Learning with Misattribution of Reference Dependence

Benjamin Bushong
Michigan State University

Tristan Gagnon-Bartsch*
Harvard University

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Abstract

This paper explores how an individual whose utility depends on expectations may develop biased beliefs when learning from experience. We consider an agent who neglects how the sensation from a positive or negative surprise contributes to her overall utility, and who wrongly attributes this component of her utility to the intrinsic value of an outcome. Our model provides a number of implications that accord with evidence on belief updating in dynamic environments. First, a misattributor’s expectations are over-influenced by recent experiences and under-influenced by earlier ones. Second, long-run beliefs grow pessimistic and undervalue prospects in proportion to their variability, leading a decision maker to abandon some risky options that are optimal. Third, when outcomes are autocorrelated, a misattributor persistently forms extrapolative and volatile forecasts about future payoffs. Applying the model, we show that (i) uncertain availability of a good tends to increase its perceived value, (ii) a misattributor may exert too much effort when searching for consumption goods, and (iii) a misattributing principal may overestimate the ability of a manipulative agent who initially suppresses expectations so as to exceed them thereafter.

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*E-mails: bbushong@msu.edu and gagnonbartsch@fas.harvard.edu. We thank Ned Augenblick, Stefano DellaVigna, Ben Enke, Erik Eyster, Xavier Gabaix, Ben Golub, David Laibson, Devin Pope, Matthew Rabin, Gautam Rao, Josh Schwartzstein, Andrei Shleifer, Kelly Shue, Dmitry Taubinsky and seminar audiences at Berkeley, Caltech, Cornell, Harvard, Harvard Business School, LSE, the Norwegian School of Economics (NHH), Princeton, SITE, Stanford, and UCSD for comments.
1 Introduction

Learning from personal experience guides a wide range of economic decisions—it shapes, for instance, our preferences over consumer products, adoption of new technologies, and impressions of others. This paper explores an intuitive mechanism through which learning from experience can bias beliefs. The bias stems from the well-known idea that people evaluate experiences relative to a reference point: the utility from an outcome depends on both its intrinsic value and how that value compares to expectations (e.g., Kahneman and Tversky 1979; Bell 1985; Kőszegi and Rabin 2006). Such reference dependence complicates learning since these two components of utility must be disentangled in order to properly forecast future utility. That is, when trying to learn the intrinsic value of an outcome, a person must distinguish this intrinsic (“reference-free”) value from the sensation of elation or disappointment it generated. Intuition and recent research suggests that people may fail to fully separate these different sources of utility, and may incorrectly attribute the sensation of surprise (i.e., elation or disappointment) to their intrinsic taste for an outcome.

To illustrate, imagine a consumer trying a new service for the first time (e.g., a traveler flying on a new airline, or a new bidder on eBay as in the related empirical paper by Backus et al. 2018). If her experience falls short of expectations, she will feel unhappy both because of the subpar service and the negative surprise. A rational consumer will understand her bad experience derived, in part, from her high expectations. A less introspective consumer, however, might misattribute her disappointment to the underlying quality of the service, and consequently underestimate how much she would enjoy that service in the future. Alternatively, consider a researcher collaborating with a new colleague. If the colleague finishes an unexpectedly large share of the project on his own, the researcher will feel happy both because her workload has diminished and because this came as a pleasant surprise. If she misattributes this latter feeling to her partner’s performance, she may recall an exaggerated perception of his contribution. As these examples suggest, surprises can distort perceived outcomes: exceeding expectations inflates perceptions, and falling short deflates them.

In this paper, we model an individual who neglects how sensations of positive and negative surprise

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1 Our companion experimental paper, Bushong and Gagnon-Bartsch (2018), provides evidence that people fail to account for their former expectations when inferring from past experience. A related literature explores other forms of misattribution and demonstrates how the effects of extraneous situational factors may be attributed to the inherent characteristics of a good or person. For instance, Haggag and Pope (2018) demonstrate that, when assessing the value of a good, people have difficulty separating state-dependent utility caused by temporary circumstances, such as thirst, from the quality of the good. Dutton and Aron (1974) find that subjects who form opinions about people they meet for the first time exhibit judgments dependent on unrelated factors (e.g., their current state of fear or excitement). We discuss this literature at greater length in Section 2.

2 Backus et al. (2018) find that new eBay users who have greater expectations of winning an auction (measured by time spent in the lead) are more likely to quit using the platform if they unexpectedly lose their first auction. Specifically, bidders who surprisingly lose near the end of the auction are 6 percentage points more likely to exit the platform for every additional day they previously held the lead. This suggests that misattribution of sensations of disappointment can have important consequences for customer perceptions and retention, independent of material outcomes.
influenced her experienced utility, and who misattributes these sensations to the underlying quality of the outcomes she faced. While a large literature has explored how reference points can affect preferences, we suggest an intuitive channel through which reference points can additionally influence beliefs. We contribute to this literature and the literature on biased learning (discussed below) by developing a formal model of this mechanism and demonstrating how it captures many observed patterns in beliefs. Specifically, we consider a misattributing agent who is learning about the average payoff of an action with stochastic returns (e.g., a consumer service, a risky project, or an employee with variable productivity). Misattribution leads to a dynamic misinterpretation of outcomes: a distorted perception of the current outcome leads to biased expectations, which further distorts the evaluation of later outcomes. These dynamics generate several known errors such as contrast effects—the current outcome appears better the worse was the previous one—and a recency bias—expectations overweight recent outcomes. We also characterize how misattribution affects long-run learning and explore in a variety of applications how misattribution distorts incentives.

Section 2 introduces the model. We consider dynamic-choice contexts where the outcome from each action may be stochastic and the decision maker is initially uncertain about the distribution of outcomes. Each period, the decision maker selects an action and then experiences utility composed of two parts: consumption utility—the classical notion of payoffs—and reference-dependent utility, which depends on the difference between her realized consumption utility and what she expected. She sequentially updates her beliefs about the distribution of outcomes based on each experience. To preview the model, suppose utility from outcome \( x \in \mathbb{R} \) when expecting \( r \in \mathbb{R} \) is \( u(x|r) = x + \eta n(x|r) \), where the reference-dependent component \( n(x|r) \) is proportional to the difference between \( x \) and \( r \) and parameter \( \eta > 0 \) measures the weight that elation and disappointment carry on total utility. We assume that a misattributor wrongly infers from past utility as if she weighted elations and disappointments by a diminished factor \( \hat{\eta} < \eta \): she correctly recalls her total utility, but under-appreciates the extent to which elation or disappointment contributed to this total. She thus infers a distorted value of each outcome \( x \). More specifically, when \( x \) surpasses expectations, she infers a value \( \hat{x} > x \); when \( x \) falls short, she infers \( \hat{x} < x \). The decision maker is otherwise rational and updates according to Bayes’ Rule as if outcome \( \hat{x} \) truly occurred. This simple model suggests that beliefs overreact to surprising outcomes and that a misattributor over-infers from first-hand experiences—those that incite elation and disappointment—relative to second-hand observations that contain identical information.\(^3\) Furthermore, if the misattributor is loss averse, disappointments will distort beliefs by

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\(^3\)A distinct but related literature shows that people rely too heavily on their own past experience when making decisions. Personal successes and failures play an important role in IPO subscription (Kaustia and Knüpf 2008; Chiang et al. 2011), risk taking and stock-market participation (Malmendier and Nagel 2011), insurance take-up (Gallagher 2014), college-major choice (Xia 2016), and compliance with deadlines (Haselhuhn, Pope, and Schweitzer 2012). Beyond misattribution, there are several plausible factors that may cause people to overweight personal experiences when making decisions. For instance, both the availability heuristic (Tversky and Kahneman 1973) and under-inference from large samples (Benjamin, Rabin, and Raymond 2016) may play a role. Our model predicts that such overweighting of personal
more than commensurate elations (e.g., Skinner and Sloan 2002; Kuhnen 2015).

In Section 3, we illustrate how misattribution biases beliefs in a simple example. We consider a manager learning about the ability of a newly-hired employee, such as an assistant, whose performance directly affects the manager’s utility. If the employee’s performance is identical each day, a misattributing manager will perceive false variation in performance over the short run, but will eventually learn the employee’s ability. Contrastingly, mistaken perceptions will persist when performance randomly varies over time. If the employee’s productivity is either high or low on any given day, then the manager will persistently overestimate the quality of the employee’s performance on productive days and underestimate it on unproductive days. Furthermore, the manager forms more distorted perceptions of the value of an outcome when that outcome is less likely—e.g., she overestimates the performance on good days by a greater extent when the employee has good days less often. We discuss implications of these results for consumer’s beliefs about the quality of products with uncertain availability.

Section 4 explores more generally how misattribution influences perceptions about a fixed action after a person tries it a large number of times. The interplay between beliefs and realized utility prevents a misattributor from reaching correct expectations despite ample experience. We characterize steady-state beliefs about the mean and variance of outcomes in terms of the true mean and variance. A misattributor will perceive a distribution that is excessively variable and, due to loss aversion, negatively skewed relative to the truth. Hence, she underestimates a prospect’s mean outcome. Furthermore, increasing the true variability of the prospect causes a misattributor to underestimate its mean by a greater extent, as increased variability amplifies sensations of disappointment on average. Therefore, a misattributor will excessively avoid risk and may accordingly incur large welfare losses.4

In Section 5 we analyze how the order in which a misattributor experiences outcomes can distort her perceived benefit of an action. For instance, consider the manager who updates her beliefs about an employee’s ability each day. Even when the employee’s performance is i.i.d., misattribution can generate a recency bias—recent outcomes influence beliefs more than older ones.5 This stems from the “contrast effect” inherent in our form of misattribution: high initial outcomes raise expectations

experience is more pronounced for surprising episodes that carry utility consequences.

4Malmendier and Nagel (2011) document increased apparent risk aversion as investors exit the stock market in response to adverse personal experiences. They argue this stems from biased beliefs rather than altered preferences, and later work (Malmendier and Nagel 2016) provides more direct support for the belief channel. A related literature attempts to explain such phenomena by positing that risk preferences depend directly on a decision maker’s history of elations and disappointments. However, these models (e.g., Dillenberger and Rozen 2015) predict a primacy effect rather than a recency effect: early experiences have a greater impact on behavior than later experiences. This implication stands at odds with the recency effect documented by Malmendier and Nagel (2011).

5Our model endogenously generates a recency effect, which has been documented in a range of economic decisions, such stock-market participation and hiring decisions (Highhouse and Gallo 1997). Malmendier, Pouzo, and Vanasco (2018) and Ehling, Graniero, and Heyerdahl-Larsen (2018) incorporate an exogenous recency effect into learning models, demonstrating how it helps explain phenomena such as excessive volatility in asset prices and trend-chasing behavior.
and cause later outcomes to be judged more harshly, while low initial outcomes lower expectations
and cause later outcomes to be judged more favorably. We provide comparative statics on the strength
of signals and priors that predict when a recency bias emerges. In such cases, we also show that a
misattributor is most optimistic about an action when, ceteris paribus, its outcomes are arranged in
an improving order. That is, fixing the total amount of work the employee completes, the manager is
most optimistic about his ability when each of his performances is better than the last.\footnote{A literature studying the role of improving sequences on perceptions demonstrates that, fixing the set of outcomes, people tend to form the most optimistic impressions when they experience outcomes in increasing order (e.g., Ross and Simonson 1991; Haisley and Loewenstein 2011). Although such a recency bias stands at odds with confirmation bias (Rabin and Schrag 1999; Fryer, Harms, and Jackson 2017), the mechanisms underlying these two errors are not mutually exclusive. Indeed, Hogarth and Einhorn’s (1992) meta-study on order effects finds strong support for both recency and confirmatory effects. Which effect dominates depends on the type of feedback received: confirmatory effects dominate as evidence becomes more ambiguous and difficult to interpret.}

Additionally, Section 5 demonstrates how misattribution generates overly-extrapolative forecasts
in environments with autocorrelated outcomes. When today’s outcome beats expectations, a misat-
tributor exaggerates its value and hence expects unreasonably high values in the future. With such
high expectations, however, the next outcome typically disappoints and leads to overly-pessimistic
expectations. This pattern will continue over time: the person forms an exaggerated forecast in the
direction of the most recent outcome, which leads to a subsequent reversal.\footnote{Our basic prediction of extrapolative and volatile forecasts accords with a range of evidence, including Greenwood and Shleifer (2014) and Gennaioli, Ma, and Shleifer (2015) who find that investors’ and managers’ predictions of their future earnings exhibit forecast errors that negatively relate with past performance. Importantly, our model applies to such settings involving money so long as earnings carry immediate hedonic consequences—a notion discussed in Kőszegi and Rabin (2009). While other models give rise to extrapolative beliefs and systematic reversals (e.g., Barberis, Shleifer, and Vishny 1998; Bordalo, Gennaioli, and Shleifer 2017a), our model provides additional predictions that may help empirically disentangle the mechanisms at play. For instance, we predict that these patterns are more pronounced when forecasting about one’s own earnings (i.e., outcomes generate elation and disappointment) and that beliefs respond more to bad news than good.}

In Section 6, we explore the implications of misattribution in two applications. We first consider
the effect on repeated search problems in which a decision maker can exert costly effort each period
to increase the chance of a good outcome that round (e.g., a consumer who spends time comparison
shopping before each purchase). Taking such measures will lead a misattributor to exaggerate the
value of further effort. Since increased effort raises expectations, bad outcomes (e.g., purchases that
end up being lower quality than expected) seem even worse when they happen. This drives her to
perceive a greater dispersion in outcomes, which causes her to eventually settle on inefficiently high
effort. In our second application, we explore how a sophisticated agent would strategically manipulate
the beliefs of a misattributing evaluator. We consider a career-concern setting where a misattributing
(but otherwise rational) principal hires an agent and sequentially updates her beliefs about that agent’s
ability based on his output. While classical models like Holmström (1999) predict that the agent’s
effort inefficiently declines over time, biased evaluations introduce new incentives that oppose the
classical prediction. Although the agent could pleasantly surprise the principal with high effort today

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to increase tomorrow’s compensation, doing so would raise the principal’s expectations and cause her to judge later output more harshly. A manipulative agent hired for a fixed duration may thus under-perform relative to the principal’s expectations early in the relationship but consistently beat them later.  

We conclude in Section 7 by noting ways that researchers can distinguish misattribution from other biases that share similar qualitative predictions. We also present some natural extensions of our model. For instance, our model can be reframed as a bias in social learning where an observer neglects how expectations shape the experiences of others. A student reading reviews for a class may fail to appreciate that a bad rating could reflect the reviewer’s high expectations rather than a low-quality professor. Failing to account for others’ expectations may also have important implications for how policy makers interpret surveys measuring satisfaction. For instance, James (2009) and Van Ryzin (2004) find that reported satisfaction with public services declines with increased expectations. If policy makers neglect the role of expectations in these reports, they may wrongly attribute such a decline to poor quality or changing tastes and consequently suggest ill-suited reforms. Moreover, misattribution captures a common intuition about why informational campaigns can backfire. If agencies tout the benefits of adopting, say, health practices or agricultural technologies, perceived outcomes may be biased downward because of high expectations, leading patients or farmers to prematurely abandon new practices.

Our paper connects to a growing literature that explores mistaken learning when agents hold misspecified models of the world. For instance, our modeling approach is similar to that of Heidhues, Kőszegi, and Strack (2018), who study how agents who overestimate their ability mislearn the value of a fundamental that influences how their effort translates to output. In their model, an agent misattributes poor performance to situational factors when in reality it was driven by lower-than-expected ability. More broadly, our model fits into the general framework provided by Esponda and Pouzo (2016) for assessing the long-run beliefs and behavior of agents with misspecified models. Finally,

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8This result is reminiscent of common strategies of “expectations management” in a variety of fields. In marketing, Kopalle and Lehmann (2006) study how a firm should optimally set quality expectations through advertisements when consumers have preferences that depend on expectations (known as “disconfirmation” or the “gap model” in marketing; see, for example, Anderson 1973, Oliver 1977, or Ho and Zheng 2004). Fixing the realized quality of a good, a consumer in their model who is pleasantly surprised in the first period is more likely to buy again in the second period. In finance, firms commonly use a variety of mechanisms to “walk down” investors’ expectations prior to earnings announcements—strategic accounting of working capital and cash flow (Burgstahler and Dichev 1997), sales (Roychowdhury 2006), or distorting analyst forecasts (e.g., Richardson, Teoh, and Wycoki 2004). Furthermore, Bartov, Givoly and Hayn (2002) argue that such efforts to meet or beat analyst expectations could yield significant excess stock returns. Indeed, Teoh, Yang and Zhang (2009) find that firms are rewarded for beating expectations even when they actively suppress analyst forecasts. Our model provides one plausible mechanism underlying these incentives to restrain expectations.

9In addition to the models discussed throughout the paper, examples include Rabin (2002) and Rabin and Vayanos (2010) on the law of small numbers; DeMarzo, Vayanos, Zweibel (2003), Eyster and Rabin (2010) and Bohren (2016) on misinferring from others’ behavior in social-learning contexts; Madarász (2012) on information projection; Schwartzstein (2014) on selective attention; and Spiegler (2016) on biases in causal reasoning.
our model extends a literature studying distorted beliefs where those beliefs directly enter an agent’s utility function.\footnote{In contrast to many models in this literature (e.g., Bénabou and Tirole 2002; Brunnermeier and Parker 2005; see Bénabou 2015 for a review), a misattributor does not purposefully manipulate her beliefs to maximize her belief-based utility. Rather, she forms distorted beliefs mechanically as a result of the bias in her perceptions.} We are unaware of existing models that explore the feedback between belief-based utility and mechanically distorted beliefs.

2 A Model of Misattribution of Reference-Dependent Utility

This section presents our model. We first introduce the decision maker’s reference-dependent utility function and then describe the consumption and learning environment. Section 2.3 formalizes our novel assumption: the decision maker misattributes sensations of elation and disappointment to the intrinsic value of an outcome, which leads to biased beliefs. We also discuss motivating evidence for the main assumptions of our model.

2.1 Reference-Dependent Utility

We begin by specifying the agent’s reference-dependent utility from consumption $c \in \mathbb{R}$.\footnote{We extend the model to multiple dimensions in Appendix C.} As in Kőszegi and Rabin (2006; henceforth KR), we assume that overall utility has two components. The first component, “consumption utility”, corresponds to the outcome-based payoff traditionally studied in economics. Let $x \in \mathbb{R}$ denote the consumption utility from $c$.\footnote{We interpret $x$ as if it derives from a classical Bernoulli utility function $m: \mathbb{R} \to \mathbb{R}$ such that $x = m(c)$, but we work directly with consumption utility $x$ to reduce notational clutter.} The second component, “gain-loss utility”, derives from comparing $x$ to a reference level of utility, denoted by $r \in \mathbb{R}$. When the reference point is deterministic, gain-loss utility, denoted by $n(x|r)$, is given by

$$n(x|r) = \begin{cases} 
  x - r & \text{if } x \geq r \\
  \lambda (x - r) & \text{if } x < r.
\end{cases}$$

(1)

We assume gain-loss utility is piecewise linear with weight $\lambda \geq 1$ on losses.\footnote{This gain-loss utility function follows the specification of KR under their Assumption A3’. While this shares many similarities with Kahneman and Tversky’s (1979) value function, we abstract from other elements of prospect theory—diminishing sensitivity and probability weighting—to focus on the role of reference points and loss aversion.} Although we often assume loss aversion with $\lambda > 1$, we sometimes consider $\lambda = 1$ to highlight results independent from loss aversion. The person’s total utility from $x$ given reference point $r$ is

$$u(x|r) = x + \eta n(x|r),$$

(2)
where $\eta > 0$ is the weight given to sensations of gain and loss relative to absolute outcomes.

Our formulation of misattribution, introduced below, is general in that it can be applied using any definition of the reference point. Given our focus on learning from experience, however, we assume the person’s reference point is her recent expectation about consumption utility, as expectations naturally serve as a reference point when a person is uncertain of the outcome. Specifically, suppose the person believes $x_t$ is distributed with c.d.f. $F$. We follow Bell (1985) and assume $x_t$ represents a gain whenever $x_t \geq \mathbb{E}_F[x_t]$ and a loss otherwise. That is, the reference point $r$ is given by $\mathbb{E}_F[x_t]$, and Equation 1 becomes

$$n(x|F) = \begin{cases} 
  x - \mathbb{E}_F[x] & \text{if } x \geq \mathbb{E}_F[x] \\
  \lambda(x - \mathbb{E}_F[x]) & \text{if } x < \mathbb{E}_F[x].
\end{cases}$$

(3)

While there are several ways to model expectations-based reference points, we adopt the specification from Equation (3) primarily for tractability. We additionally explore the stochastic reference points introduced by KR in Appendix D, which yields similar results.

### 2.2 Consumption and Learning Environment

We analyze a decision maker who experiences a series of payoffs over time. While much of the paper focuses on learning about a single action—that is, the decision maker’s payoffs derive from a single source of utility—the following presents the complete model. Each period $t \in \{1, 2, \ldots, T\}$ is structured as follows. First, the person chooses an action $a_t$ from set $A \equiv \{1, \ldots, N\}$. Second, an outcome $x_t = (x_t(1), \ldots, x_t(N)) \in \mathbb{R}^N$ is realized, where $x_t(a_t)$ is the consumption utility from choice $a_t$ that period. Third, the person earns total utility $u_t \in \mathbb{R}$ that depends on both the outcome $x_t(a_t)$ and her expectations of $x_t(a_t)$. In terms of the introductory examples, $a_t$ might be a consumer’s chosen service in period $t$ and $x_t(a_t)$ is the experienced quality of service, or $a_t$ might be a researcher’s chosen collaborator and $x_t(a_t)$ is the benefit from that colleague’s contribution.

For each action $a \in A$, the outcome $x_t(a)$ depends an unknown payoff-relevant parameter $\theta(a)$ drawn from set $\Theta(a) \subseteq \mathbb{R}$. These parameters are the object of learning. Aside from our treatment

14Several experimental studies find evidence of expectations-based reference points, though the totality of evidence is mixed (for example, favoring expectations-based reference points are Abeler et al. 2011; Ericson and Fuster 2011; Gill and Prowse 2012; Banerji and Gupta 2014; Karle, Kirchsteiger, and Peitz 2015; against are Heffetz and List 2014; Gneezy et al. 2017; Goette, Harms, and Sprenger 2018). There is additional evidence of expectations-based reference points from the field, spanning labor supply among taxi drivers (Crawford and Meng 2011; Thakral and To 2018), platform exit from an online marketplace (Backus et al. 2018), domestic violence resulting from unexpected football losses (Card and Dahl 2011), and decisions in game shows and sports (Post et al. 2008; Pope and Schweitzer 2011; Markle et al. 2015).

15Whether or not it is natural to assume that the decision maker observes elements of $x_t$ beyond $x_t(a_t)$ depends on the particular application. In some scenarios, observing the outcomes of counterfactual decisions seems realistic (e.g., betting markets), while in others it does not (e.g., experimental consumption). To avoid confusion, we highlight this assumption whenever necessary.

16For convergence results in Section 4, we will assume $\Theta$ is compact.
of autocorrelated outcomes in Section 5.2, we assume that, conditional on \( \theta(a) \), \( x_t(a) \) is i.i.d. with c.d.f. \( F^a(\cdot | \theta(a)) \) and has full support over \( \mathbb{R} \). For example, we often consider environments with \( x_t(a) = \theta(a) + \varepsilon_t(a) \) where \( \varepsilon_t(a) \) are i.i.d. normal shocks. In this case, the collection of parameters \( \theta \equiv (\theta(1), \ldots, \theta(N)) \) might represent, for instance, the average quality of each service the consumer considers, or the mean productivity of each potential colleague.

We assume each parameter \( \theta(a) \) is independent of \( \theta(b) \) for all \( b \in \mathcal{A}, b \neq a \), and the person begins with a full-support prior \( \pi^a_1 \in \Theta(\theta(a)) \) for each \( a \in \mathcal{A} \). She updates this belief after each experience and \( \pi^a_t \in \Theta(\theta(a)) \) denotes her beliefs entering period \( t \). Critically, these beliefs form her expectations, and hence reference point, in round \( t \): given action \( a_t = a \), her belief \( \pi^a_t \) over possible values of \( \theta(a) \) determines her perceived distribution of \( x_t(a) \), denoted by \( \tilde{F}_t(x|a) = \int_{\theta(a)} F^a(x|\theta(a)) \pi^a_t(\theta(a)) \), and she then earns total utility \( u(x_t(a)|\tilde{F}_t(\cdot|a)) \) given by Equations 2 and 3.

### 2.3 Misattribution of Gain-Loss Utility

We now turn to the central assumption of our model: the decision maker neglects how her past experiences were influenced by reference dependence and misattributes her gain-loss utility to an action’s underlying consumption utility.

After taking action \( a \), the decision maker seeks to learn the unknown payoff-relevant parameter \( \theta(a) \) that governs the distribution of outcomes. A rational updater faced with signal \( u_t = x_t(a) + \eta n(x_t(a)|\tilde{F}_t(\cdot|a)) \) understands this signal is “contaminated” by a gain-loss term and properly accounts for this when using \( u_t \) to update her beliefs about \( \theta(a) \). We assume that a misattributor errs in this step: she infers from \( u_t \) as if her utility function weights gains and losses by a diminished factor \( \hat{\eta} \in [0, \eta) \). Hence, a misattributor treats signal \( u_t \) as if \( u_t = x_t(a) + \hat{\eta} n(x_t(a)|\tilde{F}_t(\cdot|a)) \equiv \hat{u}(x_t(a)|\tilde{F}_t(\cdot|a)) \), where \( \hat{\eta} < \eta \). After each period, the person uses her (correct) memory of \( u_t \) along with her misspecified model of utility \( \hat{u} \) to infer the outcome \( x_t(a) \) she must have received. We denote this encoded outcome by \( \hat{x}_t(a) \) and the misinference described above implies that \( \hat{x}_t(a) \) solves

\[
u(x_t(a)|\tilde{F}_t(\cdot|a)) = u_t = \hat{u}(\hat{x}_t(a)|\tilde{F}_t(\cdot|a))._{17}
\]

Roughly put, the person’s incorrect model of her past utility understates the degree to which \( u_t \) derives from gain-loss utility. Any gain-loss utility the decision maker fails to account for is attributed to the intrinsic consumption value associated with her choice. Finally, we assume the person is unaware of her misencoding and uses \( \hat{x}_t(a) \) along with Bayes’ rule to update her beliefs \( \pi^a_t \) over \( \theta(a) \).

