APPENDIX

A Numerical Methods Theory

This paper uses finite difference methods to solve for the equilibrium of the IG agent (equations (7) – (8)). This section follows Achdou et al. (2020), who provide an excellent set of resources on these methods. I assume knowledge of finite difference methods here.

Barles and Souganidis (1991) show that a finite difference scheme converges to the unique viscosity solution of an HJB equation as long as certain conditions hold. As detailed below, these conditions do not necessarily hold when $\beta < 1$. This failure means that one cannot directly solve the Bellman of the IG agent. The key algorithmic insight of this paper is that the following two-step approach can be used to solve for the IG agent's equilibrium. First, solve the HJB equation of the dynamically consistent $\hat{u}$ agent. Second, compute the IG agent's equilibrium directly from the $\hat{u}$ agent using Propositions 10 and 11.

Failure of Monotonicity. Here I present a brief description of the problem: the Bellman equation of the IG agent fails to satisfy a monotonicity property. I follow Tourin (2013)'s treatment of Barles and Souganidis (1991). For simplicity, I assume here that income is deterministic, with $y_t = y$.

Let $\mathcal{G}$ denote the discretized grid over wealth a on which $v(a)$ is solved numerically. Assume this grid is uniformly spaced, and let Δ_a denote the size of the grid increment. At each gridpoint $g \in \mathcal{G}$, define:

$$S_g = \gamma v_g - u(c_g) - \frac{v_{g+1} - v_g}{\Delta_a}(y + ra_g - c_g)^+ - \frac{v_g - v_{g-1}}{\Delta_a}(y + ra_g - c_g)^-.$$

v_g, v_{g+1}, and v_{g-1} represent the value function at gridpoints $g, g+1$, and $g-1$, respectively. a_g is the wealth value at gridpoint g, and c_g is the consumption choice at gridpoint g.

For monotoncity to hold, S_g must be weakly decreasing in v_g, v_{g+1}, and v_{g-1}. To show that monotonicity fails when $\beta < 1$, assume that $y + ra_g - c_g < 0$. In this case, c_g is defined implicitly by $u'(c_g) = \beta \frac{v_g - v_{g-1}}{\Delta_a}$. Consider an increase in v_{g-1}:

$$\begin{aligned}\frac{\partial S_g}{\partial v_{g-1}} &= -u'(c_g)\frac{\partial c_g}{\partial v_{g-1}} + \frac{1}{\Delta_a}(y + ra_g - c_g)^- + \frac{v_g - v_{g-1}}{\Delta_a}\frac{\partial c_g}{\partial v_{g-1}} \\ &= (1-\beta)\frac{v_g - v_{g-1}}{\Delta_a}\frac{\partial c_g}{\partial v_{g-1}} + \frac{1}{\Delta_a}(y + ra_g - c_g)^-,\end{aligned}$$

where the last line uses the property that $u'(c_g) = \beta \frac{v_g - v_{g-1}}{\Delta_a}$.

If $\beta = 1$ then monotonicity holds: $\frac{\partial S_g}{\partial v_{g-1}} < 0$ since the first term drops out, and $y + ra_g - c_g < 0$ by assumption.

If $\beta < 1$ then monotonicity may not hold. Since $\frac{\partial c_g}{\partial v_{g-1}} > 0$ the term $(1-\beta)\frac{v_g - v_{g-1}}{\Delta_a}\frac{\partial c_g}{\partial v_{g-1}} > 0$ whenever $\beta < 1$. Now, it is possible for $\frac{\partial S_g}{\partial v_{g-1}} > 0$, in which case monotonicity does not hold.

The above algebra points to the difficulty of using finite difference methods to solve for the Bellman of the IG agent. Since this difficulty only arises when $\beta < 1$, finite difference methods can still be used to solve for the equilibrium of the $\hat{u}$ agent. Given a solution to the $\hat{u}$ agent, the equilibrium of the IG agent can then be backed out: Proposition 10 implies that $v_j(a) = \hat{v}_j(a)$, and Proposition 11 defines $c_j(a)$ given $\hat{c}_j(a)$.

B Present Bias and Policy: A Simple Example

This section provides a simple example to show how government intervention can improve the equilibrium of an economy with present-biased agents.[48] I study a simple "Eat-the-Pie" model of consumption-saving behavior. I assume that there is a single representative agent with initial wealth a_0. The model is deterministic, with $y_t \equiv \bar{y}$. The borrowing limit is set to the natural borrowing constraint of $\underline{a} = \frac{-\bar{y}}{r}$.

In this simple model, Proposition 5 implies that the IG agent consumes $c(a) = \frac{\rho-(1-\gamma)r}{\gamma-(1-\beta)}(a+\frac{\bar{y}}{r})$. However, the first-best consumption level is $\check{c}(a) = \frac{\rho-(1-\gamma)r}{\gamma}(a+\frac{\bar{y}}{r})$.[49]

For simplicity, I assume that $\rho = r$, $\gamma = 1$, and $a_0 > 0$. With these three assumptions, the first-best consumption level is $\check{c}(a_0) = ra_0 + \bar{y}$ and the first-best savings level is $\check{s}(a_0) = 0$. In other words, it is optimal for the agent to consume the annuity value of their wealth plus their deterministic income flow.

I now introduce a social planner to improve the consumption-saving decisions of the representative IG agent. This social planner is allowed to use a combination of interest rate subsidies and consumption taxes, subject to a balanced-budget constraint. Interest rate subsidies encourage saving, while consumption taxes are a means of financing these subsidies.

Denote the consumption tax by ϕ_t, and the subsidized interest rate by r_t^s. The social planner runs a balanced budget for all t, so the interest rate subsidy of $(r_t^s - r)a_t$ must equal the total tax revenue collected at each point in time.

With the introduction of consumption taxes, I will now use c_t to denote *gross* consumption expenditures at time t. However, the agent only gets to consume share $1 - \phi_t$ of gross expenditures, with the rest going to taxes.

The social planner can recover the first-best equilibrium using a constant consumption tax and interest rate subsidy. To implement the first-best equilibrium, the planner needs to

[48] I thank David Laibson for suggesting this example. A similar result is presented in Laibson (1998).

[49] As discussed in Section 5.4, IG preferences feature a single welfare criterion even though they are time-inconsistent.

choose r^s and ϕ such that:

$$(1-\phi)\frac{ra_0 + \frac{r\bar{y}}{r^s}}{\beta} = ra_0 + \bar{y} \tag{36}$$

$$\phi\frac{ra_0 + \frac{r\bar{y}}{r^s}}{\beta} = (r^s - r)a_0 \tag{37}$$

Under the simple calibration studied here, the IG agent will choose a gross consumption expenditure of $c(a) = \frac{ra+\frac{r\bar{y}}{r^s}}{\beta}$.[50] However, actual consumption is only $(1-\phi)c(a)$. Equation (36) imposes that realized consumption is at its first-best level: $(1-\phi)c(a_0) = ra_0 + \bar{y}$. Equation (37) is the balanced-budget condition. It says that tax revenues of $\phi c(a_0)$ must equal the interest rate subsidy of $(r^s - r)a_0$.

