Gov 2000 Section 1: Introduction to Probability

Mayya Komisarchik

Harvard University

September 8, 2016

These slides are based on Matt Blackwell's 2015 probability lecture slides.

Announcements

- Lecture Room Change: We are now in the Belfer Case Study Room in the South Building (S020)
- Problem Set 0 is Posted.
  - When is it due?
    - **Thursday, September 15th, by 6:00pm**
  - Where can I find it?
    - Look in the "Assignments" tab of the Gov 2000 page in Canvas
  - What should I submit?
    - A beautifully formatted writeup with your solutions in .pdf format
    - A working R script file (.R) that actually produces the solutions in your writeup
    - Rmarkdown (.rmd) files are fine too; we've set Canvas up to accept those.
- TF Office Hours
  - Mayya: Wednesdays, 4:00pm - 6:00pm, CGIS Cafe
  - David: Tuesdays, 4:00pm - 6:00pm, CGIS Cafe

Announcements

- Lecture Room Change: We are now in the Belfer Case Study Room in the South Building (S020)
- Problem Set 0 is Posted.
  - When is it due?
    - **Thursday, September 15th, by 6:00pm**
  - Where can I find it?
    - Look in the "Assignments" tab of the Gov 2000 page in Canvas
  - What should I submit?
    - A beautifully formatted writeup with your solutions in .pdf format
    - A working R script file (.R) that actually produces the solutions in your writeup
    - Rmarkdown (.rmd) files are fine too; we've set Canvas up to accept those.
- TF Office Hours
  - Mayya: Wednesdays, 4:00pm - 6:00pm, CGIS Cafe
  - David: Tuesdays, 4:00pm - 6:00pm, CGIS Cafe

Intuition

- Most of us have encountered probability at some point. When we "think in probabilistic terms", we're usually imagining something like this:

$$\frac{\text{\# of possibilities I'm interested in}}{\text{total \# of possibilities}}$$

- What's the probability that I flip "heads" on a single coin toss? $\frac{1}{2}$
- What's the probability that I win the lottery? Also $\frac{1}{2}$, right? I mean you either win or you lose, am I right?

Intuition

- Most of us have encountered probability at some point. When we "think in probabilistic terms", we're usually imagining something like this:

$$\frac{\text{\# of possibilities I'm interested in}}{\text{total \# of possibilities}}$$

- What's the probability that I flip "heads" on a single coin toss? $\frac{1}{2}$
- What's the probability that I win the lottery? Also $\frac{1}{2}$, right? I mean you either win or you lose, am I right?

Intuition

- Most of us have encountered probability at some point. When we "think in probabilistic terms", we're usually imagining something like this:

$$\frac{\#\text{ of possibilities I'm interested in}}{\text{total }\#\text{ of possibilities}}$$

- What's the probability that I flip "heads" on a single coin toss? $\frac{1}{2}$
- What's the probability that I win the lottery? Also $\frac{1}{2}$, right? I mean you either win or you lose, am I right?

Intuition

- Most of us have encountered probability at some point. When we "think in probabilistic terms", we're usually imagining something like this:

$$\frac{\text{\# of possibilities I'm interested in}}{\text{total \# of possibilities}}$$

- What's the probability that I flip "heads" on a single coin toss? $\frac{1}{2}$
- What's the probability that I win the lottery? Also $\frac{1}{2}$, right? I mean you either win or you lose, am I right?

Intuition

- Most of us have encountered probability at some point. When we "think in probabilistic terms", we're usually imagining something like this:

$$\frac{\text{\# of possibilities I'm interested in}}{\text{total \# of possibilities}}$$

- What's the probability that I flip "heads" on a single coin toss? $\frac{1}{2}$
- What's the probability that I win the lottery? Also $\frac{1}{2}$, right? I mean you either win or you lose, am I right?

Intuition

- Most of us have encountered probability at some point. When we "think in probabilistic terms", we're usually imagining something like this:

$$\frac{\#\text{ of possibilities I'm interested in}}{\text{total }\#\text{ of possibilities}}$$

- What's the probability that I flip "heads" on a single coin toss? $\frac{1}{2}$
- What's the probability that I win the lottery? Also $\frac{1}{2}$, right? I mean you either win or you lose, am I right?