To explore the implications of misattribution, we first demonstrate that encoded outcomes take a simple form. Letting \( \hat{\mathbb{E}}_t[x_t(a)] \) denote the person’s expectation of \( x_t(a) \) with respect to subjective

\[_{17} \text{Note that } \hat{x}(a) \text{ is well defined and unique: fixing any belief } \tilde{F}, \text{ the utility function } \hat{u} \text{ is strictly increasing in } \hat{x}(a). \]
beliefs $\hat{F}(\cdot|a)$, Equation 4 yields

$$\hat{x}_t(a) = \begin{cases} 
  x_t(a) + \kappa^G \left( x_t(a) - \hat{E}_t[x_t(a)] \right) & \text{if } x_t(a) \geq \hat{E}_t[x_t(a)] \\
  x_t(a) + \kappa^L \left( x_t(a) - \hat{E}_t[x_t(a)] \right) & \text{if } x_t(a) < \hat{E}_t[x_t(a)],
\end{cases} \tag{5}$$

where

$$\kappa^G \equiv \left( \eta - \hat{\eta} \right) \quad \text{and} \quad \kappa^L \equiv \lambda \left( \eta - \hat{\eta} \right). \tag{6}$$

The parameters $\kappa^G$ and $\kappa^L$ represent the extent that elations and disappointments, respectively, distort encoded outcomes. Intuitively, $\kappa^G$ and $\kappa^L$ increase in the degree of misattribution; i.e., as $\hat{\eta}$ decreases.

The simple specification above (Equation 5) yields several immediate implications. First, the misattributor overreacts to surprising outcomes.

**Observation 1.** Outcomes that beat expectations are distorted upward, while those that fall short are distorted downward: If $x_t(a) > \hat{E}_t[x_t(a)]$, then $\hat{x}_t(a) > x_t(a)$, and if $x_t(a) < \hat{E}_t[x_t(a)]$, then $\hat{x}_t(a) < x_t(a)$.

Second, when $\lambda > 1$, disappointments and elations distort encoded outcomes (and hence beliefs) asymmetrically.

**Observation 2.** Losses are misencoded by more than equivalently sized gains: Suppose $\lambda > 1$. Consider outcomes $g(a) \equiv \hat{E}_t[x_t(a)] + \epsilon$ and $l(a) \equiv \hat{E}_t[x_t(a)] - \epsilon$. For any $\epsilon > 0$, $|\hat{l}(a) - l(a)| > |\hat{g}(a) - g(a)|$.

Third, misattribution can generate “sequential contrast effects”: fixing the value of today’s outcome, its perceived value seems higher the lower was yesterday’s. When the previous experience lowers expectations, the current outcome is assessed against a lower benchmark and thus generates a larger elation (or a smaller disappointment).\footnote{Sequential contrast effects have been documented in numerous settings, including sequential decisions made by teachers (Bhargava 2007) and speed daters (Bhargava and Fisman 2014). In a financial setting, Hartzmark and Shue (2017) demonstrate that prior-day earnings announcements of other firms negatively correlate with stock-price reactions to contemporaneous announcements.} Contrast effects occur whenever there is an increasing relationship between the most recent observation and the resulting posterior expectation, which holds for many familiar distributions $F^a$.\footnote{Sequential contrast effects have been documented in numerous settings, including sequential decisions made by teachers (Bhargava 2007) and speed daters (Bhargava and Fisman 2014). In a financial setting, Hartzmark and Shue (2017) demonstrate that prior-day earnings announcements of other firms negatively correlate with stock-price reactions to contemporaneous announcements.}

**Observation 3.** Sequential contrast effects: Suppose the decision maker takes action $a$ in periods $t$ and $t-1$ and $\hat{E}_t[x_t(a)]$ is strictly increasing in $x_{t-1}(a)$. Then $\frac{\partial \hat{x}_t(a)}{\partial x_{t-1}(a)} < 0$. 


Finally, our model implies a difference in learning from outcomes with direct utility consequences versus observations that contain identical information but do not influence payoffs directly. In particular, given that misattribution stems from a misunderstanding of the source of utility, only those outcomes that incite sensations of elation or disappointment are prone to misencoding.\footnote{Relatedly, Charness and Levin (2005) find significantly greater errors in updating about the distribution of balls in an urn when participants observe a sample of draws that have payoff consequences relative to the case in which these signals have no payoff consequences. Their experiment suggests that the affect induced by payments is a critical factor in deviations from Bayesian updating.}

To illustrate these observations, consider a person experimenting with a new medical treatment to reduce pain. Let $x$ measure the effectiveness of the treatment (in utils), and suppose the patient expects $x = 50$. First, imagine $x = 60$; the treatment works better than expected. To decide whether to use this treatment again, the patient infers its efficacy $x$ from her experienced utility $u = 60 + \eta n(60|50) = 60 + \eta 10$. While she correctly recalls a pleasant experience, she fails to properly disentangle the consumption value of the treatment from the elation due to surprise. From Equation 5, the patient recalls value $\hat{x}$ such that

$$\hat{x} = 60 + \left( \frac{\eta - \hat{\eta}}{1 + \hat{\eta}} \right) 10 > x.$$ \hspace{1cm} (1)

If, for instance, $\eta = 1$ and $\hat{\eta} = 1/3$, then $\hat{x} = 65$. Contrastingly, imagine $x = 40$; the treatment works worse than expected. She then encodes a value

$$\hat{x} = 40 - \lambda \left( \frac{\eta - \hat{\eta}}{1 + \lambda \hat{\eta}} \right) 10 < x.$$ \hspace{1cm} (2)

Again, if $\eta = 1$, $\hat{\eta} = 1/3$ and $\lambda = 3$, then $\hat{x} = 30$. Finally, to demonstrate a contrast effect, suppose the patient tries the treatment a second time. Fixing this second experience at $x = 50$ and assuming the Bayesian expectation of $x$ is increasing in observed outcomes, she will encode a higher perceived value on the second trial if she initially experienced $x = 40$ rather than $x = 60$.

To close the model, we assume the decision maker maximizes her true utility function conditional on her erroneous beliefs. Since we assumed the person’s expectations about consumption utility in round $t$ adjust according to her preceding action, she maximizes expected utility believing that, conditional on choice $a_t$, her reference point will be $\hat{F}_t(\cdot|a_t)$.\footnote{This solution concept corresponds to KR’s (2007) notion of “choice-acclimating personal equilibrium” aside from the fact that we do not impose rational expectations: equilibrium is with respect to the misattributor’s biased subjective beliefs.} More formally, when the person observes counterfactual outcomes—implying that the optimal dynamic strategy calls for the myopically-optimal action each round—each action $a_t \in A$ satisfies

$$\hat{E}_t \left[ u(x(a_t)|\hat{F}_t(\cdot|a_t)) \right] \geq \hat{E}_t \left[ u(x(b)|\hat{F}_t(\cdot|b)) \right] \forall b \in A.$$ \hspace{1cm} (7)
The solution concept above implies that the reference point in each round depends solely on the chosen action that round and not on foregone alternatives. The full menu of options may naturally influence the reference point, however, in environments that introduce randomness over the implemented action. The model can easily extend to capture such instances of stochastic choice. For instance, consider a consumer who buys brand $A$ when it is available and otherwise buys brand $B$. The choice to buy this product thus yields a probability distribution $p_t \in \Delta(\mathcal{A})$ over the brand she will ultimately receive that round. In such cases, we assume the reference point is given by expectations with respect to the distribution $\hat{F}_t(x|p_t) \equiv \sum_{a=1}^N p_t(a) \hat{F}_t(x|a)$—the compound lottery comprised of the distribution she would face for each realized option $a$, weighted by $p_t(a)$.

There are at least two natural interpretations for the misencoding in our model. One is that the person observes her experienced utility $u_t$ each period, but cannot directly observe the underlying outcome $x_t$. Hence, $x_t$ must be inferred from $u_t$, and the misattributor errs in doing so. Another interpretation is that misattribution occurs even when the person can observe $x_t$. Because the decision maker is unaware that she mistakenly infers $x_t$ from $u_t$, she may treat $u_t$ as a sufficient statistic for updating her beliefs. Hence, she may “rationally” (from the point of view of her misspecified model) ignore or forget outcome $x_t$ even though it was observable. Under this interpretation, the decision maker learns based on how an experience feels without attending to specific outcomes.

While few studies empirically explore misattribution of reference dependence, our companion paper (Bushong and Gagnon-Bartsch 2018) provides supporting evidence. In that paper’s main experiment, participants completed one of two unfamiliar tasks: either a neutral task or the same task with an unpleasant noise played in the background. Hours after participants sampled the task and formed initial impressions, we elicited their willingness to continue working (WTW) on the same task for additional pay. Identification came from manipulating participants’ expectations about which task they would face: participants in the control group had no uncertainty over their task, while each participant in the treatment group determined her task by flipping a coin just before working. In terms of our model, treatment participants held reference points that combined both the neutral and bad tasks, meaning the job they ultimately faced came as either a positive or negative surprise. Contrastingly, control participants faced no surprise. Relative to the control group, the treatment group exhibited greater WTW on the neutral task and decreased WTW on the unpleasant task. When the stakes were

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$^{21}$We and others interpret reference-dependent utility as reflective of an underlying hedonic sensation experienced by the decision maker. For instance, Bell (1985) and Loomes and Sugden (1982) both posit that regret and rejoice (relative to a reference point) are hedonic sensations. Furthermore, there is direct evidence of reference-dependent hedonics in neuroscience, psychology, and economics. For instance, Rutledge et al. (2014) shows that a reference-dependent model predicts self-reported happiness during a simple gambling experiment.

$^{22}$This interpretation is similar to Gagnon-Bartsch, Rabin, and Schwartzstein (2018), who argue that misspecified models may persist when an agent uses her erroneous theory to guide how she allocates attention. If the decision maker wrongly believes some variable is redundant given other data she has noticed, then she may rationally (from the viewpoint of her misspecified model) ignore it and become unable to identify her error.
highest, the WTW of treatment participants was roughly 20% higher (neutral task) and 25% lower (unpleasant task) than the WTW of control participants. These results (along with various robustness checks) are consistent with misattribution of positive surprise and disappointment to the intrinsic enjoyment of the task.

Other forms of misattribution have been discussed in the economics literature. Many resemble the “fundamental attribution error” or “correspondence bias” in psychology (e.g., Ross 1977; Gilbert and Malone 1995), where transient situational factors are incorrectly attributed to underlying, stable characteristics of a person or good. For example, Haggag and Pope (2018) show that experimental participants valued an unfamiliar drink more when they first experienced it while thirsty. Additionally, they find that frequent patrons of an amusement park whose most recent visit was during good weather are more likely to return. In two papers, Simonsohn (2007, 2010) explores the effect of a transient shock (weather) on the subsequent behavior of prospective college students and admissions officers. Simonsohn (2007) demonstrates that applicants with particularly strong academic qualities were evaluated more highly by admissions officers when the weather on the evaluation day was poor. Relatedly, Simonsohn (2010) shows that incoming freshmen are more likely to matriculate at an academically rigorous school when the weather on their visit day was cloudy versus sunny. The author interprets both results as a form of attribution bias, though the specific channel is unclear. Relatedly, a series of papers show that CEOs (Bertrand and Mullainathan 2003) and politicians (Wolfers 2007; Cole, Healy, and Werker 2012) are rewarded for luck as if it was wrongly attributed to skill. Our model shares a common structure with these forms of misattribution, whereby transient sensations (elation and disappointment in our case) are misattributed.

3 Illustrative Examples

In this section, we present two simple examples to demonstrate how misattribution can bias learning in dynamic settings. We first consider a decision maker who repeatedly experiments with an option that returns the same outcome each period. While a rational actor learns quickly in this environment, a misattributor will perceive fictitious variation in outcomes. We next consider a stochastic environment in which outcomes take one of two possible values each period, and the decision maker attempts to learn these values. A misattributor will form persistently exaggerated views of both values: she overestimates the value of the better outcome and underestimates the value of the worse one. Furthermore, the probability of each outcome determines the extent of these misperceptions. Namely, the less likely an outcome is, the more she misperceives its value.

To fix ideas, consider a manager assessing the ability of a newly hired employee, such as a personal assistant. We assume the manager receives utility based on the quality of her employee’s work. Each period, the employee generates consumption value $x_t = \theta + \epsilon_t$ for the manager, where $\epsilon_t$ is mean-zero
i.i.d. noise. The manager seeks to learn parameter $\theta$.  

We first consider the case where there is no variation in the employee’s performance: that is, $\text{Var}(\varepsilon_t) \to 0$, so each day he generates the same consumption value $x_t = \theta$ for the manager.\footnote{We assume negligible rather than zero variance to meet our running assumption that outcomes have full support over $\mathbb{R}$.} To demonstrate how misattribution distorts learning, imagine that $x_1 = \theta$ surpasses the manager’s initial expectations. When the manager misattributes the sensation of positive surprise to the employee’s quality of work, she perceives a value $\hat{x}_1 > x_1 = \theta$. This biased assessment lifts her expectations. Thus, in the second period, this same output, $x_2 = x_1 = \theta$, seems less impressive relative to these higher expectations, and the manager perceives a drop in performance $\hat{x}_2 < \hat{x}_1$. Hence, even with constant outcomes, a misattributor errs by perceiving variation. In truth, the manager’s utility changes solely due to her fluctuating expectations, yet she attributes this variation to fluctuating performance. This perceived variability, however, vanishes with ample experience, so a misattributor would correctly learn $\theta$ in the long run.

The fact that a misattributor would eventually learn in the setting above hinges on the (nearly) deterministic nature of outcomes. Any non-negligible variation in outcomes will cause a misattributor to persistently mislearn. To illustrate, suppose that each day the employee is highly productive with probability $p$, and in this case $x_t = \theta(H) + \varepsilon_t$. Otherwise, with probability $1 - p$, he is less productive and $x_t = \theta(L) + \varepsilon_t$, where $\theta(L) < \theta(H)$. Again, we assume that $\text{Var}(\varepsilon_t) \to 0$, so the employee’s daily performance $x_t$ is essentially an i.i.d. binary random variable with support $\{\theta(L), \theta(H)\}$. Suppose that the manager begins with unbiased priors over the parameters $(\theta(L), \theta(H))$ and updates these beliefs following each experience with the employee.

Despite unbiased priors, the fact that outcomes come as a surprise each period will cause the manager’s estimates of $\theta(H)$ and $\theta(L)$ to polarize over time.\footnote{We discuss limit beliefs heuristically in this section and establish formal results for this environment in Section 6.1. There we assume (1) i.i.d. normally-distributed shocks $\varepsilon_t$ with variance $\sigma^2$, and (2) normally-distributed priors over $\theta(a)$ for each $a \in \{H, L\}$ with mean $\theta_0(a)$ and variance $\rho^2$. Letting $i_t \in \{H, L\}$ indicate the realized state on day $t$ and letting $N_t(a) \equiv \sum_{k=1}^{t} 1\{i_k = a\}$, her estimate of $\theta(a)$ after $t$ rounds is thus

$$
\hat{\theta}_t(a) = \frac{\rho^2}{N_t(a)\rho^2 + \sigma^2} \sum_{k : i_k = a} \hat{x}_k(a) + \frac{\sigma^2}{N_t(a)\rho^2 + \sigma^2} \theta_0(a).
$$

Fixing $\rho$ and taking $\sigma \to 0$ implies that the estimate above converges to the sample average of performances conditional on state $a$.} The manager’s initial expectations incorporate a $p$ chance of high productivity and a $1 - p$ chance of low productivity—i.e., she expects consumption utility $p\theta(H) + (1 - p)\theta(L)$. Thus, if the employee’s first day is productive, $x_1 = \theta(H)$, the manager is pleasantly surprised, and she encodes an inflated value $\hat{x}_1 = \theta(H) + (1 - p)\kappa^G[\theta(H) - \theta(L)] > \theta(H)$. This raises her estimate of the employee’s upside, $\theta(H)$. If instead the employee’s first day is unproductive, $x_1 = \theta(L)$, the manager is disappointed, and she encodes a deflated value...
\[ \hat{x}_1 = \theta(L) - p\kappa^L[\theta(H) - \theta(L)] < \theta(L) \]. This lowers her estimate of \( \theta(L) \). Hence, no matter the first outcome, the manager will overestimate the quality difference between the good and bad states.\(^{25}\)

This initial mistake will reinforce itself: the manager will persistently overestimate the employee’s variance. As the perceived quality difference increases, either outcome will generate larger sensations of surprise, which leads to greater distortion in beliefs. The manager converges to steady-state estimates, denoted \( \hat{\theta}(H) \) and \( \hat{\theta}(L) \), such that \( \hat{\theta}(H) > \theta(H) \) and \( \hat{\theta}(L) < \theta(L) \). These estimates satisfy a “fixed-point” property wherein a \( p \) chance of \( \hat{\theta}(H) \) and a \( 1 - p \) chance of \( \hat{\theta}(L) \) lead the misattributor to encode the outcomes \( \theta(H) \) and \( \theta(L) \) as \( \hat{\theta}(H) \) and \( \hat{\theta}(L) \), respectively. These steady-state conditions yield the following long-run beliefs:

\[
\begin{align*}
\hat{\theta}(H) &= \theta(H) + G(p, \eta, \lambda, \hat{\eta})[\theta(H) - \theta(L)] \\
\hat{\theta}(L) &= \theta(L) - L(p, \eta, \lambda, \hat{\eta})[\theta(H) - \theta(L)],
\end{align*}
\]

(8)

where

\[
G(p, \eta, \lambda, \hat{\eta}) \equiv \frac{(1 - p)\kappa^G(1 + \kappa^L)}{1 + p\kappa^G + (1 - p)\kappa^L} \quad \text{and} \quad L(p, \eta, \lambda, \hat{\eta}) \equiv \frac{p\kappa^L(1 + \kappa^G)}{1 + p\kappa^G + (1 - p)\kappa^L}.
\]

(9)

If the manager is loss averse, her perceived expected value of the employee, \( p\hat{\theta}(H) + (1 - p)\hat{\theta}(L) \), is strictly less than the true expected value. These findings foreshadow our results in the next section: a misattributor perceives a distribution of outcomes that is more disperse and negatively-skewed relative to the true distribution, causing her to undervalue risky prospects.\(^{26}\)

Notably, the manager forms more biased estimates about an outcome when it happens less frequently (and is hence more surprising). For instance, she overestimates the quality of a good performance, \( \theta(H) \), by a greater extent when it is less likely (i.e., as \( p \) decreases), as is evident from Equation 9. More generally, misattribution provides a mechanism through which the probability of an outcome naturally shapes its perceived value. As such, the error has implications for consumers learning about goods allocated via random processes (e.g., auctions, bargaining, or scenarios where products have uncertain availability). For instance, misattributors learning about a product that is at times randomly unavailable will come to overvalue that product. Simultaneously, they will

\(^{25}\)This stands in contrast to the predictions of Haggag and Pope (2018), where misattributors tend to underestimate the payoff difference between two outcomes. Furthermore, unlike mistakes driven by misattribution of reference dependence, biased forecasts in Haggag and Pope’s formulation vanish with experience. These distinctions stem from the fact that Haggag and Pope rule out complementaries where past experiences influence current consumption utility. Reference dependence clearly introduces this complementarity, as past experiences form the reference point against which current consumption is evaluated.

\(^{26}\)To show explicitly that the manager undervalues the prospect, note that her perceived expected utility is \( \hat{E}[u] = p\hat{\theta}(H) + (1 - p)\hat{\theta}(L) + p(1 - p)\eta(\lambda - 1)[\hat{\theta}(H) - \hat{\theta}(L)] \). Using the solutions for \( \hat{\theta}(H) \) and \( \hat{\theta}(L) \) above (Equation 8), \( \hat{E}[u] = E[u] - p(1 - p)(\kappa^L - \kappa^G)[\theta(H) - \theta(L)]/(1 + p\kappa^G + (1 - p)\kappa^L) \), where \( E[u] \) is the expected utility under rational expectations. The bias vanishes only as uncertainty vanishes (\( p \to 0 \) or \( p \to 1 \)) or \( \lambda \to 1 \).
undervalue their fall-back option. Consequently, a firm may choose to limit supply when first introducing a high-quality product with unit demand: those lucky enough to receive the good early may overstate its quality, thereby increasing demand among the second wave of consumers.

4 Long-Run Beliefs and Behavior

The previous examples highlighted how stochastic outcomes can distort a misattributor’s beliefs. This section analyzes more generally the beliefs and behavior of a misattributor following ample experience with actions that yield stochastic outcomes. For instance, imagine a person learning about the typical quality of an oft-used service, the returns from an investment, or the productivity of an employee. In the first subsection, we characterize a misattributor’s steady-state beliefs about the mean and variance of such outcomes when repeatedly taking a fixed action—thus abstracting from experimentation motives. We highlight how these beliefs depend on the true distribution and the person’s preferences, and show that a misattributor will underestimate the prospect’s mean outcome and overestimate its variability. These misperceptions lead a decision maker to undervalue prospects in proportion to their true variance, and, in the second subsection, we demonstrate how the agent may thus turn down some risky-but-optimal actions. Additionally, since only those outcomes with utility consequences distort a misattributor’s beliefs, she may perpetually cycle between actions in settings where she observes the counterfactual outcomes of foregone choices.

4.1 Long-Run Pessimism and Exaggerated Variance

Suppose each action $a \in \mathcal{A}$ returns consumption utility $x_t(a) = \theta(a) + \sigma(a)z_t$, where each $z_t$ is an i.i.d. realization of a mean-zero, unit-variance random variable $Z$ with a continuously differentiable distribution $F_Z$ and density $f_Z$. Parameters $\theta(a)$ and $\sigma(a)$ denote the true mean and standard deviation of outcomes, respectively. The agent begins with correct priors $\pi_1^a$ about $(\theta(a), \sigma(a))$ on $\mathbb{R} \times \mathbb{R}_+$ and updates these beliefs based on experienced outcomes. We first examine biased learning when there is a single action available; as such, we drop reference to $a$ from parameters $\theta(a)$ and $\sigma(a)$. Given our assumptions that the reference point corresponds to expectations about the chosen action and that outcomes are independent across actions, learning about one action does not affect a person’s long-run beliefs about another. Thus, deriving the long-run beliefs about a single action will allow us to further analyze choices over actions, assuming the decision maker has received ample feedback about each action.

How does a misattributor’s estimate of $\theta$ evolve following outcome $x_t$? Given an estimated mean $\hat{\theta}_{t-1}$ entering round $t$, a misattributor encodes outcome $\hat{x}_t = x_t + \kappa_t(x_t - \hat{\theta}_{t-1})$, where $\kappa_t \equiv \kappa^G \mathbb{I}\{x_t \geq \hat{\theta}_{t-1}\} + \kappa^L \mathbb{I}\{x_t < \hat{\theta}_{t-1}\}$. To describe the dynamics of $\hat{\theta}_t$, we define the function $G : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ as
the expectation of \( \hat{x}_t - \hat{\theta}_{t-1} \) conditional on the true parameter values:

\[
G(t, \hat{\theta}_{t-1}) \equiv \mathbb{E}[\hat{x}_t | \theta] - \hat{\theta}_{t-1}.
\] (10)

\( G(t, \hat{\theta}_{t-1}) \) corresponds to the misattributor’s expected “surprise” resulting from outcome \( x_t \) as measured by an outside observer who knows both the true parameter values and the misattributor’s subjective beliefs. **Steady-state beliefs** are thus a zero of \( G_\infty(\hat{\theta}) \equiv \lim_{t \to \infty} G(t, \hat{\theta}) \), corresponding to a belief at which the agent’s average surprise is zero; that is, at \( \hat{\theta} \), the encoded outcome is equal to \( \hat{\theta} \) on average. Intuitively, the misattributor cannot hold a long-run belief \( \hat{\theta} \) outside of \( \Gamma \equiv \{ \hat{\theta} : G_\infty(\hat{\theta}) = 0 \} \), as such a belief would generate encoded outcomes that push her systematically away from \( \hat{\theta} \): \( G_\infty(\hat{\theta}) > 0 \) implies her estimate of \( \theta \) would drift upward, and \( G_\infty(\hat{\theta}) < 0 \) implies it would drift downward. We call \( \hat{\theta} \in \Gamma \) *stable* when encoded outcomes at beliefs near \( \hat{\theta} \) push a misattributor’s expectations toward \( \hat{\theta} \). Formally, \( \hat{\theta} \in \Gamma \) is stable if there exists \( \varepsilon > 0 \) such that \( G_\infty(\theta') < 0 \) for all \( \theta' \in (\hat{\theta}, \hat{\theta} + \varepsilon) \) and \( G_\infty(\theta') > 0 \) for all \( \theta' \in (\hat{\theta} - \varepsilon, \hat{\theta}) \).

Our first result establishes that there exists a unique steady-state belief and it is stable. It also presents key comparative statics on the mean and variance of perceived outcomes in the steady state.

**Proposition 1.** Suppose the agent faces a single action with outcomes distributed according to parameters \( \theta \) and \( \sigma > 0 \). For all \( \lambda \geq 1 \):

1. \( \Gamma \) is a singleton, and the unique steady-state mean belief \( \hat{\theta} \) is stable.

2. (Underestimated mean.) \( \hat{\theta} \leq \theta \), and the inequality is strict if and only if \( \lambda > 1 \). If \( \lambda > 1 \), then \( \hat{\theta} \) is strictly decreasing in \( \sigma \), \( \lambda \), and \( \eta \), and is strictly increasing in \( \hat{\eta} \).

3. (Overestimated variation.) Let \( \hat{\sigma} \) be the standard deviation of encoded outcomes at the steady-state mean belief, \( \hat{\theta} \). \( \hat{\sigma} > \sigma \), and \( \hat{\sigma} \) is strictly increasing in \( \sigma \), \( \lambda \) and \( \eta \), and is strictly decreasing in \( \hat{\eta} \).

