatives can be ranked) and that they be *transitive* (if alternative *a* is socially preferred to *b*, and *b* is preferred to *c*, then *a* must be preferred to *c*).

socially preferred to *T*, because a majority (35%+32%) prefer *B*. Clearly, majority rule doesn't give rise to a well-defined social ranking in this case.

As far as we know, it was Condorcet who first noted the possibility of a "Condorcet cycle," in which majorities prefer *N* to *B, T* to *N* and *B* to *T* (though he was himself a strong proponent of majority rule). Condorcet cycles were Arrow's starting point in his thinking about social choice[9]. Interestingly, he was, at that time, unaware of Condorcet's work but rediscovered the above problem with majority rule for himself. It led him to wonder whether there is some other reasonable way of determining social preferences that *does* succeed as a SWF.

By "reasonable," Arrow first required that the SWF should always work. That is, it should determine the social ranking no matter what preferences individuals happen to have. This is called the *Unrestricted Domain* condition (UD). It is the UD condition that majority rule violates.

Second, he insisted that if all individuals prefer alternative *a* to *b*, then society should rank *a* above *b*. After all, it would be quite perverse for society to choose *b* when everyone thinks that *a* is better. This is called the *Pareto* condition (P).

Third, Arrow required that the social preference between two alternatives *a* and *b* should depend only individuals' preferences between *a* and *b,* and not on their views about some third alternative *c*. He argued that *c* is *irrelevant* to society's choice between *a* and *b*, and so that choice should be *independent* of *c*. This is called the *Independence of Irrelevant Alternatives* condition (IIA)[10].

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[9] See K. Arrow (2014)

[10] Here, in his own words, is how Arrow (Arrow 1951) motivated IIA: "Suppose that an election is held, with a certain number of candidates in the field, each individual filing his list of preferences, and then one of the

These three conditions –UD, P, IIA – all seem quite natural and, on the face of it, not terribly demanding. But remarkably, Arrow showed that the only SWF satisfying all three conditions is a *dictatorship*, a highly extreme sort of SWF in which there is a single individual – the dictator – who always gets his way: if he prefers alternative *a* to *b*, then society prefers *a* to *b*, regardless of other individuals' preferences. Thus, if we introduce an additional requirement – a *Nondictatorship* condition (ND), which demands that the SWF should not have a dictator – we obtain Arrow's *Impossibility Theorem*: with three or more possible social alternatives, there is no SWF that satisfies UD, P, IIA, and ND.

The Impossibility Theorem – Arrow's most famous discovery – is truly a landmark in twentieth century thought. As a crude measure of its influence, I note that *Social Choice and Individual Values* has close to 19,000 Google Scholar citations (as of July 2018).

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candidates dies. Surely the social choice should be made by taking each of the individual's preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining names in going through the procedure of determining the winner. That is, the choice … should be independent of the preferences for candidates [who have not survived]."

In Maskin (2018), I suggest two other arguments in favor of IIA: (1) it prevents the phenomena of "vote splitting" (where two similar candidates divide votes between them, allowing a much different sort of candidate to win) and "spoilers" (where an independent candidate like Ralph Nader in 2000 U.S. Presidential election takes enough votes away from mainstream candidate – in this case Al Gore – to allow the other mainstream candidate George W. Bush to win) in elections, and (2) it is closely connected to the requirement that voters should be willing to submit their true preferences rather than voting strategically.

2*. *Social Choice – Formalities*

2A*. *Basic Formulation*

Let us consider a society consisting of *n* individuals (indexed $i = 1,\ldots,n$) and a set of social alternatives *A*. For each individual *i*, let $\Re_i$ be a set of possible orderings[11] of *A* for individual *i*, then a SWF *F* is a mapping

$$F : \Re_1 \times \cdots \times \Re_n \to \Re,$$

where $\Re$ is also a set of orderings.

The Arrow conditions on SWFs are:

Unrestricted Domain (UD): The SWF must determine social preferences for all possible preferences that individuals might have. Formally, for all $i = 1,\ldots,n,$ $\Re_i$ must consist of *all* orderings of *A*.

Pareto Property[12] (P): If all individuals strictly prefer *a* to *b*, then *a* must be strictly socially preferred. Formally, for all $(\succsim_1,\ldots,\succsim_n) \in \Re_1 \times \cdots \times \Re_n$ and for all *a*, *b* $\in A$, if $a \succ_i b$[13] for all *i*, then $a \succ_s b$, where $\succsim_s = F(\succsim_1,\ldots,\succsim_n)$.

Independence of Irrelevant Alternatives (IIA): Social preferences between *a* and *b* should depend only on individuals' preferences between *a* and *b*, and not on their preferences concerning some

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[11] An ordering $\succsim$ is a binary relation that satisfies three properties: (i) *completeness*: for all *a* and *b*, either $a \succsim b$ or $b \succsim a$; (ii) *reflexivity*: for all *a*, $a \succsim a$; (iii) and *transitivity*: for all *a*, *b*, *c*, if $a \succsim b$ and $b \succsim c$, then $a \succsim c$.

[12] This is sometimes called the *weak* Pareto property because it applies only in cases where all individuals have a strict preference for *a* over *b*.

[13] $a \succ_i b$ means that $a \succsim_i b$ and $b \not\succsim_i a$

third alternative. Formally, for all $(\succsim_1,\ldots,\succsim_n), (\succsim_1',\ldots,\succsim_n') \in \Re_1 \times\cdots\times \Re_n$ and all $a, b \in A$, if, for all i, $\succsim_i$ ranks a and b the same way that $\succsim_i'$ does, then $\succsim_s$ ranks a and b the same way that $\succsim_s'$ does, where $\succsim_s = F(\succsim_1,\ldots,\succsim_n)$ and $\succsim_s' = F(\succsim_1',\ldots,\succsim_n')$.

Nondictatorship (ND): There exists no individual who always gets his way in the sense that if he prefers a to b, a must be socially preferred to b, regardless of others' preferences. Formally, there does *not* exist i^* such that for all $(\succsim_1,\ldots,\succsim_n) \in \Re_1 \times\cdots\times \Re_n$ and all $a, b \in A$, if $a \succ_{i^*} b$, then $a \succ_s b$, where $\succsim_s = F(\succsim_1,\ldots,\succsim_n)$.

We now have:

Impossibility Theorem: If A contains at least 3 alternatives, there exists no SWF satisfying UD, P, IIA, and ND.

There are many proofs of the Impossibility Theorem in the literature. I will provide one showing that the result continues to hold even when the SWF takes account of cardinal information, as long as interpersonal comparisons are ruled out. The proof also provides a simple geometrical interpretation of the Theorem. To present this proof I first generalize the concept of a SWF somewhat.

2B*. *A More General Formulation*

To allow for cardinal information, let's redefine a SWF to be a mapping

$$F: \mathcal{U}_1 \times\cdots\times \mathcal{U}_n \rightarrow \Re$$

where $\mathcal{U}_i$ is the set of possible utility functions for individual i. This is a more general (i.e., more permissive) concept of a SWF than before, because it allows for the possibility that two n-tuples of utility functions $(u_1, \ldots, u_n)$ and $(u'_1, \ldots, u'_n)$ in $\mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ could correspond to the same ordinal preferences[14] $(\succsim_1, \ldots, \succsim_n)$ but lead to different social preferences. Indeed, this redefined SWF is – so far – consistent with both Benthamite utilitarianism and the Rawlsian maximin criterion. In the former case $a \succsim_s b$ if and only if $\sum_{i=1}^{n} u_i(a) \geq \sum_{i=1}^{n} u_i(b)$. In the latter case $a \succsim_s b$ if and only if $\min_i u_i(a) \geq \min_i u_i(b)$. However, we will rule out such interpersonal comparisons with the following condition:

No Interpersonal Comparisons (NIC): For all $(u_1, \ldots, u_n) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ and all constants $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$, where $\alpha_i > 0$ for all *i*,

$$F(u_1, \ldots, u_n) = F(\alpha_1 u_1 + \beta_1, \ldots, \alpha_n u_n + \beta_n)$$

That is, social preferences are unaffected when each u_i is replaced by a positive affine transformation.