Intuition

- Most of us have encountered probability at some point. When we "think in probabilistic terms", we're usually imagining something like this:

$$\frac{\#\text{ of possibilities I'm interested in}}{\text{total }\#\text{ of possibilities}}$$

- What's the probability that I flip "heads" on a single coin toss? $\frac{1}{2}$
- What's the probability that I win the lottery? Also $\frac{1}{2}$, right? I mean you either win or you lose, am I right?

Counting

- The trick often rests in figuring out how to correctly count the number of possibilities in the numerator and denominator.
- Sometimes counting is easy. For instance:
  - Suppose the lottery is just a raffle. Everyone who purchases a ticket puts a duplicate with their ticket number into a giant hat. 1,000,000 people buy tickets. When the lottery closes, the tickets in the hat are thoroughly shuffled and exactly 1 is drawn. The owner of that ticket wins the jackpot. If you purchased one ticket, your probability of winning the lottery is:

$$\frac{1}{1,000,000}$$

Counting

- The trick often rests in figuring out how to correctly count the number of possibilities in the numerator and denominator.
- Sometimes counting is easy. For instance:
  - Suppose the lottery is just a raffle. Everyone who purchases a ticket puts a duplicate with their ticket number into a giant hat. 1,000,000 people buy tickets. When the lottery closes, the tickets in the hat are thoroughly shuffled and exactly 1 is drawn. The owner of that ticket wins the jackpot. If you purchased one ticket, your probability of winning the lottery is:

$$\frac{1}{1,000,000}$$

Counting

- The trick often rests in figuring out how to correctly count the number of possibilities in the numerator and denominator.
- Sometimes counting is easy. For instance:
  - Suppose the lottery is just a raffle. Everyone who purchases a ticket puts a duplicate with their ticket number into a giant hat. 1,000,000 people buy tickets. When the lottery closes, the tickets in the hat are thoroughly shuffled and exactly 1 is drawn. The owner of that ticket wins the jackpot. If you purchased one ticket, your probability of winning the lottery is:

$$\frac{1}{1,000,000}$$

Counting

- The trick often rests in figuring out how to correctly count the number of possibilities in the numerator and denominator.
- Sometimes counting is easy. For instance:
    - Suppose the lottery is just a raffle. Everyone who purchases a ticket puts a duplicate with their ticket number into a giant hat. 1,000,000 people buy tickets. When the lottery closes, the tickets in the hat are thoroughly shuffled and exactly 1 is drawn. The owner of that ticket wins the jackpot. If you purchased one ticket, your probability of winning the lottery is:

$$\frac{1}{1,000,000}$$

Counting

- The trick often rests in figuring out how to correctly count the number of possibilities in the numerator and denominator.
- Sometimes counting is easy. For instance:
  - Suppose the lottery is just a raffle. Everyone who purchases a ticket puts a duplicate with their ticket number into a giant hat. 1,000,000 people buy tickets. When the lottery closes, the tickets in the hat are thoroughly shuffled and exactly 1 is drawn. The owner of that ticket wins the jackpot. If you purchased one ticket, your probability of winning the lottery is:

$$\frac{1}{1,000,000}$$

Counting

- But sometimes counting is hard (seriously). For instance:
  - Suppose the lottery works more like the Powerball. 5 of the 6 winning numbers are drawn, without replacement, from integers 1-59. The 6th can be any integer 1-35. Winning tickets have the correct 6 numbers (first 5 in any order). If you purchased one ticket, what is your probability of winning the lottery now (even if you're not the sole winner)?
  - Well, you've got your 1 ticket so the numerator is still 1. But what's the denominator?
  - The denominator should represent **all** of the possible unique combinations of 6 numbers that could be drawn. That's equal to:

$$\frac{1}{\left(\frac{59!}{(59-5)!5!}\right)(35)} = \frac{1}{175,223,510} = \text{Yea, good luck w/ that}$$