Appendix A shows that beliefs indeed converge to these steady-state values for a class of distributions \( F_Z \) under which mean beliefs update in the direction of each encoded outcome (e.g., \( x_t \) are normally distributed). Although outcomes are truly i.i.d., convergence does not follow directly from a basic law of large numbers because encoded outcomes are serially correlated: prior outcomes shift a misattributor’s reference point and thus influence the current encoded outcome. As in Heidhues, Kőszegi, and Strack (2018) and Esponda and Pouzo (2016), we use techniques from stochastic-approximation theory to establish convergence. For brevity, we relegate these details to Appendix A, and the results in this section apply to those situations where beliefs do converge.27

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27This includes the familiar setting where outcomes are normally distributed and the agent is learning about both \( \theta \) and \( \sigma \). Assuming the canonical environment where priors over \( \theta \) and \( \sigma \) follow Normal and Inverse-Gamma distributions, respectively, then \( \hat{\theta}_t \) converges almost surely to the steady-state beliefs characterized in Proposition 1.
Part 2 of Proposition 1 shows that a loss averse misattributor underestimates the mean outcome. Intuitively, loss aversion causes her to encode a distribution of outcomes that is negatively skewed relative to the true distribution—she underestimates bad experiences more than she overestimates good ones. While this force drives down perceptions of $\theta$, it is not immediate that such pessimistic expectations will persist given that they generate additional pleasant surprises. The steady-state belief $\hat{\theta}$ balances these two forces: a misattributor underestimates $\theta$ to an extent that the resulting excess of positive surprises exactly offsets the downward bias stemming from loss aversion. Furthermore, stability of $\hat{\theta}$ (Part 1 of Proposition 1) follows from the belief-based nature of utility. Specifically, if she had low expectations $\tilde{\theta} < \hat{\theta}$, then outcomes would typically be encoded as elations and drive her expectations upward. Conversely, if she had expectations above $\hat{\theta}$, then an increased frequency of disappointments would push her expectations downward.

Additionally, Part 2 of Proposition 1 demonstrates that greater variability in the underlying consumption utility of outcomes, $\sigma$, causes a misattributor to underestimate $\theta$ by a larger amount. That is, a misattributor develops more pessimistic beliefs about actions that are riskier. Increased variance implies that the decision maker experiences greater sensations of elation and disappointment. And since loss aversion implies that such gain-loss utility is negative on average, encoded outcomes tend to decrease in $\sigma$. For example, if a misattributing consumer faces two logistics companies (e.g., UPS and FedEx) with identical mean delivery times, she will come to believe the more variable company typically takes longer. As we show below, such mistakes can lead to poor decisions when balancing risk and return, as the decision maker is systematically biased against risky actions.

Not only does the true variance negatively influence beliefs about $\theta$, but a misattributor also overestimates the variance in outcomes (Part 3 of Proposition 1). Given expectation $\hat{\theta}$, the long-run distribution of perceived outcomes, denoted $\hat{F}$, is readily derived from Equation 5:

$$\hat{F}(x) = \begin{cases} F_Z \left( x - \frac{[\theta + \kappa G(\theta - \hat{\theta})]}{(1 + \kappa G)\sigma} \right) & \text{if } x \geq \hat{\theta} \\ F_Z \left( x - \frac{[\theta + \kappa L(\theta - \hat{\theta})]}{(1 + \kappa L)\sigma} \right) & \text{if } x < \hat{\theta}, \end{cases} \quad (11)$$

while the true distribution is $F_Z \left( \frac{x - \theta}{\sigma} \right)$. Relative to the true distribution, the perceived distribution has fatter tails about $\hat{\theta}$—the misattributor overestimates the probability of unlikely events—and is more negatively skewed. Figure 1 displays the results described above, depicting both the long-run path of beliefs and the density of perceived outcomes for two different variances: $\sigma^2 = 1$ (top panels) and $\sigma^2 = 5$ (bottom panels).

To analyze decisions and welfare in the steady state, we now consider the misattributor’s (biased) expected utility of an action given the statistical misperceptions described in Proposition 1. With misattribution, choice and welfare analysis requires three distinct notions of (per-period) expected
utility. First, forecasted utility is what a misattributor expects to earn given her biased estimates $\hat{\theta}$ and $\hat{\sigma}$ and is the object that directs choice. Second, average experienced utility is what she actually receives, on average, given $\hat{\theta}$. Because the misattributor holds biased beliefs, her experienced utility will diverge from her forecasted utility. Finally, because expectations directly influence utility in this model, the average experienced utility differs depending on whether the agent holds biased versus rational expectations. Thus, to facilitate comparisons with the rational benchmark, we refer to the average utility received absent misattribution as hypothetical utility.\footnote{Formally, forecasted utility is the expectation of $u(x|\hat{\theta})$ with respect to the misattributor’s perceived distribution of $x$ given $\hat{\theta}$ and $\hat{\sigma}$. Experienced utility is the expectation of $u(x|\hat{\theta})$ with respect to the true distribution of $x$. Hypothetical utility is the expectation of $u(x|\theta)$ with respect to the true distribution.}

Our next result fixes the long-run action and compares forecasted, average experienced, and hypothetical utility. Let $\hat{v}$, $\bar{v}$, and $v$ denote the person’s per-period forecasted, average experienced, and hypothetical utility, respectively, in the steady state characterized by Proposition 1.

**Proposition 2.** Suppose the agent faces a single action with outcomes distributed according to parameters $\theta$ and $\sigma > 0$. If $\lambda > 1$, then:
1. Forecasted utility $\hat{v}$ is strictly less than hypothetical utility $v$. The difference $|\hat{v} - v|$ is strictly increasing in $\sigma$.

2. Average experienced utility $\bar{v}$ is strictly greater than hypothetical utility $v$. The difference $|\bar{v} - v|$ is strictly increasing in $\sigma$.

Part 1 of Proposition 2 shows that a misattributor comes to underestimate the payoff of an action whenever it yields variable outcomes. This follows immediately from the fact that a misattributor underestimates the mean and overestimates the variance of outcomes: due to loss aversion, both of these erroneous beliefs diminish her perceived benefit of action $a$. Part 2 shows that, when taking a fixed action, a misattributor experiences an average payoff that exceeds her hypothetical utility: since she forms overly pessimistic expectations $\hat{\theta} < \theta$, she realizes pleasant surprises more often than she would absent misattribution. It would be wrong, however, to conclude that the misattributor necessarily benefits from her mislearning. The proposition above holds the long-run action fixed, but a misattributor’s biased beliefs can cause her to settle on an action different from (and inferior to) the one she would choose if rational. In fact, Proposition 2’s comparative static on $\sigma$ suggests that actions with the greatest additional benefit to a misattributor relative to the rational benchmark (those with high $\sigma$) are precisely those that the misattributor undervalues the most.

4.2 Bias Against Risky Choices

We now show that a misattributor may continually choose a relatively safe action even when superior (but riskier) actions are available. Importantly, this bias against risk is on top of the agent’s intrinsic risk preferences—amplifying any existing distaste for risk—and stems from the pessimistic long-run beliefs described above. To illustrate, we consider a person who experiments with option $A$ for $t^*$ periods, and then commits (once and for all) to either $A$ or a known alternative $B$ for all remaining periods $t > t^*$. To abstract from bandit problems where insufficient experimentation can lead to incomplete learning even among rational agents, suppose the person experiments with option $A$ for an arbitrarily long horizon before deciding whether to switch (i.e., $t^* \to \infty$). To simplify matters further, we also assume $B$ yields a fixed, known consumption utility $v_B$.

Corollary 1 (below) shows that there always exists some risky action $A$ that a rational learner would choose, yet a misattributor would wrongly abandon in favor of $B$. For example, imagine a farmer updating about the payoffs she receives from her yield, which varies due to factors such as weather. Eventually, she must decide whether to continue farming (option $A$) or sell the land for a fixed price (option $B$). Unless the returns to farming are sufficiently large, the misattributor will choose to sell the land even when it is optimal to continue farming. Since the misattributor’s forecasted value of $A$ falls below its true value (Proposition 2), she will erroneously sell the land whenever its forecasted value, $\hat{v}_A$, falls below $v_B$ and the hypothetical utility, $v_A$, exceeds $v_B$. Furthermore, the range between
these values is larger when returns have greater variance: fixing the true expected utility of an action, the more variable is its output, the more likely the decision maker will wrongly abandon it.

**Corollary 1.** Consider the binary-choice setting described above. Suppose option A has outcomes distributed according to parameters \( \theta \) and \( \sigma > 0 \), and suppose \( \lambda > 1 \). Let \( \theta^* \) be the value of \( \theta \) that leaves the decision maker indifferent between A and B under full information. There exists a threshold \( \bar{\theta} > \theta^* \) such that if \( \theta \in (\theta^*, \bar{\theta}) \), then the misattributor wrongly switches to action B almost surely as \( t^* \to \infty \). The difference in the misattributor’s threshold and the rational threshold, \( \bar{\theta} - \theta^* \), is strictly increasing in \( \sigma \).

Our next proposition demonstrates that the type of error identified in Corollary 1—a bias toward safe actions—can result in arbitrarily costly mistakes each period.

**Proposition 3.** Consider the binary-choice setting described above. Suppose option A has outcomes distributed according to parameters \( \theta \) and \( \sigma > 0 \), and suppose \( \lambda > 1 \). For any value \( v > v_B \), there exist parameter values \( (\theta, \sigma) \) such that:

1. Under rational expectations, A yields (per-period) hypothetical utility \( v \).

2. The misattributor chooses B almost surely as \( t^* \to \infty \).

No matter how much the expected utility of A exceeds that of B (given rational beliefs), a misattributor ultimately settles on the safe option B whenever A is sufficiently volatile. Intuitively, the excessive weighting of occasional losses swamps the benefit of a high mean outcome as the magnitude of these losses grows large. Such biased learning may help explain why individuals tend to overly avoid risk based on their personal experiences, as shown by Malmendier and Nagel (2011).29

The result above—that a misattributor will permanently abandon optimal-but-risky actions—stems in part from the assumption that once an action is abandoned, the misattributor receives no additional information about that action. When the person can observe the counterfactual outcomes of actions she did not take, the misattributor may fall into a perpetual cycle of alternating between the optimal action and an inferior one. Since she misencodes outcomes only when they have utility consequences, the misattributor will update correctly about the (truly) optimal action once it is abandoned. This

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29 The errors characterized here would also arise in experimentation settings, though the analysis of such bandit problems is much more involved. To provide some intuition on how our results extend, suppose the agent is uncertain about both A and B and can switch between them over time. Fixing \( (\theta(B), \sigma(B)) \), there exist parameter values \( (\theta(A), \sigma(A)) \) such that option A would yield arbitrarily greater experienced utility than B on average, yet a fully patient misattributor would fail to settle on A. The proof of Proposition 3 shows that while holding the hypothetical utility of A fixed, A’s forecasted utility decreases unboundedly in \( \sigma(A) \). Hence, no matter how large a benefit A may provide, high variability can cause a misattributor to persistently take an inferior action. Additionally, in a bandit setting with discounting—where a rational decision maker may settle on the inferior option with positive probability—misattribution would increase this likelihood.
tends to correct her overly-pessimistic beliefs, and she may periodically return to the optimal action while never settling on it.

To illustrate this point, consider a variant of the workplace example. Each period, a manager chooses between two potential collaborators, \( A \) and \( B \), and learns about their productivity over time. Suppose that she observes the output of both colleagues no matter which she is currently working with. Given she observes counterfactuals—and hence has no experimentation motives—the manager’s optimal strategy is to simply choose whoever provides the highest expected benefit given her current beliefs. Suppose in truth that \( A \) is the superior choice by a small margin and the manager begins by selecting \( A \). Since working with \( A \) carries utility consequences, the variability in \( A \)’s output will cause the manager to grow pessimistic about \( A \). Contrastingly, information about \( B \) has no utility consequences, so she correctly updates about \( B \). When pessimism about \( A \) is sufficiently strong, the misattributor will switch to \( B \). At this point, \( B \)’s productivity will appear to deteriorate, while \( A \)’s will appear to improve. This will go on until the misattributor switches back to \( A \), at which point the logic above repeats. As such, the decision maker persistently vacillates between the two options. This intuition mirrors the psychology that “the grass is always greener on the other side of the fence”: regardless of what the person chooses today, a different option will grow more appealing. For example, a common (mis)perception is that a person’s chosen queue moves more slowly than an alternative, no matter which queue is chosen.

5 Order Effects and Extrapolative Beliefs

While the previous section abstracted from short-run dynamics in beliefs, we now explore how the order in which a misattributor experiences outcomes determines her perceived benefit of an action. We show that, despite i.i.d. outcomes, misattribution can generate a “recency bias”: expectations weight recent outcomes more heavily than older ones. This result further implies that—so long as the extent of misattribution is not too extreme—a misattributor forms the highest estimate of an action’s benefit following an improving sequence of outcomes. Additionally, we show that when outcomes are autocorrelated, a misattributor forms extrapolative forecasts of future payoffs even when there is nothing to learn about the data-generating process. Importantly, unlike the long-run results above, these short-run biases arise even without loss aversion (i.e., \( \lambda = 1 \)).

Throughout this section, we consider the sequence of beliefs a person forms as she takes action \( a \in \mathcal{A} \) for \( T \) periods. Each period, the action yields consumption utility \( x_t \in \mathbb{R} \). For tractability, we assume \( x_t = \theta + \varepsilon_t \), where \( \varepsilon_t \) are i.i.d. mean-zero Gaussian shocks with known variance \( \sigma^2 \).

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\( 30 \)The cycling behavior described here emerges when the two options are sufficiently similar in value, so that the biased agent undervalues \( A \) enough to switch actions. If instead \( A \) dominates \( B \) by a significant margin, then the misattributor’s pessimistic steady-state impression of \( A \) will not incite a switch. Fixing \( \sigma(\lambda) \) and the hypothetical utility of option \( B \), the range of \( \theta(\lambda) \) that leads to cycling behavior matches the interval derived in Corollary 1.
misattributor begins with a prior \( \theta \sim \mathcal{N}(\theta_0, \rho^2) \) and sequentially updates about \( \theta \) following each encoded outcome \( \hat{x}_t \). This Gaussian model allows us to simply demonstrate comparative statics on the variance of signals and priors without, we suspect, sacrificing much generality.\(^{31}\) Given this setup, the misattributor’s posterior over \( \theta \) after \( t \) observations is normally distributed with mean \( \hat{\theta}_t \), where

\[
\hat{\theta}_t = \left( \frac{\sigma^2}{\rho^2 + \sigma^2} \right) \theta_0 + \left( \frac{\rho^2}{\rho^2 + \sigma^2} \right) \sum_{\tau=1}^{t} \hat{x}_\tau.
\]

(12)

Section 5.1 shows how misattribution distorts the evolution of \( \hat{\theta}_t \), and Section 5.2 extends this analysis to cases where outcomes are autocorrelated.

### 5.1 Order Effects in Belief Updating

Although outcomes are i.i.d. and should therefore be treated exchangeably, the order of outcomes influences a misattributor’s beliefs. Early outcomes determine whether later outcomes are assessed as elations and hence overestimated, or as disappointments and thus underestimated. To build intuition, consider a misattributing manager who works with a new assistant for a week. Suppose the employee is productive every day except one bad day. If that bad day comes first, it will lower the manager’s expectations and the remaining days may seem surprisingly productive. Alternatively, if the bad day comes last (after the manager has developed high expectations) it may seem surprisingly unproductive. Even though the two sequences are permutations of the same outcomes, the fact that one generates subsequent gains whereas the other ends with a loss can cause the misattributor to reach different final beliefs.

We first delineate how a misattributor’s beliefs depart from rational beliefs. Following encoded outcome \( \hat{x}_t \), the misattributor updates her prior estimate \( \hat{\theta}_{t-1} \) to reach posterior \( \hat{\theta}_t = \alpha_t \hat{x}_t + (1 - \alpha_t) \hat{\theta}_{t-1} \), where \( \alpha_t \equiv \rho^2 / (\rho^2 + \sigma^2) \) is the proper weight a Bayesian would attach to a new observation (see Equation 12). Because a misattributor encodes \( \hat{x}_t = x_t + \kappa_t (x_t - \hat{\theta}_{t-1}) \) (Equation 5), she reaches a biased estimate

\[
\hat{\theta}_t = \alpha_t (1 + \kappa_t) x_t + [1 - \alpha_t (1 + \kappa_t)] \hat{\theta}_{t-1}.
\]

(13)

Since rational beliefs put weight \( \alpha_t \) on \( x_t \), Equation 13 immediately reveals that, relative to her rational counterpart, a misattributor “overreacts” to the latest outcome \( x_t \).\(^{32}\) Moreover, \( \hat{\theta}_t \) assigns the

\[^{31}\]Our qualitative results extend beyond the case of normally distributed outcomes and priors. For instance, many of the results in Sections 5 and 6 require only that these distributions are symmetric and quasiconcave (i.e., unimodal). These assumptions guarantee that a rational agent’s updated estimate of \( \theta \) falls between her previous estimate and the most recent observation. See Chambers and Healy (2012) for a complete characterization of when beliefs “update toward the signal”.

\[^{32}\]More formally, consider both a misattributing and rational learner who share a common prior expectation \( \theta_0 \). Let \( \hat{\theta} \) and \( \hat{\theta}^* \) be the biased and rational estimates of \( \theta \), respectively. Then, following outcome \( x \in \mathbb{R} \), \( |\hat{\theta} - \theta_0| \geq |\hat{\theta}^* - \theta_0| \). Furthermore, the misattributor’s reaction \( |\hat{\theta} - \theta_0| \) is decreasing in \( \eta \): she overreacts more as the extent of misattribution
wrong weight to each previously experienced outcome. Iterating Equation 13, we can express $\hat{\theta}_t$ as a weighted sum of the true outcomes:

**Lemma 1.** Following any sequence $(x_1, \ldots, x_t) \in \mathbb{R}^t$, a misattributor forms an estimate

$$\hat{\theta}_t = \xi_t^0 \theta_0 + \alpha_t \sum_{\tau=1}^{t} \xi_t^\tau x_\tau,$$

where

$$\xi_t^\tau = \begin{cases} 
\prod_{j=1}^{\tau} [1 - \alpha_j (1 + \kappa_j)] & \text{if } \tau = 0, \\
(1 + \kappa_\tau) \prod_{j=\tau}^{t-1} [1 - \alpha_j (1 + \kappa_{j+1})] & \text{if } \tau \in \{1, \ldots, t-1\}, \\
1 + \kappa_t & \text{if } \tau = t.
\end{cases}$$

Rational updating, however, assigns an equal weight $\xi_t^\tau = 1$ to each outcome.

To characterize the implications of a misattributor’s incorrectly weighted estimates, we introduce two definitions:

**Definition 1.** Beliefs are convex in period $t$ if, given any prior estimate $\hat{\theta}_{t-1} \in \mathbb{R}$ and any $x_t \in \mathbb{R}$, there exists $\tilde{\alpha} \in [0, 1]$ such that $\hat{\theta}_t = \tilde{\alpha} x_t + (1 - \tilde{\alpha}) \hat{\theta}_{t-1}$. We call beliefs convex if they are convex in period $t$ for all $t \geq 1$.

Convexity means that a misattributor’s posterior does not overreact by “too much”: $\hat{\theta}_t$ falls between the true outcome $x_t$ and her prior $\hat{\theta}_{t-1}$. Moreover, convexity is equivalent to the posterior $\hat{\theta}_t$ being an increasing function of the prior, $\hat{\theta}_{t-1}$.\footnote{From Equation 13, if beliefs are not convex, then $\hat{\theta}_t$ overweights $x_t$ and negatively weights the prior. A higher prior may, for instance, mean that $x_t$ comes as a larger loss. If beliefs overreact enough, then this exaggerated loss more than offsets the obvious positive effect of a higher prior on the person’s posterior. Note that rational beliefs in this setting are convex given that $\theta$ and $\varepsilon_t$ are independently drawn from normal distributions. This holds more generally assuming the distributions of $\theta$ and $\varepsilon_t$ are symmetric and quasiconcave. For a sense of when a misattributor’s beliefs are convex in terms of the underlying parameters, if $\eta = 1$, $\lambda = 3$ and $\sigma^2/\rho^2 = 1$, then beliefs are convex for $\hat{\eta} > 1/3$.}

To assess, for instance, whether a manipulative party would want to lower a misattributor’s prior in order to maximize her posterior beliefs, the question boils down to assessing convexity. With our Gaussian assumptions, convexity holds if and only if $\kappa_L < \sigma^2/\rho^2$—that is, outcomes are relatively informative about $\theta$. In settings where convexity fails because outcomes are particularly revealing, lowering a misattributor’s prior will increase her posterior. Otherwise, lowering her prior will decrease her posterior just like with rational beliefs. Many of our results below focus on settings where beliefs are convex and therefore highlight that misattribution can have important implications even when biased beliefs share some of the same basic properties as rational beliefs.

We also consider a weaker concept that holds whenever beliefs are convex:
Definition 2. Let $x'_{-k}$ denote $(x_1, \ldots, x_t)$ excluding the $k^{th}$ element. Beliefs are monotonic if for all $t \geq 1$ and all $x'_{-k} \in \mathbb{R}^{t-1}$, $\hat{\theta}_t$ is increasing in $x_k$ conditional on $x'_{-k}$ for each $k \leq t$.

Monotonicity is equivalent to a posterior that increases in the true data and holds if and only if $\kappa \lambda < 1 + \sigma^2 / \rho^2$.

If beliefs are monotonic, a misattributor exhibits a recency bias: her beliefs weight a recent gain more than any preceding gain and weight a recent loss more than any preceding loss.

Proposition 4. Consider any sequence $(x_1, \ldots, x_T) \in \mathbb{R}^T$ and suppose beliefs are monotonic. For any two outcomes $x_t, x_\tau$ that are both gains or both losses (i.e., $\kappa_t = \kappa_\tau$), a misattributor’s expectations place greater weight on the more recent outcome: $\xi_T^t > \xi_T^\tau$ if and only if $t > \tau$. Furthermore, expectations overweight recent outcomes and underweight early outcomes: $\xi_T^T > 1$ and for all $t < T$, $\xi_T^t \to 0$ as $T - t \to \infty$.

Since losses influence beliefs more than gains (Observation 2), it is possible that a loss in period $t - 1$ has a larger influence on beliefs than a more recent gain in period $t$. However, for any two outcomes that fall in the same domain (i.e., both gains or both losses), the more recent outcome has a greater influence on beliefs than the earlier one. Additionally, relative to the weight on the recent outcome, the weight on the early outcome vanishes as the time between the two grows large.

The proposition above reveals how sequential contrast effects (as illustrated in Observation 3) distort beliefs over several periods of learning. Namely, in terms of their influence on final beliefs, early outcomes are “self limiting”: high early outcomes raise the person’s expectations, and subsequent outcomes are thus underestimated by more (or overestimated by less), offsetting the positive effect of the high initial outcomes. Similarly, low early outcomes depress expectations, and subsequent outcomes are overestimated by more (or underestimated by less), offsetting the initial reduction in beliefs. As the horizon $T$ increases, an early outcome $x_t$ exerts this countervailing force on a larger number of subsequent outcomes, which pushes its weight $\xi_T^t$ to zero.

A recency bias suggests that a misattributor will form a higher estimate of $\theta$ when her best experiences happen at the end of the horizon. To characterize when this is the case, suppose the misattributor experiences some arbitrary set of outcomes $\mathcal{X} = \{x_1, \ldots, x_T\}$. Among all permutations of $\mathcal{X}$, when does the sequence with an increasing profile maximize the misattributor’s perception of $\theta$?

To develop intuitions, we first consider sequences with just two outcomes, where we can provide the weakest conditions sufficient for such order effects. The misattributor reaches a higher estimate of $\theta$ when she receives the worse outcome first, so long as $\lambda$ is not too large. Roughly put, an increasing sequence minimizes disappointments while maximizing elations.

Proposition 5. Suppose $a, b \in \mathbb{R}$ with $a > b$. Let $\hat{\theta}_2^d$ denote the mean belief following the decreasing sequence $(a, b)$, and let $\hat{\theta}_2^i$ denote that following the increasing sequence $(b, a)$. 

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1. If beliefs are convex, then \( \hat{\theta}_i^2 > \hat{\theta}_d^2 \) for all \( a, b \in \mathbb{R} \).

2. If beliefs are not convex, then there exists a threshold \( \bar{\lambda} > 2 \) such that \( \hat{\theta}_i^2 > \hat{\theta}_d^2 \) for some \( a, b \in \mathbb{R} \) only if \( \lambda > \bar{\lambda} \).

Hence, either convex beliefs or \( \lambda < 2 \) guarantees that \( \hat{\theta}_i^2 > \hat{\theta}_d^2 \). Furthermore, even when beliefs are not convex and \( \lambda > \bar{\lambda} \), the range of values \( (a, b) \) for which the decreasing sequence maximizes beliefs is limited. This happens only if both outcomes come as a loss relative to the person’s prior—\( \theta_0 > a > b \)—and \( b \) is sufficiently close to \( a \) (see Equation B.18 for a precise condition).\(^{34}\) In light of this caveat, the following corollary provides a sufficient condition for a recency effect that may be useful for empirical tests.

**Corollary 2.** Suppose \( T = 2 \). If at least one of the two realized outcomes beats initial expectations, then a misattributor forms the highest estimate of \( \theta \) when the two outcomes are experienced in increasing order.

The logic of Proposition 5 extends to sequences of any length so long as beliefs are convex.\(^{35}\)

**Proposition 6.** Consider any set of \( T \) distinct outcomes, \( \mathcal{X} \). If beliefs are convex, then among all possible orderings of the outcomes in \( \mathcal{X} \), the misattributor’s estimate \( \hat{\theta}_T \) is highest following the sequence in which the elements are ordered from least to greatest.