A SWF that depends only on ordinal rankings (as in the basic formulation) automatically satisfies NIC (because a positive affine transformation of a utility function corresponds to the same ordinal preference ranking as before). But NIC allows for making use of cardinal information (specifically, note that if u_i represents individual *i*'s von Neumann-Morgenstern preferences over lotteries, then so does $\alpha_i u_i + \beta_i$). Still, it is strong enough to rule out interpersonal comparisons in social preferences. For example, suppose that $u_1(a) > u_2(b)$, i.e.,

[14] A utility function u_i for individual *i* is a function $u_i : A \rightarrow \Re$. It corresponds to *i*'s ordinal preferences $\succsim_i$ if, for all *a*, *b*, $a \succsim_i b$ if and only if $u_i(a) \geq u_i(b)$. Notice that u_i potentially provides more information than $\succsim_i$. For example, u_i could be a von Neumann-Morgenstern utility function for *i*, in which case it also determines *i*'s risk preferences.

individual 1 in alternative *a* is better off than 2 in alternative *b*. Notice that this interpersonal comparison will be reversed by replacing u_1 with $\alpha_1 u_1 + \beta_1$ where $\alpha_1 = 1$ and $\beta_1 < u_2(b) - u_1(a)$. But NIC says that social preferences are the same before and after the transformation. So, under NIC, the social ranking cannot take into account the interpersonal comparison.

We can now restate the Arrow conditions in the reformulated model:

Unrestricted Domain* (UD*): For all *i*, $\mathcal{U}_i$ consists of all utility functions on *A*.

Pareto* (P*): For all $(u_1, \ldots, u_n)$ and all *a*, *b*, if $u_i(a) > u_i(b)$ for all *i*, then $a \succ_s b$, where $\succsim_s = F(u_1, \ldots, u_n)$.

Independence of Irrelevant Alternatives* (IIA*): for all $(u_1, \ldots, u_n)$, $(u_1', \ldots, u_n')$ and all *a*, *b*, if, for all *i*, $u_i(a) = u_i'(a)$ and $u_i(b) = u_i'(b)$, then $a \succsim_s b$ if and only if $a \succsim_s' b$, where $\succsim_s = F(u_1, \ldots, u_n)$ and $\succsim_s' = F(u_1', \ldots, u_n')$.

Nondictorship* (ND*): There does not exist i^* such that, for all $(u_1, \ldots, u_n)$ and all *a*, *b*, if $u_{i^*}(a) > u_{i^*}(b)$, then $a \succ_s b$, where $\succsim_s = F(u_1, \ldots, u_n)$.

Here is the more general version of Arrow's theorem:

Impossibility Theorem* (IT*): If *A* contains at least 3 alternatives, there exists no SWF satisfying NIC, UD*, P*, IIA*, and ND*[15].

[15] To simplify the argument, I will prove IT* for a slightly stronger version of P*, which also requires that if everyone is indifferent between *a* and *b*, then society should also be indifferent:
Pareto** (P**): P* holds, and, in addition, for all $(u_1, \ldots, u_n) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ and all *a*, $b \in A$, if $u_i(a) = u_i(b)$ for all *i*, then $a \sim_s b$, where $\succsim_s = F(u_1, \ldots, u_n)$.

I begin the proof of IT* with the following preliminary result:

Welfarism: If A contains at least 3 alternatives and F satisfies U*, P**, and IIA*, then for all $(u_1,\ldots,u_n),\ (u_1',\ldots,u_n') \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$, and all $a, b, w, z \in A$, if $u_i(a) = u_i'(w)$ and $u_i(b) = u_i'(z)$ for all i, then

$$a \succsim_s b \Leftrightarrow w \succsim_s' z,$$

where $\succsim_s = F(u_1,\ldots,u_n)$ and $\succsim_s' = F(u_1',\ldots,u_n')$

Proof: From U*, we can choose $(u_1'',\ldots,u_n'')$ such that

$$u_i(a) = u_i''(a) = u_i'(w) = u_i''(w)$$

and

$$u_i(b) = u_i''(b) = u_i'(z) = u_i''(z)$$ [16]

Then,

$$a \succsim_s b \Leftrightarrow a \succsim_s'' b, \text{ where } \succsim_s'' = F(u_1'',\ldots,u_n'') \text{ (from IIA}^*\text{)}$$

$$\Leftrightarrow w \succsim_s'' z \text{ (from P}^{**}\text{)}$$

$$\Leftrightarrow w \succsim_s' z \text{ (from IIA}^*\text{)}$$ Q.E.D.

This result establishes that the social ranking of *a* and *b* doesn't depend on anything about these alternatives other than the *utilities* they generate (hence, the term *welfarism,* which implies that

[16] I am arguing as though *a*, *b*, *w*, and *z* are all distinct, but the argument easily extends to the case where there is overlap (as long as *A* has at least 3 alternatives).

only welfares matter and not the physical nature of the alternatives[17]). That is, we can describe social preferences as rankings of n-tuples of utilities, e.g.,

(*) $(v_1,\ldots,v_n) \succsim_s (v'_1,\ldots,v'_n)$,

where $(v_1,\ldots,v_n),\ (v'_1,\ldots,v'_n) \in \mathbb{R}^n$ [18].

Now, for convenience, let's suppose that $n = 2$. Thanks to welfarism, we can describe a SWF F by its "social indifference curves" (a social indifference curve is a set of utility pairs (v_1, v_2) among which society is indifferent) as in Figure 1[19]:

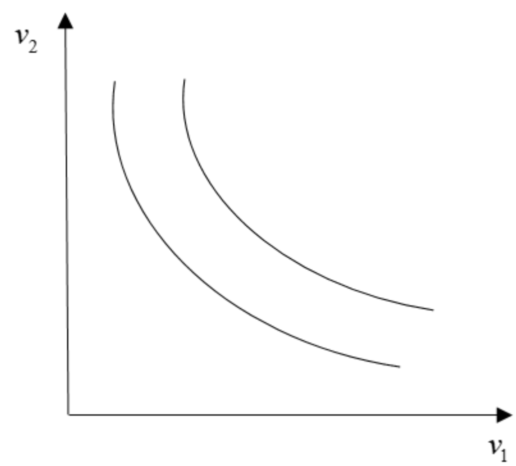

Figure 1
Two social indifference curves in utility space

From P*, the indifference curves in Figure 1 must be weakly downward sloping. To complete the proof of IT*, we need to show that the indifferences curves are either *vertical* (in which case, individual 1 is a dictator) or *horizontal* (2 is a dictator). Suppose, to the contrary, that

[17] Other terms for "welfarism" include "neutrality" and "consequentialism."
[18] We shall go back and forth between expressing social preferences over alternatives (e.g., $a \succsim_s b$) and expressing them over utility n-tuples as in (*).
[19] Strictly speaking, social preferences must be continuous in individuals' utilities for this to be true, but we will take a slight liberty here.

indifference curves are neither vertical nor horizontal. Then we can choose (v_1, v_2), (v_1', v_2'), and (v_1'', v_2''), as in Figure 2, so that (v_1', v_2') that is on a higher social indifference curve than (v_1, v_2) or (v_1'', v_2'') and $v_1 < v_1' < v_1''$ and $v_2 > v_2' > v_2''$.

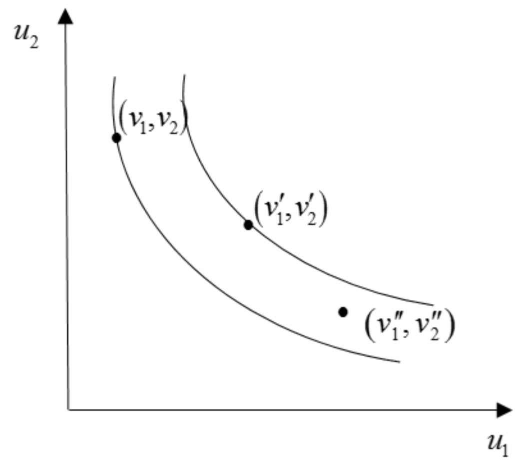

Figure 2

Let a and b be alternatives and (u_1, u_2) utility functions such that $(u_1(a), u_2(a)) = (v_1, v_2)$ and $(u_1(b), u_2(b)) = (v_1', v_2')$. Then, because (from Figure 2) $(v_1', v_2') \succ_s (v_1, v_2)$, we have

(**) $\quad b \succ_s a$.

Now for i =1, 2 choose $\alpha_i = \dfrac{v_i'' - v_i'}{v_i' - v_i}$ and $\beta_i = v_i' - \dfrac{v_i'' - v_i'}{v_i' - v_i} v_i$. Notice that

$$\alpha_i v_i + \beta_i = v_i' \text{ and } \alpha_i v_i' + \beta_i = v_i''$$

Thus, if we take $(u_1', u_2') = (\alpha_1 u_1 + \beta_1 \ , \ \alpha_2 u_2 + \beta_2)$, we have

(***) $\quad (u_1'(a), u_2'(a) = (v_1', v_2')$ and $(u_1'(b), u_2'(b) = (v_1'', v_2'')$

From NIC,

$$F(u_1,u_2)=F(u_1',u_2'),$$

and so, from (**) and (***)

(****) $b \succ_s' a$, i.e., $(v_1'',v_2'') \succ_s' (v_1',v_2')$.