Counting

- But sometimes counting is hard (seriously). For instance:
  - Suppose the lottery works more like the Powerball. 5 of the 6 winning numbers are drawn, without replacement, from integers 1-59. The 6th can be any integer 1-35. Winning tickets have the correct 6 numbers (first 5 in any order). If you purchased one ticket, what is your probability of winning the lottery now (even if you're not the sole winner)?
  - Well, you've got your 1 ticket so the numerator is still 1. But what's the denominator?
  - The denominator should represent **all** of the possible unique combinations of 6 numbers that could be drawn. That's equal to:

$$\frac{1}{\left(\frac{59!}{(59-5)!5!}\right)(35)} = \frac{1}{175,223,510} = \text{Yea, good luck w/ that}$$

Counting

- But sometimes counting is hard (seriously). For instance:
  - Suppose the lottery works more like the Powerball. 5 of the 6 winning numbers are drawn, without replacement, from integers 1-59. The 6th can be any integer 1-35. Winning tickets have the correct 6 numbers (first 5 in any order). If you purchased one ticket, what is your probability of winning the lottery now (even if you're not the sole winner)?
  - Well, you've got your 1 ticket so the numerator is still 1. But what's the denominator?
  - The denominator should represent **all** of the possible unique combinations of 6 numbers that could be drawn. That's equal to:

$$\frac{1}{\left(\frac{59!}{(59-5)!5!}\right)(35)} = \frac{1}{175,223,510} = \text{Yea, good luck w/ that}$$

Counting

- But sometimes counting is hard (seriously). For instance:
  - Suppose the lottery works more like the Powerball. 5 of the 6 winning numbers are drawn, without replacement, from integers 1-59. The 6th can be any integer 1-35. Winning tickets have the correct 6 numbers (first 5 in any order). If you purchased one ticket, what is your probability of winning the lottery now (even if you're not the sole winner)?
  - Well, you've got your 1 ticket so the numerator is still 1. But what's the denominator?
  - The denominator should represent **all** of the possible unique combinations of 6 numbers that could be drawn. That's equal to:

$$\frac{1}{\left(\frac{59!}{(59-5)!5!}\right)(35)} = \frac{1}{175,223,510} = \text{Yea, good luck w/ that}$$

Counting

- But sometimes counting is hard (seriously). For instance:
  - Suppose the lottery works more like the Powerball. 5 of the 6 winning numbers are drawn, without replacement, from integers 1-59. The 6th can be any integer 1-35. Winning tickets have the correct 6 numbers (first 5 in any order). If you purchased one ticket, what is your probability of winning the lottery now (even if you're not the sole winner)?
  - Well, you've got your 1 ticket so the numerator is still 1. But what's the denominator?
  - The denominator should represent **all** of the possible unique combinations of 6 numbers that could be drawn. That's equal to:

$$\frac{1}{\left(\frac{59!}{(59-5)!5!}\right)(35)} = \frac{1}{175,223,510} = \text{Yea, good luck w/ that}$$

Counting

- But sometimes counting is hard (seriously). For instance:
  - Suppose the lottery works more like the Powerball. 5 of the 6 winning numbers are drawn, without replacement, from integers 1-59. The 6th can be any integer 1-35. Winning tickets have the correct 6 numbers (first 5 in any order). If you purchased one ticket, what is your probability of winning the lottery now (even if you're not the sole winner)?
  - Well, you've got your 1 ticket so the numerator is still 1. But what's the denominator?
  - The denominator should represent **all** of the possible unique combinations of 6 numbers that could be drawn. That's equal to:

$$\frac{1}{\left(\frac{59!}{(59-5)!5!}\right)(35)} = \frac{1}{175,223,510} = \text{Yea, good luck w/ that}$$

Counting

How did we know that? Suppose you have n objects and you're trying to count ways to choose a subset of size k from those objects. Use this as a guide when counting possibilities:

	Order Matters ($aab \neq aba$)	Order Doesn't Matter ($aab = aba$)
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Where:

- $n! = n(n-1)(n-2)(n-3)...(1)$
- There are $n!$ ways to arrange n objects
- $\binom{n}{k}$ can be read as "n choose k" and indicates the formula $\frac{n!}{(n-k)!k!}$