The results above accord with a large literature demonstrating that people both prefer improving sequences—fixing the outcomes they experience—and retrospectively form the most optimistic evaluations thereafter. For example, Ross and Simonson (1991) allow participants to sample two video games and find that willingness to pay for the bundle is significantly higher for those who sampled the better game second. Similarly, Haisley and Loewenstein (2011) demonstrate that advertising promotions are most effective when sequenced in increasing order of value—that is, the high-value promotional item is given last. Several authors suggest that such assessments stem from adaptation and subsequent contrast, which has a similar intuition to our formal model (see, e.g., Tversky and Griffin 1990; Loewenstein and Prelec 1993; Baumgartner, Sujan, and Padgett 1997).

That said, our order effects seemingly stand at odds with confirmation bias, wherein new evidence is wrongly interpreted as conforming to one’s expectations (e.g., Rabin and Schrag 1999; To see the rationale, suppose beliefs are not convex and hence excessively react to new observations. If \( b \) is sufficiently close to \( a \), then \( b \) is perceived as a gain when experienced after \( a \): beliefs become so pessimistic after the initial loss \( a \) that the truly worse outcome \( b \) feels like a gain. Furthermore, if losses distort beliefs sufficiently more than gains (i.e., \( \lambda > \bar{\lambda} \)), then \( \hat{\theta}_i^2 \) is maximized (roughly) by minimizing experienced losses. Thus, because the first outcome necessarily comes as a loss while the second comes as a gain, losses are minimized when the better outcome happens first.\(^{35}\) Even when beliefs are not initially convex, they will eventually become convex if the horizon is sufficiently long: over time, each new observation has less impact on beliefs as the total number of observations accumulates. There exists a period \( t^* = \lfloor 1 + \kappa L - \sigma^2 / \rho^2 \rfloor \) beyond which beliefs are convex in each period \( t > t^* \). Therefore, \( \hat{\theta}_T \) is maximized by having an increasing profile of payoffs beyond period \( t^* \).
Fryer, Harms, and Jackson 2017): outcomes deviating from expectations are encoded as closer to expectations, so early outcomes are overweighted relative to later outcomes. As highlighted above, misattribution makes the opposite prediction. Despite this tension, the mechanisms are not mutually exclusive. Indeed, empirical tests of order effects in belief updating find support for both confirmatory and recency effects (see Hogarth and Einhorn 1992 for a meta analysis). Which effect prevails seems to depend on the nature of the learning problem: confirmatory effects tend to dominate as evidence becomes more ambiguous and difficult to interpret.

While the results above show how the order of outcomes can bias a misattributor’s beliefs, distortions will arise even when she faces the same outcome repeatedly. In this case, she will perceive a fictitious trend in outcomes. Whether outcomes appear to improve or deteriorate depends on how the true outcome compares to initial expectations: an outcome that is better than average appears to deteriorate, while one that is worse than average appears to improve. A misattributing consumer who tries a surprisingly good product a second time will not experience the same elation she did at first, so she will wrongly perceive a drop in quality.\footnote{The basic psychology described here may play a role in perceptions of a “sophomore slump”—the idea that impressive debuts are often followed by seemingly less-stellar performances. While regression to the mean surely contributes to this phenomenon, our mechanism amplifies the perceived drop in quality between the debut and follow-up performances, making the “slump” more pronounced.}

\footnote{We restrict attention to the case of positive autocorrelation solely for the sake of exposition. Analogous results hold for the case of $\phi \in [-1, 0)$.}

The next proposition formalizes this idea, first outlined in Section 3.

**Proposition 7.** Suppose $x_t = x$ for each $t = 1, \ldots, T$ and that beliefs are convex. If $x > \theta_0$, then $\hat{x}_t$ is strictly decreasing in $t$. If $x < \theta_0$, then $\hat{x}_t$ is strictly increasing in $t$. In both cases, $\lim_{T \to \infty} x_T = x$.

This implies, for instance, that a misattributing consumer may abandon high-quality products due to a false perception that they are on the decline.

### 5.2 Over-Extrapolation and Belief Reversals in Autocorrelated Environments

The order effects discussed above extend in interesting ways when outcomes are autocorrelated. In such settings, a misattributor persistently forms overly-extrapolative forecasts of future payoffs even when she has nothing to learn about the data generating process. Furthermore, the misattributor makes a predictable forecasting error: if she overestimates today’s outcome, she tends to underestimate tomorrow’s outcome (and vice versa).

Building on the setup above, we assume $x_t = \theta + \phi x_{t-1} + \epsilon_t$, where $\epsilon_t$ are again i.i.d. Gaussian shocks with variance $\sigma^2$, and the parameter $\phi \in (0, 1]$ measures the extent of autocorrelation.\footnote{We also assume the person knows both $\phi$ and $\theta$, and normalize $\theta$ to zero. This highlights the persistent error introduced by autocorrelation independent of the order effects generated by uncertainty over}
θ (Section 5.1). Given this setting, the person’s forecast following \( x_{t-1} \) about outcome \( x_t \), denoted \( \hat{E}_t[x_t] \), is simply \( \hat{E}_t[x_t] = \varphi \hat{x}_{t-1} \).

To build intuitions, imagine a misattributor who suffers a chronic medical condition (e.g., arthritis, back pain) that varies in severity over time. Suppose that on Monday she experiences worse pain than expected. As usual, misattribution causes her to exaggerate this pain; she incorrectly attributes her disappointment to the pain itself. She thus forms an overly-pessimistic forecast for Tuesday, meaning that, on average, her condition on Tuesday comes as a pleasant surprise. As such, she overestimates the quality of her condition on Tuesday and forms an overly-optimistic forecast going forward. This oscillating pattern will continue over time: the person forms exaggerated forecasts in the direction of the most recent outcome, which lead to subsequent “surprises” in the opposite direction.

Formalizing these intuitions, we show that a misattributor’s forecasts over-respond to the most recent outcome and are therefore excessively volatile and exhibit predictable errors. Let \( d_t \equiv \hat{x}_t - \hat{E}_t[x_t] \) denote the misattributor’s forecast error realized on date \( t \).

**Proposition 8.** A misattributor’s forecast entering date \( t + 1 \) is \( \hat{E}_{t+1}[x_{t+1}] = \varphi \hat{x}_t \), where

\[
\hat{x}_t = (1 + \kappa_t)x_t + \sum_{j=1}^{t-1} (1 + \kappa_j) \left( \frac{1}{\varphi} \prod_{i=j+1}^{t} \kappa_i \right) x_j.
\]

Hence, forecasts exhibit:

1. Excessive extrapolation and volatility: \( \hat{E}_{t+1}[x_{t+1}] \) overweights the outcome on date \( t \) by a factor \( (1 + \kappa_t) \), and conditional on \( (x_1, \ldots, x_{t-1}) \), \( \text{Var}(\hat{E}_{t+1}[x_{t+1}]) = (1 + \kappa_t)^2 \text{Var}(\hat{E}_t[x_t]) \).

2. Predictable errors and reversals: Forecast errors follow a negatively-correlated process given by

\[
d_t = (1 + \kappa_t) \left\{ -\varphi \left( \frac{\kappa_{t-1}}{1 + \kappa_{t-1}} \right) d_{t-1} + \varepsilon_t \right\}.
\]

The most recent outcome is overweighted by a factor \( 1 + \kappa_t \), implying that forecasts overreact to recent losses more than gains. Additionally, a misattributor’s forecast wrongly depends on all past outcomes, while the rational forecast is independent of outcomes prior to \( t \) after accounting for \( x_t \). Consistent with the oscillating logic in the example above, Equation 15 reveals that the misattributor’s forecast negatively weights outcomes that occurred an odd number of periods ago and positively weights those that happened an even number of periods ago.

While rational predictions generate uncorrelated forecast errors, Part 2 of Proposition 8 highlights the negative relationship between a misattributor’s errors: overly optimistic forecasts are typically

\footnote{We define the forecast error as the difference between the person’s perceived outcome and her expectations—this is the decision maker’s own perception of her forecast error. Our prediction of a negative relationship between today’s forecast error and tomorrow’s holds if we alternatively define the forecast error as the difference between the true outcome and expectations.}
followed by overly pessimistic forecasts. The strength of this relationship is increasing in both the extent of misattribution and the extent of autocorrelation. Figure 2 below uses a simulated time series to depict a misattributor’s overly extrapolative forecasts (top panel) and the negative relationship in her forecast errors (bottom panel).

![Figure 2: The top panel displays both the rational and biased forecasts for a simulated process with \( \varphi = 0.7, \sigma = 5, \eta = 1, \lambda = 3 \) and \( \hat{\eta} = 1/3 \). Using that same data, the bottom panel depicts the negative relationship between forecast errors on date \( t \) and \( t + 1 \).](image)

Our basic prediction of overly-extrapolative and volatile forecasts accords with a range of evidence. For example, Gennaioli, Ma, and Shleifer (2015) and Greenwood and Shleifer (2014) find that managers and investors form extrapolative, volatile predictions of their future earnings and that their forecast errors are negatively correlated with past performance. While alternative models give rise to this general pattern of extrapolative beliefs and systematic reversals—e.g., Bordalo, Gennaioli, and Shleifer’s (2017a) model of diagnostic expectations based on Kahneman and Tversky’s (1972) representativeness heuristic—our model provides additional predictions that may help empirically disentangle these mechanisms. In particular, we predict that these patterns are more pronounced when outcomes carry utility consequences for the forecaster and that beliefs respond more to bad
6 Applications

6.1 Escalating Effort in Repeated Tasks

In our first application, we analyze a repeated “search” environment where each period a person can exert effort to improve the likelihood of a better outcome that round. For example, consider a consumer who researches before each new purchase, a business traveler who investigates hotels before each trip, or an athlete who trains before each competition. The person seeks to learn the optimal effort to provide, but exerting effort will lead a misattributor to exaggerate the value of further effort. Over time, this causes her to settle on an inefficiently high level of work each round. Intuitively, increased effort raises expectations, which causes bad outcomes (e.g., purchases that end up being lower quality than expected) to seem even worse when they happen. This drives the misattributor to perceive greater dispersion in the possible outcomes, which increases her perceived benefit from working and thus leads to her to work harder. That is, behavior exhibits a form of sunk-cost fallacy: the more effort a person has already exerted, the more she feels compelled to try even harder going forward. This process may underlie, for instance, common intuitions that people exert excessive effort when comparison shopping.

For clarity and continuity, we focus on a binary-outcome example similar to the one in Section 3: each period \( t \), the person receives consumption utility of either \( \theta(H) + \varepsilon_t \) with probability \( p \) or \( \theta(L) + \varepsilon_t \) with probability \( 1 - p \), where \( \theta(L) < \theta(H) \); the person has normally distributed priors over these means; and \( \varepsilon_t \sim \mathcal{N}(0, \sigma^2) \) with \( \sigma \to 0 \). Suppose additionally that the person can exert effort at the start of each round to increase \( p \), the probability of the better outcome. Formally, at the start of each period \( t \), the person chooses this probability, denoted by \( p_t \in [p_0, \bar{p}] \), and incurs a cost \( c(p_t - p_0) \), where \( \bar{p} \in (p_0, 1) \) and \( c(\cdot) \) is convex, minimized at zero, and admits a continuous marginal cost that is weakly convex with \( c'(0) = 0 \). We assume that \( p_0 = 1/2 \)—that is, the default probability of the better outcome is 1/2 when the person exerts no effort.\(^{39}\) In terms of the example above in which a business traveler seeks to book hotels for her trips, without researching, the traveler faces a coin flip between a good or bad hotel. However, exerting effort comparing different options increases the chance of getting a good hotel, which yields consumption utility \( \theta(H) \). When bad luck strikes and the hotel is instead low quality, the traveler receives \( \theta(L) < \theta(H) \).

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\(^{39}\)The assumptions on \( c(\cdot) \) simplify the exposition, as they guarantee a unique optimal choice each round. The assumption that \( \bar{p} < 1 \) reflects the idea that it is not possible to eliminate all uncertainty. Moreover, \( \bar{p} < 1 \) rules out paths where the person reaches \( p_t = 1 \) and remains there simply because she lacks feedback about \( \theta(L) \). Finally, a “default” value \( p_0 = 1/2 \) ensures that the marginal benefit of increasing \( p \) is always positive. This rules out pathological cases where the decision-maker actually prefers a smaller chance of the better outcome, which stem from the large aversion to risk inherent in the KR model. Such cases can happen for some values of \( \eta \) and \( \lambda \) when \( p_0 \) is sufficiently small.
A misattributor fails to optimize effort in this setting because she does not account for how her perceptions of outcomes are shaped by the expectations she sets through her chosen effort. To loosely illustrate, suppose the misattributor faced a fixed probability $\tilde{p}$ in the past, so her perceptions of $\theta(H)$ and $\theta(L)$ approximately match the steady-state values $\tilde{\theta}(H; \tilde{p})$ and $\tilde{\theta}(L; \tilde{p})$ derived in Equation 8 of Section 3, respectively. Given these perceptions (which depend on the expectations induced by $\tilde{p}$), the person’s expected benefit from choosing a different probability $p$, denoted $\hat{U}(p; \tilde{p})$, is

$$\hat{U}(p; \tilde{p}) \equiv p\tilde{\theta}(H; \tilde{p}) + (1-p)\tilde{\theta}(L; \tilde{p}) - p(1-p)\eta(\lambda - 1)[\tilde{\theta}(H; \tilde{p}) - \tilde{\theta}(L; \tilde{p})].$$  \hspace{1cm} (17)

She naively maximizes $\hat{U}(p; \tilde{p})$ with respect to $p$, treating her perceptions of $\theta(H)$ and $\theta(L)$ as independent of her new effort choice, $p^*$. That is, she ignores the fact that changing from $\tilde{p}$ to $p^*$ will eventually alter her perceptions of $\theta(H)$ and $\theta(L)$, at which point $p^*$ will no longer maximize her expected utility.

More specifically, once the misattributor exerts effort, and thus increases $p$, her seemingly optimal choice will eventually feel inadequate. Recall from Section 3 that the more the misattributor expects a good outcome (i.e., the higher is $p$), the worse a bad outcome will seem and hence the more she underestimates $\theta(L)$. While higher expectations also cause the misattributor to overestimate $\theta(H)$ by less, loss aversion ensures that distortions caused by higher expectations are stronger for the bad outcome than the good. Thus, overall, increasing $p$ causes a misattributor to overestimate the payoff difference between the good and bad outcomes by more. Since the optimal effort level is increasing in her perceived payoff difference, these new beliefs inspire her to further increase $p$. This pattern of escalating effort will continue as the person converges to an inefficiently high steady-state level.

**Proposition 9.** Consider the repeated search setting described above. If $\lambda > 1$, then a misattributor perpetually exerts excessive effort: if the full-information rational effort level, $p^r$, is interior ($p^r < \bar{p}$), then $p_t$ converges almost surely to a long-run value that strictly exceeds $p^r$. Otherwise, if $p^r = \bar{p}$, then $p_t$ converges almost surely to $\bar{p}$.

Moreover, this excessive effort is costly to a misattributor: her average experienced utility at the long-run level is strictly less than what she would earn if she correctly inferred $\theta(H)$ and $\theta(L)$.

### 6.2 Expectations Management and Reputation

In our second application, we explore how a sophisticated agent may strategically manipulate the beliefs of a misattributor. Specifically, we consider a career-concern setting where a misattributing (but otherwise rational) principal sequentially updates her beliefs about an agent’s ability and offers wages based on those inferences. For instance, imagine a manager assessing her assistant, or a homeowner assessing a contractor. Unlike the rational signal-jamming logic where high effort is interpreted as un-
informative, a misattributor wrongly attributes outcomes that deviate from expectations to the agent’s ability, providing a way for the agent to deceive the principal. Thus, an agent aware of the principal’s misattribution faces new incentives that push against the classical intuitions like Holmström (1999) that predict declining effort over time. Namely, we show that the sophisticated agent follows an effort path that initially under-performs relative to the principal’s expectations but consistently beats them thereafter. By initially setting the bar low, he can supply a series of elations that are interpreted as favorable signals of his ability.

Following Holmström (1999), a principal (she) hires an agent (he) with uncertain ability to exert effort over $T$ periods (i.e., the principal and agent have agreed ex ante to a fixed-duration relationship.) Each period $t$, the agent supplies effort $e_t \in \mathbb{R}_+$ leading to output $x_t = \theta + e_t + \varepsilon_t$, where $\theta \in \mathbb{R}$ is the agent’s ability. Continuing the Gaussian setup in Section 5, we assume a common prior $\theta \sim \mathcal{N}(0, \rho^2)$ and that $\varepsilon_t$ are i.i.d. with $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$. We take $x_t$ (net of paid wages) as the principal’s consumption value in period $t$—she directly benefits from the agent’s performance. Additionally, the principal cannot observe effort, so she updates about the agent’s ability based on output. These beliefs determine the agent’s wage in the subsequent period. We assume the market perfectly competes for the agent’s labor, so the principal pays a wage $w_t$ at the start of each round equal to her current expectation of $x_t$. As such, the agent maximizes the principal’s perception of his ability subject to effort costs. We assume these costs are separable across periods and given by a flow disutility function $c(\cdot)$ that is strictly increasing and convex.

We consider a principal who suffers misattribution while inferring the agent’s ability, and the agent—aware of this error—best responds to it. The principal accordingly misencodes output. Given her mistaken perceptions of output, the principal’s beliefs follow the mechanics derived in Section 5 given reference points based on her expectations of $\theta$. Moreover, because the principal is unaware of her mistake, she neglects the agent’s incentive to exploit it. Thus, she additionally develops incorrect beliefs about the agent’s strategy.\footnote{To solve the model, we assume the misattributing principal (wrongly) presumes common knowledge of rationality: she believes the agent follows the Bayesian Nash Equilibrium strategy that he would play when facing a fully rational principal, and she best responds to this presumed behavior of the agent. In this case, the naive principal mispredicts the agent’s effort in expectation and misattributes realized discrepancies to ability and noise. We further assume that the agent knows (1) the principal’s incorrect beliefs about his strategy and (2) her misspecified updating rule, and accordingly best responds these distorted beliefs. Our qualitative predictions are robust to other natural assumptions about the principal’s anticipated effort. For instance, we could alternatively assume the principal correctly predicts the agent’s effort profile despite lacking a good theory as to why the agent deviates from the Bayesian-Nash strategy.}

The sophisticated agent’s optimal effort path will fall short of the principal’s expectations early and then consistently beat expectations later in the relationship. Let $e_t^*$ and $e_t^R$ denote the agent’s optimal effort in round $t$ when facing a misattributing and rational principal, respectively.

**Proposition 10.** Consider the expectations-management setting described above and assume $\lambda = 1$. If beliefs are convex, then there exists a period $t^*$, $1 \leq t^* < T$, such that the agent’s optimal effort
falls short of the rational benchmark \((e^*_t < e^r_t)\) for all \(t < t^*\) and exceeds this benchmark \((e^*_t > e^r_t)\) for all \(t \geq t^*\).\(^{41}\)

When facing a misattributing principal, early effort by the agent imposes a cost on his future selves: hard work in period one increases the principal’s expectations in all future periods, which means that subsequent output will be judged more harshly. The agent therefore restrains early effort and pleasantly surprises the principal in subsequent periods.\(^{42,43}\)

Additionally, the proposition highlights that while eventually beating expectations is beneficial, the agent prefers to do so sufficiently late in the horizon as to not set the bar too high too early. Intuitively, the longer is the horizon of the relationship, the greater is the “externality” that early effort imposes on future selves. Indeed, the extent to which the agent under-performs at the onset is increasing in the horizon.\(^{44}\)

**Corollary 3.** Consider the expectations-management setting described above. If beliefs are convex, then there exists a horizon \(\bar{T}\) such that \(T > \bar{T}\) implies \(e^*_1 < e^r_1\). Furthermore, \(e^*_1 - e^r_1\) is strictly decreasing in \(T\) conditional on \(T > \bar{T}\).

While moral hazard leads the agent to supply inefficiently little effort when the principal is rational, misattribution can lead the agent to provide greater effort in total and thus increase efficiency. As in the rational case, an agent working for a misattributing principal still has some incentive to provide high initial effort. However, he has greater incentive to maintain high effort under misattribution since he is unduly penalized for falling short of expectations. In a sense, the principal’s biased evaluations impose an informal contract under which the agent feels committed to uphold the precedent for high effort he sets early in the relationship.

Although we framed this application as a familiar career-concern model, the analysis directly extends to other settings where one party has incentive to build a positive reputation. Furthermore,

\(^{41}\)We restrict attention to \(\lambda = 1\) and relegate the case of \(\lambda > 1\) to Appendix E. Loss aversion yields qualitatively similar results but complicates the analysis.

\(^{42}\)This intuition shares similarities with “ratcheting effects” studied in the literature on contracts and regulation (e.g., Freixas, Guesnerie, and Tirole 1985; Laffont and Tirole 1988). In those settings, the agent is reluctant to reveal positive private information about his efficiency early in the relationship so that he can demand higher compensation. In the setting above, the agent would want to reveal positive information about \(\theta\) if he could credibly do so. Misattribution, however, complicates the dynamics of revealing such information.

\(^{43}\)Because Proposition 10 describes the agent’s deviations from the rational path, it does not necessarily imply that \(e^*_t\) increases over time. In fact, convexity rules out an increasing effort path in this particular setting where the agent earns a reward each round. That said, misattribution can cause the agent to supply an increasing profile of effort in alternative settings even when beliefs are convex. For example, consider a setting with one payment period that follows multiple rounds of effort and evaluation. If the principal were rational, the agent would smooth his effort across rounds, since each is a perfect substitute for another in terms of the principal’s posterior beliefs. In contrast, when the principal suffers misattribution, the optimal pattern of effort follows a low-to-high profile (see Proposition 6).

\(^{44}\)These results apply to environments where the two parties work together for a fixed duration, so the agent does not have additional incentive to provide a positive first impression. While allowing for the principal to fire the agent would complicate the analysis, it would not change the qualitative conclusion that the agent undercuts the rational benchmark early in the relationship.
our results may speak to forms of expectations management used in diverse settings ranging from politics, to marketing, to finance. Politicians and firms often strategically “walk down” expectations only to later surpass them. Additionally, research from empirical finance shows that firms attempt to lower investors’ expectations prior to earnings announcements. Bartov, Givoly and Hayn (2002) demonstrate that meeting or beating analyst expectations yields significant excess stock returns. Similarly, Teoh, Yang and Zhang (2009) show that firms are rewarded for beating expectations even when those analyst forecasts are walked down by firm guidance. Although such expectations management may be prevalent in a number of domains, previously considered models struggle to explain why this technique might influence beliefs.

7 Discussion

We conclude by contrasting our approach with alternative models and by highlighting ways that future empirical work could explore the implications of our model. Additionally, we discuss two natural extensions of our framework: (1) incorporating misattribution of news utility (Kőszegi and Rabin 2009) and (2) extending misattribution to social-learning environments.

In addition to the models of mistaken learning noted in the introduction, we build on an emerging literature that examines limited or distorted memory. Wilson (2014) follows a rational approach with bounded memory and examines the optimal coarsening of information given a memory constraint. This approach yields predictions distinct from misattribution as it implies first impressions dominate subsequent evaluations. Mullainathan (2002) provides a model of limited rationality which can generate a form of over-reaction to information through memory associations. Relatedly, Bordalo, Gennaioli, and Shleifer (2017b) consider a model of limited attention in which events are more memorable when they are “salient”. In some settings, these two models predict an effect similar to ours: good or bad experiences “stand out”. However, we predict this effect even for first-time experiences that simply stand out relative to expectations (e.g., Backus et al. 2018).

A natural avenue for empirical exploration is our prediction of belief-based contrast effects—a fixed outcome will seem better when the previous one was worse. We predict that contrast effects increase when the perceived correlation between today’s outcome and tomorrow’s is stronger. Further, Hartzmark and Shue (2017) find that contrast effects among investors stemming from prior-day earnings

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45 Such “expectations management” can be enacted through a number of channels, including strategic accounting of working capital and cash flow from operations (Burgstahler and Dichev 1997), real activities such as sales (Roychowdhury 2006), or through indirect channels such as managing analyst forecasts (e.g., Richardson, Teoh, and Wycoki 2004).

46 Our model defines “surprising or “salient” events as those with utility consequences that deviate from expectations. An alternative model could define these notions based on small-probability events irrespective of utility consequences. In some cases these approaches would yield similar predictions, but we adopt the current model to more precisely pin down what we mean by surprise, building on existing models. A deeper analysis of alternative approaches is beyond the scope of this paper.

47 Relatedly, Hartmark and Shue (2017) find that contrast effects among investors stemming from prior-day earnings
thermore, in order to separate effects generated by our mechanism from other potential explanations—e.g. the Gambler’s Fallacy (Chen, Moskowitz, and Shue 2016)—we suggest that researchers examine circumstances where decisions have utility consequences versus those without. Our model predicts that contrast effects will be enhanced the more that a person cares about the outcomes she faces. A similar empirical strategy could help distinguish our mechanism from base-rate neglect (e.g. Benjamin, Bodoh-Creed, and Rabin 2016) or the representativeness heuristic (e.g. Bordalo, Gennaioli, and Shleifer 2017a), which both predict recency effects and extrapolative beliefs. For instance, one could test whether investors’ forecasts are more extrapolative about companies they hold a stake in relative to those they do not. Additionally, testing whether such forecasts respond differently to losses versus gains may provide yet another way to distinguish these mechanisms.

Finally, we note two potential extensions of our model. First, throughout this paper we have omitted the notion of “news utility” (Kősze gi and Rabin 2009), in which a person experiences elations and disappointments from changes in beliefs about future consumption. News utility provides a channel for monetary outcomes to directly influence contemporaneous experienced utility, and thus incorporating misattribution of news utility would naturally extend our predictions to settings involving money or earnings. Moreover, this extension introduces novel comparative statics. To illustrate, consider a worker who agrees to a new position or project for a pre-specified amount of time, and imagine that her first encounter with the new job is worse than expected. With misattribution of news utility, her evaluation of that first experience will be worse the longer she committed to the job; essentially, that first episode provides worse news about the future as the duration of her contract grows longer.

