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The Unity of Auction Theory: Milgrom's Masterclass[1]

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1. *Introduction*

By any standard measure, auction theory has been an enormous success. Even after 25 years of intensive work, the literature continues to grow at a prodigious, even accelerating rate;[3] it has spawned much empirical and experimental research;[4] its tentacles have spread into other disciplines; [5] and auction theorists have been influential in the design of mechanisms for the privatization of public assets (such as spectrum bands) and for the allocation of electricity and other goods (they have also often served as consultants to the bidders in those mechanisms).

One explanation for this success is good timing. Many researchers started working seriously on auctions in the late 1970s and early 1980s,[6] just when the right game-theoretic methods for studying this subject—games of incomplete information (John Harsanyi 1967–68) and perfect equilibrium (Reinhard Selten 1975)—were becoming widely known. Of course, numerous other fields, e.g., industrial organization (I.O.), benefited from the same symbiosis of technique and application; collectively, they resulted in the game theory revolution. But the study of auctions has had more staying power than many other applications of game theory. Whereas enthusiasm for theoretical industrial organization has cooled somewhat since the heady days of twenty years ago, research on auctions, as I have noted, continues apace. There are, I believe, several reasons why auction theory has fared comparatively well.

First, theorists of I.O. and other applied fields labor under the constraint that they do not know the games that the players they

[1] Paul Milgrom, *Putting Auction Theory to Work*. Cambridge, Churchill Lectures in Economics: Cambridge University Press, 2004. NY, Melbourne, Pp. *xxi*, 368. ISBN 0-521-55184-6.

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[3] To give just one indication: at the August 2004 joint meeting of the Econometric Society and European Economic Association, there were seven separate sessions on auction theory, a figure well beyond that for any other sort of theory.

[4] Again, to cite only one, conference-related datum, the organizers of the 2005 Econometric Society World Congress, who attempt to invite special talks on the most lively and interesting developments in recent economics, are planning a set of talks on empirical auctions work.

[5] So, for example, there is now a sizeable computer-science literature on auction theory, often focusing on computational issues.

[6] Auction theory actually began well before then. Indeed, the seminal contribution was William Vickrey (1961). But until game theory came into its own fifteen years later, Vickrey's work—as well as that of other early pioneers such as James Griesmer, Richard Levitan, and Martin Shubik (1967), Armando Ortega Reichert (1968), and Robert Wilson (1969)—remained largely ignored.

study (e.g., firms or consumers) are *actually* playing; models are at best approximations of reality. By contrast, auction theorists typically know the rules that *their* players follow *precisely*. If, for example, a high-bid auction is the object of study, the theorist *knows* that (i) the bidders submit nonnegative real numbers as sealed bids; (ii) the winner is the bidder submitting the highest bid; and (iii) the winner pays his bid (of course, there may still be uncertainty about how the buyers *behave* under these rules). This precision helps put the auction theorist's findings on a relatively strong footing; it also simplifies the job of the experimentalist or empiricist.

Second, auction theory appeals to economists' "social engineering" instincts. Many people go into economics at least in part because they want to improve the world. The mechanism-design [7] aspect of auction theory—tinkering with the rules of the game in order to achieve a better outcome—helps gratify that urge. No doubt, one reason Vickrey's work is so celebrated is that his famous creation, the Vickrey auction, provides an attractive solution to an important social problem: designing an efficient allocation mechanism.

Third, the basic transaction of an auction—the transfer of a good from seller to buyer in exchange for a monetary payment—is fundamental to all of economics, and so auction theory has been nourished by its connection with other theoretical areas. For example, it has sometimes been used as a foundation for understanding the workings of competitive markets.[8] Of course, there are important differences: competitive theory usually supposes that there are large numbers of buyers and sellers, whereas in most auction theory, numbers are small (indeed, one implication of the papers of footnote 8 is that, as numbers grow, most reasonable sorts of auctions converge in performance; only in the small-numbers case do the differences between auction forms come into their own).

Finally, auction theory is a genuinely beautiful edifice:[9] many of its major propositions deliver remarkably powerful conclusions from apparently modest hypotheses.

Despite all these attractions, auctions might have disappeared from the economic theory scene had they not received an important rejuvenating boost from the worldwide impulse toward privatization that began in the early 1990s. This trend was brought on by the fall of communism in the East—and the consequent need to sell off state assets—and the disenchantment with public ownership in the West. But state bureaucracy's loss proved to be auction theory's gain, as auction mechanisms, to great public acclaim, were increasingly invoked to implement the transfer of resources. More recently, online auction enterprises such as eBay have provided further impetus for the theory.

2. *Milgrom's Unified View*

Paul Milgrom has played a starring role in auction theory's success story. Not only has he been a seminal contributor to the theoretical literature (e.g., Milgrom 1981; and Milgrom and Robert Weber 1982), but together with Robert Wilson, he had a major hand in designing the simultaneous ascending auction format used by the Federal Communication Commission to sell off much of the radio spectrum in the United States. Thus, his book *Putting Auction Theory to Work* has been eagerly awaited

[7] Of course, auction theory is only a small part of a vast mechanism design/implementation theory literature. For recent surveys of the literature from a general perspective see Thomas Palfrey (2002); Roberto Serrano (2004); and Eric Maskin and Tomas Sjöström (2002).

[8] See for example Milgrom (1981), Wolfgang Pesendorfer and Jeroen Swinkels (1997); Mark Satterthwaite and Steven Williams (1989); and Wilson (1977).

[9] Economists, being a hard-boiled lot, sometimes deny that esthetics have anything to do with what they are up to. But this sentiment belies the fact that the most important economic ideas, e.g., the first welfare theorem of competitive theory or the principle of comparative advantage, are things of real beauty.

dec04_Article 5 12/14/04 2:27 PM Page 1104

since his 1995 Churchill lectures, on which it is based.