But, from Figure 2, (v_1',v_2') is on a higher social indifference curve than (v_1'',v_2''), contradicting (****).[20] Q.E.D.

This argument makes clear that we can interpret the Impossibility Theorem as saying simply that the only social indifference curves for which the social ranking remains invariant to positive affine transformations of utilities are (i) vertical curves and (ii) horizontal curves.

3. *General Equilibrium*

Economics has been obsessed with markets at least since Adam Smith (Smith 1776) and with good reason - - they provide a powerful and decentralized way of organizing an economy. One might have thought that a system in which consumers and firms are all pursuing their individual goals with no central coordinating mechanism would lead to chaos. But through Smith's "invisible hand," markets harness self-interest and generate not just order, but a remarkable degree of efficiency (c.f., the First Welfare Theorem, discussed below). As Smith famously put it, "It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own interest."

Most economic analyses concentrate on a single market at a time. Indeed, *The Wealth of Nations* contains long disquisitions on the markets for labor, land, and gold. Often these analyses focus on the equilibrating effect of the market price. For example, if a blight wipes out much of

[20] This proof is based on ideas in Geanakoplos (2005) and Roberts (1980).

the potato crop in Ireland, we would expect the price of potatoes to rise sharply. This will have the effect of bringing demand into line with the suddenly reduced supply of potatoes, but it will also induce potato growers in other countries to increase their outputs, thus mitigating the initial shortage.

This story considers the potato market alone, but, in reality, it is not isolated. For example, potato prices affect demand and supply for wheat, rice, and meat. And their prices in turn affect still other markets. In short, it is not enough to look at one market at time when studying the equilibration of supply and demand; we must consider all markets simultaneously.

This point was understood by the nineteenth-century economists Léon Walras (Walras 1874-1877) and Vilfredo Pareto (Pareto 1896-1897). However, their approaches to supply and demand in multiple markets were far from careful. Indeed, Walras seemed to think that in a system with *m* markets, there must exist prices for which supply equals demand in every market – i.e., a *general equilibrium* – because there are as many prices to solve for as there are supply-and-demand equations (viz., *m*). A major contribution of Arrow and Debreu (1954) – henceforth AD – was to rigorously establish existence of general equilibrium in a general model of a private-ownership economy.[21]

3A. *The AD Model*

The AD model consists of *consumers*, who consume goods, and *firms*, which produce goods. Each firm is described by its *production set* comprising the combinations of inputs and

[21] Abraham Wald (1936) and John von Neumann (1937) provided the first mathematically acceptable proofs of equilibrium existence, but in models much more special than that of AD. Lionel McKenzie (1954) published a quite general existence proof in the same issue of *Econometrica* as AD. McKenzie's model, however, is not as "fundamental" as that of AD because it starts with demand functions rather than with consumers' endowments and their shares in firms' profits.

outputs that are feasible for the firm, i.e., its feasible *production plans*. Each consumer is described by (i) her *consumption set*, consisting of her feasible *consumption plans* (a consumption plan specifies how much of each good she intends to consume); (ii) her *preferences* over consumption plans; (iii) her initial *endowment*, which specifies how much of each good she starts with; and (iv) her *shares* of the different firms' profits. An *allocation* is a specification of feasible production and consumption plans such that supply equals demand for each good (demand for a good is the total amount of that good consumers intend to consume; supply equals the total initial endowment of the good plus the additional amounts produced by firms minus the amounts they use as inputs).

Each good entailed in a firm's production plan or a consumer's consumption plan can be bought or sold at a publicly known *price*; this is called the *complete markets* assumption. Firms and consumers take all prices as *given*. This is simply assumed by AD, but gets at the idea that individual firms and consumers are too small relative to the market to affect prices.[22] For given prices, a firm's profit from a particular production plan is the value of the outputs in the plan minus the value of the inputs. A consumer's income for given prices is the value of her endowment plus the sum of her shares of the various firms' profits. A *competitive equilibrium* is an allocation and a specification of prices such that, given those prices, (i) each firm's production plan maximizes its profit within its production set; and (ii) each consumer's consumption plan maximizes her preferences within her consumption set, given her income (i.e., she prefers no

[22] There has been a great deal of subsequent work on *deriving* price-taking behavior in a model with a large number of consumers or firms. See for example Roberts and Postlewaite (1976).

other consumption plan that is also feasible and affordable).[23] A competitive equilibrium is the fulfilment of Walras's and Pareto's aspiration: simultaneous equilibration in all markets.

The AD model is remarkably general, broad enough to include as special cases nearly every general equilibrium models up to 1954 and most subsequent models too.[24] Indeed, it is considerably broader than it may initially appear to be. This is because it allows the same physical good in different locations or at different dates to be treated as different goods. Thus, the AD model readily accommodates settings in which geography or time plays an important role.

If the AD model is applied to a setting with multiple dates, it requires the implicit assumption that consumers and firms make *once-and-for-all* decisions (i.e., at the first date, consumers choose their consumption not only for that date, but for all subsequent dates; firms do the same with production). This means, in particular, that there must be *futures markets* for all goods, so that, for instance, a consumer can sell her future endowments to finance her current consumption.[25]

3B. *Equilibrium Existence*[26]

AD establishes the existence of competitive equilibrium by turning the competitive model into a noncooperative game in which the players are firms (who choose production plans as strategies to maximize profits), consumers (who choose consumption plans to maximize

[23] In an allocation, supply equals demand for each good. But, in fact, AD allows supply to *exceed* demand for a good in competitive equilibrium as long as the equilibrium price for the good is zero (this means that, even at zero price, consumers don't want more of it).
[24] The exceptions are usually models that violate the complete market assumption - - but, even then, the AD formulation is often the point of departure.
[25] An alternative to commodity futures markets is a *capital market* that allows a consumer or firm to borrow or lend future income. See the discussion of Arrow securities in subsections 3E and 3E* for more details.
[26] Existence is a technical matter, but I'm omitting most of the details in this subsection; see 3B* for a fuller account.

preferences), and an artificial player, the "auctioneer" (who chooses prices to maximize the value of excess demand – the difference between demand and supply).[27] AD show that this game[28] has a Nash equilibrium (Nash 1950), which they then establish is a competitive equilibrium.

To make this argument, they make certain assumptions about production sets, consumption sets, and consumer preferences. First, they impose some technical conditions ensuring that there are solutions to firms' and consumers' maximization problems.[29]

More interestingly, they suppose that (i) each firm's production set is convex, which means that production has constant or decreasing returns to scale; (ii) each consumer's consumption set is convex – which means that, for any two feasible consumption plans, a weighted average of the two plans is also feasible; (iii) the consumer is endowed with a positive quantity of each good[30]; (iv) preferences are convex, which means that if consumer prefers consumption plan *A* to plan *B*, then she also prefers a weighted average of *A* and *B* to *B*; (v) consumers are unsatiated, which means that no matter which consumption plan a consumer chooses, there is another that she prefers to it; and (vi) the set of allocations is bounded.

Assumption vi says that unlimited production and consumption are infeasible. From a mathematical perspective, assumptions i-v serve to ensure that supply and demand are "well behaved" (e.g., that they vary continuously with prices), but they have real economic content too.

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[27] This objective for the auctioneer gets at the idea that if the system is out of equilibrium, prices should rise for goods in excess demand and fall for those in excess supply.
[28] Actually, the game is not quite standard because a consumer's strategy space (her set of affordable consumption plans) itself depends on other players' strategies, viz., on prices and production plans. See subsection 3B* for details.
[29] Specifically, they assume that production and consumption sets are closed and that preferences are continuous.
[30] I explain the role of this assumption in subsection 3B*.

In particular, assumption i rules out increasing returns to scale – where doubling inputs *more* than doubles outputs – a phenomenon that is not just technically problematic but in conflict with the premise that firms take prices as given. That is, we would expect that, with significant increasing returns, firms will be large relative to the market and so should be able to affect prices appreciably.

Assumption iv, requiring that preferences be convex, is standard in the theory of consumer behavior and expresses the principle of diminishing marginal rates of substitution: a consumer requires increasingly larger amounts of one good to make up for successive losses of another.

Assumption v is a weak version of the notion that more of a good is always better for a consumer.

3C. *The Welfare Theorems*

The idea that a competitive equilibrium might have attractive optimality properties is a major reason why economists have been so attracted to competitive environments. Optimality is usually expressed by the two welfare theorems, which go back – in one form or another – to Walras and Pareto, if not earlier.