Second, our model can be reframed as an interpersonal bias where an observer neglects how expectations shape the experiences of others. For instance, a person reading online reviews (e.g., Yelp) for a product may fail to appreciate that a bad rating could simply reflect the reviewer’s high expectations rather than poor quality. In scenarios where consumers form their expectations based on predecessors’ reviews, misattribution—that is, taking others’ ratings at face value without accounting for their expectations—might hinder social learning. Additionally, social-learning settings may be data-rich environments to explore the empirical implications of our model. If such social misattribution occurs, we would expect ratings to demonstrate the dynamic patterns described in this paper.

announcements are larger for within-sector peers than across industries.
References


Appendix

A Convergence of Mean Beliefs

This section describes sufficient conditions for the convergence of the sequence of mean beliefs $\langle \hat{\theta}_t \rangle$ studied in Section 4. Our convergence arguments, which rely on stochastic approximation theory, are similar to those in Esponda and Pouzo (2016) and Heidhues, Kőszeği, and Strack (2018). Stochastic approximation theory is used to describe the asymptotic behavior of beliefs by a deterministic ordinary differential equation. While encoded outcomes are not independent ($\hat{x}_t$ is a function of $\hat{\theta}_{t-1}$, which depends on $x_1, \ldots, x_{t-1}$), they become approximately independent as $t$ grows large and hence $\hat{\theta}_t$ changes a small amount (on average) in response to any new outcome. Intuitively, we could imagine a new process $\hat{\theta}(\tau)$ on a time scale $\tau$ that is redefined such that (1) the person experiences a large number, $N(\tau)$, of outcomes holding a fixed expectation $\hat{\theta}(\tau)$, and then (2) updates her belief $\hat{\theta}(\tau)$ based on those $N(\tau)$ independent observations. Under such a rescaling that keeps expected steps in $\hat{\theta}(\tau)$ constant, the process $\hat{\theta}(\tau)$ corresponds to a continuous-time, “sped up” analog of $\hat{\theta}_t$. As $\tau$ grows large, a unit of time in the process $\hat{\theta}(\tau)$ corresponds to many i.i.d. outcomes, which means the resulting change in $\hat{\theta}(\tau)$ is nearly deterministic. The limiting behavior of $\hat{\theta}(\tau)$ is thus approximated by the deterministic ordinary differential equation $\hat{\theta}'(\tau) = G_\infty(\hat{\theta}(\tau))$, where $G_\infty$ is the average deviation of $\hat{x}_t$ from $\hat{\theta}_t$ (Equation 10) in the limit $\tau \to \infty$.

We apply this method to demonstrate convergence of $\langle \hat{\theta}_t \rangle$ in the specific case where (1) $x_t = \theta + \sigma z_t$ and $z_t$ are i.i.d. normally-distributed mean-zero random variables with unit variance, and (2) the person begins with a prior $\theta \sim \mathcal{N}(\theta_0, \rho^2)$.\(^{48}\) While convergence obtains more generally whenever the conditions below are met, it is particularly straightforward to verify these conditions for the normal case given our derivation of $\hat{\theta}_t$ for normally-distributed outcomes in Section 5. From Equation 13, the misattributor’s beliefs update according to $\hat{\theta}_t = \hat{\theta}_{t-1} + \alpha_t [\hat{x}_t - \hat{\theta}_{t-1}]$, where $\alpha_t \equiv (1 + \kappa_t)\alpha_0$ and $\alpha_0 = \rho^2/(\tau \rho^2 + \sigma^2)$. In this case, we can appeal to Theorem 5.2.1 in Kushner and Yin (2003), who provide sufficient conditions for the convergence of dynamic systems that take this form. Specifically, when the 4 conditions below are met, $\langle \hat{\theta}_t \rangle$ converges almost surely to the unique element of $\Gamma$ characterized in Proposition 1:

A1. $\sum_{t=1}^{\infty} \alpha_t = \infty$ and $\lim_{t \to \infty} \alpha_t = 0$.

A2. $\sum_{t=1}^{\infty} (\alpha_t)^2 < \infty$.

A3. $\sup_t \mathbb{E}[|\hat{x}_t - \hat{\theta}_{t-1}|^2 | \theta] < \infty$, where the expectation is taken at time $t = 0$.

A4. Finally, we require the existence of a continuous function $G$ and a sequence of random variables $\langle \gamma_t \rangle$ such that $\mathbb{E}[\hat{x}_t - \hat{\theta}_{t-1} | \theta, \hat{\theta}_{t-1}] = G(\hat{\theta}_t) + \gamma_t$ and $\sum_{t=1}^{\infty} \alpha_t |\gamma_t| < \infty$ w.p. 1. We take $G$ to be the function $G_\infty$ defined following Equation 10, and $\gamma_t = G(t, \hat{\theta}_{t-1}) - G_\infty(\hat{\theta}_{t-1})$. Given this definition, $\sum_{t=1}^{\infty} \alpha_t |\gamma_t| < \infty$ w.p. 1, as required. Furthermore, from Equation 10, it is straightforward that $G_\infty(\cdot)$ is continuous given that $F_Z$ and $f_Z$ are continuous.

\(^{48}\)The result trivially extends to the case where the person is also uncertain about the variance of outcomes, $\sigma^2$, so long as her beliefs about the variance do not influence how she updates about $\theta$. This is the case, for instance, in the canonical example where priors over $\theta$ and $\sigma^2$ follow independent Normal and Inverse-Gamma distributions, respectively.
The following proposition establishes that the 4 sufficient conditions hold for any arbitrary collection of distributional parameters \((\theta, \theta_0, \rho, \sigma)\).

**Proposition A.1.** Let \(G_\infty\) be defined as in Equation 10, and let \(\hat{\theta}_\infty \in \mathbb{R}\) be the unique value derived in Proposition 1 such that \(G_\infty(\hat{\theta}_\infty) = 0\). For all \(\eta, \lambda, \text{ and } \hat{\eta} \in [0, \eta]\), \(\hat{\theta}_t\) converges a.s. to \(\hat{\theta}_\infty\).

**Proof.** Since Condition 4 is argued in the text above, we need only show that Conditions 1-3 hold.

**Condition 1.** Note that

\[
\sum_{t=1}^{\infty} \hat{\alpha}_t = \sum_{t=1}^{\infty} (1 + \kappa_t) \alpha_t \geq (1 + \kappa^G) \sum_{t=1}^{\infty} \alpha_t = (1 + \kappa^G) \sum_{t=1}^{\infty} \frac{p^2}{t^2 + \sigma^2}. \tag{A.1}
\]

Since the final sum diverges to \(\infty\), \(\sum_{t=1}^{\infty} \hat{\alpha}_t\) must as well. Furthermore, it is clear that \(\lim_{t \to \infty} \hat{\alpha}_t = 0\).

**Condition 2.** Note that

\[
\sum_{t=1}^{\infty} (\alpha_t)^2 = \sum_{t=1}^{\infty} (1 + \kappa_t)^2 \alpha_t^2 \leq (1 + \kappa^L)^2 \sum_{t=1}^{\infty} \alpha_t^2. \tag{A.2}
\]

From the definition of \(\alpha_t\), \(\sum_{t=1}^{\infty} (\alpha_t)^2 < \sum_{t=1}^{\infty} \frac{1}{t^2} < \infty\). Thus, \(\sum_{t=1}^{\infty} (\hat{\alpha}_t)^2 < \infty\).

**Condition 3.** We must show \(\sup_{t} \mathbb{E}[(\hat{x}_t - \hat{\theta}_{t-1})^2 | \theta] < \infty\). Note that \(\hat{x}_t - \hat{\theta}_{t-1} = x_t - \kappa_t (x_t - \hat{\theta}_{t-1}) - \hat{\theta}_{t-1} = (1 + \kappa_t)(x_t - \hat{\theta}_{t-1})\). Letting \(\theta_{t-1}\) be the rational estimate of \(\theta\) following \(t - 1\) rounds, we have

\[
\sup_{t} \mathbb{E}[(\hat{x}_t - \hat{\theta}_{t-1})^2 | \theta] \leq (1 + \kappa^L) \sup_{t} \mathbb{E}[(x_t - \theta_{t-1}) + (\theta_{t-1} - \hat{\theta}_{t-1})]^2 | \theta]. \tag{A.3}
\]

From Minkowski’s Inequality,

\[
\sqrt{\mathbb{E}[(x_t - \theta_{t-1}) + (\theta_{t-1} - \hat{\theta}_{t-1})]^2 | \theta]} \leq \sqrt{\mathbb{E}[(x_t - \theta_{t-1})^2 | \theta]} + \sqrt{\mathbb{E}[(\theta_{t-1} - \hat{\theta}_{t-1})^2 | \theta]} \tag{A.4}
\]

Since \(\mathbb{E}[(x_t - \theta_{t-1})^2 | \theta]\) is finite, we need only examine the second term on the right-hand side of of Equation A.4. Using Lemma 1, we can write

\[
\theta_{t-1} - \hat{\theta}_{t-1} = \alpha_{t-1} \sum_{k=1}^{t-1} x_k - \alpha_{t-1} \sum_{k=1}^{t-1} \xi_{k}^{t-1} x_k = \alpha_{t-1} \sum_{k=1}^{t-1} (1 - \xi_{k}^{t-1}) x_k, \tag{A.5}
\]

where \(\xi_{k}^{t-1}\), defined in Lemma 1, are functions of \(\kappa_j\) and \(\alpha_j\) for \(j \in \{k, \ldots, t - 1\}\). Thus

\[
\sqrt{\mathbb{E}[(\theta_{t-1} - \hat{\theta}_{t-1})^2 | \theta]} \leq \alpha_{t-1} \sum_{k=1}^{t-1} \sqrt{\mathbb{E}[(1 - \xi_{k}^{t-1}) x_k]^2 | \theta]} \tag{A.6}
\]

We now argue that for all \(t \geq 2\) and all \(k \leq t - 1\), the value \(|1 - \xi_{k}^{t-1}|\) is bounded from above by some finite constant \(Q\). Given that \(\kappa_t \in \{\kappa^G, \kappa^L\}\) and the definition of \(\alpha_j\), it is clear that such a \(Q\) exists for
any finite \( t \). Thus, we need only consider the case where \( t \to \infty \). In this case, we have

\[
\lim_{t \to \infty} \xi_{k}^{t-1} = (1 + \kappa_k) \lim_{t \to \infty} \prod_{j=k}^{t-2} [1 - \alpha_j \kappa_{j+1}].
\]

For sufficiently large \( j, |1 - \alpha_j \kappa_{j+1}| < 1 \). This means that, fixing \( k \), there exists some \( \bar{t} \geq k \) such that \( |\xi_{k}^{t-1}| \) is decreasing in \( t \) on \( t \geq \bar{t} \). Thus, given that \( |1 - \xi_{k}^{t-1}| \) is bounded by some finite \( Q \),

\[
\sqrt{\mathbb{E}[|\theta_{t-1} - \hat{\theta}_{t-1}|^2 | \theta]} \leq Q \alpha_{t-1} \sum_{k=1}^{t-1} \sqrt{\mathbb{E}[|x_k|^2 | \theta]}
= Q \alpha_{t-1} \sum_{k=1}^{t-1} \sqrt{\sigma^2 + \theta^2}
= Q \frac{\rho^2}{\rho^2 + \sigma^2/(t-1)} \sqrt{\sigma^2 + \theta^2}
\leq Q \sqrt{\sigma^2 + \theta^2}, \tag{A.7}
\]

where the first equality follows from the fact that \( \mathbb{E}[x_k^2] = \text{Var}(x_k) - \mathbb{E}[x_k]^2 \), and the second equality follows from the fact that for all \( t \geq 2 \), \( \alpha_{t-1} = \rho^2/((t-1)\rho^2 + \sigma^2) \). Thus \( \sqrt{\mathbb{E}[|\theta_{t-1} - \hat{\theta}_{t-1}|^2 | \theta]} \) is finite, which completes the proof.

\[\blacksquare\]

B \hspace{1em} Proofs of Results in the Main Text

Proof of Proposition 1.

Proof. Part 1. Let \( G_\infty(\hat{\theta}; \theta, \sigma) \) be defined as in the text following Equation 10 conditional on the true parameter values \((\theta, \sigma)\). Thus, \( G_\infty(\hat{\theta}; \theta, \sigma) = \mathbb{E}[\hat{\xi}_t] - \hat{\theta} \), where \( \mathbb{E}[\hat{\xi}_t] \) is conditional on the mis-attributor holding expectation \( \hat{\theta}_{t-1} = \hat{\theta} \) and taken with respect to the true distribution conditional on \((\theta, \sigma)\). This implies \( \mathbb{E}[\hat{\xi}_t] = \theta + \kappa^G \mathbb{Pr}(x \geq \hat{\theta})(\mathbb{E}[x|x \geq \hat{\theta}] - \hat{\theta}) + \kappa^L \mathbb{Pr}(x < \hat{\theta})(\mathbb{E}[x|x < \hat{\theta}] - \hat{\theta}) = \theta - k \mathbb{Pr}(x < \hat{\theta})(\hat{\theta} - \mathbb{E}[x|x < \hat{\theta}]) \) where

\[
k \equiv \frac{\kappa^L - \kappa^G}{1 + \kappa^G} = \frac{(\lambda - 1)(\eta - \hat{\eta})}{(1 + \eta)(1 + \hat{\eta} \lambda)}. \tag{B.1}
\]

Hence, \( \hat{\theta} \in \Gamma \) implies \( \hat{\theta} \) solves \( 0 = G_\infty(\hat{\theta}; \theta, \sigma) = \hat{\theta} - \theta + kH(\hat{\theta}; \theta, \sigma) \) where

\[
H(\hat{\theta}; \theta, \sigma) \equiv \mathbb{Pr}(x < \hat{\theta})\left(\hat{\theta} - \mathbb{E}[x|x < \hat{\theta}]\right). \tag{B.2}
\]

Note that \( H(\cdot; \theta, \sigma) \) is a positive and strictly increasing function of \( \hat{\theta} \): \( H(\hat{\theta}; \theta, \sigma) = \hat{\theta} F_\mathcal{Z}\left(\frac{\theta - \hat{\theta}}{\sigma}\right) - \mathbb{E}[x|x < \hat{\theta}] \).
\[ \int_{-\infty}^{\hat{\theta}} x \frac{1}{\sigma} f_Z \left( \frac{x - \theta}{\sigma} \right) \, dx, \] hence

\[ \frac{\partial}{\partial \hat{\theta}} H(\hat{\theta}; \theta, \sigma) = \frac{1}{\sigma} f_Z \left( \frac{\hat{\theta} - \theta}{\sigma} \right) + F_Z \left( \frac{\hat{\theta} - \theta}{\sigma} \right) - \frac{1}{\sigma} f_Z \left( \frac{\theta - \theta}{\sigma} \right) = F_Z \left( \frac{\hat{\theta} - \theta}{\sigma} \right) > 0. \quad (B.3) \]

Hence \( G_{\infty}(\hat{\theta}; \theta, \sigma) \) is a strictly increasing function of \( \hat{\theta} \) with range \( \mathbb{R} \). Since \( \hat{\theta} \) is defined by \( G_{\infty}(\hat{\theta}; \theta, \sigma) = 0 \), the solution \( \hat{\theta} \) exists and is unique.

**Part 2.** We first establish underestimation. For sake of contradiction, suppose \( \lambda > 1 \) and \( \hat{\theta} \geq \theta \). From Part 1, \( \hat{\theta} \) solves \( \hat{\theta} + kH(\hat{\theta}; \theta, \sigma) = \theta \). Because \( H(\hat{\theta}; \theta, \sigma) > 0 \), \( \hat{\theta} + kH(\hat{\theta}; \theta, \sigma) \) exceeds \( \theta \leftrightarrow k > 0 \), which holds \( \leftrightarrow \lambda > 1 \), implying a contradiction. Turning to comparative statics, since \( \hat{\theta} \) satisfies \( G_{\infty}(\hat{\theta}; \theta, \sigma) = 0 \), the implicit function theorem implies that, for any parameter \( w \in \{ \eta, \hat{\eta}, \lambda, \sigma \} \),

\[ \frac{\partial \hat{\theta}}{\partial w} = -\left( \frac{\partial G_m(\hat{\theta}; \theta, \sigma)}{\partial \hat{\theta}} \right)^{-1} \frac{\partial G_m(\hat{\theta}; \theta, \sigma)}{\partial w}. \]

Since \( \frac{\partial G_m(\hat{\theta}; \theta, \sigma)}{\partial \hat{\theta}} = 1 + k \frac{\partial^2}{\partial \hat{\theta}^2} H(\hat{\theta}; \theta, \sigma) > 0 \), \( \frac{\partial \hat{\theta}}{\partial w} \) has the opposite sign as \( \frac{\partial G_m(\hat{\theta}; \theta, \sigma)}{\partial w} \). It thus follows from the definition of \( k \) that \( \hat{\theta} \) is decreasing in \( \lambda \) and \( \eta \), and increasing in \( \hat{\eta} \). To show that variance has a decreasing effect, note that

\[
\frac{\partial G_{\infty}(\hat{\theta}; \theta, \sigma)}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left( \frac{\partial}{\partial \sigma} \left[ \theta + \sigma u \right] f_Z(u) \right) - \int_{-\infty}^{\hat{\theta}} \frac{\partial}{\partial \sigma} \left[ \theta + \sigma u \right] f_Z(u) \, du
\]

\[
= -\left( \frac{\hat{\theta} - \theta}{\sigma^2} \right) \frac{\partial f_Z}{\partial \sigma} \left( \frac{\hat{\theta} - \theta}{\sigma} \right) + \left( \frac{\hat{\theta} - \theta}{\sigma^2} \right) \left[ \theta + \sigma u \right] f_Z(u) \left. \right|_{u = \frac{\theta - \hat{\theta}}{\sigma}} - \int_{-\infty}^{\hat{\theta}} \frac{\partial}{\partial \sigma} \left[ \theta + \sigma u \right] f_Z(u) \, du
\]

\[
= -\int_{-\infty}^{\hat{\theta}} u f_Z(u) \, du
\]

\[
> 0,
\]

where the second line follows from Leibniz’s Rule and the final inequality follows from the fact that the integral is negative given \( \hat{\theta} < \theta \).

**Part 3.** Fix \( \hat{\theta}_{t-1} = \hat{\theta} \), so \( \hat{x}_t = x_t + \kappa_t(x_t - \hat{\theta}) \). Hence, \( \text{Var}(\hat{x}_t) = \text{Var}(x_t) + \text{Var}(\kappa_t(x_t - \hat{\theta})) + 2\text{Cov}(x_t, \kappa_t(x_t - \hat{\theta})) \). Note that \( \text{Cov}(x_t, \kappa_t(x_t - \hat{\theta})) = \mathbb{E}[x_t \kappa_t(x_t - \hat{\theta})] - \theta \mathbb{E}[\kappa_t(x_t - \hat{\theta})] \), where

\[
\mathbb{E}[\kappa_t(x_t - \hat{\theta})] = [1 - F(\hat{\theta})] \kappa^G \mathbb{E}[x_t - \hat{\theta}|x_t \geq \hat{\theta}] + F(\hat{\theta}) \kappa^L \mathbb{E}[x_t - \hat{\theta}|x_t < \hat{\theta}]
\]

\[
= \kappa^G(\theta - \hat{\theta}) + F(\hat{\theta}) (\kappa^L - \kappa^G) \mathbb{E}[x_t - \hat{\theta}|x_t < \hat{\theta}], \quad (B.4)
\]

and

\[
\mathbb{E}[x_t \kappa_t(x_t - \hat{\theta})] = [1 - F(\hat{\theta})] \kappa^G \mathbb{E}[x_t(x_t - \hat{\theta})|x_t \geq \hat{\theta}] + F(\hat{\theta}) \kappa^L \mathbb{E}[x_t(x_t - \hat{\theta})|x_t < \hat{\theta}]
\]

\[
= \kappa^G \mathbb{E}[x_t(x_t - \hat{\theta})] + F(\hat{\theta}) (\kappa^L - \kappa^G) \mathbb{E}[x_t(x_t - \hat{\theta})|x_t < \hat{\theta}]
\]

\[
= \kappa^G(\sigma^2 + \theta^2 - \hat{\theta} \hat{\theta}) + F(\hat{\theta}) (\kappa^L - \kappa^G) \mathbb{E}[x_t(x_t - \hat{\theta})|x_t < \hat{\theta}], \quad (B.5)
\]
where the last line follows from the fact that $\sigma^2 \equiv \text{Var}(x_t)$ and $\theta \equiv \mathbb{E}[x_t]$. Hence,

$$
\text{Cov}(x_t, \kappa_t(x_t - \hat{\theta})) = \kappa^G \sigma^2 + F(\hat{\theta})(\kappa^L - \kappa^G)\mathbb{E}[x_t(x_t - \hat{\theta}) - (x_t - \hat{\theta})|x_t < \hat{\theta}]
$$

$$= \kappa^G \sigma^2 + F(\hat{\theta})(\kappa^L - \kappa^G)\mathbb{E}[(x_t - \theta)(x_t - \hat{\theta})|x_t < \hat{\theta}]. \quad (B.6)
$$

Since $\hat{\theta} < \theta$ (Part 2), $x_t < \hat{\theta}$ implies $x_t < \theta$, meaning the expectation in Equation B.6 is always positive. Thus, $\text{Var}(\hat{x}_t) = \text{Var}(x_t) + \text{Var}(\kappa_t(x_t - \hat{\theta})) + 2\text{Cov}(x_t, \kappa_t(x_t - \hat{\theta})) > \text{Var}(x_t)$. The relevant comparative statics follow from the dependence of $\kappa_t$ on $\eta$, $\lambda$, and $\hat{\eta}$.

**Proof of Proposition 2.**

*Proof. Part 1.* Let $u(\cdot|\theta)$ denote utility given expectation $\theta$. Hypothetical utility is $v \equiv \mathbb{E}[u(x|\theta)]$, where the expectation $\mathbb{E}[\cdot]$ is taken with respect to the distribution implied by the true parameter values $(\theta, \sigma)$. Thus

$$
v = \mathbb{E}[u(x|\theta)] = \mathbb{E}[x] + \eta \Pr(x \geq \theta) (\mathbb{E}[x|x \geq \theta] - \theta) + \eta \lambda \Pr(x < \theta) (\mathbb{E}[x|x < \theta] - \theta)
$$

$$= \theta - \eta(\lambda - 1) \Pr(x < \theta) (\theta - \mathbb{E}[x|x < \theta])
$$

$$= \theta - \eta(\lambda - 1)H(\theta; \theta, \sigma), \quad (B.7)
$$

where $H$ is defined as in Equation B.2. Analogously, forecasted utility is $\hat{v} \equiv \hat{\mathbb{E}}[u(x|\hat{\theta})]$, where $\hat{\mathbb{E}}[\cdot]$ is with respect to the distribution implied by parameter values $(\hat{\theta}, \hat{\sigma})$. Thus, $\hat{v} = \theta - \eta(\lambda - 1)H(\hat{\theta}; \hat{\theta}, \hat{\sigma})$. For any $\hat{\theta}$ and $\hat{\sigma}$, we can write $H(\hat{\theta}; \hat{\theta}, \hat{\sigma})$ simply as

$$
\tilde{H}(\hat{\theta}, \hat{\sigma}) \equiv \hat{\theta} F_Z(0) - \int_{-\infty}^{0} [\hat{\theta} + \hat{\sigma} u] f_Z(u) du = |\tilde{z}|, \quad (B.8)
$$

where $\tilde{z} \equiv \int_{-\infty}^{0} u f_Z(u) du < 0$. Therefore $\frac{\partial}{\partial \hat{\theta}} \tilde{H}(\hat{\theta}, \hat{\sigma}) = 0$ and $\frac{\partial}{\partial \hat{\sigma}} \tilde{H}(\hat{\theta}, \hat{\sigma}) = |\tilde{z}| > 0$. Hence $\tilde{H}(\hat{\theta}, \hat{\sigma}) > \tilde{H}(\theta, \sigma)$ given that $\hat{\sigma} > \sigma$ (Proposition 1, Part 3). Together with the fact that $\hat{\theta} < \theta$ (Proposition 1, Part 2), this implies that $v - \hat{v} = (\theta - \hat{\theta}) - \eta(\lambda - 1)(\tilde{H}(\theta; \theta, \sigma) - \tilde{H}(\hat{\theta}; \hat{\theta}, \hat{\sigma})) > 0$.

*Part 2.* Average experienced utility is $\bar{v} \equiv \mathbb{E}[u(x|\hat{\theta})]$, where $\mathbb{E}[\cdot]$ is again with respect to parameter values $(\theta, \sigma)$. Thus

$$
\bar{v} = \mathbb{E}[u(x|\hat{\theta})] = \mathbb{E}[x] + \eta \Pr(x \geq \hat{\theta}) (\mathbb{E}[x|x \geq \hat{\theta}] - \hat{\theta}) + \eta \lambda \Pr(x < \hat{\theta}) (\mathbb{E}[x|x < \hat{\theta}] - \hat{\theta})
$$

$$= \theta + \eta(\hat{\theta} - \hat{\theta}) - \eta(\lambda - 1) \Pr(x < \hat{\theta}) (\hat{\theta} - \mathbb{E}[x|x < \theta])
$$

$$= \theta + \eta(\hat{\theta} - \hat{\theta}) - \eta(\lambda - 1)H(\hat{\theta}; \theta, \sigma).
$$

The difference between average utility experienced and hypothetical utility (Equation B.7) is thus

$$
\bar{v} - v = \mathbb{E}[u(x|\hat{\theta})] - \mathbb{E}[u(x|\theta)] = \eta(\theta - \hat{\theta}) + \eta(\lambda - 1)(H(\theta; \theta, \sigma) - H(\hat{\theta}; \hat{\theta}, \hat{\sigma})) > 0. \quad (B.9)
$$

The inequality follows given that $\hat{\theta} < \theta$ and $H(\cdot; \theta, \sigma)$ is strictly increasing (see Equation B.3). Finally, using properties of $H$ derived in the proof of Proposition 1.
\[
\frac{d}{d\sigma}(\mathbb{E}[u(x|\hat{\theta})] - \mathbb{E}[u(x|\theta)]) = -\eta \frac{\partial \hat{\theta}}{\partial \sigma} + \eta(\lambda - 1) \left( \frac{\partial}{\partial \sigma} H(\theta; \theta, \sigma) - \frac{d}{d\sigma} H(\hat{\theta}; \theta, \sigma) \right)
\]

\[
= -\eta \frac{\partial \hat{\theta}}{\partial \sigma} \left( 1 + (\lambda - 1) \frac{\partial}{\partial \sigma} H(\hat{\theta}; \theta, \sigma) \right)
\]

> 0.