One can show that the following set of policy tools produces the first-best equilibrium:

$$r^s = \frac{r}{\beta}$$

$$\phi = \frac{ra_0(1-\beta)}{ra_0 + \beta\bar{y}}$$

For example, consider the calibration $\beta = 0.75$, $r = 3\%$, $\bar{y} = 1$, and $a_0 = 3$. The optimal consumption tax is $\phi = 2.67\%$ and the optimal subsidized interest rate is $r^s = 4\%$.

Welfare and Implementability when $\beta = 1$. Proposition 21 highlights the channels through which present bias can matter for policymakers: present bias does not matter for determining whether a policy is welfare-improving, but does matter for determining whether a policy is feasible. This toy model can be used to formalize this discussion.

Proposition 21 implies that this interest rate subsidy plus consumption tax policy will be welfare-improving for the $\beta = 1$ agent. However, this policy is not implementable in an economy populated by a representative $\beta = 1$ agent. At time 0, the $\beta = 1$ agent will consume $r(a_0 + \frac{\bar{y}}{r^s})$, which will be too low to generate the requisite taxes needed to support the interest rate subsidy of $(r^s - r)a_0$.

[50] This consumption function uses the property that a constant consumption tax does not change the gross expenditure of the IG agent. This property is derived formally in the proof of Proposition 20.

C Proofs

Throughout this appendix, it is assumed that the reader understands the construction of the $\hat{u}$ agent. Details of the $\hat{u}$ construction are given in Section 4, and in Harris and Laibson (2013).

C.1 Proof of Proposition 1

The proof of Proposition 1 relies on Proposition 10.

Lemma 22. *The value function of the $\hat{u}$ agent, denoted $\hat{v}_j(a)$, is unique.*

Proof. See Harris and Laibson (2013) for full details. Intuitively, the $\hat{u}$ agent is an exponential discounter who optimally choses consumption to maximize $\hat{v}_j(a)$. Since there is only one maximal value function, $\hat{v}_j(a)$ must be unique. □

The proof of Proposition 1 follows immediately from Lemma 22 and Proposition 10. In particular, Lemma 22 says $\hat{v}$ is unique, and Proposition 10 says $v = \hat{v}$. Hence, v is unique.

C.2 Proof of Corollary 3

This proof extends the $\beta = 1$ case of Achdou et al. (2020). A similar result is given in Harris and Laibson (2004). Taking a derivative of (7) with respect to a and applying the first-order condition (8) gives

$$\left[(\rho - r) + (1-\beta)c_j'(a)\right] u'(c_j(a)) = u''(c_j(a))c_j'(a)s_j(a) + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a))).$$

Applying Ito's Lemma to $u'(c_j(a_t))$ gives $\mathbb{E}_t du'(c_j(a_t)) = u''(c_j(a_t))c_j'(a_t)da_t + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a)))dt$. Since $da_t = s_j(a_t)dt$, we have $\mathbb{E}_t du'(c_j(a_t)) = u''(c_j(a))c_j'(a)s_j(a)dt + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a)))dt$. Rearranging gives (13).

C.3 Proof of Proposition 4

For full details, see Theorem 21 of Harris and Laibson (2004). The value function for the IG agent is given by (see equation (7)):

$$\rho v_j(a) = u(c_j(a)) + v_j'(a)(y_j + ra - c_j(a)) + \lambda_j(v_{-j}(a) - v_j(a)).$$

If the constraint binds at $\underline{a}$ for income state j, then $c_j(\underline{a}) = y_j + r\underline{a}$. Thus,

$$\rho v_j(\underline{a}) = u(y_j + r\underline{a}) + \lambda_j(v_{-j}(\underline{a}) - v_j(\underline{a})).$$

Since the value function is continuous, $\rho v_j(\underline{a}) = \lim_{a \to^+ \underline{a}} \rho v_j(a)$. Therefore:

$$u(y_j + r\underline{a}) + \lambda_j(v_{-j}(\underline{a}) - v_j(\underline{a})) = \lim_{a \to^+ \underline{a}} \left[u(c_j(a)) + v_j'(a)(y_j + ra - c_j(a)) + \lambda_j(v_{-j}(a) - v_j(a)) \right],$$

or simply

$$u(y_j + r\underline{a}) = \lim_{a \to^+ \underline{a}} \left[u(c_j(a)) + v_j'(a)(y_j + ra - c_j(a)) \right].$$

Using equation (8) gives

$$u(y_j + r\underline{a}) = \lim_{a \to^+ \underline{a}} \left[u(c_j(a)) + \frac{1}{\beta} u'(c_j(a))(y_j + ra - c_j(a)) \right].$$

This equation can be rearranged to yield:

$$\lim_{a \to^+ \underline{a}} u'(c_j(a)) = \beta \frac{\lim_{a \to^+ \underline{a}} u(c_j(a)) - u(y_j + r\underline{a})}{\lim_{a \to^+ \underline{a}} c_j(a) - (y_j + r\underline{a})}.$$

C.4 Proof of Proposition 5

The proof of Achdou et al. (2020)'s Proposition 2 applies to the $\hat{u}$ agent, giving

$$\lim_{a \to \infty} \hat{c}_j(a) = \frac{\rho - (1 - \gamma) r}{\gamma} a.$$

Since the IG agent sets $c_j(a) = \frac{1}{\psi}\hat{c}_j(a)$ (see Proposition 11), this gives

$$\begin{aligned}\lim_{a\to\infty} c_j(a) &= \frac{1}{\psi}\frac{\rho-(1-\gamma)r}{\gamma} \\ &= \frac{\rho-(1-\gamma)r}{\gamma-(1-\beta)}a\end{aligned}$$

The proof for $\lim_{s\to\infty} s_j(a)$ is similar.

C.5 Proof of Corollary 6

Using Itô's Lemma, $\mathbb{E}_t\frac{dc_j(a_t)/dt}{c_j(a_t)} = \frac{c_j'(a)s_j(a)+\lambda_j(c_{-j}(a_t)-c_j(a_t))}{c_j(a)}$. Equations (15) and (16) give $\lim_{a\to\infty}\mathbb{E}_t\frac{dc_j(a_t)/dt}{c_j(a_t)} = \frac{r\beta-\rho}{\gamma-(1-\beta)}$. Taking a derivative with respect to r completes the proof.

C.6 Proof of Proposition 10

In this section I prove value function equivalence between the IG agent and the $\hat{u}$ agent. This proof is similar to Harris and Laibson (2013) and Laibson and Maxted (2020), and is included for complete detail.