To state the theorems, let's define an allocation to be *Pareto optimal* if there exists no other allocation that every consumer finds at least as preferable as the original allocation and some consumer finds strictly preferable. The *First Welfare Theorem* (FWT) then asserts that a competitive equilibrium allocation is Pareto optimal, whereas the *Second Welfare Theorem* (SWT) establishes that any Pareto optimal allocation can be "decentralized" in the sense that it arises in a competitive equilibrium when consumers have incomes equal to the values of their consumption plans in this allocation (their "target consumption plans").

Arrow (1951a) and Debreu (1951) separately gave the first general statements and proofs of the welfare theorems. Versions of these theorems had previously appeared in many places, but always with unnecessary extra assumptions. For example, earlier proofs of the FWT typically assumed that the boundaries of firms' production sets are differentiable, equilibrium consumptions plans are in the interior of the consumption set, and consumers' utility functions (representing their preferences) are differentiable and concave.[31] But as Arrow (1951a) and Debreu (1951) show, this result holds without *any* assumptions on production sets, consumption sets, or preferences except for the requirement that preferences be unsatiated.[32]

It is fair to say that the FWT has been far more important in economics than the SWT. Indeed, the FWT is the primary intellectual underpinning of policies aimed at making markets more competitive. The SWT may be satisfying theoretically, but since policies ensuring that consumers have incomes equal to the values of their target consumption plans are difficult to imagine, it seems of little practical use.

3D. *Uncertainty*

I already mentioned in subsection 3A that the AD model allows for goods to be distinguished by location and date. Arrow (1953) points out that goods can also be distinguished according to the *resolution of uncertainty*.

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[31] These assumptions allow one to differentiate to obtain the first-order conditions for a competitive equilibrium and then show that they are the same as those for maximizing a weighted sum of consumers' utilities.

[32] AD need closedness, convexity, and continuity assumptions to guarantee existence of equilibrium, but such a guarantee isn't needed for the FWT. See the proof in subsection 3C*.

Imagine, for example, an economy in which wheat, umbrellas, and labor are the only goods. Labor can be employed today to produce wheat and umbrellas tomorrow. Let's suppose that the quantity of wheat produced depends not just on labor but on whether it is rainy or sunny tomorrow, and that consumers' preferences over the three goods also depend on the weather tomorrow (say, consumers want more umbrellas when it rains).

Arrow shows that such an economy can easily be embedded in a two-date AD framework. Specifically, let us define two *states of nature*: "rain" and "sun." At the first date ('today'), firms buy labor and choose production plans[33]; they also sell date-2 wheat and umbrellas *contingent* on the state (there is a rain-contingent wheat price and a sun-contingent wheat price; the same for umbrellas). Consumers do just the opposite: they sell labor and buy state-contingent wheat and state-contingent umbrellas. At the second date ('tomorrow'), umbrellas and wheat are realized (depending on the state of nature) and firms deliver these goods to consumers as dictated by the date-1 trades and the realized state. Notice that although this is a two-date model, all exchange takes place at the first date. The only economic activities at date 2 are the realizations of wheat and umbrella production and delivery[34]. Observe too that the ability to trade on contingent markets implies that there is *no uncertainty* about firms' profits and consumers' incomes, despite the uncertainty about the state of nature. Of course, consumers' consumption plans and firms' production plans *do* entail risk.[35]

[33] A production plan is a *random variable* specifying, given an input of labor, how much wheat and how many umbrellas get produced for each state of nature.
[34] In this simple example, there are no production decisions at date 2. But in a more general model, a firm may also deploy resources at date 2 (see the model of subsection 3D*).
[35] Previously, we interpreted the convexity assumptions about consumers' preferences as reflecting *decreasing marginal rates of substitution*. But with uncertainty, convexity also reflects *risk aversion*. Consider, for example, two consumptions plans, *A* and *B*: in *A*, a consumer gets 1 unit of a good in state 1 and none in state 2; in *B*, she gets 1 unit in state 2 and nothing in state 1. If she is risk averse and the states have equal probabilities, she will prefer the convex combination in which she gets 1/2 unit in both states to either *A* or *B*; i.e., her preferences are convex.

Under the same assumptions as in subsection 3B, a competitive equilibrium for the labor market together with the four state-contingent markets exists (call this a *contingent-markets equilibrium*).

Now, once the number of goods and possible states of nature becomes realistically large, the number of *contingent* markets becomes unrealistically huge. Hence, another crucial insight of Arrow (1953) is that contingent commodity markets can be replaced entirely by trade in *securities* and ordinary *spot* markets. I will lay out this idea – a foundation for much of the modern literature in finance – in the next subsection.

For a contingent-markets trading system to work satisfactorily, it is important that states of nature be *verifiable* after the fact. To see what can go wrong otherwise, imagine that some consumer has bought an umbrella contingent on rain but not one contingent on sun. If the state of the weather can't be proved, the firm that is supposed to deliver the umbrella can always claim that the weather is sunny regardless of whether it really is and simply hold onto the umbrella. This illustrates the general problem that lack of verifiable information can interfere with the complete markets assumption.

3E. *Arrow Securities*

Consider the wheat/umbrella/labor economy of subsection 3D. Let's imagine that, at date 1, a firm sells two kinds of *securities* rather than state-contingent wheat and umbrellas. One security pays a dollar in the rainy state (and nothing in the sunny state); the other pays a dollar in the sunny state (and nothing in the rainy state). In effect, the firm is giving a security purchaser the wherewithal to buy wheat and umbrellas at date 2 rather than selling her these goods directly at date 1. It is not hard to see that the contingent market equilibrium of subsection 3D can be replicated by a competitive equilibrium for an economy consisting of the labor market, the two securities markets, and the date-2 wheat and umbrellas markets (which are spot markets, rather than contingent future markets)[36].

Specifically, in this new equilibrium, a firm will sell enough of the rainy-state security to match its income from selling rain-contingent wheat and umbrellas in the original equilibrium, and similarly for its sale of the sunny-state security. Correspondingly, a consumer will purchase enough of the rainy-state security to afford buying the same amounts of wheat and umbrellas in the date-2 rainy state as she bought contingently in the original equilibrium (and analogously for her purchase of the sunny-state security).

In other words, there is no need for contingent commodity markets at all; security markets plus spot markets can fully substitute for them. Furthermore, the securities/spot route entails, in general, far fewer markets. If, for example, there are 10 date-2 goods and 10 possible states of nature, contingent markets require $10 \times 10 = 100$ markets. By contrast, only 10 securities plus 10 spot markets – for a total of 20 markets in all – are needed via security and spot markets[37,38].

A security that, as in our example, pays off just in one state is called an *Arrow security*. In the example, there is a *complete set* of Arrow securities: one for each state. They give rise to a payoff matrix as in Table 2.

[36] Thus, unlike in the standard multiperiod AD model, not all economic exchange occurs at date 1.

[37] There is, however, a downside, to using spot markets: firms and consumers need to forecast date-2 spot prices correctly; by contrast, no forecasting is needed with contingent markets since all trade occurs at date 1.

[38] We have been discussing securities in the context of uncertainty, but observe that they are useful even in dynamic models with complete certainty. We noted in subsection 3A that a multiperiod AD model assumes futures markets for all goods. But such futures markets can be avoided if there are securities that enable consumers and firms to transfer wealth across periods.

		States	
		rain	sun
Securities	rain-contingent	1	0
	sun-contingent	0	1

Table 2

More generally, one can show that, in a model with *m* states, any set of securities will serve as well as the set of *m* Arrow securities as long as the corresponding payoff matrix has rank *m*.

3F. *Externalities*

The standard AD model presumes that no firm's choice of production plan and no consumer's choice of consumption plan affects any other firm or consumer. If this presumption is violated, we say that the firm or consumer in question creates an *externality* for the affected parties.

A classic example of an externality is *pollution*. Suppose that one of the outputs in a steel firm's production is atmospheric smoke. This smoke interferes in other firms' production (think of a laundry located near the steel factory) and also harms consumers (e.g., their health may be damaged).

Externalities do not ordinarily affect the *existence* of competitive equilibrium, but they do affect its *optimality*. This is because the First Welfare Theorem relies critically on the *complete markets assumption* – the requirement that consumers and firms be able to buy and sell *all* the goods that enter their objective functions – and this assumption *fails* when there are externalities: e.g., the laundry cannot buy a reduction in the steel producer's smoke output.