**Proof of Corollary 1.**

**Proof.** Suppose the distribution from \( A \) has parameter values \((\theta, \sigma)\), and let \( v(\theta, \sigma) \) denote the hypothetical (i.e., fully informed) expected utility from \( A \). Likewise, let \( \hat{v}(\theta, \sigma) \) denote the long-run forecasted utility following \( t^* \to \infty \) rounds of experimentation. From Proposition 2, \( \hat{v}(\theta, \sigma) < v(\theta, \sigma) \), and from Equation B.8 \( \hat{v}(\theta, \sigma) = \hat{\theta} - \eta(\lambda - 1)\hat{\sigma}|\hat{z}^-| \), where \( \hat{\theta} \) and \( \hat{\sigma} \) are the steady-state values characterized by Proposition 1. Note from the proof of Proposition 1 that \( \hat{\theta} \) is a continuous and strictly increasing function of \( \theta \) with range \( \mathbb{R} \). As such, \( \hat{v}(\cdot, \sigma) \) shares these properties and there exists a well-defined value \( \hat{\theta} \) such that \( \hat{v}(\hat{\theta}, \sigma) = v_B \). Since the person selects \( A \) at \( t^* \to \infty \) if and only if \( \hat{v}(\theta, \sigma) > v_B \) (assuming indifference is broken in favor of \( B \)), she settles on \( A \leftrightarrow \theta > \hat{\theta} \). Additionally, note that \( \theta^* \) is defined by \( v(\theta^*, \sigma) = v_B \). Proposition 2 thus implies that \( \hat{v}(\theta, \sigma) = v(\theta^*, \sigma) > \hat{v}(\theta^*, \sigma) \). Since \( \hat{v}(\hat{\theta}, \sigma) > \hat{v}(\theta^*, \sigma), \hat{\theta} > \theta^* \). Hence, \( \theta \in (\theta^*, \hat{\theta}) \) implies \( \hat{v}(\theta, \sigma) < v_B \), meaning the person switches to \( B \) at \( t^* \to \infty \).

We now show \( \hat{\theta} - \theta^* \) is increasing in \( \sigma \). Consider any two positive values of \( \sigma \) such that \( \sigma_2 > \sigma_1 \). For \( i = 1, 2 \), define \( \theta_i^* \) and \( \bar{\theta}_i \) as the values that solve \( v(\theta_i^*, \sigma_i) = v_B \) and \( \hat{v}(\bar{\theta}_i, \sigma_i) = v_B \), respectively. From Equations B.7 and B.8, \( \theta_2^* = \theta_2^* + V \) where \( V = \eta(\lambda - 1)(\sigma_2 - \sigma_1)|\bar{z}^-| \). Since \( \bar{\theta}_2 - \theta_2^* > \bar{\theta}_1 - \theta_1^* \Leftrightarrow \bar{\theta}_2 = \bar{\theta}_1 + V \) for sake of contradiction. In this case, Equation B.7 implies \( v(\bar{\theta}_2, \sigma_2) = v(\bar{\theta}_1, \sigma_1) \). Finally, by Part 1 of Proposition 2, \( v(\bar{\theta}_2, \sigma_2) - v(\bar{\theta}_2, \sigma_2) > v(\bar{\theta}_1, \sigma_1) - v(\bar{\theta}_1, \sigma_1) \). This implies \( \hat{v}(\bar{\theta}_2, \sigma_2) < \hat{v}(\bar{\theta}_1, \sigma_1) \) since \( v(\bar{\theta}_2, \sigma_2) - v(\bar{\theta}_1, \sigma_1) = 0 \). However, \( \hat{v}(\bar{\theta}_2, \sigma_2) \neq \hat{v}(\bar{\theta}_1, \sigma_1) \) contradicts the definition of \( \bar{\theta}_i \).

**Proof of Proposition 3.**

**Proof.** Let \( \mathcal{P}(\nu) \) denote the set of parameter values \((\theta, \sigma)\) for option \( A \) such that the hypothetical utility of \( A \) is even though all prospects with \((\theta, \sigma) \in \mathcal{P}(\nu) \) have hypothetical utility \( v \), we show that, constrained to \((\theta, \sigma) \in \mathcal{P}(\nu) \), \( \lim_{\sigma \to \infty} \hat{v}(\theta, \sigma) = -\infty \).

First, we show that for any prospect with \((\theta, \sigma) \in \mathcal{P}(\nu) \), the steady-state perceived mean \( \hat{\theta} \) of that prospect is a linearly decreasing function of \( \sigma \). Recall that \( \hat{\theta} \) solves \( \hat{\theta} - \theta + kH(\hat{\theta}; \theta, \sigma) = 0 \). Since \( H(\hat{\theta}; \theta, \sigma) = \hat{\theta}F_Z\left(\frac{\hat{\theta} - \theta}{\sigma}\right) - \int_{-\infty}^{\hat{\theta}} \frac{1}{\sigma} f_Z\left(z - \frac{\hat{\theta}}{\sigma}\right) \) \( dx \), we can define \( \hat{z} \equiv (\hat{\theta} - \theta)/\sigma \) and rewrite \( H(\hat{\theta}; \theta, \sigma) \) as

\[
H(\hat{\theta}; \theta, \sigma) = \hat{\theta}F_Z(\hat{z}) - \int_{-\infty}^{\hat{z}} [\theta + \sigma z] f_Z(z) \, dz = \sigma \left( \hat{z}F_Z(\hat{z}) - \sigma \int_{-\infty}^{\hat{z}} z f_Z(z) \, dz \right). \tag{B.10}
\]
Hence, the steady-state value \( \hat{\theta} \) is defined by the value \( \hat{z} \) that solves

\[
\hat{z} + k \left( \hat{F}_Z(\hat{z}) - \int_{-\infty}^{\hat{z}} z f_Z(z) \, dz \right) = 0. \tag{B.11}
\]

By virtue that \( \hat{\theta} \) is unique and finite, there exists a unique, finite \( \hat{z} \) that solves Equation B.11, which we denote by \( z^* \). Clearly \( z^* \) depends solely on \( F_Z, f_Z, \) and \( k, \) and is thus independent of \( \theta \) and \( \sigma \). As such, \( z^* = (\hat{\theta} - \theta)/\sigma \) implies the steady-state estimate is \( \hat{\theta} = \theta + z^* \sigma \). Furthermore, the fact that \( \hat{\theta} < \theta \) implies that \( z^* < 0 \). Thus \( \hat{\theta} = \theta - |z^*|\sigma \).

Now consider a prospect with \( (\theta, \sigma) \in \mathcal{P}(v) \). From Equations B.7 and B.8, the hypothetical and forecasted utilities from this prospect are

\[
v = \theta - \eta(\lambda - 1)\sigma|\tilde{z}^-| \quad \text{and} \quad \hat{v} = \hat{\theta} - \eta(\lambda - 1)\hat{\sigma}|\tilde{z}^-|,
\]

respectively, where \( \tilde{z}^- \equiv \int_{-\infty}^{0} uf_Z(u) \, du < 0 \) as before. Substituting the linear specification of \( \hat{\theta} \) into the above equation for \( \hat{v} \) and writing \( \hat{\theta} \) in terms of \( v \) yields

\[
\hat{v}(\theta, \sigma) = \theta - |z^*|\sigma - \eta(\lambda - 1)\hat{\sigma}|\tilde{z}^-| = v - |z^*|\sigma - \eta(\lambda - 1)|\hat{\sigma} - \sigma||\tilde{z}^-| < v - |z^*|\sigma.
\]

where the last line follows given that \( \hat{\sigma} > \sigma \) (Proposition 1, Part 3). Thus \( \hat{v}(\theta, \sigma) - v \) diverges to \( -\infty \) as \( \sigma \to \infty \) along the locus of parameter values defining \( \mathcal{P}(v) \).

\section*{Proof of Lemma 1.}

\textbf{Proof.} Without loss of generality, suppose \( \theta_0 = 0 \). Thus \( \hat{\theta}_t = \alpha_t \sum_{k=1}^{t} \xi_k = \alpha_t \sum_{k=1}^{t} \left[ x_k + \kappa_t (x_t - \theta_{t-1}) \right] \). We prove the following claim by induction: \( \hat{\theta}_t = \alpha_t \sum_{k=1}^{t} \xi'_k x_t \) where \( \xi'_k = (1 + \kappa_t) \Pi_{j=k}^{t-1} (1 - \alpha_t \kappa_{j+1}) \) for \( k < t \) and \( \xi'_t = (1 + \kappa_t) \). To establish the base case, note \( \hat{\theta}_1 = \alpha_1 \left[ x_1 + \kappa_1 (x_1 - \theta_0) \right] = \alpha_1 (1 + \kappa_1) x_1 \).

Now suppose the claim holds for period \( t > 1 \). Then

\[
\hat{\theta}_{t+1} = \alpha_{t+1} \sum_{k=1}^{t+1} \xi'_{k} = \alpha_{t+1} \left\{ (1 + \kappa_{t+1}) x_{t+1} - \kappa_{t+1} \hat{\theta}_t + \frac{1}{\alpha_t} \hat{\theta}_t \right\} = \alpha_{t+1} \left\{ (1 + \kappa_{t+1}) x_{t+1} + \sum_{k=1}^{t} (1 + \kappa_k) \left( \Pi_{j=k}^{t-1} (1 - \alpha_j \kappa_{j+1}) \right) x_k \right\} = \alpha_{t+1} \left\{ (1 + \kappa_{t+1}) x_{t+1} + \sum_{k=1}^{t} (1 + \kappa_k) \left( \Pi_{j=k}^{t-1} (1 - \alpha_j \kappa_{j+1}) \right) x_k \right\}.
\]

Hence, the induction step holds, establishing the claim.

\section*{Proof of Proposition 4.}

\textbf{Proof.} The results follow from Lemma 1. If beliefs are monotonic, then \( \kappa^L < 1 + \sigma^2/\rho^2 \). This implies \( 1 - \alpha_t \kappa_{t+1} \in (0, 1) \) for all \( t \in \{1, \ldots, T - 1\} \). Consider \( t, \tau \in \{1, \ldots, T\} \) such that \( \tau < t \) and
suppose $\kappa_t = \kappa_T$. From Lemma 1, $\xi_T / \xi_T = \prod_{t=1}^{T-1} [1 - \alpha_j \kappa_{j+1}] < 1$. Also from Lemma 1, $\xi_T^T = 1 + \kappa_T > 1$ and $\lim_{T \to \infty} \xi_T^T = (1 + \kappa_T) \lim_{T \to \infty} \prod_{t=1}^{T-1} [1 - \alpha_j \kappa_{j+1}] \leq (1 + \kappa_T) \lim_{T \to \infty} \prod_{t=1}^{T-1} [1 - \alpha_j \kappa^G_T]$. Since $\sum_{j=1}^{\infty} \alpha_j$ diverges, $\prod_{j=1}^{\infty} [1 - \alpha_j \kappa^G_T] = 0$, completing the proof.

Proof of Proposition 5.

Proof. From Equation 12, we can write $\hat{\theta}_1^d = \alpha_2 (\hat{b}_1^d + \hat{d}_2^d) + (1 - 2 \alpha_2) \theta_0$ where $\hat{b}_1^d$ and $\hat{d}_2^d$ are the encoded values of $b$ and $a$ respectively when facing the increasing sequence $(b, a)$. Likewise, $\hat{\theta}_2^d = \alpha_2 (\hat{a}_1^d + \hat{b}_2^d) + (1 - 2 \alpha_2) \theta_0$, where $\hat{a}_1^d$ and $\hat{b}_2^d$ are the encoded values when facing the decreasing sequence $(a, b)$. Let $\kappa_1^d = \kappa^G \{ b \geq \theta_0 \} + \kappa^L \{ b < \theta_0 \}$, and $\kappa_2^d = \kappa^G \{ a > \theta_1^d \} + \kappa^L \{ a < \theta_1^d \}$ where $\theta_1^d = \alpha_1 (1 + \kappa_1^d) (b - \theta_0) + \theta_0$. Similarly, let $\kappa_1^d = \kappa^G \{ a \geq \theta_0 \} + \kappa^L \{ a < \theta_0 \}$, and $\kappa_2^d = \kappa^G \{ b \geq \theta_1^d \} + \kappa^L \{ b < \theta_1^d \}$ where $\theta_1^d = \alpha_1 (1 + \kappa_1^d) (a - \theta_0) + \theta_0$. Hence $\hat{a}_1^d = a + \kappa_1^d (a - \theta_0), \hat{b}_1^d = b + \kappa_1^d (b - \theta_0), \hat{a}_2^d = a + \kappa_2^d (a - \theta_0 - \alpha_1 [1 + \kappa_1^d] (b - \theta_0)), \hat{b}_2^d = b + \kappa_2^d (b - \theta_0 - \alpha_1 [1 + \kappa_1^d] (a - \theta_0)).$ This implies $\hat{\theta}_1^d > \hat{\theta}_2^d$ if and only if

$$\kappa_1^d (b - \theta_0) + \kappa_1^d (a - \theta_0 - \alpha_1 [1 + \kappa_1^d] (b - \theta_0)) > \kappa_2^d (b - \theta_0 - \alpha_1 [1 + \kappa_1^d] (a - \theta_0)) \quad (B.12)$$

Letting $\tilde{a} = (a - \theta_0)$ and $\tilde{b} = (b - \theta_0)$, Condition B.12 reduces to

$$\kappa_1^d \tilde{b} + \kappa_1^d (\tilde{a} - \alpha_1 [1 + \kappa_1^d] \tilde{b}) > \kappa_2^d \tilde{a} + \kappa_2^d (\tilde{b} - \alpha_1 [1 + \kappa_1^d] \tilde{a}). \quad (B.13)$$

There are three cases to consider, depending on whether $\tilde{a}$ and $\tilde{b}$ have the same sign. When $\tilde{a}$ and $\tilde{b}$ have the same sign, then $\kappa_1^d = \kappa_1^d$ and condition B.13 reduces as follows, which is useful for checking the various cases: $\hat{\theta}_1^d > \hat{\theta}_2^d$ if and only if

$$\kappa_1^d (1 + \alpha_1 [1 + \kappa_1^d]) (\tilde{a} - \tilde{b}) - (\kappa_2^d - \kappa_2^d) (\tilde{b} - \alpha_1 [1 + \kappa_1^d] \tilde{a}) > \kappa_1^d (\tilde{a} - \tilde{b}). \quad (B.14)$$

**Case 1:** $\theta_0 < b < a$. This implies $\kappa_1^d = \kappa_1^d = \kappa^G$. There are 3 sub-cases to consider:

**Case 1.a.** Suppose both $a$ and $b$ come as gains if received in period 2. This implies $\kappa_1^d = \kappa_1^d = \kappa^G$. Hence, Condition B.14 amounts to $\kappa^G (1 + \alpha_1 [1 + \kappa^G]) (\tilde{a} - \tilde{b}) > \kappa^G (\tilde{a} - \tilde{b})$, which is true given $\tilde{a} > \tilde{b}$.

**Case 1.b.** Suppose both $a$ and $b$ come as losses if received in period 2. This implies $\kappa_1^d = \kappa_1^d = \kappa^L$. Hence, Condition B.14 amounts to $\kappa^L (1 + \alpha_1 [1 + \kappa^G]) (\tilde{a} - \tilde{b}) > \kappa^L (\tilde{a} - \tilde{b})$, which is true given $\tilde{a} > \tilde{b}$ and $\kappa^L > \kappa^G$.

**Case 1.c.** Suppose only $a$ comes a gain if received in period 2. This implies $\kappa_1^d = \kappa^G$ and $\kappa_1^d = \kappa^L$. Hence, Condition B.14 amounts to $\kappa^G (1 + \alpha_1 [1 + \kappa^G]) (\tilde{a} - \tilde{b}) - (\kappa^L - \kappa^G) (\tilde{b} - \alpha_1 [1 + \kappa^G] \tilde{a}) > \kappa^G (\tilde{a} - \tilde{b})$, which reduces to $\kappa^G (\alpha_1 [1 + \kappa^G]) (\tilde{a} - \tilde{b}) - (\kappa^L - \kappa^G) (\tilde{b} - \alpha_1 [1 + \kappa^G] \tilde{a}) > 0$. Since $\tilde{b}_2^d$ comes as a loss, it must be that $\tilde{b} - \alpha_1 [1 + \kappa^G] \tilde{a} < 0$, meaning the condition above holds.

**Case 2:** $b < a < \theta_0$. This implies $\kappa_1^d = \kappa_1^d = \kappa^L$. There are 3 sub-cases to consider:
**Case 2.a.** Suppose both $a$ and $b$ come as losses if received in period 2. This implies $\kappa^L_2 = \kappa^d_2 = \kappa^L$. Hence, Condition B.14 amounts to $\kappa^L(1 + \alpha_1[1 + \kappa^L])(\tilde{a} - \tilde{b}) > \kappa^L(\tilde{a} - \tilde{b})$, which is true given $\tilde{a} > \tilde{b}$.

**Case 2.b.** Suppose both $a$ and $b$ come as gains if received in period 2. This implies $\kappa^d_2 = \kappa^G$. Hence, Condition B.14 amounts to $\kappa^G(1 + \alpha_1[1 + \kappa^L])(\tilde{a} - \tilde{b}) > \kappa^L(\tilde{a} - \tilde{b})$, which holds if and only if $\kappa^G(1 + \alpha_1[1 + \kappa^L]) > \kappa^L$. Using the definitions of $\kappa^G$ and $\kappa^L$ and simplifying reveals that this condition fails only if $\lambda - 1 > \alpha_1(1 + \eta \lambda)$. Furthermore, for both $a$ and $b$ to come as gains in period 2 implies that $\alpha_1(1 + \kappa^L) > 1$. Thus, there exist values of $(a, b)$ meeting the conditions of subcase 2.b for which recency fails only if $\lambda - 1 > \alpha_1(1 + \eta \lambda)$ and $\alpha_1(1 + \kappa^L) > 1$.

**Case 2.c.** Suppose only $a$ comes as a gain if received in period 2. This implies $\kappa^d_2 = \kappa^G$ and $\kappa^L_2 = \kappa^L$. Hence, Condition B.14 amounts to $\kappa^G(1 + \alpha_1[1 + \kappa^L])(\tilde{a} - \tilde{b}) - (\kappa^L - \kappa^G)(\tilde{b} - \alpha_1[1 + \kappa^L] \tilde{a}) > \kappa^L(\tilde{a} - \tilde{b})$. This condition reduces to

$$\kappa^G \alpha_1(1 + \kappa^L)(\tilde{a} - \tilde{b}) > (\kappa^L - \kappa^G)(1 - \alpha_1(1 + \kappa^L)) \tilde{a}. \quad (B.15)$$

The left-hand side of Condition B.15 is always positive, while the right-hand side is positive if and only if $\alpha_1(1 + \kappa^L) > 1$ (i.e., beliefs are not convex). Thus, Condition B.15 always holds if $\alpha_1(1 + \kappa^L) < 0$, but may fail otherwise. To see when it fails, notice that B.15 fails when

$$\tilde{b} > \frac{1}{\kappa^G} \left( \kappa^L - \frac{\kappa^L - \kappa^G}{\alpha_1(1 + \kappa^L)} \right). \quad (B.16)$$

Since this subcase assumes that $\tilde{b}^d$ comes as a loss, it must be that $\tilde{b} < \alpha_1(1 + \kappa^L)$. Hence, for there to exist a value $\tilde{b} < 0$ that falls into case 2.c and satisfies Condition B.16, we require

$$\frac{1}{\kappa^G} \left( \kappa^L - \frac{\kappa^L - \kappa^G}{\alpha_1(1 + \kappa^L)} \right) > \alpha_1(1 + \kappa^L). \quad (B.17)$$

Using the definitions of $\kappa^G$ and $\kappa^L$ and simplifying reveals that Condition B.17 is equivalent to $\lambda - 1 > \alpha_1(1 + \eta \lambda)$. In summary, there exist values of $(a, b)$ meeting the conditions of subcase 2.c for which recency fails only if $\lambda - 1 > \alpha_1(1 + \eta \lambda)$ and $\alpha_1(1 + \kappa^L) > 1$. Assuming these conditions hold, values $(a, b)$ fail to generate recency only if

$$\frac{1}{\kappa^G} \left[ \kappa^L - \frac{\kappa^L - \kappa^G}{\alpha_1(1 + \kappa^L)} \right] a < b < \alpha_1(1 + \kappa^L) a. \quad (B.18)$$

**Case 3: $b < \theta_0 < a$.** This implies $\kappa^L = \kappa^L_1 = \kappa^d_1$, $\kappa^L_2 = \kappa^G$, and $\kappa^d_2 = \kappa^L$. Hence, Condition B.13 amounts to $\kappa^L \tilde{b} + \kappa^G(\tilde{a} - \tilde{a}[1 + \kappa^L] \tilde{b}) > \kappa^G \tilde{a} + \kappa^L(\tilde{b} - \alpha_1[1 + \kappa^G] \tilde{a}) \iff -\alpha_1 \kappa^G[1 + \kappa^L] \tilde{b} > -\alpha_1 \kappa^L[1 + \kappa^G] \tilde{a}$. This condition always holds given $\tilde{a} > 0 > \tilde{b}$.

In summary, the only cases where recency could fail are 2.b and 2.c. As noted, recency holds in both of these ambiguous cases whenever $\alpha_1(1 + \kappa^L) < 1$, i.e., beliefs are convex. This completes Part 1 of the proposition. Turning to Part 2, if beliefs are not convex, then recency fails in cases 2.b and 2.c only when $\lambda > 1 + \alpha_1(1 + \eta \lambda) \equiv \hat{\lambda}$. Since beliefs are not convex in this case, $1 < \alpha_1(1 + \kappa^L) <
\[ \alpha_t(1 + \eta\lambda), \text{ implying } \tilde{\lambda} > 2. \]

**Proof of Corollary 2.**

**Proof.** The claim follows from Cases 1 and 3 of the proof of Proposition 5.

**Proof of Proposition 6.**

**Proof.** Let \( \mathcal{X} = \{x_1, \ldots, x_T\} \) be an arbitrary set of \( T \) distinct elements of \( \mathbb{R} \). Let \( S(\mathcal{X}) \) be the set of all distinct sequences formed from elements of \( \mathcal{X} \). Consider \( x \in S(\mathcal{X}) \) and let \( \hat{\theta}_T(x) \) be the misattributor’s estimate following sequence \( x \). We say \( x \) is increasing if \( x_i < x_{i+1} \) for all \( i = 1, \ldots, T - 1 \). Toward a contradiction, suppose \( x \) is not increasing but \( x = \arg\max_{x' \in S(\mathcal{X})} \hat{\theta}_T(x') \). Hence, there must exist adjacent \( x_i, x_{i+1} \) such that \( x_i > x_{i+1} \). Fix \( \hat{\theta}_{i-1} \) entering round \( i \). From Proposition 5, permuting \( x_i \) and \( x_{i+1} \) would generate a higher estimate \( \hat{\theta}_{i+1} \) than if the person experiences \((x_i, x_{i+1})\). Hence, following this permutation, the person has a higher belief entering round \( i + 2 \) than under the original sequence. Again from Proposition 5, convexity implies that each \( \hat{\theta}_{i+1} \) is increasing in \( \hat{\theta}_{i-1} \), and hence \( \hat{\theta}_T \) must increase in \( \hat{\theta}_{i+1} \). Thus permuting \( x_i \) and \( x_{i+1} \) increases \( \hat{\theta}_T \), implying a contradiction.

**Proof of Proposition 7.**

**Proof.** Suppose \( x > \theta_0 \). Since beliefs are convex, \( \hat{\theta}_1 \in [\theta_0, x] \). Convexity again implies \( \hat{\theta}_2 \in [\hat{\theta}_1, x] \). Continuing forward, it’s clear that \( \hat{\theta}_t \in [\hat{\theta}_{t-1}, x] \) for all \( t = 1, 2, \ldots \). Thus, \( \hat{\theta}_t \) is increasing toward \( x \), and therefore \( \hat{x}_t = (1 + \kappa^t)x - \kappa^t \hat{\theta}_{t-1} \) is decreasing in \( t \). Conversely, if \( x < \theta_0 \), then \( \hat{\theta}_1 \in [x, \theta_0] \). Similar to the logic above, \( \hat{\theta}_t \in [x, \hat{\theta}_{t-1}] \) for all \( t = 1, 2, \ldots \), implying that \( \hat{x}_t \) is increasing in \( t \).