Remark 23. *Most of the complexity in this proof arises in allowing for $\underline{a}$ to bind. The proof simplifies when $\underline{a}$ is the natural borrowing limit.*

The $\hat{u}$ Agent's HJBVI. To begin, I express the $\hat{u}$ agent's value function recursively. In the model of Section 3.1, $\hat{v}_j(a)$ is a viscosity solution to the following Hamilton-Jacobi-Bellman Variational Inequality (HJBVI):[51]

$$0 = \min\left\{\rho\hat{v}_j(a) - \max_{\hat{c}}\hat{u}^+(\hat{c}) + \hat{v}_j'(a)(y_j + ra - \hat{c}) + \lambda_j(\hat{v}_{-j}(a) - \hat{v}_j(a)),\ \hat{v}_j(a) - \hat{v}_j^*\right\}, \quad (38)$$

[51] For details on HJBVIs, and for additional economic applications, see the Stopping Time Problems at https://benjaminmoll.com/codes/.

where $\hat{v}_j^* = \frac{u(y_j + r\underline{a}) + \lambda_j \hat{v}_{-j}(\underline{a})}{\rho + \lambda_j}$.[52] Additionally, HJBVI equation (38) is subject to the following boundary condition at $\underline{a}$:

$$0 \leq \left[\hat{v}_j'(\underline{a}) - \frac{\partial \hat{u}^+(\psi(y_j + r\underline{a}))}{\partial \hat{c}} \right] \left(\hat{v}_j(\underline{a}) - \hat{v}_j^* \right). \quad (39)$$

This boundary condition is a shorthand way to restrict the consumption decision of the $\hat{u}$ agent at the borrowing limit $\underline{a}$. In particular, it ensures that the $\hat{u}$ agent either sets $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$ or else sets $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$ (details below).

The "variational inequality" structure of equation (38) captures the option value that is inherent in the $\hat{u}$ utility function. The left branch of equation (38) is the standard HJB of an exponential discounter with utility function $\hat{u}^+$. The right branch of equation (38) captures the agent's ability to "stop." Stopping yields the value $\hat{v}_j^* = \frac{u(y_j + r\underline{a}) + \lambda_j \hat{v}_{-j}(\underline{a})}{\rho + \lambda_j}$. This stopping condition can be thought of as giving the $\hat{u}$ agent the option to trade away all wealth above $\underline{a}$ in exchange for a utility flow of $u(y_j + r\underline{a})$ that persists until the agent's income state changes.[53]

For $a > \underline{a}$ it will always be optimal to select the left branch of equation (38) (i.e., do not "stop"). When $a > \underline{a}$ the $\hat{u}$ agent can always choose $\hat{c}$ such that $\hat{u}^+(\hat{c}) = u(y_j + r\underline{a})$, and this utility flow is attained without trading away all wealth above $\underline{a}$.

For $a = \underline{a}$, the $\hat{u}$ agent faces a choice. If the agent chooses the left branch of equation (38) then the lower utility function $\hat{u}^+$ is obtained. But, the benefit of choosing the left branch is that $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$ and therefore wealth is accumulated since $\hat{s}_j(\underline{a}) > 0$. If the agent chooses the right branch of equation (38) at $a = \underline{a}$ then the agent earns a larger utility flow of $u(y_j + r\underline{a})$, but remains stuck at wealth level $\underline{a}$. Thus, equation (38) captures the tradeoff that the $\hat{u}$ agent faces at $a = \underline{a}$.

The boundary condition in equation (39) is a way to restrict the behavior of the $\hat{u}$ agent at $\underline{a}$. The $\hat{u}$ agent must choose either to "continue" by setting $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$, or else to "stop." Stopping implies $\hat{v}_j(\underline{a}) = \hat{v}_j^*$, in which case equation (39) holds. If the agent chooses to continue, meaning that $\hat{v}_j(\underline{a}) \geq \hat{v}_j^*$, boundary condition (39) imposes that

[52] This implicitly assumes that $\underline{a} > \frac{-y_1}{r}$. If $\underline{a} = \frac{-y_1}{r}$ then $\hat{v}_j^* = -\infty$. In this case, $\hat{v}_j^*$ is never chosen.

[53] If the agent chooses this alternate "stopping" option, their value function is given by $\rho \hat{v}_j(\underline{a}) = u(y_j + r\underline{a}) + \lambda_j(\hat{v}_{-j}(\underline{a}) - \hat{v}_j(\underline{a}))$. Solving for $\hat{v}_j(\underline{a})$ gives the formula for $\hat{v}_j^*$.

$\hat{v}_j'(\underline{a}) \geq \frac{\partial \hat{u}^+(\psi(y_j+r\underline{a}))}{\partial \hat{c}}$. Since the $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(a))}{\partial \hat{c}} = \hat{v}_j'(a)$ in the continuation region, this lower bound on $\hat{v}_j'(\underline{a})$ ensures that $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$.

Proof Intuition. The intuition for this proof is as follows. Assume that $v_j(a) = \hat{v}_j(a)$ and $a > \underline{a}$. Then, differential equations (7) and (38) can be combined to yield:

$$u(c_j(a)) - v_j'(a)c_j(a) = \hat{u}^+(\hat{c}_j(a)) - \hat{v}_j'(a)\hat{c}_j(a).$$

Utility function $\hat{u}$ is reverse-engineered so that this condition holds.

The IG Agent: A Modified Bellman Equation. Following Theorem 2 of Harris and Laibson (2013), let $f^+(\alpha)$ be the unique value of c satisfying $u'(c) = \alpha$. Let $h^+(\alpha) = u(f^+(\beta\alpha)) - \alpha f^+(\beta\alpha)$. Since $u'(c_j(a)) = \beta v_j'(a)$ for $a > \underline{a}$, it is the case that $h^+(v_j'(a)) = u(f^+(\beta v_j'(a))) - v_j'(a) f^+(\beta v_j'(a)) = u(c_j(a)) - v_j'(a)c_j(a)$.

Next, let $\underline{f}_j(\alpha)$ be the unique value of c satisfying $u'(c) = \max\{u'(y_j + r\underline{a}), \alpha\}$. Let $\underline{h}_j(\alpha) = u(\underline{f}_j(\beta\alpha)) - \alpha \underline{f}_j(\beta\alpha)$.

Define:

$$h_j(\alpha, a) = \begin{cases} h^+(\alpha) & \text{if} \quad a > \underline{a} \\ \underline{h}_j(\alpha) & \text{if} \quad a = \underline{a} \end{cases}.$$

Function h can be used to rewrite equation the Bellman equation of the IG agent (equations (7) and (8)) as follows:

$$\rho v_j(a) = v_j'(a)(y_j + ra) + \lambda_j(v_{-j}(a) - v_j(a)) + h_j(v_j'(a), a). \tag{40}$$

The $\hat{u}$ Agent: A Modified Bellman Equation. At $a = \underline{a}$, the $\hat{u}$ agent faces a choice to "continue" or to "stop". Continuing attains utility function $\hat{u}^+$ with consumption $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$. Stopping attains utility function u with consumption $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$.