The wait has clearly been worth it. The book covers a great deal of theoretical material and does so with extraordinary economy (without sacrificing rigor). This economy derives from Milgrom's conception of auction theory as a subspecies of demand theory, in which a few key tools—the envelope theorem in particular—do most of the work. Indeed, once these tools are in place, he establishes most theorems with just a few lines of proof. As the title suggests, he also discusses the extent to which the results bear on the design of real auctions.

Admittedly, the monograph is not the only current volume of reflections by a leading auction expert on theory and practice.[10] Nor, despite its unfailing clarity, is it the most likely candidate for a graduate text on the subject.[11] Rather, its signal contribution is to lay out Milgrom's unified view of the theory. This vision is notably distinct from that of other major auction scholars. For example, he must be nearly alone in deliberately avoiding the revelation principle as an auction-theoretic technique. But we quickly learn to enjoy seeing things his way. In this sense, the book is more a "master class" (to quote Al Roth's blurb on the back cover) than a text. And, of course, a master class is more fun.

As for his ideas on how to apply (or not to apply) the theory to actual auctions, these are certainly most welcome and enlightening. But, as they sometimes depend as much on judgment (albeit very well-informed judgment) as logic, they occasionally contrast jarringly with the authority and precision of the theory. For example, in Milgrom's opinion, the Vickrey auction (more precisely, its multigood generalization due to Theodore Groves 1973, and Edward Clarke 1971) is "unsuitable for most applications"—a conclusion that is far from being a theorem and that I will come back to in section 7.

But putting such quibbles aside, I should emphasize that Milgrom is completely persuasive on the general point that auction theory matters in practice. In chapter one, he shows that the 1990 New Zealand spectrum auction's failure to raise the revenue anticipated can be traced to its seriously flawed design: separate simultaneous sealed-bid auctions for each license. Specifically, he points out why this auction form cannot properly accommodate substitutability or complementarity across licenses. And he responds to those who argue that how government assets are sold off is irrelevant for efficiency (because, in their view, the "market" will correct any misallocation afterwards) with the theoretical riposte that, under incomplete information, there exists *no* nonconfiscatory mechanism (market-based or otherwise) capable of attaining efficiency, once the assets are in private hands (see Proposition 5 below).

The heart of chapters 2–8 consists of a succession of formal results, almost all proved in detail. I will try to reinforce the book's important lesson that auction theorems are easy to prove by stating and proving some of them below (although I will not attempt to replicate Milgrom's rigor or generality).

3. *Vickrey Auctions*

In chapter two, Milgrom turns to the most famous example of modern auction design, the Vickrey (or "second-price") auction (and its Groves–Clarke extension). Suppose that there is one unit of an indivisible good for

[10] Coincidentally, Paul Klemperer—like Milgrom, a theorist of the first rank and also a principal architect of the United Kingdom's 3G mobile-phone auction—has almost simultaneously produced his own take on the subject (Klemperer 2004). The two books differ markedly in style and substance. Milgrom's is primarily a compendium of theorems and proofs, together with less-formal observations about their application to actual auctions. Except for the appendices of chapters 1 and 2, Klemperer's is almost wholly nontechnical and consists largely of his views on the design of large-scale auctions in practice (although these views are informed by theory).

[11] Indeed, Vijay Krishna, yet another prominent auction theorist, has recently produced a beautifully lucid treatment (Krishna 2002), that in its organization and coverage may be more suitable as an introductory textbook.

dec04_Article 5 12/14/04 2:27 PM Page 1105

sale. There are n potential buyers, indexed by $i = 1, \ldots, n$, and each buyer i has a valuation v_i for the good (the most he is willing to pay for it). Thus if he pays p, his net payoff is $v_i - p$.

An *auction* is a game in which (i) buyers make "bids" for the good (for now we will be permissive about what a bid can be), on the basis of which (ii) the good is allocated to (at most) one of the buyers, and (iii) buyers make payments (which can in principle be negative) to the seller. An auction is *efficient* if, in equilibrium (we need not worry about the precise concept of equilibrium at this point), the winner is the buyer i with the highest valuation.[12]

Vickrey discovered that efficiency is attained by a *second-price auction*: an auction in which buyers submit nonnegative numbers as bids, the winner is the high bidder (ties can be broken randomly), and the winner pays the second-highest bid (nobody else pays anything). Formally, we have:

Proposition 1 (Vickrey 1961; Theorem 2.1 in Milgrom 2004): In a second-price auction, it is (weakly) dominant for each buyer i to bid his valuation v_i (i.e., regardless of how other buyers bid, it is optimal for buyer i to set a bid of $b_i = v_i$). Furthermore, the auction is efficient.

Proof: Suppose that buyer i bids $b_i < v_i$. The only circumstance in which the outcome for i is changed by his bidding b_i rather than v_i is when the highest bid b by other bidders satisfies $v_i > b > b_i$. In that event, buyer i loses by bidding b_i (for which his net payoff is 0) but wins by bidding v_i (for which his net payoff is $v_i - b$). Thus, he is *worse off* bidding $b_i < v_i$. By symmetric argument, he can only be worse off bidding $b_i > v_i$. We conclude that bidding his valuation (truthful bidding) is weakly dominant. Because it is optimal for buyers to bid truthfully and the high bidder wins, the second-price auction is efficient.

Q.E.D.

The key to the second-price auction's dominant-strategy property is the fact that a winning buyer's payment does not depend on his bid. Next, we show that, under mild hypotheses, the second-price auction is the *only* efficient auction with this property (modulo adding a term not depending on b_i to buyer i's payment):

Proposition 2 (Jerry Green and Jean-Jacques Laffont 1977; Bengt Holmström 1979; Laffont and Maskin 1980; Milgrom's Theorem 2.3): Suppose that, for all i, v_i can assume any value in [0, 1]. Then an auction is efficient and bidding truthfully is weakly dominant if and only if (a) the high bidder wins and (b) for all i, buyer i's payment p_i satisfies

$$p_i = \begin{cases} \max\limits_{j \neq i} b_j + t_i(b_{-i}), \text{ if buyer } i \text{ wins} \\ t_i(b_{-i}), \text{ if buyer } i \text{ loses} \end{cases}$$

for some function t_i, where b_{-i} is the vector of bids other than b_i.