Recognizing this point, Arrow (1969) imagines expanding the set of markets so that agents *can* buy and sell external effects. So, in the smoke example, the steel producer will sell *smoke reduction* and each of the affected parties will buy smoke reduction. In this exchange, the producer will receive the *sum* of the parties' payments. Of course, in equilibrium, the amount of smoke reduction must be *same* for everyone. Thus, since different parties may not all value smoke reduction equally, they may have to pay different amounts for it (in effect, they face *personalized* prices)[39]. If markets are created for all external effects, the FWT is thereby restored; competitive equilibrium is once again Pareto optimal.

Arrow's expanded economy is illuminating conceptually, but he didn't intend it as a practical solution to externalities. Indeed, there are at least two considerable obstacles to instituting such a scheme in reality. First, if each affected party has its own personalized price (so that there is just one trader on each side of the market), then the standard assumption that consumers and firms take prices *as given* strains credulity. Second, in Arrow's conceptual framework, each affected party buys the *entire* smoke reduction on its own - - in effect, it expects that exactly the reduction it buys will be implemented. But other parties are doing the same thing, and so even if a given party stays out of this market itself, there will still be smoke reduction, contrary to its expectation[40,41].

3G. *Stability*

One issue left out of our discussion so far is how equilibrium is *reached*. Presumably, if supply and demand aren't equal, prices and quantities will change to bring them into line. In other words, equilibrium is the convergence point of some adjustment process. Since Walras, one intensely studied adjustment process has been the *tâtonnement*. In such a process, the price for

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[39] This is the heart of Lindahl (1919)'s solution for dealing with public goods.

[40] The Arrow scheme may be impractical as a competitive market but it gets at the Coasean idea (Coase 1960) that externality problems can often be solved if the parties concerned get together and reach a bargain (Arrow's personalized prices can be interpreted as the terms of trade reached in the bargain). Indeed, Coase was skeptical of the Pigouvian approach (Pigou 1920) in which taxes and subsidies are used to correct all external effects.

[41] Moreover, as Starrett (1972) shows the convexity assumptions needed for equilibrium existence may be violated.

good i will rise in proportion to excess demand if demand exceeds supply. Similarly, the price falls proportionately when excess demand is negative. If, starting near enough competitive equilibrium prices, the tâtonnement converges to that equilibrium, then the equilibrium is *locally stable*. If regardless of the starting point, the process converges to the equilibrium, the equilibrium is *globally stable* (and therefore unique).

Arrow, Block, and Hurwicz (1959) give three alternative sufficient conditions for global stability[42]: (i) all pairs of goods are *gross substitutes*[43]; (ii) the economy satisfies the *weak axiom of revealed preference*[44]; and (iii) the matrix of partial price derivatives of excess demand has a dominant diagonal[45].

3*. *General Equilibrium – Formalities*

3A*. *The Arrow -Debreu Model*

There is a set of *goods*, indexed by g, $g = 1, \ldots, G$; a set of *firms*, indexed by f $f = 1, \ldots, F$; and a set of *consumers*, indexed by h[46], $h = 1, \ldots, H$.

Each firm f has a *production set* $Y^f \subseteq \mathbb{R}^G$. For each *production plan* $y^f = (y_1^f, \ldots, y_G^f) \in Y^f$, positive components correspond to *outputs* and negative components to *inputs*. Each consumer h is described by her *consumption set* $X^h \subseteq \mathbb{R}^G$; her preferences $\succsim^h$

[42] Scarf (1960) shows that without such conditions there might be no equilibrium that is even locally stable.
[43] I.e., an increase in the price of one good causes an increase in consumers' demand for the other good.
[44] I.e., if the vector of aggregate demand is x at prices for which x' is affordable (i.e., consumers' aggregate incomes are sufficient to purchase x'), then x' is never the aggregate demand at any prices for which x is affordable.
[45] I.e., for every row, the magnitude of the diagonal entry is greater than or equal to the sum of the magnitudes of the other entries.
[46] h is a mnemonic for "household."

over X^h; her *initial endowment* $\omega^h \in \mathbb{R}_+^G$ (specifying how much of each good she starts with);

and, for all f, her share in firm f's profit s^{hf}, where $0 \le s^{hf} \le 1$ for all h and f and $\sum_{h=1}^{H} s^{hf} = 1$ for all f. For all consumption plans $x^h = (x_1^h, \ldots, x_G^h) \in X^h$, we have $x_g^h \ge 0$ for all g (the consumer can't consume negative quantities of goods).

An *allocation* is a specification of production plans $\{y^f\}_{f=1}^F$ and consumption plans $\{x^h\}_{h=1}$ – with $y^f \in Y^f$ for all f, and $x^h \in X^h$ for all h – such that

$$\sum_{h=1}^{H} x^h = \sum_{h=1}^{H} \omega^h + \sum_{f=1}^{F} y^f,$$

i.e., supply equals demand for every good.

We assume that all goods can be bought and sold at market prices $p = (p_1, \ldots, p_G) \in \mathbb{R}_+^G$, and firms and consumers take prices as given. Given prices p, firm f's *profit* from production plan y^f is

$$p \cdot y^f = \sum_{i=1}^{G} p_i y_i^f,$$

and consumer h's *income* from production plans $\{y^f\}_{f=1}^F$ is

$$I^h(p, \{y^f\}_{f=1}^F) = p \cdot \omega^h + \sum_{f=1}^{F} s^{hf} p \cdot y^f .$$

A *competitive equilibrium* consists of prices $\hat{p}$ together with an allocation $\{\hat{y}^f\}, \{\hat{x}^h\}$ such that

(a) $\hat{p} \cdot \hat{y}^f \geq \hat{p} \cdot y^f$ for all $y^f \in Y^f$ and all f

(b) $\hat{p} \cdot \hat{x}^h \leq I^h(\hat{p}, \{\hat{y}^f\})$ for all h

and

(c) $\hat{x}^h \succsim x^h$, for all $x^h \in X^h$ such that $\hat{p} \cdot \hat{x}^h \leq I^h(\hat{p}, \{\hat{y}^f\})$ and for all h.

Condition (a) says that firms are maximizing profits. Condition (b) says that consumers are staying within their incomes. And condition (c) says that consumers are maximizing their preferences subject to their incomes.

3B*. *Equilibrium Existence*

To prove existence expeditiously (while still illustrating the main ideas), I will make assumptions that are somewhat stronger than in AD. Specifically, I will assume that

(i) each production set Y^f is closed, bounded,[47] and convex and contains the point $(0,\ldots,0)$ [48] for all f

(ii) $X^h = \mathbb{R}_+^G$ for all h (any consumption plan with nonnegative components is feasible)[49]

(iii) $\omega_g^h > 0$ for all g and h (a consumer has a positive endowment of every good)

(iv) $\succsim^h$ is continuous, convex, and increasing[50] for all h

Existence Theorem: Under assumptions (i)-(iv), a competitive equilibrium exists.

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[47] AD do not require boundedness of production sets; instead they assume that the set of allocations is bounded.
[48] I.e., it is feasible for the firm to do nothing. This ensures that profit is nonnegative.
[49] AD allow for more general consumption sets.
[50] That is, if $x \geq x'$ (i.e., $x_g \geq x'_g$ for all g with at least one strict inequality), then $x \succ^h x'$. AD require only nonsatiation (see section 3B).

Proof: Let's study the game in which (i) each firm *f* chooses $y^f \in Y^f$ to maximize its profit $p \cdot y^f$, given prices *p*; each consumer *h* chooses x^h to maximize her preferences $\succsim^h$ given her income $I^h(p,\{y^f\})$; and, given $\{y^f\},\{x^h\}$, the auctioneer chooses prices $p \in \Delta^{G-1} = \{\tilde{p} \in \mathbb{R}_+^G \mid \sum \tilde{p}_g = 1\}$[51] to maximize

$$p \cdot (\sum_{h=1}^{H} x^h - \sum_{h=1}^{H} \omega^h - \sum_{f=1}^{F} y^f),$$

i.e., the auctioneer maximizes the value of excess demand (the difference between demand and supply).[52]

In order to apply the Kakutani Fixed Point Theorem---the KFPT (Kakutani 1941)--- and show that the game has a Nash equilibrium, we need each player's strategy set to be closed, convex and bounded. This is true by assumption for the auctioneer and the firms. For consumers, convexity and closedness are also automatic. But if the price of good *g* is zero, then they can buy unlimited amounts of it (even though they have a limited income). So, we will use a trick devised by AD to bound consumers' strategy spaces. For each *g*, let $\bar{x}_g$ be a quantity of good *g* so big that it is infeasible for firms to produce that much (given their bounded production sets and the finite endowments). Let $\bar{X}^h = \{x \in \mathbb{R}_+^G \mid x_g^h \le \bar{x}_g \text{ for all } g\}$. That is, we are restricting consumers to consume no more than $\bar{x}_g^h$ of any good.