Lemma 24. *The $\hat{u}$ agent will choose to continue at $\underline{a}$ when $\hat{v}_j'(\underline{a}) > \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$. The $\hat{u}$ agent will choose to stop when $\hat{v}_j'(\underline{a}) < \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$.*

Proof. If the $\hat{u}$ agent chooses to continue at $\underline{a}$ then the $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(\underline{a}))}{\partial \hat{c}} = \hat{v}_j'(\underline{a})$. This implies $\hat{c}_j(\underline{a}) = \psi(\beta \hat{v}_j'(\underline{a}))^{-\frac{1}{\gamma}}$. This consumption choice yields a value at $\underline{a}$ of

$$\rho \hat{v}_j(\underline{a}) = \hat{u}^+(\hat{c}_j(\underline{a})) + \frac{\psi^\gamma}{\beta} \hat{c}_j(\underline{a})^{-\gamma}(y_j + r\underline{a} - \hat{c}_j(\underline{a})) + \lambda_j(\hat{v}_{-j}(\underline{a}) - \hat{v}_j(\underline{a})). \tag{41}$$

If the $\hat{u}$ agent chooses to stop, the value at $\underline{a}$ is given by

$$\rho \hat{v}_j(\underline{a}) = u(y_j + r\underline{a}) + \lambda_j(\hat{v}_{-j}(\underline{a}) - \hat{v}_j(\underline{a})). \tag{42}$$

Comparing (41) and (42), one can show that the $\hat{u}$ agent is indifferent between the two choices when $\hat{v}_j'(\underline{a}) = \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$ (which implies $\hat{c}_j(\underline{a}) = \psi(y_j + r\underline{a})$). The $\hat{u}$ agent chooses value function (41) when $\hat{v}_j'(\underline{a}) > \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$, and value function (42) when $\hat{v}_j'(\underline{a}) < \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$. □

Using Lemma 24, I now proceed as in the IG case. For the $\hat{u}$-agent, let $\hat{f}^+(\alpha)$ be the unique value of $\hat{c}$ satisfying $\frac{\partial \hat{u}^+(\hat{c})}{\hat{c}} = \alpha$. Let $\hat{h}^+(\alpha) = \hat{u}^+(\hat{f}^+(\alpha)) - \alpha \hat{f}^+(\alpha)$. Since the $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(a))}{\partial \hat{c}} = \hat{v}_j'(a)$ for $a > \underline{a}$, it is the case that $\hat{h}^+(\hat{v}_j'(a)) = \hat{u}^+(\hat{f}^+(\hat{v}_j'(a))) - \hat{v}_j'(a)\hat{f}^+(\hat{v}_j'(a)) = \hat{u}_j^+(\hat{c}_j(a)) - \hat{v}_j'(a)\hat{c}_j(a)$.

Next, let $\underline{\hat{f}}_j(\alpha)$ be the unique value of $\hat{c}$ satisfying:

$$\hat{c} = \begin{cases} \psi(\beta\alpha)^{-\frac{1}{\gamma}} & \text{if} \quad \alpha \geq \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma} \\ y_j + r\underline{a} & \text{otherwise} \end{cases}.$$

Let $\underline{\hat{h}}_j(\alpha) = \hat{u}_j(\underline{\hat{f}}_j(\alpha), \underline{a}) - \alpha \underline{\hat{f}}_j(\alpha)$.

Define:

$$\hat{h}_j(\alpha, a) = \begin{cases} \hat{h}^+(\alpha) & \text{if} \quad a > \underline{a} \\ \underline{\hat{h}}_j(\alpha) & \text{if} \quad a = \underline{a} \end{cases}.$$

Using Lemma 24, function $\hat{h}$ can be used to rewrite the Bellman equation of the $\hat{u}$ agent (equation (38)) as follows:

$$\rho \hat{v}_j(a) = \hat{v}_j'(a)(y_j + ra) + \lambda_j(\hat{v}_{-j}(a) - \hat{v}_j(a)) + \hat{h}_j(\hat{v}_j'(a), a) \tag{43}$$

From inspection, we can see that equations (40) and (43) are identical if and only if h_j and $\hat{h}_j$ are the same. This can be confirmed directly.

C.7 Proof of Proposition 11

For $a > \underline{a}$, equation (8) specifies that the IG agent sets $u'(c_j(a)) = \beta v'_j(a)$. The $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(a))}{\partial \hat{c}} = \hat{v}'_j(a)$. Imposing the value function equivalence that $v_j(a) = \hat{v}_j(a)$ (Proposition 10) gives:

$$\frac{\partial u(c_j(a))}{\partial c} = \beta \frac{\partial \hat{u}^+(\hat{c}_j(a))}{\partial \hat{c}}.$$

This implies

$$c_j(a) = \frac{1}{\psi}\hat{c}_j(a).$$

For $a = \underline{a}$, consider first the case where $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$. If the $\hat{u}$ agent sets $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$ then this implies that $\hat{v}'_j(\underline{a}) \geq \frac{\partial \hat{u}^+(\psi(y_j + r\underline{a}))}{\partial \hat{c}} = \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$. Since $v_j(a) = \hat{v}_j(a)$ by value function equivalence (Proposition 10), this also means that $\beta v'_j(\underline{a}) \geq (y_j + r\underline{a})^{-\gamma}$. In this case, equation (8) specifies that the IG agent sets $u'(c_j(\underline{a})) = \beta v'_j(\underline{a})$. The argument that was just used for $a > \underline{a}$ continues to hold here.

Next, consider the case where $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$. If the $\hat{u}$ agent sets $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$ then it must be that $\hat{v}'_j(\underline{a}) \leq \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$ (Lemma 24). By optimality condition (8), the IG agent also sets $c_j(\underline{a}) = y_j + r\underline{a}$.

C.8 Proof of Proposition 13

This proof uses the property that the $\hat{u}$ agent behaves identically to a standard exponential agent when $\underline{a}$ is the natural borrowing limit (Remark 12).

First consider the case where $r \leq \rho$. The Euler equation of the standard exponential agent implies that $\check{c}(a) \geq y + ra$ for all $a \geq 0$ (see Achdou et al. (2020)). Since the IG agent sets $c(a) = \frac{1}{\psi}\hat{c}(a) = \frac{1}{\psi}\check{c}(a)$, the IG agent strictly dissaves for all $a \geq 0$ when $r \leq \rho$.

Next consider the case where $r \in (\rho, \frac{\rho}{\beta})$. The standard exponential agent consumes according to $\check{c}(a) = \frac{\rho-(1-\gamma)r}{\gamma}(a + \frac{y}{r})$ for $a \geq 0$ (see e.g. Fagereng et al. (2019) for a proof). The IG agent therefore sets $c(a) = \frac{1}{\psi}\check{c}(a) = \frac{\rho-(1-\gamma)r}{\gamma-(1-\beta)}(a + \frac{y}{r})$ for $a \geq 0$. One can show that $s(a) = y + ra - c(a) < 0$ whenever $r < \frac{\rho}{\beta}$. Thus, the IG agent strictly dissaves for all $a \geq 0$ when $r \in (\rho, \frac{\rho}{\beta})$.