Proof: Consider an efficient auction in which truthful bidding is dominant. Then, the high bidder must win (property (a)). As for (b), let $t_i^L(b_i, b_{-i})$ be buyer i's payment if he loses and the bids are (b_i, b_{-i}). If

$$t_i^L(b_i', b_{-i}) > t_i^L(b_i'', b_{-i})$$

for bids $b_i', b_i'' \leq \max\limits_{j \neq i} b_j$, then buyer i is better off bidding b_i'' when $v_i = b_i'$, contradicting the dominant-strategy property. Hence, we can write $t_i^L(b_i, b_{-i})$ as

$$t_i^L(b_i, b_{-i}) = t_i(b_{-i}). \qquad (1)$$

Similarly, we can write buyer i's payment $t_i^W(b_i, b_{-i})$ if he wins as

$$t_i^W(b_i, b_{-i}) = \hat{t}_i(b_{-i}). \qquad (2)$$

Now, if $v_i = \max\limits_{j \neq i} b_j$, buyer i's winning or losing are both efficient, and so for truthful bidding to be dominant, buyer i must be indifferent between them. From (1) and (2), we have

[12] Efficiency is often an important criterion in auction design, particularly in the case of privatization. Indeed, the U.S. Congress directed the FCC to choose an auction design for allocating spectrum licenses that (to quote Al Gore) puts "licenses into the hands of those who value them the most" (see Milgrom 2004, p. 4).

dec04_Article 5 12/14/04 2:27 PM Page 1106

$$\max_{j \neq i} b_j - \hat{t}_i(b_{-i}) = -t_i(b_{-i}).$$

Hence,

$$\hat{t}_i(b_{-i}) = \max_{j \neq i} b_j + t_i(b_{-i}),$$

i.e., (b) holds. Conversely, if (a) and (b) hold, it is immediate that the auction is efficient and, from Proposition 1, that truthful bidding is dominant. Q.E.D.

Call the auctions of Proposition 2 "generalized Vickrey" auctions. It is easy to see that there is no generalized Vickrey auction in which payments "balance," i.e., sum to zero.

Proposition 3 (Green and Laffont 1977; Laffont and Maskin 1980; Milgrom's Theorem 2.2): Under the hypotheses of Proposition 2 there exists no generalized Vickrey auction in which the payments balance, i.e., $\sum_{i=1}^{n} p_i \equiv 0$.

Proof: For convenience assume $n=2$. Consider a generalized Vickrey auction. Choose $v_1 > v_2$. From Proposition 2

$$p_1 + p_2 = v_2 + t_1(v_2) + t_2(v_1). \quad (3)$$

If the right-hand side of (3) equals zero for all v_2, then

$$t_1(v_2) = -v_2 + k_1, \quad (4)$$

where k_1 is a constant. Similarly, for $v_2 > v_1$, we obtain

$$t_2(v_1) = -v_1 + k_2, \quad (5)$$

for constant k_2. From (4) and (5), we can rewrite (3) as

$$p_1 + p_2 = k_1 + k_2 - v_1,$$

which clearly cannot equal zero for all v_1. Hence, balanced payments are impossible. Q.E.D.

As Claude d'Aspremont and Louis-André Gérard-Varet (1979) show, the failure of balance in Proposition 3 can be overcome by relaxing the solution concept from dominant-strategy to *Bayesian* equilibrium:

Proposition 4 (d'Aspremont and Gérard-Varet 1979): Suppose that, for all i, v_i is drawn independently from a distribution with c.d.f. F_i and support [0, 1]. Then there exists an efficient and payment-balanced auction in which bidding truthfully constitutes a Bayesian equilibrium.

Proof: For convenience, assume $n=2$. In an auction where the high bidder wins and buyer 1's expected payment is $P_1(b_1)$ if he bids b_1, buyer 1 will choose b_1 to maximize

$$\int_0^{b_1} v_1 dF_2(x) - P_1(b_1),$$

if buyer 2 bids truthfully. The first-order condition for this maximization is

$$b_1 F_2'(b_1) = P_1'(b_1).$$

Thus if we set

$$P_1(b_1) = \int_0^{b_1} x F_2'(x) dx,$$

buyer 1's best reply to 2 is to bid truthfully (because the first-order condition holds at $b_1 = v_1$, and so does the second-order condition: $F_2'(v_1) \geq 0$). Similarly, truthfulness is a best reply for buyer 2 if his payment function is

$$P_2(b_2) = \int_0^{b_2} x F_1'(x) dx.$$

Now take as payment functions

$$p_1(b_1, b_2) = P_1(b_1) - P_2(b_2).$$

$$p_2(b_1, P_2) = P_2(b_2) - P_1(b_1).$$

Then it is evident that the players' payments sum to zero and that truthfulness remains an equilibrium (the latter follows because subtracting $P_2(b_2)$ from buyer 1's payment does not affect his incentives and similarly for buyer 2). Q.E.D.

Although balanced payments are consistent with efficiency once we relax the solution concept, we cannot also require *individual rationality* if one of the players already owns the good. More specifically, suppose that player 1 owns the good and that $n=2$. An efficient mechanism will transfer the good to player 2 if and only if $v_2 > v_1$. Thus, in a payment-balanced and efficient mechanism, *individual rationality* for player 1 (the "seller") is the requirement that

dec04_Article 5 12/14/04 2:27 PM Page 1107

$$\int_0^1 p_2\big(b_1(v_1),b_2(v_2)\big)dF_2(v_2) - \int_{v_2=v_1}^1 v_1 dF_2(v_2)$$

$$\geq 0 \text{ for all } v_1 \text{ (since } p_2=-p_1), \quad (6)$$

whereas individual rationality for player 2 (the "buyer") is the condition

$$\int_{v_1=0}^{v_2} v_2 dF_1(v_1) - \int_0^1 p_2\big(b_1(v_1),b_2(v_2)\big)dF_1(v_1)$$

$$\geq 0 \text{ for all } v_2, \quad (7)$$

where $b_1(v_1)$ and $b_2(v_2)$ are the (Bayesian) equilibrium bids by players 1 and 2 when their valuations are v_1 and v_2 respectively.