[51] Notice that we have made the (harmless) normalization that prices sum to 1.
[52] This is actually a generalization of the usual concept of a game because consumer *h* is restricted to choosing consumption plans that are affordable given her income, i.e., her feasible strategy space depends on others' strategies. Debreu (1952) establishes existence of a Nash equilibria under the conditions of the Existence Theorem, and I follow his approach here.

Now, define the players' best-response correspondences:

$$\gamma^f(p) = \{y^f \in Y^f \,|\, p \cdot y^f \geq p \cdot \tilde{y} \text{ for all } \tilde{y} \in Y^f\} \text{ (firms maximize their profits, given } p\text{)}$$

$$\gamma^h(p,\{y^f\}) = \{x^h \in \bar{X}^h \,|\, p \cdot x^h \leq I^h(p,\{y^f\}) \text{ and } x^h \succsim^h \tilde{x}^h \text{ for all } \tilde{x}^h \in \bar{X}^h \text{ such that}$$

$$p \cdot \tilde{x}^h \leq I^h(p,\{y^f\})\} \text{ (consumers maximize their preferences, given } p \text{ and } \{y^f\})$$

$$\gamma^a(\{y^f\},\{x^h\}) = \{p \in \Delta^{G-1} \,|\, (p - \tilde{p}) \cdot (\sum x^h - \sum \omega^h - \sum y^f) \geq 0 \text{ for all } \tilde{p} \in \Delta^G\} \text{ (the auctioneer maximizes the value of excess demand, given } \{y^f\} \text{ and } \{x^h\})$$

To show that the game has a Nash equilibrium, we apply the KFPT to establish that the mapping

$$(\{y^f\},\{x^h\},p) \mapsto \prod_{f=1}^{F} \gamma^f(p) \times \prod_{h=1}^{H} \gamma^h(p,\{y^f\}) \times \gamma^a(\{y^f\},\{x^h\})$$

has a fixed point. For the KFPT to be applicable, the mapping must be convex- and nonempty-valued and upper hemicontinuous (uhc).

That the mapping is convex-valued is completely standard given Nash (1951) and follows from the convexity of $Y^f, \bar{X}^h$, and Δ^{G-1}; the linearity of the firms' and the auctioneer's objective functions; and the convexity of consumers' preferences.

That it is nonempty-valued is also completely standard and follows from the closedness and boundedness of $Y^f, \bar{X}^h$, and Δ^{G-1} and the fact that firms', consumers', and the auctioneer's objective functions are all continuous.

The part of the argument that is novel to AD is upper hemicontinuity. Even here, there are no complexities pertaining to firms and the auctioneer; their best-response correspondences are uhc simply from the closedness of the $Y^f s$ and Δ^{G-1} and the continuity of their objective functions. The complications arise instead for *consumers* because they are restricted to choosing consumption plans within their incomes. Consider a sequence $(p^m, \{y^{fm}\}, \{x^{hm}\})$ converging to $(p, \{y^f\}, \{x^h\})$ such that, for all *m* and *h*, $x^{hm} \in \gamma^h(p^m, \{y^{fm}\})$. For upper hemicontinuity, we must have $x^h \in \gamma^h(p, \{y^f\})$. If $p_{g_*} = 0$ for some good g_*, however, the concern is that there may exist $\tilde{x}^h$ with $p \cdot \tilde{x}^h \leq I^h(p, \{y^f\})$ for which $\tilde{x}^h \succ_h x^h$ because, at that limit, consumer *h* can suddenly afford the maximum amount $\overline{x}_{g_*}$ of good g_* (since its price is now zero). Indeed if $\omega^h_{g_*} = 0$, there is such a discontinuity[53] at $p_{g_*} = 0$ in the set of consumption plans that are affordable. But if $\omega^h_g > 0$ for all *g*, then, for *m* big enough, there must exist $\tilde{x}^{hm}$ near $\tilde{x}^h$ with $p^m \cdot \tilde{x}^{hm} \leq I^h(p^m, \{y^{fm}\})$ (this is why we make that assumption about endowments), and so by continuity of the consumer's preferences $\tilde{x}^{hm} \succ x^{hm}_m$, a contradiction of $x^{hm} \in \gamma^h(p^m, \{y^{fm}\})$. Hence the consumer's best-response correspondence is uhc.

We conclude that the KFPT applies and that the mapping has a fixed point $\hat{p}, \{\hat{y}^f\}, \{\hat{x}^h\}$, which is a Nash equilibrium of the game. Now, because consumers have increasing preferences, they will spend all their income, i.e.,

$$\hat{p} \cdot \hat{x}^h = \hat{p} \cdot (\omega^h + \sum_f s^{hf} \hat{y}^f) \text{ for all } h$$

We can sum this equation over consumers to obtain

$$\hat{p} \cdot (\sum \hat{x}^h - \sum \omega^h - \sum \hat{y}^f) = 0. \qquad (3B.1)$$

[53] Formally, a failure of lower hemicontinuity of the set of consumption plans that are affordable.

Suppose there is a good g_* for which excess demand is positive. Then the auctioneer can make the left-hand side of equation (3B.1) strictly positive by assigning $\hat{p}_{g_*} = 1$, a contradiction of the equation. So excess demand is nonpositive for all goods. If excess demand is strictly negative for some good g_*, then $\hat{p}_{g_*} = 0$ from (3B.1), and so consumers can afford to buy the maximum $\bar{x}_{g_*}$, which they will do since preferences are increasing. But then, excess demand for g_* is positive (since $\bar{x}_{g_*}$ is not feasible to produce), a contradiction again of (3B.1). Hence, supply equals demand for all goods. Thus, except for the artificial limitations $\{\bar{x}_g\}$ (see the next paragraph), the Nash equilibrium is a competitive equilibrium.

The last remaining step is to show that consumer *h* can't strictly improve her welfare once the constraints $x_g^h \leq \bar{x}_g$ for all *g* are removed. Suppose, to the contrary, that there exists $\tilde{x}^h$ such that $\tilde{x}^h \succ \hat{x}^h$ and $\hat{p} \cdot \tilde{x}^h \leq I^h(\hat{p}, \{\hat{y}^f\})$. But then, for all *g*, $\lambda \hat{x}_g^h + (1-\lambda)\tilde{x}_g^h < \bar{x}_g$ for λ slightly less than 1 and, from convexity of preferences, $\lambda \hat{x}^h + (1-\lambda)\tilde{x}^h \succ_h \hat{x}^h$, contradicting the preference maximality of $\hat{x}^h$. Q.E.D.

3C*. *The Welfare Theorems*

An allocation $\{y^f\},\{x^h\}$ is *Pareto optimal* if there does not exist another allocation $\{\tilde{y}^f\},\{\tilde{x}^h\}$ such that $\tilde{x}^h \succsim^h x^h$ for every consumer *h*, with strict preference for some consumer.

First Welfare Theorem: If $\{\hat{y}^f\},\{\hat{x}^h\}$ is a competitive equilibrium allocation for prices $\hat{p}$ and consumers' preferences are increasing, then the allocation is Pareto optimal.

Proof: Suppose, to the contrary, that $\{y^f\},\{x^h\}$ is an allocation that Pareto dominates $\{\hat{y}^f\},\{\hat{x}^h\}$, i.e., $x^h \succsim_h \hat{x}_h$ for all *h*, with strict preference for some *h*. From profit-maximization,

$$\hat{p}\cdot\hat{y}^f \geq \hat{p}\cdot y^f \text{ for all } f. \qquad (3C.1)$$

If $\hat{p}\cdot x^h < \hat{p}\cdot\hat{x}^h$, then consumer *h* could increase her consumption above x^h while still staying within income $I^h(\hat{p},\{\hat{y}^f\})$ (call $\tilde{x}^h$ the consumption plan with increased consumption). Because $\succsim^h$ is increasing, we have

$$\tilde{x}^h \succ_h \hat{x}^h,$$

contradicting preference maximization. Hence

$$\hat{p}\cdot x^h \geq \hat{p}\cdot\hat{x}^h \text{ for all } h\text{, with strict inequality for } h \text{ with } x^h \succ_h \hat{x}^h. \qquad (3C.2)$$

Summing over the inequalities (3C.1) and (3C.2) we obtain

$$\hat{p}\cdot(\sum x^h - \sum\omega^h - \sum y^f) > 0 = \hat{p}\cdot(\sum\hat{x}^h - \sum\omega^h - \sum\hat{y}^f),$$

contradicting the assumption that $\{y^f\},\{x^h\}$ is an allocation (i.e., that supply equals demand for each good). Q.E.D.