In both cases the IG agent strictly dissaves for all $a \geq 0$. This means that the IG agent dissaves at $a = 0$, completing the proof that $s(0) < 0$ whenever $r < \frac{\rho}{\beta}$. This holds regardless of how large $\Gamma(0)$ is.

Note that this proof does not rely on some sort of consumption discontinuity at $a = 0$. The consumption function $c(a)$ is continuous at $a = 0$. To show this, recall that the IG agent's value function is given by

$$\rho v(a) = u(c(a)) + v'(a)(\varsigma(a)a + y - c(a)).$$

The IG agent sets $u'(c(a)) = \beta v'(a)$. Therefore

$$\rho v(a) = u(c(a)) + \frac{c(a)^{-\gamma}}{\beta}(\varsigma(a)a + y - c(a)).$$

Since $v(a)$ is continuous and $\varsigma(a)a$ is continuous, $c(a)$ is also continuous at $a = 0$.

C.9 Proof of Proposition 14

Value Function Uniqueness. I first prove that the IG agent's intrapersonal game features a unique value function.

Lemma 25. *For the two-asset model described in Section 5.2, the value function of the $\hat{u}$ agent, denoted $\hat{v}_j(a,\zeta)$, is equivalent to the value function $v_j(a,\zeta)$ of the IG agent.*

Proof. In this extended model, $\hat{v}_j(a,\zeta)$ is a viscosity solution to the following Hamilton-Jacobi-Bellman Variational Inequality (HJBVI):

$$0 = \min\Bigg\{\hat{v}_j(a,\zeta) - \hat{v}_j^*(\zeta),\ \rho\hat{v}_j(a,\zeta) - \max_{\hat{c},\hat{d}}\ \hat{u}^+(\hat{c}) + \frac{\partial \hat{v}_j(a,\zeta)}{\partial a}(y_j + \varsigma(a)a - \hat{d} - \chi(\hat{d},\zeta) - \hat{c}) + \frac{\partial \hat{v}_j(a,\zeta)}{\partial \zeta}(\zeta r^{\zeta} + \hat{d}) + \lambda_j(\hat{v}_{-j}(a,\zeta) - \hat{v}_j(a,\zeta)) + \frac{1}{2}\frac{\partial^2 \hat{v}_j(a,\zeta)}{\partial \zeta^2}(\zeta\sigma^{\zeta})^2\Bigg\}, \tag{44}$$

where $\hat{v}_j^*(\zeta) = \frac{u(y_j + r\underline{a}) + \lambda_j \hat{v}_{-j}(\underline{a},\zeta)}{\rho + \lambda_j}$.[54] HJBVI equation (44) is subject to the following boundary condition at $\underline{a}$:

$$0 \leq \left[\hat{v}_j'(\underline{a},\zeta) - \frac{\partial \hat{u}^+(\psi(y_j + r\underline{a}))}{\partial \hat{c}}\right] \left[\hat{v}_j(\underline{a},\zeta) - \hat{v}_j^*(\zeta)\right]. \tag{45}$$

Given this setup, the proof of value function equivalence is the same as Proposition 10. Asset allocation choice $d_j(a,\zeta)$ adds no additional difficulty because the $\hat{u}$ agent and the IG agent both utilize the same first-order condition to choose $d_j(a,\zeta)$. □

Given Lemma 25, the proof of value function uniqueness is the same as Proposition 1.

Independence of Asset Allocation Decision and β. When $\underline{a}$ is the natural borrowing limit, the proof that $d_j(a,\zeta)$ is independent of β is as follows. By Lemma 25, $v_j(a,\zeta) = \hat{v}_j(a,\zeta)$. Given value function equivalence, equation (27) implies that the IG agent chooses the same illiquid asset policy function as the $\hat{u}$ agent: $d_j(a,\zeta) = \hat{d}_j(a,\zeta)$. When $\underline{a}$ is the natural borrowing limit the $\hat{u}$ agent behaves identically to a standard exponential agent (Remark 12). Thus, $d_j(a,\zeta) = \hat{d}_j(a,\zeta) = \check{d}_j(a,\zeta)$.

C.10 Proof of Proposition 16

The (potentially naive) IG agent sets $u'(c_j(a,\zeta)) = \beta \frac{\partial v_j^E(a,\zeta)}{\partial a}$. By Lemma 25, one can construct a $\hat{u}$ agent using β^E such that $\hat{v}_j(a,\zeta) = v_j^E(a,\zeta)$. This $\hat{u}$ agent chooses consumption such that $\frac{\partial \hat{u}^+(\hat{c}_j(a,\zeta))}{\partial \hat{c}} = \frac{\partial \hat{v}_j(a,\zeta)}{\partial a}$. This implies that $\frac{\partial \hat{v}_j(a,\zeta)}{\partial a} = \frac{(\psi^E)^\gamma}{\beta^E} \hat{c}_j(a,\zeta)^{-\gamma}$, where $\psi^E = \frac{\gamma - (1-\beta^E)}{\gamma}$.

Using the value function equivalence property that $\hat{v}_j(a,\zeta) = v_j^E(a,\zeta)$:

$$u'(c_j(a,\zeta)) = \beta \frac{(\psi^E)^\gamma}{\beta^E} \hat{c}_j(a,\zeta)^{-\gamma}.$$

Rearranging gives

$$c_j(a,\zeta) = \left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{1}{\psi^E} \times \hat{c}_j(a,\zeta).$$

[54] This implicitly assumes that $\underline{a} > \frac{-y_1}{r}$. If $\underline{a} = \frac{-y_1}{r}$ then $\hat{v}_j^*(\underline{a}) = -\infty$. In this case, $\hat{v}_j^*$ is never chosen.

To complete the proof, note that the $\hat{u}$ agent behaves identically to a standard exponential agent when $\underline{a}$ does not bind (Remark 12). This implies that the $\hat{u}$ agent sets $\hat{c}_j(a,\zeta) = \check{c}_j(a,\zeta)$ regardless of β and β^E. Therefore $c_j(a,\zeta) = \left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{1}{\psi^E} \times \check{c}_j(a,\zeta)$, as desired.

To see when consumption is increasing in naivete, consider:

$$\begin{aligned}
\frac{\partial c_j(a,\zeta)}{\partial \beta^E} &\propto \frac{1}{\gamma}\left(\frac{\beta^E}{\beta}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\beta}\frac{1}{\psi^E} - \left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{1}{\psi^E}\frac{1}{\gamma - (1-\beta^E)} \\
&\propto \frac{1}{\beta^E} - \frac{1}{\psi^E}
\end{aligned}$$

For $\beta^E < 1$, one can show that $\psi^E > \beta^E$ when $\gamma > 1$, and $\psi^E < \beta^E$ when $\gamma < 1$. Thus, consumption is increasing in naivete when $\gamma > 1$, and decreasing in naivete when $\gamma < 1$.