Proposition 5 (Laffont and Maskin 1979; Roger Myerson and Mark Satterthwaite 1983; Milgrom's Theorem 3.6): Let $n=2$. Under the hypotheses of Proposition 4, there exists no efficient and payment-balanced mechanism that is individually rational for both players when Bayesian equilibrium is the solution concept.

Proof: The proof is considerably simplified by supposing that F_1 and F_2 are uniform distributions on [0, 1]. Consider a balanced-payment and efficient mechanism for which $\big(b_1(\cdot),b_2(\cdot)\big)$ is a Bayesian equilibrium. Let

$$\hat{P}_1(v_1) = \int_0^1 p_2\big(b_1(v_1),b_2(v_2)\big)dv_2$$

and

$$\hat{P}_2(v_2) = \int_0^1 p_2\big(b_1(v_1),b_2(v_2)\big)dv_1 ,$$

where $p_2\big(b_1(v_1),b_2(v_2)\big)$ is buyer 2's equilibrium payment when valuations are (v_1,v_2). Hence, in equilibrium, player 1's and player 2's maximization problems are

$$\max_{\hat{v}_1}\left[\hat{P}_1(\hat{v}_1) - \int_{\hat{v}_1}^1 v_1 dv_2\right]$$

and

$$\max_{\hat{v}_2}\left[\int_0^{\hat{v}_2} v_2 dv_1 - \hat{P}_2(\hat{v}_2)\right].$$

In Bayesian equilibrium, $\hat{v}_1=v_1$ and $\hat{v}_2=v_2$, and so we obtain first-order conditions

$$\hat{P}_1'(v_1) + v_1 = 0$$

and

$$v_2 - \hat{P}_2'(v_2) = 0 .$$

We conclude that

$$\hat{P}_1(v_1) = -\frac{v_1^2}{2} + k_1 \quad (8)$$

$$\hat{P}_2(v_2) = \frac{v_2^2}{2} + k_2 , \quad (9)$$

where k_1 and k_2 are constants of integration. From (6) and (8) when $v_1=1$, we obtain

$$k_1 \geq \frac{1}{2} \quad (10)$$

and from (7) and (9) when $v_2=0$, we have

$$k_2 \leq 0 . \quad (11)$$

By definition of $\hat{P}_1$ and $\hat{P}_2$,

$$\int_0^1 \hat{P}_1(v_1)dv_1 = \int_0^1 \hat{P}_2(v_2)dv_2 .$$

Hence, from (8) and (9), we obtain

$$k_1 - \frac{1}{6} = k_2 + \frac{1}{6} ,$$

which contradicts (10) and (11). Q.E.D.

Notice the striking contrast between Propositions 1 and 4 on the one hand (which exhibit efficient auctions) and Proposition 5 on the other (which denies the existence of such a mechanism). The reason for the difference lies in the issue of ownership. In the former two propositions, no player yet owns the good. We can interpret the latter proposition, however, as applying to the circumstance in which there has already been an auction that player 1 won—so that he now has the opportunity to resell. Together, these two sets of propositions validate Milgrom's refutation of the claim that auctions are unnecessary for efficiency, that *ex post* free trade among the players will ensure the right allocation. According to this claim we might just as well assign assets like spectrum licenses randomly; firms can always exchange them later to correct misallocations. But Proposition 5 demonstrates that once the licenses have been distributed, efficiency may no longer be attainable. Q.E.D.

4. *Auction Equivalences*

A major achievement of auction theory is to have established equivalences between

dec04_Article 5 12/14/04 2:27 PM Page 1108

very different auction forms. Milgrom presents his view of this material in chapters 3 and 4. The central result is what he calls the *payoff-equivalence theorem* (which implies the considerably weaker but more familiar revenue-equivalence theorem):

Proposition 6 (Vickrey 1961; Myerson 1981; John Riley and William Samuelson 1981; Milgrom's Theorem 3.3): Under the hypotheses of Proposition 4, if there are two auctions such that, in Bayesian equilibrium, (a) for all i and v_i, the probability of winning for a buyer i with valuation v_i is the same in both auctions, and (b) for all i, the amount that buyer i with valuation 0 pays is the same in both auctions, then, for all i and v_i, the *equilibrium expected payoff* for buyer i with valuation v_i is the *same* in both auctions.

Proof*:* Choose one of the two auctions and let $(b_1(v_1),\ldots,b_n(v_n))$ be Bayesian equilibrium bids by the buyers when valuations are $(v_1,\ldots,v_n)$. Because buyer i has the option of behaving as though his valuation is $\hat{v}_i$ when in fact it is v_i, he, in effect, faces the maximization problem

$$\max_{\hat{v}_i}\left[G_i(\hat{v}_i)v_i - P_i(\hat{v}_i)\right], \qquad (12)$$

where $G_i(\hat{v}_i)$=buyer i's probability of winning and $P_i(\hat{v}_i)$=buyer i's expected payment if he bids $b_i(\hat{v}_i)$ and each of the other buyers j bids according to the equilibrium bid function $b_j(\cdot)$. By definition of equilibrium, the maximizing choice of $\hat{v}_i$ in (12) is $\hat{v}_i=v_i$, and so we obtain first-order condition

$$P_i'(v_i) = G_i'(v_i)v_i \text{ for all } i. \qquad (13)$$

Integrating (13), we have

$$P_i(v_i) = v_i G_i(v_i) - \int_0^{v_i} G_i(x)dx + k_i, \qquad (14)$$

where k_i is a constant of integration. Notice from (14) that buyer i's expected payment if $v_i=0$ is k_i. By hypothesis (b), this is true of the other auction as well. Furthermore, by hypothesis (a), i's probability of winning in the other auction is $G_i(v_i)$ for all v_i. Hence, from (14), buyer i's expected payment is $P_i(v_i)$ and his equilibrium expected payoff is $G_i(v_i)v_i - P_i(v_i)$ in both auctions. Q.E.D.