**Proof of Proposition 8.**

**Proof.** Part 1. Note that \( \hat{x}_t = x_t + \kappa_t(x_t - \hat{E}_t[x_t]) = x_t + \kappa_t(x_t - \varphi \hat{x}_{t-1}) \). Thus, recursively writing \( \hat{x}_t \) in terms of \((x_1, \ldots, x_t)\) yields

\[
\begin{align*}
\hat{x}_t &= (1 + \kappa_t)x_t - \varphi \kappa_t \hat{x}_{t-1} \\
&= (1 + \kappa_t)x_t - \varphi \kappa_t ((1 + \kappa_{t-1})x_{t-1} - \varphi \kappa_{t-1} \hat{x}_{t-2}) \\
&= (1 + \kappa_t)x_t - \varphi \kappa_t (1 + \kappa_{t-1})x_{t-1} + \varphi^2 \kappa_t \kappa_{t-1} \hat{x}_{t-2} \\
&= (1 + \kappa_t)x_t - \varphi \kappa_t (1 + \kappa_{t-1})x_{t-1} + \varphi^2 \kappa_t \kappa_{t-1} (1 + \kappa_{t-1})x_{t-2} - \varphi^3 \kappa_t \kappa_{t-1} \kappa_{t-2} \hat{x}_3 \\
& \vdots \\
&= (1 + \kappa_t)x_t + \sum_{j=1}^{t-1} (-\varphi)^{t-j} \prod_{i=j+1}^{t} \kappa_i (1 + \kappa_j)x_j. \quad (B.19)
\end{align*}
\]

Hence, conditional on \((x_1, \ldots, x_{t-1})\), \( \text{Var}(\hat{E}_{t+1}[x_{t+1}]) = \varphi^2 \text{Var}((1 + \kappa_t)x_t) > \varphi^2 \text{Var}(x_t) = \text{Var}(\hat{E}_{t+1}[x_{t+1}]) \), where \( \hat{E}_{t+1}[x_{t+1}] \) denotes the rational expectation.
Part 2. Let \( d_t = \hat{s}_t - \hat{E}_t[x_l] = \hat{s}_t - \varphi \hat{s}_{t-1} \). Thus

\[
d_t = (1 + \kappa_t) x_t - \varphi \kappa_t \hat{s}_{t-1} - \varphi \hat{s}_{t-1} \\
= (1 + \kappa_t) (x_t - \varphi \hat{s}_{t-1}) \\
= (1 + \kappa_t) (\varphi x_t - \varepsilon_t - \varphi((1 + \kappa_t - 1)x_t - \varphi \kappa_{t-1} \hat{s}_{t-2})) \\
= (1 + \kappa_t) (\varepsilon_t - \varphi \kappa_{t-1} (x_{t-1} - \varphi \hat{s}_{t-2})) \\
= (1 + \kappa_t) (\varepsilon_t - \varphi \kappa_{t-1} d_{t-1}).
\]

(B.20)

Proof of Proposition 9

Proof. Convergence follows along the lines of Proposition A.1. Mean beliefs \( \hat{\theta}_t(H) \) and \( \hat{\theta}_t(L) \) both follow dynamics \( \hat{\theta}_t(a) = \hat{\theta}_{t-1}(a) + \hat{a}_t(\hat{s}_t - \hat{\theta}_{t-1}(a)) \) if event \( a \in \{H, L\} \) occurs in \( t \); else \( \hat{\theta}_t(a) = \hat{\theta}_{t-1}(a) \). Thus, \( \hat{\theta}_{\mu(t)}(a) \) follows precisely the dynamics considered in Proposition A.1 on a redefined time scale \( j^\mu(t) \) that counts periods in which event \( a \) occurs (i.e., for either \( a \in \{H, L\} \), \( j^\mu(t + 1) = j^\mu(t) + 1 \) if \( a \) occurs in \( t + 1 \); otherwise \( j^H(t + 1) = j^H(t) \)). Note, however, there are two differences between this setting and the one considered in Proposition A.1. Let \( \hat{x}_t(a) \) denote the encoded outcome in \( t \) conditional on event \( a \in \{H, L\} \) occurring in \( t \). First, \( \hat{x}_t(a) \) depends on both \( \hat{\theta}_t(H) \) and \( \hat{\theta}_t(L) \). Second, \( \hat{x}_t(a) \) depends on the person’s action \( p_t \) in \( t \). As such, we establish the convergence of this two-dimensional system of beliefs. The limiting values are given by the solution of a two-dimensional system of ODEs analogous to the one-dimensional solution described in Appendix A.

We reverify conditions A1-A4 sufficient for convergence on both dimensions \( \hat{\theta}_t(a), a \in \{H, L\} \). Clearly A1 and A2 still hold. Now consider A3. Note that \( \hat{x}_t(H) = x_t(H) + \kappa_t(x_t(H) - \hat{\theta}_{t-1}(p_t)) \) where \( \hat{\theta}_{t-1}(p_t) = p_t \hat{\theta}_{t-1}(H) + (1 - p_t) \hat{\theta}_{t-1}(L) \), meaning \( \hat{x}_t(H) - \hat{\theta}_{t-1}(H) = x_t(H) + \kappa_t(x_t(H) - \hat{\theta}_{t-1}(p_t)) - \hat{\theta}_{t-1}(H) \), and thus

\[
\hat{x}_t(H) - \hat{\theta}_{t-1}(H) = (1 + \kappa_t) x_t(H) - (1 + p_t \kappa_t) \hat{\theta}_{t-1}(H) - (1 - p_t) \kappa_t \hat{\theta}_{t-1}(L) \\
= (1 + \kappa_t) (x_t(H) - \hat{\theta}_{t-1}(H)) + (1 - p_t) \kappa_t (\hat{\theta}_{t-1}(H) - \hat{\theta}_{t-1}(L)) \\
+ (1 + p_t \kappa_t) (\theta_{t-1}(H) - \hat{\theta}_{t-1}(H)) + (1 - p_t) \kappa_t (\theta_{t-1}(L) - \hat{\theta}_{t-1}(L)),
\]

(B.21)

where \( \hat{\theta}_{t-1}(H) \) and \( \hat{\theta}_{t-1}(L) \) are the rational estimates following \( t - 1 \) rounds. Thus,

\[
\text{sup}_t \mathbb{E} [||\hat{x}_t(H) - \hat{\theta}_{t-1}(H)||^2 | \theta(H), \theta(L)] < \infty
\]

by an application of Minkowski’s Inequality and noting that the expected squared absolute value of each term in the final sum of Equation B.21 is finite, as established in the proof of Proposition A.1. The only difference between these terms and those addressed in Proposition A.1 is the presence of \( p_t \). Since this action is bounded in \([0, 1]\), the inequalities from that previous result continue to hold. An analogous argument establishes

\[
\text{sup}_t \mathbb{E} [||\hat{x}_l(L) - \hat{\theta}_{l-1}(L)||^2 | \theta(H), \theta(L)] < \infty.
\]

Turning to A4, the expected deviation function \( G \) (analogous to Equation 10) will be two dimensional. Along dimension \( a \in \{H, L\} \), \( G^a \) is the expectation of \( \hat{x}_t(a) - \hat{\theta}_{t-1}(a) \) conditional on
\(a \in \{H, L\}\) and \(\hat{\theta}_{t-1} \equiv (\hat{\theta}_{t-1}(H), \hat{\theta}_{t-1}(L))\):

\[
G^a(t, \hat{\theta}_{t-1}) = \mathbb{E}_{x_t(a)}[\hat{x}_t(a, p^*_t(t, \hat{\theta}_{t-1})) \bigg| \theta(H), \theta(L) \bigg] - \hat{\theta}_{t-1}(a),
\]

where \(\hat{x}_t(a, p^*_t(t, \hat{\theta}_{t-1}))\) is the encoded outcome given mean beliefs \(\hat{\theta}_{t-1}\) entering round \(t\) and \(p^*_t(t, \hat{\theta}_{t-1})\) is the optimal action given those beliefs. This choice of \(p_t^*\) maximizes Equation 17 given his current estimates \(\hat{\theta}_{t-1}\), so \(p_t^*\) solves

\[
\left[ \hat{\theta}_{t-1}(H) - \hat{\theta}_{t-1}(L) \right] \left( 2\eta \lambda p^*_t + 1 - \eta \lambda \right) = c'(p^*_t - p_0).
\]

Our assumptions on the cost function are relevant here: (1) \(p_0 = 1/2\) implies that the LHS of Equation B.23 is positive, so \(c'(0) = 0\) implies that \(p_t^* \in (p_0, \bar{p})\); and (2) since the LHS of Equation B.23 is linearly increasing in \(p_t^*\), \(c'\) increasing and weakly convex generically implies that \(p_t^*\) is unique. Furthermore, from Equation B.23, it is clear that \(p_t^*\) is independent of \(t\) conditional on \(\hat{\theta}_{t-1}\), so we can simply write the optimal choice in \(t\) as \(p^*(\hat{\theta}_{t-1})\). Thus

\[
G^a(t, \hat{\theta}_{t-1}) = \theta(a) + \kappa^G(1 - F^a(\hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1})))(\mathbb{E}[x_t(a)|x_t(a) \geq \hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1}))] - \hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1})))
\]

\[
+ \kappa^L F^a(\hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1})))(\mathbb{E}[x_t(a)|x_t(a) < \hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1}))] - \hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1}))) - \hat{\theta}_{t-1}(a)
\]

\[
= \theta(a) + \kappa^G \left( \theta(a) - \hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1})) \right)
\]

\[
+ (\kappa^L - \kappa^G) F(\hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1})))(\mathbb{E}[x_t(a)|x_t(a) < \hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1}))] - \hat{\theta}_{t-1}(p^*(\hat{\theta}_{t-1}))) - \hat{\theta}_{t-1}(a).
\]

(B.24)

It is clear from Equation B.23 that \(p^*(\hat{\theta}_{t-1})\) is continuous in \(\hat{\theta}_{t-1}(a)\) and \(\hat{\theta}_{t-1}(p^*)\) is continuous in \(p^*\). Hence, \(G^a(t, \hat{\theta}_{t-1})\) is continuous in \(\hat{\theta}_{t-1}(a)\) for each \(a \in \{H, L\}\), as required. Finally, following the verification of A4 in Appendix A, we let the sequence of random variables \((\gamma_{j(t)}^a)\) be again defined by \(\gamma_{j(t)}^a = G^a(j(t), \hat{\theta}_{j(t)-1}) - G_{\infty}^a(\hat{\theta}_{j(t)-1})\), where \(j(t)\) counts the number of times event \(a\) occurs through \(t\) rounds and \(G_{\infty}^a(\hat{\theta}) = \lim_{t \to \infty} G^a(j(t), \hat{\theta})\). Since the time period is irrelevant for \(G^a(j, \hat{\theta}_{j-1})\) conditional on \(\hat{\theta}_{j-1}\), the condition \(\sum_{j(t)=1}^{\infty} |\hat{\theta}_{j(t)}| |\gamma_{j(t)}^a| < \infty\) holds for the sequence \(j(t)\) as in A4. The limiting beliefs \(\hat{\theta} = (\hat{\theta}(H), \hat{\theta}(L))\) hence satisfy

\[
\begin{bmatrix}
G_{\infty}^H(\hat{\theta}(H), \hat{\theta}(L)) \\
G_{\infty}^L(\hat{\theta}(H), \hat{\theta}(L))
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

For simplicity, we focus on the case where \(\sigma \to 0\). From Equation B.24, \(G_{\infty}^H\) and \(G_{\infty}^L\) reduce in this case to

\[
G_{\infty}^H(\hat{\theta}) = \theta(H) + \kappa^G(\hat{\theta}(H) - \hat{\theta}(p^*(\hat{\theta}))) - \hat{\theta}(H)
\]

(B.25)

\[
G_{\infty}^L(\hat{\theta}) = \theta(L) + \kappa^L(\hat{\theta}(L) - \hat{\theta}(p^*(\hat{\theta}))) - \hat{\theta}(L).
\]

(B.26)
Fixing $p^*$, the solution to this system is given by Equation 8 in Section 3. At this solution, $\hat{\theta}(H) - \hat{\theta}(L) > \theta(H) - \theta(L)$ regardless of $p^*$. Thus, Equation B.23 implies that the long-run action under misattribution exceeds the full-information optimal action whenever the full-information action is interior. Otherwise, the misattributor settles on the highest possible action, which coincides with the full-information action ($p^* = \bar{p}$).

**Proof of Proposition 10.**

*Proof.* We begin by deriving the wage and effort strategies of the principal and agent, respectively, assuming that the principal suffers misattribution and that the agent is aware of this. Assuming the principal presumes common knowledge of rationality, she believes the agent follows the Bayesian Nash Equilibrium strategy that he would play when facing a fully rational principal. In each round $t$, let $\hat{e}_t$ and $\hat{\theta}_{t-1}$ denote the principal’s expectation of the agent’s effort and ability, respectively, under this false presumption that the agent follows the rational Bayesian Nash Equilibrium of the game. The principal best responds to these beliefs by offering a wage in each round $t$ equal to her misspecified updating rule, and accordingly best responds these distorted beliefs. Thus, following Holmström’s (1999) model, the agent’s effort maximizes the sum of his discounted expected utility given discount factor $\delta \in (0, 1]$. In period $t$, he faces an objective function

$$U_t \equiv \sum_{\tau=t}^{T} \delta^{\tau-t} [w_{\tau} - c(e_{\tau})] = \sum_{\tau=t}^{T} \delta^{\tau-t} [\hat{\theta}_{t-1} + \hat{e}_t - c(e_{\tau})].$$

To derive the optimal effort, we can isolate the part of $U_t$ that depends on $e_t$: since we assume the Gaussian setup from Section 5, Lemma 1 implies $\hat{\theta}_{\tau} = \alpha_{\tau} \sum_{k=1}^{\tau} \xi_{k} (x_{k} - \hat{e}_{k})$, where $\xi_{\tau} = (1 + \kappa^G)$ and $\xi_{k} = (1 + \kappa^G) \prod_{j=k}^{\tau-1} [1 - \kappa^G \alpha_{j}]$ for all $k < \tau$, which yields the objective

$$-c(e_t) + \sum_{\tau=t+1}^{T} \delta^{\tau-t} \alpha_{\tau-1} \xi_{\tau} e_t = -c(e_t) + e_t (1 + \kappa^G) \left\{ \delta \alpha_t + \sum_{\tau=t+2}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right) \right\}.$$ 

Letting $c_e(\cdot)$ denote the first derivative of $c(\cdot)$, the optimal effort in period $t$ is thus $e_t^* \equiv c_e^{-1}(M_t)$, where

$$M_t \equiv (1 + \kappa^G) \left\{ \delta \alpha_t + \sum_{\tau=t+2}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right) \right\}. \tag{B.28}$$

Given this derivation of the agent’s optimal effort path, we can now show the result of Proposition 10, which compares $e_t^*$ to the effort provided in the alternative case where the principal is fully rational (and this was common knowledge). In this fully rational case, the agent’s optimal effort in period $t$ is $e_t^* \equiv c_e^{-1}(M_t')$, where $M_t' \equiv \sum_{\tau=t+1}^{T} \delta^{\tau-t} \alpha_{\tau-1}$. Because $c$ is strictly convex, it follows that $e_t^* < e_t^* \Leftrightarrow M_t < M_t'$. Notice that one can write $M_t$ in terms of $M_t'$ as follows:

$$M_t = M_t' + \kappa^G \delta \alpha_t - \sum_{\tau=t+2}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( 1 - (1 + \kappa^G) \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right). \tag{B.29}$$

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It’s clear that $M_t > M'_t$ for $t = T - 1$. Furthermore, since $\prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j]$ decreases to 0 as $\tau - 2 - t$ grows large, sufficiently large $T$ implies there exists a period $t^* < T$ such that $M_t < M'_t$ for $t < t^*$. If $T$ is not sufficiently large, then $t^* = 1$. More formally, let $D_t \equiv M_t - M'_t$. Equation B.29 implies

$$D_t = \kappa^G \delta \alpha_t - \delta^2 \alpha_{t+1} [1 - (1 + \kappa^G)(1 - \kappa^G \alpha_t)] - \sum_{\tau=t+3}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left(1 - (1 + \kappa^G) [1 - \kappa^G \alpha_t] \prod_{j=t+1}^{\tau-2} [1 - \kappa^G \alpha_j]\right) \quad (B.30)$$

Hence $D_t = \delta(D_{t+1} + \kappa^G \alpha_t [1 - M_{t+1}])$. Thus, $D_t > 0$ implies $D_{t+1} > 0$ so long as $D_{t+1} > -\kappa^G \alpha_t [1 - M_{t+1}] = -\kappa^G \alpha_t [1 - D_{t+1} - M'_t]$, which is equivalent to $[1 - \kappa^G \alpha_t] D_{t+1} > -\kappa^G \alpha_t [1 - M'_t]$. By convexity of beliefs, $[1 - \kappa^G \alpha_t] > 0$, so the preceding inequality holds so long as $M'_t$ is sufficiently small. Since $M'_t$ decreases in $t$ to a value strictly less than one, there exists a $\bar{t}$ such that $M'_t < 1$ for $t \geq \bar{t}$. Thus, $D_{t+1}$ remains positive for $t$ sufficiently large.

**Proof of Corollary 3.**

Proof. The existence of $\bar{T}$ follows from the proof of Proposition 10, where we establish that there exists a value $t^*$ such that $M_t < M'_t$ whenever $t < t^*$ and $t^* > 1$ when $T$ is sufficiently large. Suppose $T > \bar{T}$ and let $M_1(T)$ and $M'_1(T)$ denote the biased and rational marginal benefit of effort in period 1 as a function of the horizon, $T$. From Equation B.29, $M'_1(T + 1) - M_1(T + 1) > M'_1(T) - M_1(T)$ if and only if

$$\delta^T \alpha_{T-1} \left[1 - (1 - \kappa^G) \prod_{j=1}^{T-1} [1 - \kappa^G \alpha_j]\right] > 0, \quad (B.31)$$

which holds iff $(1 - \kappa^G) \prod_{j=1}^{T-1} [1 - \kappa^G \alpha_j] < 1$. If this condition fails, then $M_1(T) > M'_1(T)$, contradicting $T > \bar{T}$.

**C**  Misattribution with Multiple Dimensions

In this section, we extend our model of misattribution to settings where consumption utility is multidimensional. This extension requires an additional assumption on how surprises along one dimension influence encoded outcomes on other dimensions. While there are a range of plausible assumptions, we propose a specific model to close this degree of freedom and to provide a starting point for potential empirical exploration. Namely, we assume that the encoded outcome on one dimension depends entirely on sensations of elation or disappointment felt on that dimension—it is independent of the surprise experienced along any other dimension.

Following KR’s multidimensional model, suppose consumption vector $c \in \mathbb{R}^K$ generates consumption utility $x$ that is additively separable across $K$ dimensions; that is, $x = (x_1, \ldots, x^K)$ where $x_k \in \mathbb{R}$ denotes consumption utility on dimension $k$. Let $F$ denote the decision maker’s joint c.d.f. over $x$. The person’s total utility from $x$ given reference distribution $F$ is then $u(x|F) = \sum_{k=1}^{K} u_k(x^k|F)$, where $u_k(x|F) \equiv x^k + \eta n(x^k|F)$ is the total utility along dimension $k$ and $n(x^k|F)$ is the unidimensional
gain-loss utility from the main text (Equation 3) on dimension $k$:

$$n(x^k|F) = \begin{cases} 
    x^k - \mathbb{E}_F[x^k] & \text{if } x \geq \mathbb{E}_F[x^k] \\
    \lambda (x^k - \mathbb{E}_F[x^k]) & \text{if } x < \mathbb{E}_F[x^k].
\end{cases} \quad (C.1)$$

Our notion of misattribution generally extends to this setting: following outcome $x$ and total utility level $u = u(x|F)$, a misattributor encodes a distorted value $\hat{x}$ that would have generated the same total utility level $u$ if she instead held a utility function $\hat{u}(\cdot|F)$ that weights each $n(\cdot|F)$ by $\hat{\eta} \in [0, \eta]$. That is, the person encodes $\hat{x}$ that solves $\hat{u}(\hat{x}|F) = u(\hat{x}|F)$. To further pin down the misencoded outcome on each dimension, we assume that each $\hat{x}^k$ depends solely on gains and losses experienced on dimension $k$: $\hat{x}^k$ is defined by $\hat{u}_k(\hat{x}^k|F) = \hat{x}^k + \eta n(\hat{x}^k|F) = x^k + \eta n(x^k|F) = u_k(x^k|F)$, so $\hat{x}^k = \hat{u}_k^{-1}(u_k|F)$. While we suspect that the more general psychology of “attribution bias” may lead to across-dimension misencoding (e.g., Haggag and Pope 2018), we believe this formulation provides a tractable stepping stone for empiricists.

**D Stochastic Reference Points**

In this section, we consider misattribution under an alternative specification of the reference point. Here, we assume the stochastic specification formalized by Kőszegi and Rabin (2006). Suppose the decision maker believes that outcomes are distributed according to $F$. In KR’s model, the sense of gain or loss derives from comparing outcome $x$ with each counterfactual outcome that was possible under $F$. Thus, gain-loss utility is no longer given by Equation 3, but instead defined as:

$$n(x|F) = \int_{\hat{x} < x} (x - \hat{x}) \, dF(\hat{x}) + \lambda \int_{\hat{x} \geq x} (x - \hat{x}) \, dF(\hat{x}), \quad (D.1)$$

meaning outcome $x$ is compared against each hypothetical outcome and this comparison is weighted by the probability of that hypothetical outcome. With a stochastic reference point, the encoded outcome $\hat{x}$ following true outcome $x$ is defined exactly as in the main text (Equation 4) aside from this alternative specification of $n(\cdot|F)$. Since $n(x|F)$ in Equation D.1 is strictly increasing in $x$ conditional on $F$, the encoded outcome $\hat{x}$ is still well defined and unique.

Many of the long-run results presented in Section 4 extend to stochastic reference points (Equation D.1), albeit with the loss of tractability. Here, we present results analogous to Proposition 1 for the case of stochastic reference points assuming normally-distributed outcomes. Similar to Proposition 1, a loss-averse misattributor underestimates the mean and overestimates the variance of an action’s outcomes. Additionally, she underestimates the mean in proportion to the true variance in outcomes, again implying a bias against risky actions.

Consider a misattributor with a stochastic reference point (Equation D.1) who is learning the parameter values of an action with normally-distributed outcomes. In truth, suppose that $x_t \sim \mathcal{N}(\theta, \sigma^2)$. Throughout, we let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard-normal c.d.f. and p.d.f., respectively. The following lemma specifies the gain-loss utility function (Equation D.1) assuming the person believes $x$ is normally distributed. When the person believes $x \sim \mathcal{N}(\hat{\theta}, \hat{\sigma}^2)$, we denote gain-loss utility by

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49 If either $\lambda = 1$ or if the reference distribution $F$ is degenerate, then this stochastic reference-point model reduces to the model used in the main text (Equation 3).
Lemma D.1. Suppose that the person believes \( x \sim \mathcal{N}(\hat{\theta}, \hat{\sigma}^2) \). With a stochastic reference point,

\[
n(x|\hat{\theta}, \hat{\sigma}) = \hat{\sigma} \lambda z - \hat{\sigma}(\lambda - 1) [z \Phi(z) + \phi(z)],
\]

where \( z = (x - \hat{\theta}) / \hat{\sigma} \).

We now assess how beliefs about \( \theta \), denoted \( \hat{\theta}_t \), evolve over time when a misattributor faces outcomes \( x_t \). Relative to Section 4, stochastic reference points complicate the analysis. With deterministic reference points, the person’s perceived distribution in the steady state is entirely specified by her perceived mean outcome. This is because her reference point depends solely on this single moment of the outcome distribution. With stochastic reference points, however, gain-loss utility depends on the entire perceived distribution. As such, solving for the steady-state perception with stochastic reference points in general would entail finding a “fixed point” in the space of distributions: the steady-state distribution is such that, if believed, the person’s encoded outcomes follow that distribution. We leave a full treatment of this case to future work and focus here on the normal-distribution model where the agent fits parameters \( \theta \) and \( \sigma \) to the long-run distribution of perceived outcomes.

What beliefs does a misattributor reach when she is uncertain about both \( \theta \) and \( \sigma \)? Unlike the case with deterministic reference points, stochastic reference points imply that perceptions of \( \theta \) depend on perceptions of \( \sigma \). This implies that the steady-state perceptions, \( \hat{\theta} \) and \( \hat{\sigma} \), must satisfy the following system of equations:

\[
\begin{align*}
\hat{\theta} &= \mathbb{E}[\hat{x}|\hat{\theta}, \hat{\sigma}] & (D.2) \\
\hat{\sigma}^2 &= \text{Var}[\hat{x}|\hat{\theta}, \hat{\sigma}], & (D.3)
\end{align*}
\]

where \( \mathbb{E}[\cdot|\hat{\theta}, \hat{\sigma}] \) and \( \text{Var}[\cdot|\hat{\theta}, \hat{\sigma}] \) are with respect to the true data generating process fixing the person’s perceptions \( \hat{\theta} \) and \( \hat{\sigma} \). The remainder of this section solves these steady-state equations and shows that the solution is such that a misattributor overestimates \( \sigma \) and underestimates \( \theta \) in proportion to \( \hat{\sigma} \)—her perception of \( \sigma \).