C.11 Proof of Corollary 19

In the model of Section 3, the expected continuation-value function $v_j^E(a)$ is characterized as follows:

$$\rho v_j^E(a) = u(c_j^E(a)) + \frac{\partial v_j^E(a)}{\partial a}(y_j + ra - c_j^E(a)) + \lambda_j(v_{-j}^E(a) - v_j^E(a)), \tag{46}$$

$$u'(c_j^E(a)) = \begin{cases} \beta^E \frac{\partial v_j^E(a)}{\partial a} & \text{if } a > \underline{a} \\ \max\{\beta^E \frac{\partial v_j^E(\underline{a})}{\partial a}, u'(y_j + r\underline{a})\} & \text{if } a = \underline{a} \end{cases}. \tag{47}$$

Equations (46) – (47) are identical to (7) – (8) except that the true short-run discount factor β is replaced by the perceived discount factor β^E. The agent's actual consumption decision is given by:

$$u'(c_j(a)) = \begin{cases} \beta \frac{\partial v_j^E(a)}{\partial a} & \text{if } a > \underline{a} \\ \max\{\beta \frac{\partial v_j^E(\underline{a})}{\partial a}, u'(y_j + r\underline{a})\} & \text{if } a = \underline{a} \end{cases}.$$

Let $s_j^E(a) = y_j + ra - c_j^E(a)$ denote the perceived savings rate. Taking a derivative of

(46) with respect to a and applying the first-order condition $u'(c_j^E(a)) = \beta^E \frac{\partial v_j^E(a)}{\partial a}$ gives

$$\left[(\rho - r) + \left(1 - \beta^E\right) \frac{\partial c_j^E(a)}{\partial a}\right] \frac{\partial v_j^E(a)}{\partial a} = \frac{\partial^2 v_j^E(a)}{\partial a^2} s_j^E(a) + \lambda_j \left(\frac{\partial v_{-j}^E(a)}{\partial a} - \frac{\partial v_j^E(a)}{\partial a}\right).$$

Multiplying through by β and using the property that $u'(c_j(a)) = \beta \frac{\partial v_j^E(a)}{\partial a}$ gives:

$$\left[(\rho - r) + \left(1 - \beta^E\right) \frac{\partial c_j^E(a)}{\partial a}\right] u'(c_j(a)) = u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} s_j^E(a) + \lambda_j \left(u'(c_{-j}(a)) - u'(c_j(a))\right).$$

Applying Ito's Lemma gives $\mathbb{E}_t du'(c_j(a_t)) = u''(c_j(a_t))c_j'(a_t)s_j(a_t) + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a)))dt$:

$$\begin{aligned}\left[(\rho - r) + \left(1 - \beta^E\right) \frac{\partial c_j^E(a)}{\partial a}\right] u'(c_j(a)) &= \mathbb{E}_t[du'(c_j(a_t))/dt] + u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} (s_j^E(a) - s_j(a)) \\ &= \mathbb{E}_t[du'(c_j(a_t))/dt] + u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} (c_j(a) - c_j^E(a)).\end{aligned}$$

The first-order conditions of $u'(c_j^E(a)) = \beta^E \frac{\partial v_j^E(a)}{\partial a}$ and $u'(c_j(a)) = \beta \frac{\partial v_j^E(a)}{\partial a}$ imply that $c_j^E(a) = \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} c_j(a)$. Thus,

$$\left[(\rho - r) + \left(1 - \beta^E\right) \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} \frac{\partial c_j(a)}{\partial a}\right] u'(c_j(a)) = \mathbb{E}_t[du'(c_j(a_t))/dt] + u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} c_j(a) \left(1 - \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}}\right)$$

Dividing through by marginal utility and using the property that $\gamma = \frac{-cu''(c)}{u'(c)}$:

$$\left[(\rho - r) + \left(1 - \beta^E\right) \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} \frac{\partial c_j(a)}{\partial a}\right] = \frac{\mathbb{E}_t[du'(c_j(a_t))/dt]}{u'(c_j(a))} - \gamma \frac{\partial c_j(a)}{\partial a} \left(1 - \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}}\right).$$

Rearranging yields the desired result.

C.12 Proof of Proposition 20

Step 1: Value Function Equivalence for the Naive Agent ($\gamma \neq 1$). Recall that the $\hat{u}$ utility function is constructed so that the value function of the sophisticated IG agent is equivalent to the value function of an exponential agent with utility function $\hat{u}$. The first step of this proof generalizes this construction to allow for naivete. I construct a utility function, denoted $\hat{\hat{u}}$, such that the *realized* value function of the (potentially naive) IG is equivalent to the value function of an exponential agent with utility function $\hat{\hat{u}}$. I refer to this agent as the $\hat{\hat{u}}$ agent.

Note that when $\beta^E \neq \beta$ the realized value function of the naive IG agent does not equal the naif's expected value function. As given in the main text, the expected continuation-value function is $v_t^E = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s^E) ds\right]$. However, the realized value function is

$$v_t^R = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s) ds\right].$$

v^R is based on the naif's realized consumption, while v^E is based on their perceived consumption.

Let $\hat{\hat{u}}(c) = \frac{xc^{1-\gamma}-1}{1-\gamma}$. This is a positive affine transformation of CRRA utility function $u(c)$ whenever $x > 0$. When this is the case, the $\hat{\hat{u}}$ agent will behave identically to a standard exponential agent. Thus, I will use $\check{c}_j(a, \zeta)$ to refer to the consumption of the $\hat{\hat{u}}$ agent.

In order to generate value function equivalence between the (possibly naive) IG agent and the $\hat{\hat{u}}$ agent, I construct $\hat{\hat{u}}$ so that the following condition holds for all $a > \underline{a}$:

$$u(c_j(a, \zeta)) - \frac{\partial v_j^R(a, \zeta)}{\partial a} c_j(a, \zeta) = \hat{\hat{u}}(\check{c}_j(a, \zeta)) - \frac{\partial v_j^R(a, \zeta)}{\partial a} \check{c}_j(a, \zeta). \tag{48}$$

This condition ensures that $v_j^R(a, \zeta) = \hat{\hat{v}}_j(a, \zeta)$ whenever $\underline{a}$ does not bind in equilibrium. See the proof of Proposition 10 for details.

I want to solve for x such that equation (48) holds. From Proposition 16, note that $\check{c}_j(a, \zeta) = \alpha c_j(a, \zeta)$, where $\alpha = \psi^E \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}}$. Additionally, the $\hat{\hat{u}}$ agent sets $\check{c}_j(a, \zeta)$ such that $x\check{c}_j(a, \zeta)^{-\gamma} = \frac{\partial v_j^R(a, \zeta)}{\partial a}$. Using these properties in equation (48) gives:

$$\frac{c_j(a, \zeta)^{1-\gamma}}{1-\gamma} - x\alpha^{-\gamma} c_j(a, \zeta)^{1-\gamma} = \frac{x(\alpha c_j(a, \zeta))^{1-\gamma}}{1-\gamma} - x(\alpha c_j(a, \zeta))^{1-\gamma}.$$

This can be rearranged to yield:

$$x = \frac{\alpha^\gamma}{1 - \gamma + \alpha\gamma}.$$

Note that $x = \frac{\psi\gamma}{\beta}$ in the case of sophistication ($\beta^E = \beta$), in which case $\hat{\hat{u}}(c) = \hat{u}(c)$.