Clearly, $G_i(v_i)$ must be lie between 0 and 1, but there are other restrictions on it as well. In particular, it must be nondecreasing.

Proposition 7 (Myerson 1981, Riley and Samuelson 1981, Milgrom's Theorem 4.1): In any Bayesian equilibrium a buyer's probability of winning is a nondecreasing function of his valuation.

Proof*:* From (12) and (13), the derivative of buyer i's equilibrium expected payoff if his valuation is v_i but he bids as though it were $\hat{v}_i$ is

$$G'(\hat{v}_i)(v_i - \hat{v}_i). \qquad (15)$$

But if $G'(v_i)<0$ for some v_i then from (15) the second-order condition for a maximum $(G'(v_i)\geq 0)$ is violated at $\hat{v}_i=v_i$, a contradiction. Q.E.D.

Notice that

$$G_i(v_i) = \times_{j\neq i} F_j(v_i) \qquad (16)$$

in equilibrium of the second-price auction because, from Proposition 1, buyers bid truthfully, and so a buyer's probability of winning is simply the probability that the other buyers all have lower valuations (the right-hand side of (16)). Furthermore,

$$P_i(0) = 0 \qquad (17)$$

in that auction. But (16) and (17) also hold for equilibrium of the English auction, the mechanism in which buyers call out bids openly, each successive bid must be higher than the previous one, and the winner is the last buyer to bid (and pays his bid). To see this, notice that a buyer will continue to bid higher in the English auction until the current price reaches his valuation, and so the high-valuation buyer will win, i.e., (16) holds. Thus we have:

Proposition 8 (Vickrey 1961): The second-price and English auctions are payoff-equivalent.

Remarkably, in the case of ex ante buyer symmetry, i.e., where $F_1=\ldots=F_n$, *all* the "standard" auctions are equivalent.

Proposition 9 (Vickrey 1961; Riley and Samuelson 1981; Myerson 1981; Milgrom's Theorems 4.6 and 4.9): When each v_i is

dec04_Article 5 12/14/04 2:27 PM Page 1109

drawn independently from a distribution with c.d.f. F and support [0, 1], then the high-bid, second-price, English, Dutch, and all-pay auctions are payoff-equivalent.

Proof: We have already described the rules of all but the Dutch and all-pay auctions. In the *Dutch auction*, the auctioneer continuously lowers the price, starting from some high level, until some buyer (the winner) agrees to buy at the current price. Notice that this is formally the same as the high-bid auction, since the price at which a buyer agrees to buy in the Dutch auction is the same as the bid he would make in the high-bid auction.[13] In the *all-pay auction*, buyers submit sealed bids and the winner is the high bidder, but all buyers pay their bids.

Consider a symmetric equilibrium $b(\cdot)$ of the high-bid auction; i.e., $b(v)$ is the bid any buyer with valuation v makes (a symmetric equilibrium exists because of the ex ante symmetry of the buyers). From Proposition 7, $b(\cdot)$ must be nondecreasing. Suppose it is not strictly increasing, i.e., suppose $b(v)=b^*$ for all $v\in[v^*,v^{**}]$. We have $v^*-b^*\geq 0$ because, thanks to the atom at b^*, a bid of b^* wins with positive probability (and thus if $v^*-b^*<0$, the buyer would have a negative payoff). Hence we obtain

$$v^{**}-b^*>0. \tag{18}$$

But if a buyer with reservation price v^{**} bids b^*, he ties for the high bid with positive probability. Thus if he slightly increases his bid, he discontinuously raises his chances of winning (since a tie then has zero probability), which is worthwhile in view of (18). We conclude that $b(\cdot)$ must be *strictly* increasing, which means that the high-valuation buyer always wins. Thus, Proposition 6 implies that the high-bid auction is equivalent to the second-price auction. This same argument applies also to the all-pay auction. Q.E.D.

[13] This equivalence relies on the assumption that buyers obey the usual axioms of expected utility; see Daisuke Nakajima (2004).

We next examine how the auctions of Proposition 9 can be modified to maximize the seller's revenue:

Proposition 10 (Riley and Samuelson 1981; Myerson 1981; Milgrom's Theorem 3.9): Assume the hypotheses of Proposition 9 and suppose that

$$J'(v)>0 \text{ for all } v, \tag{19}$$

where $J(v)=v-\dfrac{1-F(v)}{F'(v)}$. Then any of the auctions of Proposition 9 maximizes the seller's expected revenue provided that the seller sets a reserve price v^* (i.e., he refuses to sell for less than v^*), where $J(v^*)=0$.