We first derive the first-moment condition, Equation D.2. To simplify matters, we focus on the case where \( \hat{\eta} = 0 \). Hence, \( \hat{x} = x + \eta n(x|\hat{\theta}, \hat{\sigma}) \), where \( n(\cdot|\hat{\theta}, \hat{\sigma}) \) is given in Lemma D.1. Let \( z \equiv (x - \hat{\theta}) / \hat{\sigma} \) and \( \bar{z} \equiv (\theta - \hat{\theta}) / \hat{\sigma} \). The expectation of \( \hat{x} \) with respect to the true distribution, which has density \( \frac{1}{\hat{\sigma}} \phi \left( \frac{x - \theta}{\sigma} \right) \), is

\[
\mathbb{E}[x + \eta n(x|\hat{\theta}, \hat{\sigma})] = \theta + \hat{\sigma} \eta \left( \lambda \bar{z} - (\lambda - 1) \left[ \int z \Phi(z) \frac{1}{\sigma} \phi((x - \theta)/\sigma) dx + \int \phi(z) \frac{1}{\sigma} \phi((x - \theta)/\sigma) dx \right] \right),
\]

While many of our punchlines from Section 4 extend with stochastic reference points, our result that average experienced utility exceeds hypothetical utility (Proposition 2) does not necessarily extend. With stochastic reference points, realized utility depends explicitly on the agent’s perceived variance. Fixing the true variance in outcomes, a person who faces normally-distributed outcomes receives a lower utility on average when she anticipates exaggerated variance. This force counters the increase in average experienced utility earned from additional “pleasant surprises” generated by overly-pessimistic beliefs about the mean.
Let \( w = \frac{x-\theta}{\sigma} \), which implies \( z = a + bw \) where \( a = \hat{\theta} = \frac{\theta - \hat{\theta}}{\sigma} \) and \( b = \frac{\sigma}{\sigma} \). Hence,

\[
\mathbb{E}[x + \eta n(x|\hat{\theta}, \hat{\sigma})] = \theta + \hat{\sigma} \eta \left( \lambda a - (\lambda - 1) \left[ \int (a + bw) \Phi(a + bw) \phi(w) dw + \int \phi(a + bw) \phi(w) dw \right] \right)
\]

Thus,

\[
\mathbb{E}[x + \eta n(x|\hat{\theta}, \hat{\sigma})] = \theta + \hat{\sigma} \eta \left( \lambda a - (\lambda - 1) [aI_1 + bI_2 + I_3] \right), \tag{D.4}
\]

where

\[
I_1 \equiv \int \Phi(a + bw) \phi(w) dw = \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right), \tag{D.5}
\]

\[
I_2 \equiv \int w \Phi(a + bw) \phi(w) dw = \frac{b}{\sqrt{1 + b^2}} \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right), \tag{D.6}
\]

\[
I_3 \equiv \int \phi(a + bw) \phi(w) dw = \frac{1}{\sqrt{1 + b^2}} \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right). \tag{D.7}
\]

Hence, the first equation of the steady-state system, Equation D.2, amounts to

\[
0 = a + \eta \left\{ \lambda a - (\lambda - 1) \left[ a \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) + \sqrt{1 + b^2} \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \right] \right\} \tag{D.8}
\]

We now turn to the second-moment equation of the steady-state system, Equation D.3. Note that

\[
\text{Var}[x|\hat{\theta}, \hat{\sigma}] = \mathbb{E}[x^2] - \mathbb{E}[x]^2, \quad \text{where} \quad \mathbb{E}[x^2] = \sigma^2 + \theta^2 + 2\eta \mathbb{E}[xn(x|\hat{\theta}, \hat{\sigma})] + \eta^2 \mathbb{E}[n(x|\hat{\theta}, \hat{\sigma})^2].
\]

We derive \( \mathbb{E}[n(x|\hat{\theta}, \hat{\sigma})^2] \) and \( \mathbb{E}[xn(x|\hat{\theta}, \hat{\sigma})] \) in turn (\( \mathbb{E}[x] \) is derived above). From Lemma D.1,

\[
n(x|\hat{\theta}, \hat{\sigma})^2 = \hat{\sigma}^2 \{ \lambda^2 z^2 - [2\lambda (\lambda - 1)] (z^2 \Phi(z) + z \phi(z)) \}
\]

\[
+ (\lambda - 1)^2 (z^2 \Phi(z)^2 + 2z \Phi(z) \phi(z) + \phi(z)^2). \tag{D.9}
\]

We now take the expectation of each of these terms with respect to the true distribution. To do so, we first rewrite \( \mathbb{E}[n(x|\hat{\theta}, \hat{\sigma})^2] \) in terms of several Gaussian integrals, and then evaluate those integrals.

\[
\mathbb{E}[n(x|\hat{\theta}, \hat{\sigma})^2] = \hat{\sigma}^2 \{ \lambda^2 (a^2 + b^2 I_4) - [2\lambda (\lambda - 1)] (a^2 I_1 + 2ab I_2 + b^2 I_5 + a I_3 + b I_6) \}
\]

\[
+ (\lambda - 1)^2 (a^2 I_{10} + 2ab I_{11} + b^2 I_{12} + 2(a I_8 + b I_9) + I_7) \} \tag{D.10}
\]

where

\[
I_1 = \int \Phi(a + bw) \phi(w) dw \quad I_2 = \int w \Phi(a + bw) \phi(w) dw \quad I_3 = \int \phi(a + bw) \phi(w) dw
\]

\[
I_4 = \int w^2 \phi(w) dw \quad I_5 = \int w \phi(a + bw) \phi(w) dw \quad I_6 = \int w^2 \Phi(a + bw) \phi(w) dw
\]

\[
I_7 = \int \phi(a + bw)^2 \phi(w) dw \quad I_8 = \int \Phi(a + bw) \phi(a + bw) \phi(w) dw
\]

\[
I_9 = \int w \Phi(a + bw) \phi(a + bw) \phi(w) dw \quad I_{10} = \int \phi(a + bw)^2 \phi(w) dw
\]

\[
I_{11} = \int w \Phi(a + bw)^2 \phi(w) dw \quad I_{12} = \int w^2 \Phi(a + bw)^2 \phi(w) dw.
\]

We now evaluate each of these integral terms. Note that \( I_1, I_2, \) and \( I_3 \) were derived above, and \( I_4 = \mathbb{E}[w^2] = \sigma^2 - \mathbb{E}[x]^2 = 1 \). We now turn to the remaining terms:
Letting \( f(\cdot|m, s) \) denote a generic normal p.d.f. with mean \( m \) and standard deviation \( s \), the following identity will be useful:
\[
f(w|m_1, s_1)f(w|m_2, s_2) = Sf(w|\bar{m}, \bar{s})
\]
(D.11)
where
\[
\bar{m} = \frac{m_1s_2 + m_2s_1}{s_1^2 + s_2^2} \quad \text{and} \quad \bar{s} = \sqrt{\frac{s_1^2s_2^2}{s_1^2 + s_2^2}},
\]
and \( S \equiv f(m_1m_2, \sqrt{s_1^2 + s_2^2}) \) is a scaling factor. Using this identity with \( \bar{m} = -a/b \left( \frac{1}{b^2} + 1 \right) \),
\[
I_5 = \frac{1}{b} \frac{1}{\sqrt{1 + \frac{1}{b^2}}} \phi \left( -\frac{a/b}{\sqrt{1 + \frac{1}{b^2}}} \right) \int w f(w|\bar{m}, \bar{s})dw = \frac{1}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) -ab \left( \frac{1}{1 + b^2} \right),
\]
(D.12)
which follows from the fact that \( \phi \) is symmetric: \( \phi(-y) = \phi(y) \).

Using integration by parts,
\[
I_6 = \int w^2 \Phi(a + bw) \phi(w) dw = \int w \Phi(a+bw) [w \phi(w)] dw
\]
\[= -w \Phi(a+bw) \phi(w) \bigg|_{-\infty}^{\infty} + \int \phi(w) [bw \phi(a+bw) + \Phi(a+bw)] dw
\]
\[= b \int \phi(a+bw) \phi(w) dw + \int \Phi(a+bw) \phi(w) dw \quad \text{(D.13)}
\]
\[= bI_5 + I_1 \quad \text{(D.14)}
\]

Using our identity for the product of two normal densities above (Equation D.11), \( \phi(z)^2 = Sf(z|\bar{m}, \bar{s}) \), where \( \bar{m} = 0, \bar{s} = 1/\sqrt{2} \) and \( S = \phi(0)/\sqrt{2} \). Hence, \( I_7 = \phi(0) \int \phi(\bar{a} + \bar{b}z) \phi(w) dw \), where \( \bar{a} = \sqrt{2}a \) and \( \bar{b} = \sqrt{2}b \). Thus, using the derivation of \( I_5 \):
\[
I_7 = \frac{\phi(0)}{\sqrt{1 + b^2}} \phi \left( \frac{\bar{a}}{\sqrt{1 + b^2}} \right) = \frac{\phi(0)}{\sqrt{1 + 2b^2}} \phi \left( \frac{\sqrt{2}a}{\sqrt{1 + 2b^2}} \right).
\]
(D.15)

\( aI_8 + bI_9 = \int (a+bw) \Phi(a+bw) \phi(a+bw) \phi(w) dw \), which can be simplified using our formula for the product of two normal density functions: \( \phi(a+bw) \phi(w) = Sf(w|\bar{m}, \bar{s}^2) \) where \( f(\cdot|\bar{m}, \bar{s}^2) \) is the p.d.f. of a normal random variable with mean \( \bar{m} = -ab/(1 + b^2) \), standard deviation \( \bar{s} = 1/(\sqrt{1 + b^2}) \), and scaling factor \( S = 1/\sqrt{1 + b^2} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \). Thus
\[
aI_8 + bI_9 = S \left( (a+b\bar{m}) \int \Phi(a+bw) f(w|\bar{m}, \bar{s}^2) dw + b\bar{s} \int y \Phi(a+bw) f(w|\bar{m}, \bar{s}^2) dw \right),
\]
where \( y = \frac{w-\bar{m}}{\bar{s}} \). Finally, using the derivation of \( I_1 \) (Equation D.5), \( \int \Phi(a+bw) f(w|\bar{m}, \bar{s}^2) dw = \int \Phi(a'+b'y) \phi(y) dy = \Phi \left( \frac{a'}{\sqrt{1 + b'^2}} \right) \), where \( a' = a + b\bar{m} = a/(1 + b^2) \) and \( b' = b\bar{s} = b/\sqrt{1 + b^2} \). Likewise, using the derivation of \( I_2 \) (Equation D.6), \( \int y \Phi(a'+b'y) \phi(y) dy = \frac{b'}{\sqrt{1 + b'^2}} \phi \left( \frac{a'}{\sqrt{1 + b'^2}} \right) \). Combining
ing all the terms above yields

\[ aI_8 + bI_9 = \frac{1}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \left[ \frac{a}{1 + b^2} \Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) + \frac{b^2}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) \right]. \quad (D.16) \]

\[ I_{10} : \text{Note that} \]

\[ I_{10} = \int \Phi(a + bw)^2 \phi(w) dw = \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) - 2T \left( \frac{a}{\sqrt{1 + b^2}}, \frac{1}{\sqrt{1 + 2b^2}} \right), \]

where \( T(h, q) = \phi(h) \int_0^q \frac{\phi(hx)}{1 + x^2} dx \) is Owen’s T function.

\[ I_{11} : \text{Note that} \]

\[ I_{11} = \int w\Phi(a + bw)^2 \phi(w) dw = \frac{2b}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right). \]

\[ I_{12} : \text{Integration by parts yields} \]

\[ I_{12} = \int [\Phi(a + bw)^2 + 2bw \Phi(a + bw) \phi(a + bw)] \phi(w) dw = I_{10} + 2b \int w\Phi(a + bw) \phi(a + bw) \phi(w) dw. \quad (D.17) \]

The integral \( \int w\Phi(a + bw) \phi(a + bw) \phi(w) dw \) was calculated in the derivation of \( aI_8 + bI_9 \) above. It follows that

\[ I_{12} = I_{10} + \frac{2b^2}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \left[ \frac{-a}{1 + b^2} \Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) + \frac{1}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) \right]. \quad (D.18) \]

Using \( I_1 \) through \( I_{12} \) derived above, tedious algebra allows us to write \( \mathbb{E}[n(x \mid \hat{\theta}, \hat{\sigma})^2] \) (Equation D.10) as \( \hat{\sigma}^2 N_2(a, b) \), where

\[ N_2(a, b) = \lambda^2(a^2 + b^2) \]

\[ - [2\lambda(1 - \lambda)] \left[ (a^2 + b^2) \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) + a \sqrt{1 + b^2} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \right] \]

\[ + (\lambda - 1)^2 \left\{ (a^2 + b^2) \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) - 2(a^2 + b^2)T \left( \frac{a}{\sqrt{1 + b^2}}, \frac{1}{\sqrt{1 + 2b^2}} \right) \right\} \]

\[ + 2(1 + b^2)S \left[ a\Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) + \frac{b^2}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) \right] \]

\[ + \frac{\phi(0)}{\sqrt{1 + 2b^2}} \phi \left( \frac{\sqrt{2a}}{\sqrt{1 + 2b^2}} \right), \quad (D.20) \]
and $S = \frac{1}{\sqrt{1+b^2}} \phi \left( \frac{a}{\sqrt{1+b^2}} \right)$. We now derive $E[xn(x|\hat{\theta}, \hat{\sigma})]$. Using Lemma D.1 and the change of variables introduced above,

$$xn(x|\hat{\theta}, \hat{\sigma}) = \hat{\sigma} \lambda (\theta a + [\theta b + \sigma a]w + \sigma bw^2) - \hat{\sigma}(\lambda - 1)(\theta a + [\theta b + \sigma a]w + \sigma bw^2)\Phi(a + bw) - \hat{\sigma}(\lambda - 1)(\theta - \sigma w)\phi(a + bw). \quad (D.21)$$

Taking expectations with respect to $w$ and using the integral identities above yields

$$E[xn(x|\hat{\theta}, \hat{\sigma})] = \hat{\sigma} \{ \lambda [\theta a + \sigma b] - (\lambda - 1)(\theta [aI_1 + bI_2 + I_3] + \sigma bI_1) \}. \quad (D.22)$$

Putting these components together,

$$\text{Var}[xn|\hat{\theta}, \hat{\sigma}] = \sigma^2 + \theta^2 + 2\eta E[xn(x|\hat{\theta}, \hat{\sigma})] + \eta^2 \hat{\sigma}^2 N_2(a,b) - (\theta + \hat{\sigma} \eta N_1(a,b))^2, \quad (D.23)$$

where the last term follows from Equation D.4 with $N_1(a,b) \equiv \lambda a - (\lambda - 1)[aI_1 + bI_2 + I_3]$. Combining Equations D.22 and D.23 yields

$$\text{Var}[xn|\hat{\theta}, \hat{\sigma}] = \sigma^2 + 2\eta \hat{\sigma} (E[xn(x|\hat{\theta}, \hat{\sigma})] - N_1(a,b)) + \eta^2 \hat{\sigma}^2 [N_2(a,b) - N_1(a,b)]$$

$$= \sigma^2 + 2\eta \sigma^2 (\lambda - (\lambda - 1)I_1) + \eta^2 \hat{\sigma}^2 [N_2(a,b) - N_1(a,b)]$$

$$= \sigma^2 + 2\eta \sigma^2 N_0(a,b) + \eta^2 \hat{\sigma}^2 [N_2(a,b) - N_1(a,b)], \quad (D.24)$$

where $N_0(a,b) \equiv \lambda - (\lambda - 1)I_1$. Thus $\hat{\sigma}^2 = \text{Var}[xn|\hat{\theta}, \hat{\sigma}]$ is equivalent to

$$0 = b^2 + 2\eta b^2 N_0(a,b) + \eta^2 [N_2(a,b) - N_1(a,b)]. \quad (D.25)$$

Finally, from Equations D.8 and D.25, $\hat{\theta}$ and $\hat{\sigma}$ are implicitly defined by the values $a$ and $b$ that solve the system

$$0 = a + \eta N_1(a,b) \quad (D.26)$$

$$0 = b^2 + 2\eta b^2 N_0(a,b) + \eta^2 [N_2(a,b) - N_1(a,b)]. \quad (D.27)$$

One can show that this system has a unique solution. Moreover, both of the equations in the system above depend solely on the p.d.f. and c.d.f. of the normal distribution and parameters $\eta$ and $\lambda$. Thus, the solution, denoted by $(a^*, b^*)$, characterizes $\hat{\theta}$ and $\hat{\sigma}$ as follows: $\hat{\sigma} = \sigma/b^*$, which implies $\hat{\theta} = \theta - a^* \hat{\sigma} = \theta - \frac{a^*}{b^*} \sigma$. Again, since $a^*$ and $b^*$ are independent of $\sigma$, the perceived mean is linearly decreasing in the true variance of outcomes.

### E Reputation Model with Loss Aversion

This section explores the career-concern model of Section 6.2 when the principal is loss averse ($\lambda > 1$). In this case, because gains and losses affect the principal’s beliefs asymmetrically, the agent faces uncertainty over whether any future outcome will be weighted as a gain or as a loss. Given that the agent’s current effort choice influences this uncertainty, his optimal strategy must account for this. Technically speaking, the marginal effect of today’s effort on future wages—$M_t$ from Equation B.28—is no longer deterministic when $\lambda > 1$: the $\kappa^G_t$ terms are replaced by the expected value of $\kappa_t$.
for $\tau \geq t$, which lies in the interval $[\kappa^G, \kappa^L]$. As argued below, this change does not alter the predicted shape of the effort profile in expectation.

Specifically, we derive the agent’s optimal effort policy when $T = 3$, highlighting that (1) it is qualitatively similar to the case without loss aversion ($\lambda = 1$) analyzed in the main text, and (2) the intuition behind Proposition 10 continues to hold. In particular, in the three-period model, $e_1$ will fall below the rational benchmark, while $e_2$ exceeds it. Finally, we argue that these qualitative similarities between the $\lambda = 1$ and $\lambda > 1$ case hold more generally for any $T \geq 3$.

Suppose $T = 3$, and for ease of exposition, let $\delta = 1$. Since $e_3^s = 0$, the agent initially seeks to maximize $\Pi_1 \equiv \mathbb{E}_{x_1} [\hat{\theta}_1 - c(e_1) + \mathbb{E}_{(x_1,x_2)} [\hat{\theta}_2 - c(e_2)]]$. Note that $\hat{\theta}_2 = \hat{\theta}_1 + \alpha_2(1 + \kappa_2)[x_2 - \hat{e}_2 - \hat{\theta}_1] = \hat{\theta}_1 + \alpha_2(1 + \kappa_2)d_2$, where $d_2 \equiv x_2 - \hat{e}_2 - \hat{\theta}_1$. Hence, $\Pi_1 = \mathbb{E}_{(x_1,x_2)} [2\hat{\theta}_1 + \alpha_2(1 + \kappa_2)d_2 - c(e_2)] - c(e_1)$. Because $e_2$ is a function of $e_1$, optimizing $\Pi_1$ with respect to $e_1$ requires that we first derive how $e_2$ depends on $e_1$ at the optimum.

The period-2 objective is $\Pi_2 = \mathbb{E}_{x_2} [\hat{\theta}_2 - c(e_2)] = \hat{\theta}_1 + \alpha_2 \mathbb{E}_{x_2} [(1 + \kappa_2)d_2|x_1] - c(e_2)$, implying $e_2$ must satisfy $c'(e_2) = \alpha_2 \frac{\partial}{\partial \hat{e}_2} \mathbb{E}_{x_2} [(1 + \kappa_2)d_2|x_1]$. Note that $d_2 \sim \mathcal{N}(\hat{\theta}_1 + e_2 - \hat{\theta}_1 - \hat{\theta}_1 - \hat{\theta}_1 - \hat{\theta}_1)$, where $\hat{\theta}_1$ is the rational estimate of $\theta$ following $x_1$ and $\sigma^2$ is the variance of $\hat{\theta}_1 + e_2$, which is independent of $e_1$. Let $p_2 \equiv \Pr(d_2 > 0|x_1) = 1 - \Phi\left(-\frac{\bar{d}_2}{\sigma_1}\right)$, where $\Phi$ is the standard normal c.d.f. and $\bar{d}_2 = \mathbb{E}[d_2|x_1]$. Thus, $\mathbb{E}_{x_2} [(1 + \kappa_2)d_2|x_1] = \bar{d}_2 + p_2 \kappa^G \mathbb{E}_{x_2} [d_2|x_2 > 0, x_1] + (1 - p_2) \kappa^G \mathbb{E}_{x_2} [-d_2|x_2 < 0, x_1]$. Since $\mathbb{E}_{x_2} [d_2|x_2 > 0] = \bar{d}_2 + \sigma_1 \phi\left(-\frac{\bar{d}_2}{\sigma_1}\right)/p_2$ and $\mathbb{E}_{x_2} [d_2|x_2 < 0] = \bar{d}_2 - \sigma_1 \phi\left(-\frac{\bar{d}_2}{\sigma_1}\right)/(1 - p_2)$, it follows that $\mathbb{E}_{x_2} [(1 + \kappa_2)d_2|x_1] = [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \bar{d}_2 - (\kappa^L - \kappa^G) \sigma_1 \phi\left(-\frac{\bar{d}_2}{\sigma_1}\right)$. This implies

$$
\frac{\partial}{\partial e_2} \mathbb{E}_{x_2} [(1 + \kappa_2)d_2|x_1] = - (\kappa^L - \kappa^G) \frac{\partial p_2}{\partial e_2} \bar{d}_2 + [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \frac{\partial \bar{d}_2}{\partial e_2} - (\kappa^L - \kappa^G) \sigma_1 \phi\left(-\frac{\bar{d}_2}{\sigma_1}\right) \left(\frac{\bar{d}_2}{\sigma_1}\right) \left(\frac{\sigma_1}{\sigma_1}\right) \frac{\partial \sigma_1}{\partial e_2}.
$$

(E.1)

From the definition of $p_2$, $\frac{\partial p_2}{\partial e_2} = \phi\left(-\frac{\bar{d}_2}{\sigma_1}\right) \left(\frac{1}{\sigma_1}\right) \frac{\partial \bar{d}_2}{\partial e_2}$, and since $\frac{\partial \bar{d}_2}{\partial e_2} = 1$, Equation E.1 reduces to

$$
c'(e_2) = \alpha_2 [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] - (\kappa^L - \kappa^G) \sigma_1 \phi\left(-\frac{\bar{d}_2}{\sigma_1}\right) \left(\frac{\sigma_1}{\sigma_1}\right) \frac{\partial \sigma_1}{\partial e_2}.
$$

(E.2)

Returning to the optimal effort choice in period 1, note that $e_1$ must satisfy first-order condition

$$
\mathbb{E}_{x_1} \left[2 \frac{\partial \hat{\theta}_1}{\partial e_1} + \frac{\partial}{\partial e_1} \mathbb{E}_{x_2} [\alpha_2 (1 + \kappa_2)d_2 - c(e_2)|x_1] \right] = c'(e_1).
$$

(E.3)

Analogous to the derivation of Equation E.1, $\frac{\partial}{\partial e_1} \mathbb{E}_{x_2} [(1 + \kappa_2)d_2|x_1] = [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \frac{\partial \bar{d}_2}{\partial e_1}$, where $\frac{\partial \bar{d}_2}{\partial e_1} = \frac{\partial e_2}{\partial e_1} - \frac{\partial \hat{\theta}_1}{\partial e_1}$, given that $\bar{d}_2 = \hat{\theta}_1 + e_2 - \hat{\theta}_1 - \hat{\theta}_1$. Hence, the first-order condition for $e_1$ amounts to

$$
\mathbb{E}_{x_1} \left[2 \frac{\partial \hat{\theta}_1}{\partial e_1} + \alpha_2 [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \left(\frac{\partial e_2}{\partial e_1} - \frac{\partial \hat{\theta}_1}{\partial e_1}\right) - c'(e_2) \frac{\partial e_2}{\partial e_1}\right] = c'(e_1).
$$

(E.4)
Since the agent’s choice of $e_2$ conditional on $x_1$ must satisfy Equation E.2, Equation E.4 yields

$$
\mathbb{E}_{x_1} \left[ 2 \frac{\partial \hat{\theta}_1}{\partial e_1} - \alpha_2 [1 + p_2 \kappa G + (1 - p_2) \kappa L] \frac{\partial \hat{\theta}_1}{\partial e_1} \right] = c'(e_1).
$$

(E.5)

Since $\hat{\theta}_1 = \alpha_1 (1 + \kappa_1) d_1$ where $d_1 = x_1 - \hat{e}_1$, $\frac{\partial \hat{\theta}_1}{\partial e_1} = \alpha_1 (1 + \kappa_1)$. Thus, the first-order condition for $e_1$ reduces further to $c'(e_1) = \alpha_1 \mathbb{E}_{x_1} \left[ (1 + \kappa_1) (2 - \alpha_2 - \alpha_2 [p_2 \kappa G + (1 - p_2) \kappa L]) \right] = \mathbb{E}_{x_1} \left[ (1 + \kappa_1) (\alpha_1 + \alpha_2 - \alpha_1 \alpha_2 [p_2 \kappa G + (1 - p_2) \kappa L]) \right]$, where the second equality follows from the fact that $\alpha_1 (1 - \alpha_2) = \alpha_2$. Note that $[p_2 \kappa G + (1 - p_2) \kappa L] = \mathbb{E} [\kappa_2 | x_1]$ given the optimal policy. Hence, the first-order condition for $e_1$ can be written as $c'(e_1) = \mathbb{E}_{x_1, x_2} \left[ ((1 + \kappa_1) (\alpha_1 + \alpha_2 [1 - \kappa_2 \alpha_1]) \right]$. This first-order condition is equivalent to the one with $\lambda = 1$ (see Equation B.28) aside from the expected value over $\kappa_t$ on the right-hand side: with $\lambda = 1$, each $\kappa_t = \kappa^G$ deterministically.

Given the similarity between the solutions with $\lambda = 1$ and $\lambda > 1$, the predictions of Proposition 10 continue to hold with $\lambda > 1$. We can see this directly in the analysis above. The first-order condition for $e_2$ requires that $c'(e_2) = \alpha_2 [1 + p_2 \kappa G + (1 - p_2) \kappa L]$. Rational effort, however, solves $c'(e_2) = \alpha_2$. Since $\alpha_2 [1 + p_2 \kappa G + (1 - p_2) \kappa L] > \alpha_2$, second-round effort under misattribution exceeds the rational benchmark. Contrastingly, first-round effort may fall short: rational effort solves $c'(e_1) = \alpha_1 + \alpha_2$, while effort under misattribution solves $c'(e_1) = \mathbb{E}_{x_1, x_2} \left[ ((1 + \kappa_1) (\alpha_1 + \alpha_2 [1 - \kappa_2 \alpha_1]) \right]$.

One could continue this backward induction argument for arbitrary $T$ and show that, in general, $e_t = c_e^{-1}(\mathbb{E}[M_t | e_t, h_{t-1}])$, where $M_t \equiv (1 + \kappa_t) \left\{ \delta \alpha_t + \sum_{\tau=t+2}^T \delta^{\tau-t} \alpha_{\tau-1} \left( \prod_{j=t}^{\tau-2} [1 - \kappa_j \alpha_j] \right) \right\}$ and $\mathbb{E}[M_t | e_t, h_{t-1}]$ is the expected value of $M_t$ conditional on the history, the agent’s current choice, and her policy going forward. Recall that when $\lambda = 1$, $e_t = c_e^{-1}(M_t)$, where $M_t = (1 + \kappa^G) \left\{ \delta \alpha_t + \sum_{\tau=t+2}^T \delta^{\tau-t} \alpha_{\tau-1} \left( \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right) \right\}$ (see Equation B.28). Hence, the key difference between the solution with $\lambda > 1$ and the one with $\lambda = 1$ is that the expected values of $\kappa_t$ in $M_t$ will fall in the interval $[\kappa^G, \kappa^L]$ rather than remain constant at $\kappa^G$ deterministically. This change does not alter the qualitative pattern of the effort profile in expectation.