Second Welfare Theorem: Suppose that assumptions (i), (ii), and (iv) of the Existence Theorem hold (see subsection 3B*). If $\{\tilde{y}^f\},\{\tilde{x}^h\}$ is a Pareto optimal allocation such that $\tilde{x}_g^h > 0$ for all *g* and *h*, then there exist prices $\tilde{p}$ such that $\tilde{p},\{\tilde{y}^f\},\{\tilde{x}^h\}$ is a competitive equilibrium assuming each consumer *h*'s income is $\tilde{p}\cdot\tilde{x}^h$.

Proof: If given prices *p*, consumer *h* is assigned income $p \cdot \tilde{x}^h$, we can largely repeat the argument of the Existence Theorem to conclude that there exists a competitive equilibrium $\hat{p}, \{\hat{y}^f\}, \{\hat{x}^h\}$[54]. Because each consumer *h* can afford $\tilde{x}^h$, we have, from preference maximization,

$$\hat{x}^h \succsim^h \tilde{x}^h \text{ for all } h \tag{3C.3}$$

None of the preferences in (3C.3) can be strict because $\{\tilde{y}^f\}, \{\tilde{x}^h\}$ is assumed to be Pareto optimal. Hence,

$$\hat{x}^h \sim^h \tilde{x}^h \text{ for all } h. \tag{3C.4}$$

and so, because preferences are increasing,

$$\hat{p} \cdot \hat{x}^h = \hat{p} \cdot \tilde{x}^h \text{ for all } h \tag{3C.5}$$

From profit maximization,

$$\hat{p} \cdot \hat{y}^f \geq \hat{p} \cdot \tilde{y}^f \text{ for all } f \tag{3C.6}$$

If any inequality in (3C.6) is strict, we can sum equations (3C.5) and inequalities (3C.6) to obtain

$$\hat{p} \cdot (\sum \hat{x}^h - \sum \omega^h - \sum \hat{y}^f) < 0 = \hat{p} \cdot (\sum \tilde{x}^h - \sum \omega^h - \sum \tilde{y}^f),$$

a contradiction of the fact that $\{\hat{y}^f\}, \{\hat{x}^h\}$ is an allocation. Hence

[54] The proof of equilibrium existence for the SWT is slightly more complex than that for the Existence Theorem because in the proof of the latter, the fixed point $\hat{p}, \{\hat{y}^f\}, \{\hat{x}^h\}$ *automatically* satisfies (EB.1) (since aggregate consumer income equals the values of consumers' endowments plus firms' profits). By contrast, for the SWT a little more work is needed (see Maskin and Roberts 2008 for details).

$$\hat{p} \cdot \hat{y}^f = \hat{p} \cdot \tilde{y}^f \text{ for all } f \qquad (3C.7)$$

Take $\tilde{p} = \hat{p}$. From (3C.4) and (3C.7), $\tilde{p}, \{\tilde{y}^f\}, \{\tilde{x}^h\}$ is a competitive equilibrium. Q.E.D.

Note that the closedness, convexity, boundedness, and continuity hypotheses of the SWT are needed only to guarantee existence of competitive equilibrium. If instead we simply *assume* existence of an equilibrium for each Pareto optimal allocation $\{\tilde{y}^f\}, \{\tilde{x}^h\}$, only the assumption that preferences are increasing is required to show that $\{\tilde{y}^f\}, \{\tilde{x}^h\}$ is itself a competitive equilibrium allocation. In this sense, the SWT is the mirror image of the FWT; the critical assumption for each is increasing preferences.

3D*. *Uncertainty*

Consider a two-date version[55] of the AD model, but let's now introduce uncertainty. Specifically, at date 1, there is a set of goods indexed by $g_1 = 1, \ldots, G_1$. Just before date 2, the state of nature $\theta = 1, \ldots, \Theta$ is realized. The state θ potentially affects date-2 production, date-2 endowments, and preferences. At date 2, there is a set of goods indexed by $g_2 = 1, \ldots, G_2$. Each firm f has a production set $Y^f \subseteq \mathbb{R}^{G_1 + G_2\Theta}$. For each production plan $y^f \in Y^f$, the first G_1 components correspond to date-1 goods and the remaining $G_2\Theta$ components to state-contingent goods at date 2 (thus, given production plan y^f, $y^f_{\theta g_2}$ is firm f's output of good g_2 at date 2 contingent on state θ). Each consumer h is described by her preferences $\succsim^h$ over $\mathbb{R}_+^{G_1 + G_2\Theta}$, her initial endowment $\omega^h \in \mathbb{R}_+^{G_1 + G_2\Theta}$, and her shares $\{s^{hf}\}$ in firms' profits.

[55] See subsection 3A for a discussion of reinterpreting the AD model to include multiple dates. It is straightforward to generalize the analysis of this subsection to more than two dates.

Given prices $p \in \Delta^{G_1+G_2\Theta-1}$, each firm *f* chooses $y^f \in Y^f$ to maximize $p \cdot y^f$ and each consumer *h* chooses $x^h \in \mathbb{R}_+^{G_1+G_2\Theta-1}$ to maximize her preferences subject to

$p \cdot x^h \leq I^h(p,\{y^f\}) = p \cdot \omega^h + \sum_f s^{hf} p \cdot y^f$. As I noted in subsection 3D, firms' profits and consumers' incomes are *deterministic*, i.e., they entail no risk. A competitive equilibrium (called a *contingent-markets equilibrium*) $\hat{p},\{\hat{y}^f\},\{\hat{x}^h\}$ is defined exactly as in subsection 3A* except that the date-2 prices, outputs, and consumptions are now all state contingent.

3E*. *Arrow Securities*

Consider the two-date economy of subsection 3D* and fix a contingent-market equilibrium $\hat{p},\{\hat{y}^f\},\{\hat{x}^h\}$. For each state θ, imagine a security that is bought and sold at date 1 and yields a payoff of 1 (in the unit of account) at date 2 if and only if state θ is realized (if some other state is realized, the security pays nothing). This called *Arrow security* θ. Let π_θ be the probability of state θ.

I will show that the contingent-market equilibrium can be replicated by a competitive equilibrium $\tilde{p},\{\tilde{y}^f\},\{\tilde{x}^h\}$ of an economy in which (i) at date 1, date-1 goods are produced and traded (these are date-1 "spot" markets) and Arrow securities are traded; and (ii) at date 2, after state θ is realized, owners of Arrow security θ are paid off and there is production and trade of date-2 goods (these are the date-2 spot markets).

I will explicitly construct the competitive equilibrium. Let $\tilde{p}_{g_1} = \hat{p}_{g_1}$ for all g_1, $\tilde{p}_{\theta g_2} = \hat{p}_{\theta g_2} / \pi_\theta$ for all θ and g_2, and $\tilde{p}_\theta$ (the price of Arrow security $\theta) = \pi_\theta$.

We shall suppose that each firm *f* maximizes its expected total profit[56]. If the firm chooses production plan y^f and sells an amount z_θ^f of each Arrow security θ, then its expected profit at equilibrium prices $\tilde{p}$ is

$$\sum_{g_1=1}^{G_1} \tilde{p}_{g_1} y_{g_1}^f + \sum_{\theta=1}^{\Theta} \tilde{p}_\theta z_\theta^f + \sum_{\theta=1}^{\Theta} \pi_\theta (-z_\theta^f + \sum_{g_2=1}^{G_2} \tilde{p}_{\theta g_2} y_{\theta g_2}^f)$$

$$= \sum_{g_1=1}^{G_1} \hat{p}_{g_1} y_{g_1}^f + \sum_{\theta=1}^{\Theta} \pi_\theta z_\theta^f + \sum_{\theta=1}^{\Theta} \pi_\theta (-z_\theta^f + \sum_{g_2=1}^{G_2} (\hat{p}_{\theta g_2} / \pi_\theta) y_{\theta g_2}^f)$$

Thus, firm *f* maximizes its profit by taking $\tilde{y}^f = \hat{y}^f$ and $\tilde{z}_\theta^f = \sum \hat{p}_{\theta g_2} \hat{y}_{\theta g_2}^f$ (notice that z_θ^f does not affect *f*'s expected profit, and so any choice is optimal; however, taking $z_\theta^f = \sum \hat{p}_{\theta g_2} \hat{y}_{\theta g_2}^f$ ensures that *f*'s profit is zero except at date 1).