Step 2: The Effect of a Consumption Tax. I now introduce a constant perpetual consumption tax of $\tau \in [0, 1)$. Given consumption tax $\tau \in [0, 1)$, let $\check{c}_j(a, \zeta)$ denote the gross consumption expenditure rate of the standard exponential agent.[55] Here I show that a consumption tax of τ does not affect the exponential agent's gross consumption expenditure.

With no tax, the standard exponential agent chooses consumption to maximize $\check{v}$:

$$\check{v}_j(a, \zeta) = \max_{\check{c}} \ \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(\check{c}_s) ds \right].$$

With a consumption tax, the standard exponential agent chooses consumption to maximize:

$$\check{v}_j(a, \zeta; \tau) = \max_{\check{c}} \ \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u((1-\tau)\check{c}_s) ds \right].$$

Note that $u((1-\tau)c)$ is a positive affine transformation of $u(c)$. Thus, policy function $\check{c}_j(a, \zeta)$ is unaffected by consumption tax τ. The only effect of the tax is that for $\tau > 0$, $\check{c}_j(a, \zeta)$ denotes gross consumption expenditure. The agent only gets to consume $(1-\tau)\check{c}_j(a, \zeta)$, with the rest going to taxes.

Step 3: The Welfare Effect of Present Bias ($\gamma \neq 1$). Since $\hat{\hat{u}}$ is a positive affine transformation of u, the $\hat{\hat{u}}$ agent behaves identically to a standard exponential agent. Additionally, value function equivalence implies that the realized value function of the IG agent equals the value function of the $\hat{\hat{u}}$ agent whenever $\underline{a}$ does not bind in equilibrium: $v_j^R(a, \zeta) = \hat{\hat{v}}_j(a, \zeta)$. This was shown in Step 1 of this proof.

The final step is to derive the consumption tax τ that equates the realized value function of the IG agent ($v_j^R(a, \zeta)$) with the value function of a standard exponential agent facing a

[55] In other words, the agent spends $\check{c}$ to consume $(1-\tau)\check{c}$.

consumption tax ($\check{v}_j(a,\zeta;\tau)$). Using value function equivalence, the realized value function of the IG agent is:

$$v_j^R(a,\zeta) = \hat{\hat{v}}_j(a,\zeta) = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\hat{\hat{u}}(\hat{\hat{c}}_s)ds\right]. \tag{49}$$

The value function of a standard exponential agent facing a consumption tax is:

$$\check{v}_j(a,\zeta;\tau) = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}u((1-\tau)\check{c}_s)ds\right]. \tag{50}$$

The key to this proof is to note that $\hat{\hat{c}}_j(a,\zeta) = \check{c}_j(a,\zeta)$. Therefore the consumption path in the integral of equation (49) is identical to the gross consumption expenditure path in equation (50) (this hold state by state, so it also holds in expectation). Thus, setting equation (49) equal to equation (50) is as simple as finding the value of τ such that:

$$\hat{\hat{u}}(c) = u((1-\tau)c).$$

This implies that $x = (1-\tau)^{1-\gamma}$. Rearranging gives

$$\tau = 1 - \left(\frac{\alpha^\gamma}{1-\gamma+\gamma\alpha}\right)^{\frac{1}{1-\gamma}}.$$

Special Case: $\gamma = 1$. In the special case of $\gamma = 1$ the naif and the sophisticate behave identically (Proposition 16). The realized value function $v_j^R(a,\zeta)$ is therefore independent of β^E. So, I calculate the $\gamma = 1$ case under the assumption of sophistication, $\beta^E = \beta$.

I again derive the consumption tax τ that equates the realized value function of the IG agent ($v_j^R(a,\zeta)$) with the value function of a standard exponential agent facing a consumption tax ($\check{v}_j(a,\zeta;\tau)$). Using value function equivalence, the realized value function of the IG agent is:

$$v_j^R(a,\zeta) = \hat{v}_j(a,\zeta) = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\hat{u}(\hat{c}_s)ds\right]. \tag{51}$$

The value function of a standard exponential agent facing a consumption tax is:

$$\check{v}_j(a, \zeta; \tau) = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u((1-\tau)\check{c}_s) ds \right]. \tag{52}$$

Since $\hat{c}_j(a, \zeta) = \check{c}_j(a, \zeta)$, the consumption path in the integral of equation (51) is identical to the gross consumption expenditure path in equation (52). As above, I need to find the value of τ such that:

$$\hat{u}(c) = u((1-\tau)c).$$

When $\gamma = 1$ this implies that $-\ln(\beta) + \frac{\beta-1}{\beta} = \ln(1-\tau)$. Rearranging gives

$$\tau = 1 - \frac{\exp\left(\frac{\beta-1}{\beta}\right)}{\beta}.$$

The Effect of β and β^E. First, I show that τ is decreasing in α. The derivative

$$\frac{\partial \tau}{\partial \alpha} = \frac{-1}{1-\gamma} \left(\frac{\alpha^\gamma}{1-\gamma+\gamma\alpha} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma\alpha^{\gamma-1}}{1-\gamma+\gamma\alpha} - \frac{\gamma\alpha^\gamma}{(1-\gamma+\gamma\alpha)^2} \right)$$

implies that

$$\text{sgn}\left(\frac{\partial \tau}{\partial \alpha}\right) = \text{sgn}(\gamma - 1) \times \text{sgn}\left(1 - \frac{\alpha}{1-\gamma+\gamma\alpha}\right), \text{ or equivalently}$$
$$\text{sgn}\left(\frac{\partial \tau}{\partial \alpha}\right) = \text{sgn}(\gamma - 1)\text{sgn}(1-\gamma).$$

Thus, τ is always decreasing in α.

The derivative of α with respect to β is:

$$\frac{\partial \alpha}{\partial \beta} = \frac{\psi^E}{\gamma \beta^E} \left(\frac{\beta}{\beta^E}\right)^{\frac{1-\gamma}{\gamma}} > 0.$$

As stated in the main text, this implies that $\frac{\partial \tau}{\partial \beta} < 0$.