Proof: Given buyers' ex ante symmetry, (14) implies that, for any symmetric auction (we restrict to symmetric auctions without loss of generality), the seller's expected revenue is

$$n\int_0^1\left[vG(v)-\int_0^v G(x)dx+k\right]dF(v),$$

which can be rewritten as

$$n\int_0^1 J(v)G(v)dF(v)+nk. \tag{20}$$

We have already noted that $G(v)$ must satisfy $G'(v)>0$ and

$$0\leq G(v)\leq 1. \tag{21}$$

As Steven Matthews (1984) shows, it must also satisfy

$$n\int_v^1\left[\left(F(x)\right)^{n-1}-G(x)\right]dF(x)\geq 0 \text{ for all } v. \tag{22}$$

Consider the problem of maximizing (20) subject to (21) and (22). Note from (12) and (14) that $k\leq 0$, otherwise a buyer with zero valuation has a negative expected payoff (an impossibility, since he always has the option of not participating). Hence the maximizing choice of k is 0, i.e.,

$$P(0)=\text{payment by a 0-valuatioin buyer}=0. \tag{23}$$

From (19)–(22), the optimal choice of $G(v)$ is

$$G(v)=\begin{cases}\left(F(v)\right)^{n-1}, & \text{if } v>v^* \\ 0, & \text{if } v\leq v^*,\end{cases} \tag{24}$$

dec04_Article 5 12/14/04 2:27 PM Page 1110

where $J(v^*)=0$. But all of the auctions of Proposition 9, modified by a reserve price of v^*, satisfy (23) and (24), and so they are solutions to the seller's problem.[14] Q.E.D.

The fact that the Dutch and high-bid auctions are equivalent is obvious from the identical strategic structure of the two forms. Nor is the equivalence (Proposition 8) between the English auction and the second-price auction very deep. But the sense in which all four auctions are equivalent (Proposition 9) is more interesting, as is the idea that any of them—modified by setting a reserve price[15]—can be used to maximize the seller's revenue (Proposition 10).

5. *Departures from the Benchmark Model*

This deeper equivalence, however, relies on some restrictive hypotheses, viz., (i) buyers' risk neutrality, (ii) private values (to be defined below), (iii) independent valuations, (iv) ex ante symmetry, and (v) financially unconstrained buyers.[16] We will now relax each of (i)–(iii) in turn (for relaxation of (iv), see Milgrom's Theorems 4.24–4.27, and Maskin and Riley 2000; for relaxation of (v), see Milgrom's Theorem 4.17, and Che and Gale 1998).

Note first that in Propositions 6–10, we suppose that buyer *i*'s objective function is given by (12), i.e., that he is *risk neutral*. If we replace risk neutrality with risk aversion, then in particular Proposition 9 no longer holds.

Proposition 11 (Holt 1980; Maskin and Riley 1984; Matthews 1983; Milgrom's Theorem 4.12): Assume that buyers are *risk averse*; i.e., buyer *i*'s utility from winning is $u_i(v_i - p_i)$, where u_i is strictly concave. Suppose that buyers are ex ante symmetric, i.e., the v_i's are drawn (independently) from the same distribution with c.d.f. F and support [0, 1] and $u_1 = \ldots = u_n = u$. Then the high-bid auction generates higher expected revenue than the second-price auction.

Proof: First, observe that risk aversion does not affect behavior in the second-price auction; it is still optimal to bid truthfully. If $b(\cdot)$ is a symmetric-equilibrium bid function in the high-bid auction, a buyer with valuation v solves

$$\max_{\hat{v}} \left(F(\hat{v})\right)^{n-1} u\left(v - b(\hat{v})\right).$$

The first-order condition is therefore

$$-F^{n-1}u'b' + (n-1)F^{n-2}F'u = 0,$$

and so

$$b'(v) = \frac{(n-1)F'(v)u\left(v - b(v)\right)}{F(v)u'\left(v - b(v)\right)}. \qquad (25)$$

Now, if buyers were risk neutral, (25) would become

$$b'_{RN}(v) = \frac{(n-1)F'(v)\left(v - b_{RN}(v)\right)}{F(v)}.$$

But from Proposition 9, the high-bid and second-price auctions are payoff-equivalent when buyers are risk neutral and ex ante symmetric. Hence, $b_{RN}(v)$ is also the expected payment by a winning v-valuation buyer in the second-price auction (whether he is risk averse or not). Because $u''<0$,

$$\frac{u\left(v - b(v)\right)}{u'\left(v - b(v)\right)} > v - b(v),$$

and so

$$b'(v) > b'_{RN}(v) \text{ whenever } b(v) = b_{RN}(v).$$

Because $b(0)=b_{RN}(0)=0$, we conclude that

$$b(v) > b_{RN}(v) \text{ for all } v>0$$

which implies that, for every $v>0$, a buyer pays more in the high-bid than in the second-price auction. Q.E.D.

[14] We have ignored the constraint $G' \geq 0$ because it is satisfied by the solution to the program in which it is omitted.

[15] The reason a reserve price helps the seller is that it puts a lower bound on what buyers pay. Of course, by setting a positive reserve, the seller runs the risk of not selling at all, but this effect is outweighed by the lower-bound consideration. To see this, imagine that there is just one buyer. Then in a high-bid auction, that buyer would bid zero; the only way to get him to pay *anything* is to make the reserve positive.

[16] The efficiency of the second-price auction (Proposition 1) invokes (ii), as we will see in Proposition 12, and (v) (see Maskin 2000), although it does not demand (i), (iii), or (iv).

dec04_Article 5 12/14/04 2:27 PM Page 1111

Another important hypothesis in sections 2 and 3 is that buyers actually *know* their valuations (more to the point, that their valuations do not depend on the private information of *other* buyers). This is called the *private values* assumption. Let us now relax it to accommodate *interdependent values* (sometimes called generalized *common values*). Specifically, suppose that each buyer i receives a private signal s_i and that his valuation is a function of *all* the signals: i.e., his valuation is $v_i(s_i, s_{-i})$, where s_{-i} is the vector of other buyers' signals. In such a setting, the second-price auction will no longer be efficient (Maskin 1992); the problem is that because buyers no longer know their valuations, their bids (reflecting their *expected* valuations) do not guarantee that the high bidder actually has the highest valuation. Nevertheless, if the signals are one-dimensional (i.e., scalars), then, as Jacques Crémer and Richard McLean (1985) and Maskin (1992) show, there exist mechanisms that ensure efficiency (provided that "single crossing" holds, i.e., buyer i's signal has a greater marginal effect on his own valuation than on other buyers' valuations: $\frac{\partial v_i}{\partial s_i} > \frac{\partial v_j}{\partial s_i}$ for $j \neq i$). More concretely, Partha Dasgupta and Maskin (2000) and Motty Perry and Philip Reny (2002) show that, with single crossing, there is a way of extending the second-price auction to accommodate *contingent* bids so that efficiency is restored (in a more limited class of cases, Maskin 1992 and Krishna 2003 show that the English auction is efficient with one-dimensional signals). However, when signals are multidimensional, then efficiency is no longer possible.