Given firms' choices, consumer *h* chooses x^h and $\{z_\theta^h\}$ to maximize $\succsim^h$ to subject to

$$\sum_{g_1} \tilde{p}_{g_1} \cdot x_{g_1}^h + \sum \tilde{p}_\theta z_\theta^h \leq \sum_{g_1} \tilde{p}_{g_1} \omega_{g_1}^h + \sum_f s^{hf} \tilde{p} \cdot \hat{y}^f \quad \text{(date-1 income)}$$

and

$$\sum_{g_2} \tilde{p}_{\theta g_2} x_{\theta g_2}^h \leq z_\theta^h + \sum_{g_2} \tilde{p}_{\theta g_2} \omega_{\theta g_2}^h \quad \text{for all } \theta \text{ (date-2/state } \theta \text{ income),}$$

which can be rewritten as

$$\sum_{g_1=1}^{G_1} \hat{p}_{g_1} x_{g_1}^h + \sum_{\theta=1}^{\Theta} \pi_\theta z_\theta^h \leq \sum_{g_1=1}^{G_1} \hat{p}_{g_1} \omega_{g_1}^h + \sum_{f=1}^{F} s^{hf} \hat{p} \cdot \hat{y}^f$$

and

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[56] In this model – unlike that of 3D* – firms' profits are in principle *uncertain* because their date-2 profits are realized only after θ is realized. However, in equilibrium, profits will turn out, once again, to entail no risk; all profits at date 2 will be zero.

$$\sum_{g_2=1}^{G_2} (\hat{p}_{\theta g_2} / \pi_m) x_{\theta g_2}^h \leq z_\theta^h + \sum_{g_2=1}^{G_2} (\hat{p}_{\theta g_2} / \pi_m) \omega_{\theta g_2}^h \text{ for all } \theta.$$

Thus, consumer *h* maximizes her preferences by taking $\tilde{x}^h = \hat{x}^h$ and, for all θ,

$\tilde{z}_\theta^h = \sum_{g_2=1}^{G_2} (\hat{p}_{\theta g_2} / \pi_m)(\hat{x}_{\theta g_2}^h - \omega_{\theta g_2}^h),$ and we have constructed a competitive equilibrium that results in the same production and consumption as the contingent-markets equilibrium.

4. *Other Major Scientific Contributions*

To round out my discussion of Kenneth Arrow's work, I will briefly touch on a few other contributions that have been especially influential.

4A. *Measures of Risk Aversion*

A *lottery* is a random variable whose possible realizations are monetary gains or losses. A decision-maker (DM) is said to be *risk averse* if she prefers the expected outcome of a lottery to the lottery itself. If the DM's preferences can be expressed by a von Neumann-Morgenstern utility function $u : \mathbb{R} \to \mathbb{R}$ [57], then she is risk averse if and only *u* is concave.

Arrow (1971) and Pratt (1964) independently proposed, that given *u*, the formula $-u''(z) / u'(z)$ is a good measure of how (absolutely) risk averse the DM is, where u' and u'' are the first and second derivatives of *u*. In particular, suppose that u_1 and u_2 are von Neumann-Morgenstern utility functions and $-u_1''(z) / u_1'(z) > -u_2''(z) / u_2'(z)$ for all *z*. If DM with utility

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[57] If utility function u represents the DM's preferences , then she prefers lottery $\tilde{z}^*$ to $\tilde{z}^{**}$ if and only if $E_{\tilde{z}^*} u(z) \geq E_{\tilde{z}^{**}} u(z)$. Von Neumann and Morgenstern (1944) gave necessary and sufficient conditions on preferences under which such a utility function exists.

function u_1 prefers lottery $\tilde{z}$ to certain outcome $\bar{z}$, then DM with utility function u_2 will also prefer $\tilde{z}$ to $\bar{z}$. That is, if a more risk-averse DM prefers a lottery to a sure thing, then a less risk-averse DM will prefer the same lottery to the sure thing[58].

4B. *Asymmetric Information and Medical Care*

Arrow (1963) notes that the market for medical care is rife with informational asymmetries. In particular, a patient will not typically know exactly what a physician is proposing to do nor will she be completely informed about what the physician knows. As I discussed in subsection 3D, such lack of information interferes with the complete markets assumptions; a patient cannot purchase physician services contingent on all relevant uncertainty. Accordingly, Arrow concludes that even if a competitive equilibrium exists, it is likely to lack the optimality properties of the First Welfare Theorem. He suggests, therefore, that *nonmarket* institutions such as self-regulation by the medical profession and a code of ethics for physicians can play an important role in improving market performance.

4C. *Learning by Doing*

Arrow (1962) considers a producer whose technique improves with experience; the more it produces, the more efficient production becomes. Other firms can benefit from what the firm has learned. Thus, there is an externality that the firm doesn't take into account, and we therefore expect the producer to underproduce relative to the optimum. This idea later became a foundation of the endogenous growth literature (See Lucas 1988 and Romer 1986).

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[58] Arrow and Pratt also developed a measure of *relative* risk aversion: $-zu''(z)/u'(z)$

4D. *Invention*

Arrow (1962a) points out that an invention shares some properties of a classic public good. In particular, once a discovery is made, allowing everyone to use it for free is optimal. The catch, however, is that such free-riding may greatly diminish an inventor's incentive to innovate in the first place.

Arrow reviews the standard argument that intellectual property rights such as patents can restore incentives; by temporarily awarding a monopoly, patents permit inventors to obtain a return on their investment. But he shows that patent holders still under-innovate relative to the social optimum. This is because a monopolist undercuts itself by improving on a good for which it already is earning monopoly profit (this logic is called the "Arrow effect").

In addition to these four pioneering areas, let me also mention Arrow and Kurz (1970) (which provides conditions under which one can pin down the social discount rate), Arrow and Lind (1970) (which does the same for the social cost of risk), Arrow and Fisher (1974) (which derives the option value for environmental goods), Arrow and Dasgupta (2009) (which discusses how public policy can be designed to enhance welfare in an economy with conspicuous consumption), and Arrow, Harris, and Marschak (1951) (which derives optimal inventory policy). I could continue with references to other important Arrow articles for many more pages.

5. *Beyond Research*

Kenneth Arrow was (in Paul Samuelson's words) "the most important theorist of the twentieth century in economics." But his importance extends well beyond his research.

To begin with, he was a dedicated and extraordinarily effective teacher. Four of his students went on to win Nobels of their own, and many others also attained great prominence. In the classroom, he was unfailingly patient and kind, but because his mind worked on a different level from everyone else, he sometimes failed to understand why students couldn't follow him. Told once by a class that he needed to define his terms more slowly, he came in the next day and wrote "$f(x)$" on the board. "This is a function," he explained. "The variable x is the input, and $f(x)$ is the output." To him, apparently, functions and fixed point theorems were all at the same level.

Arrow not only contributed much to our theoretical understanding of public goods, he was a major producer of them, both inside and outside the profession. To mention just a few of these goods: He was a founding editor (with Michael Intriligator) of the Handbook Series in Economics published by Elsevier. He was also a founding editor of the *Annual Review of Economics*, the journal publishing this article. Together with Menahem Yaari, he established the Jerusalem Summer School in Economic Theory, which he then directed for 18 years. He helped found the Santa Fe Institute, an interdisciplinary research center devoted to complexity. He demonstrated on picket lines for civil rights in the 1960s. He served on the staff of the Council of Economic Advisors in the early 1960s. He participated in a panel studying whether the U.S. should build a supersonic transport (he recommended against it) and chaired another panel on whether antimalarial treatments in Africa and Asia should be subsidized (he argued in favor). At the age of 92, he was an active member of the Lancet Global Health Commission.

He was a legendary participant in seminars. He would often fall asleep for long stretches but somehow wake up in time to make the most perceptive observation of all.

No essay on Ken Arrow would be complete without mentioning his utterly unpretentious personality and his extraordinary erudition in almost every imaginable subject. Larry Summers illustrates the unpretentiousness with a story[59] from the mid-1970s about how Ken once traveled from the AEA meetings in Atlantic City to his sister Anita's home in Philadelphia. Rather than take a limo (as a man of his distinction might have been expected to do) or get a ride from one of the Penn economists (as any of them would have been thrilled to provide), Ken took the bus. When Anita pointed out that there had been other options, Ken replied that they hadn't occurred to him.

The story I like to tell about Ken's vast store of knowledge concerns a group of junior faculty who concocted a scheme for outshining their learned senior colleague. They read up on the most arcane subject they could think of: the breeding habits of gray whales. On the appointed day, they gathered at coffee time and waited until Ken arrived. Then they started talking about the elaborate theory of a marine biologist named Turner about how whales find their way back to the same breeding spot every year. Ken was silent… they had him at last! With a delicious sense of triumph, they continued to discuss Turner, while Ken looked increasingly perplexed. Finally, he couldn't hold back: "But I thought Turner's theory was discredited by Spencer who showed that the proposed homing mechanism couldn't work."

[59] in a talk given at the Arrow Memorial Symposium, Stanford University, October 9, 2017

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