The derivative of α with respect to β^E is:

$$\frac{\partial \alpha}{\partial \beta^E} = \frac{1}{\gamma}\left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} - \frac{1}{\gamma}\left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} \frac{\psi^E}{\beta^E}.$$

So, $\frac{\partial \alpha}{\partial \beta} > 0$ when $\beta^E > \psi^E$, and $\frac{\partial \alpha}{\partial \beta} < 0$ when $\beta^E < \psi^E$. Since $\beta^E > \psi^E$ when $\gamma < 1$ (and vice versa), this implies that α is increasing in β^E when $\gamma < 1$, and decreasing in β^E when $\gamma > 1$. This also implies that $\frac{\partial \tau}{\partial \beta^E} < 0$ when $\gamma < 1$, and $\frac{\partial \tau}{\partial \beta^E} > 0$ when $\gamma > 1$. As stated in the main text, naivete increases the welfare cost of present bias when $\gamma > 1$.

C.13 Proof of Proposition 21

This follows from the proof of Proposition 20, which shows that the realized continuation-value function of the IG agent is a positive affine transformation of the value function for the standard exponential agent. Accordingly, improving the (realized) welfare of the IG agent is equivalent to improving the welfare of the standard exponential agent.

D Model Solution with Naivete

This section replicates the numerical example of Section 3.3 under the assumption of complete naivete. To generate an equilibrium interest rate of 3%, I set $\beta = 0.75$, $\beta^E = 1$, and $\rho = 2.45\%$. The calibration is otherwise identical to Section 3.3.

Overall the results are qualitatively similar. The biggest difference between the consumption of the naif and the sophisticate occurs near $\underline{a}$. Though the naif still overconsumes near the borrowing constraint, Figure 5 illustrates that the naif overconsumes by less than the sophisticate. As described in the main text, when the consumer is sophisticated their present bias interacts with the effective planning horizon to increase consumption near $\underline{a}$. This effect does not arise under naivete because the naif trusts all future selves.

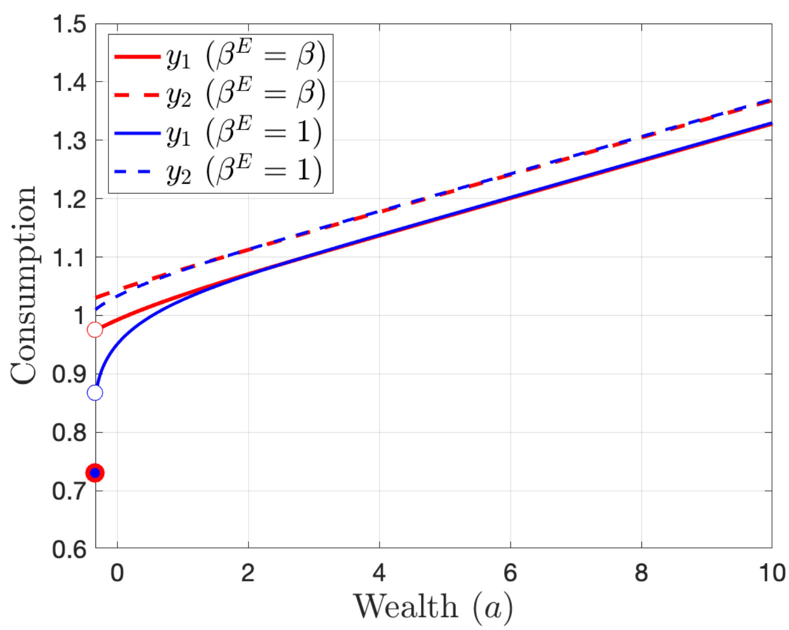

Figure 5: **Equilibrium Consumption-Saving Decisions.** The figure plots the equilibrium consumption function for the $\beta^E = \beta$ calibration (sophistication) and the $\beta^E = 1$ calibration (naivete).

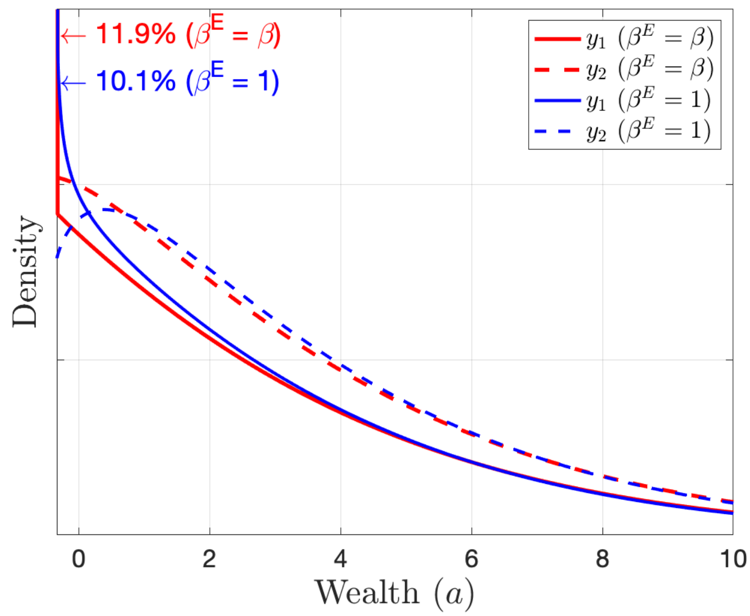

Figure 6: **The Distribution of Wealth.** This figure shows the stationary wealth distribution for the $\beta^E = \beta$ calibration and the $\beta^E = 1$ calibration.

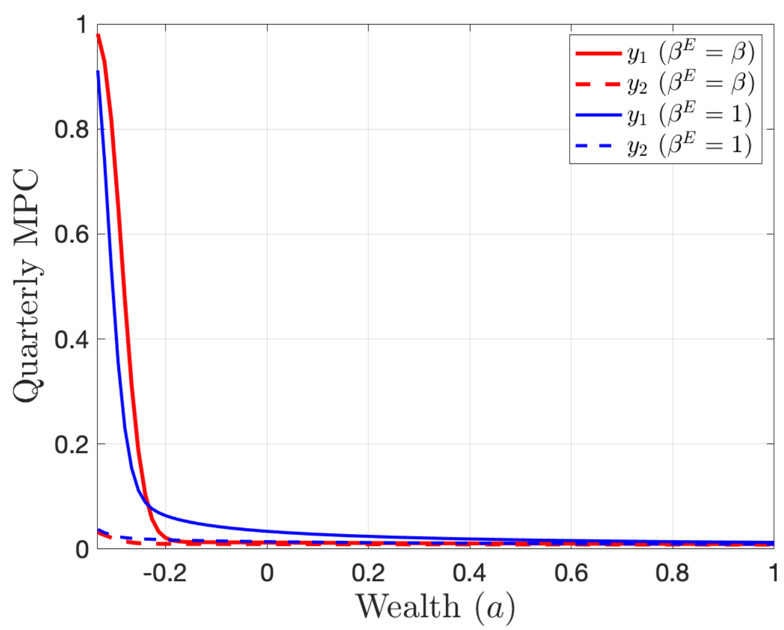

Figure 7: **MPCs.** This figure plots quarterly MPCs for the $\beta^E = \beta$ calibration and the $\beta^E = 1$ calibration.