Proposition 12 (Maskin 1992; Philippe Jehiel and Benny Moldovanu 2001; Milgrom's Theorem 3.8): Suppose that, for some buyer i, $s_i = (s_{i1}, s_{i2})$,[17] where

$$v_i(s_i, s_{-i}) = \varphi_i(s_i) + \psi_i(s_{-i}) \qquad (26)$$

and

$$\frac{\sum_{j\neq i} \frac{\partial v_j}{\partial s_{i1}}}{\sum_{j\neq i} \frac{\partial v_j}{\partial s_{i2}}} \neq \frac{\frac{\partial \varphi_i}{\partial s_{i1}}}{\frac{\partial \varphi_i}{\partial s_{i2}}}. \qquad (27)$$

Then, if Bayesian equilibrium is the solution concept, there exists no efficient auction.

Proof (sketch): Choose parameter values s_i' and s_i'' such that $\varphi_i(s_i') = \varphi_i(s_i'')$. From (26), buyer i's preferences are *identical* for $s_i = s_i'$ and $s_i = s_i''$ and so, in equilibrium, he must be indifferent between the outcomes that result from the two cases. But, from (27), which of $s_i = s_i'$ or $s_i = s_i''$ holds will in general lead to different efficient allocations—e.g., perhaps buyer i wins when $s_i = s_i'$ and loses when $s_i = s_i''$. Thus buyer i cannot be indifferent between the outcomes. Q.E.D.

Despite this negative result, matters are not as bleak as they may seem, at least in the case of *single-good* auctions. Specifically, introduce a one-dimensional "reduced" signal r_i for buyer i and, for all $j \neq i$, define

$$\hat{v}_j(r_i, s_{-i}) = E_{s_i}\left(v_j(s_i, s_{-i}) \middle| \varphi_i(s_i) = r_i\right),$$

i.e., $\hat{v}_j(r_i, s_{-i})$ is $v_j(s_i, s_{-i})$ expected over all those values s_i such that $\varphi_i(s_i) = r_i$. Because we are back to one-dimensional signals, the extended second-price auction mentioned above will be efficient with respect to the "reduced" valuations $v_i(r_i, s_{-i})$ and $\{\hat{v}_j\}$ (assuming that the single-crossing property above holds). That is, the auction is efficient subject to the constraint that buyer i behaves the same way for any signal values s_i for which $\varphi_i(s_i) = r_i$, i.e., it is *incentive efficient* (see Dasgupta and Maskin 2000). Jehiel and Moldovanu (2001) show, however, that this "reduction" technique does not generalize to more than one good.

Finally, let us explore what happens when we drop the assumption of independence in Proposition 9:

Proposition 13 (Milgrom and Weber 1982; Milgrom's Theorem 4.21): Suppose that $n = 2$,[18] and that v_1 and v_2 are jointly

[17] For convenience, suppose that all the other signals are one-dimensional.

[18] Milgrom and Weber (1982) generalize this result to $n \geq 3$.

dec04_Article 5 12/14/04 2:27 PM Page 1112

symmetrically distributed with support [0, 1] and *affiliated* (positively correlated) in the sense that

$$\frac{\partial^2 \log F(v_2|v_1)}{\partial v_1 \partial v_2} > 0\,, \tag{28}$$

where $F(v_2|v_1)$ is the c.d.f. for v_2 conditional on v_1. Then, revenue from the second-price auction exceeds that from the high-bid auction.

Proof: Let $P_1(v_1)$ be buyer 1's equilibrium expected payment when his valuation is v_1. We have

$$P_1^S(v_1) = \int_0^{v_1} v_2 dF(v_2|v_1) \tag{29}$$

and

$$P_1^H(v_1) = F(v_1|v_1)\, b(v_1)\,, \tag{30}$$

where the superscripts S and H denote second-price and high-bid auctions, respectively, and $b(\cdot)$ is the symmetric equilibrium bid function in the high-bid auction. Clearly, $P_1^S(0)=P_1^H(0)$. We wish to show that $P_1^S(v_1)>P_1^H(v_1)$ for all $v_1>0$. It suffices to show that $\frac{dP_1^S(v_1)}{dv_1} > \frac{dP_1^H(v_1)}{dv_1}$ whenever $P_1^S(v_1)=P_1^H(v_1)$.

Differentiating (29) and (30) we obtain

$$\frac{dP_1^S}{dv_1} = v_1 F^1(v_1|v_1) + \int_0^{v_1} v_2 dF^2(v_2|v_1) \tag{31}$$

and

$$\frac{dP_1^H}{dv_1} = F(v_1|v_1)\, b'(v_1) + b(v_1)F^1(v_1|v_1) + b(v_1)F^2(v_1|v_1),$$

where superscripts of F denote partial derivations. The second equation can be rewritten, using buyer 1's first-order condition, as

$$\frac{dP_1^H}{dv_1} = v_1 F^1(v_1|v_1) + b(v_1)F^2(v_1|v_1) \tag{32}$$

When $P_1^H(v_1)=P_1^S(v_1)$,

$$b(v_1) = \frac{\int_0^{v_1} v_2 dF(v_2|v_1)}{F(v_1|v_1)}. \tag{33}$$

From (31)–(33), it remains to show that

$$\int_0^{v_1} v_2 dF^2(v_2|v_1) > \frac{\int_0^{v_1} v_2 dF(v_2|v_1) F^2(v_1|v_1)}{F(v_1|v_1)}. \tag{34}$$

But (34) follows from affiliation, i.e., from (28). Q.E.D.

Notice that Propositions 11 and 13 pull in opposite directions: the former favors the high-bid auction, the latter the second-price auction. This tension illustrates one of Milgrom's introductory points: that the kind of auction a seller will want to use depends heavily on the circumstances.

6. *Theory versus Practice*

I have already suggested that some of Milgrom's observations about auctions in practice are a good deal less compelling than the book's theoretical results. But this contrast is not primarily his fault. In spite of all that it has accomplished, auction theory still has not developed far enough to be directly applicable to situations as complex as, say, the spectrum auctions.

To begin with, those auctions involve multiple goods. Observe, however, that all the results presented above are for single-good auctions. This is no coincidence; the literature on auction theory has overwhelmingly focused on the single-good case. Apart from the efficiency of the multigood second-price auction (the Vickrey-Clarke-Groves mechanism; see section 7) with private values, there are very few general results for more than one good. The auction designer can attempt to extrapolate from well-analyzed environments (one good) to new circumstances (multiple goods). But doing so is hazardous (see, for example, Perry and Reny 1999).

Another difficulty for theory is that real auctions impose constraints that are difficult to formalize. Milgrom notes that prospective bidders and sellers are typically nervous about participating in auction mechanisms that seem unfamiliar or complicated. But

dec04_Article 5 12/14/04 2:27 PM Page 1113

giving a precise meaning to "unfamiliar" or "complicated" is forbiddingly difficult.

The upshot is that giving advice on real auction design is, at this stage, far less a science than an art. And the essence of an art is far harder than a science to convey convincingly in writing.

7. *The Vickrey-Clarke-Groves Mechanism*

I have noted that Milgrom voices serious criticisms of the Vickrey-Clarke-Groves (VCG) mechanism, the generalization of the second-price auction to multiple goods. Indeed, his unhappiness with it has led him to collaborate with Lawrence Ausubel (Ausubel and Milgrom 2002) on an interesting and ingenious alternative mechanism, reported on in chapter eight.

The VCG mechanisms works as follows: (i) each buyer makes a bid not just for each good but for each combination (or "package") of goods; (ii) goods are allocated to buyers in the way that maximizes the sum of the winning bids (a bid for a package is "winning" if the buyer making that bid is allocated the package); (iii) each winning buyer *i* pays an amount equal to the difference between (a) the sum of the bids that would win if *i* were not a participant in the auction and (b) the sum of the other buyers' (actual) winning bids. Following the line of argument in the proof of Proposition 1, one can show that truthful bidding (reporting one's true valuation for each package) is dominant. Thus the auction results in an efficient allocation (an allocation that maximizes the sum of the winning valuations).

One frequent objection to the VCG mechanism is that it makes heavy demands on both buyers (placing bids on each package can be an onerous task) and the auctioneer (computing the winning allocation is a potentially difficult maximization problem). This, however, is not the shortcoming that Milgrom dwells on; in fact, the Ausubel-Milgrom paper is subject to the same sort of criticism.

Instead, Milgrom worries about the following sort of problem. Assume that there are two goods A and B, and two potential buyers 1 and 2. Suppose that each buyer wants these goods only as a *package*, i.e., his valuation for A or B alone is zero. Suppose that buyer 1 has a valuation of \$100 for A and B together, but that buyer 2 has a package valuation of \$200. If the buyers bid truthfully in the VCG mechanism, then buyer 2 will win both A and B (and pay \$100, the winning bid were 2 not present). Buyer 1 will come away empty-handed.

Suppose, however, that buyer 1 enters bids through two *different* proxy buyers, $1x$ and $1y$. As buyer $1x$ he enters a bid of \$201 for good A (and zero for both B and the package AB). As $1y$, he enters a bid of \$201 for good B (and zero for both A and AB). Then $1x$ and $1y$ will be the winners of A and B respectively, and so 1 will obtain both goods. Furthermore, notice that had $1x$ not bid at all, $1y$ would still be the winner of good B (good A would just be thrown away), and so the sum of the other buyers' winning bids is the same (namely, \$201) whether $1x$ participates or not. Thus, by VCG rules, $1x$ pays nothing at all (and, similarly, neither does $1y$), which means that the ploy of passing himself off as multiple buyers is worthwhile for 1. Unhappily, it generates no revenue for the seller and leads to an inefficient allocation (1 wins the goods, rather than 2), which is why Milgrom is led to reject the VCG auction.

But notice that having $1x$ and $1y$ enter these bids makes sense for 1 only if he is quite sure that buyer 2 does not value A and B as single goods. As soon as there is a serious risk that 2 will make single-good bids that add up to \$101 or more, buyer 1 will come out behind with this strategy (relative to truthful bidding). If, for example, buyer 2 bid \$51 for each of A and B alone (as well as \$200 for the package), $1x$ and $1y$ would still be awarded A and B with their \$201 bids but buyer 1 would now pay \$51+\$51=\$102 for a package worth only \$100 to him.

dec04_Article 5 12/14/04 2:27 PM Page 1114

Indeed, with sufficient uncertainty about how other buyers will bid, it is not hard to see that nothing other than truthful bidding makes sense for a buyer in a VCG auction. And since I would venture to say that considerable uncertainty is quite common in real auction settings, I believe that Milgrom is too harsh when he deems VCG "unsuitable" for most applications.

8. *Concluding Remarks*

This is a minor reservation about a volume that covers a cornucopia of material in magisterial fashion and gives us deep insight into the thinking of an outstanding theorist. The book is not for everybody; one needs at least enough technique to be able to follow the proofs of the propositions above. But, with that qualification, I warmly commend it to all wishing to experience the beauty and power of this remarkable theory.

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dec04_Article 5 12/14/04 2:27 PM Page 1115

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