Counting

How did we know that? Suppose you have n objects and you're trying to count ways to choose a subset of size k from those objects. Use this as a guide when counting possibilities:

	Order Matters ($aab \neq aba$)	Order Doesn't Matter ($aab = aba$)
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Where:

- $n! = n(n-1)(n-2)(n-3)...(1)$
- There are $n!$ ways to arrange n objects
- $\binom{n}{k}$ can be read as "n choose k" and indicates the formula $\frac{n!}{(n-k)!k!}$

Counting

How did we know that? Suppose you have n objects and you're trying to count ways to choose a subset of size k from those objects. Use this as a guide when counting possibilities:

	Order Matters ($aab \neq aba$)	Order Doesn't Matter ($aab = aba$)
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Where:

- $n! = n(n-1)(n-2)(n-3)...(1)$
- There are $n!$ ways to arrange n objects
- $\binom{n}{k}$ can be read as "n choose k" and indicates the formula $\frac{n!}{(n-k)!k!}$

Counting

How did we know that? Suppose you have n objects and you're trying to count ways to choose a subset of size k from those objects. Use this as a guide when counting possibilities:

	Order Matters ($aab \neq aba$)	Order Doesn't Matter ($aab = aba$)
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Where:

- $n! = n(n-1)(n-2)(n-3)...(1)$
- There are $n!$ ways to arrange n objects
- $\binom{n}{k}$ can be read as "n choose k" and indicates the formula $\frac{n!}{(n-k)!k!}$

Counting

How did we know that? Suppose you have n objects and you're trying to count ways to choose a subset of size k from those objects. Use this as a guide when counting possibilities:

	Order Matters ($aab \neq aba$)	Order Doesn't Matter ($aab = aba$)
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Where:

- $n! = n(n-1)(n-2)(n-3)...(1)$
- There are $n!$ ways to arrange n objects
- $\binom{n}{k}$ can be read as "n choose k" and indicates the formula $\frac{n!}{(n-k)!k!}$

Counting

How did we know that? Suppose you have n objects and you're trying to count ways to choose a subset of size k from those objects. Use this as a guide when counting possibilities:

	Order Matters ($aab \neq aba$)	Order Doesn't Matter ($aab = aba$)
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Where:

- $n! = n(n-1)(n-2)(n-3)...(1)$
- There are $n!$ ways to arrange n objects
- $\binom{n}{k}$ can be read as "n choose k" and indicates the formula $\frac{n!}{(n-k)!k!}$

Counting

How did we know that? Suppose you have n objects and you're trying to count ways to choose a subset of size k from those objects. Use this as a guide when counting possibilities:

	Order Matters ($aab \neq aba$)	Order Doesn't Matter ($aab = aba$)
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Where:

- $n! = n(n-1)(n-2)(n-3)...(1)$
- There are $n!$ ways to arrange n objects
- $\binom{n}{k}$ can be read as "n choose k" and indicates the formula $\frac{n!}{(n-k)!k!}$

Counting

Important tips to consider when applying these formulas:

- You'll almost never be asked to count outcomes in such certain terms (e.g. how many ways can I choose 5 objects from 60 objects without replacement?) So you'll have to think about how those formulas fit into the design of your process
- You'll often have to apply combinations of these formulas (see lottery example)
- It's helpful to think about your possible outcomes as vectors, or sequences of events. What actually constitutes a unique sequence? How many elements can they all have?
  - e.g. 3 of a kind, in poker could be $\{J\clubsuit, J\spadesuit, J\heartsuit\}$ or $\{A\clubsuit, A\spadesuit, A\heartsuit\}$, etc.
- Finally, make sure you're accounting for *all* of the variation in your process (e.g. suit *and* rank, regular lotto numbers *and* powerball, etc.)

Counting

Important tips to consider when applying these formulas:

- You'll almost never be asked to count outcomes in such certain terms (e.g. how many ways can I choose 5 objects from 60 objects without replacement?) So you'll have to think about how those formulas fit into the design of your process
- You'll often have to apply combinations of these formulas (see lottery example)
- It's helpful to think about your possible outcomes as vectors, or sequences of events. What actually constitutes a unique sequence? How many elements can they all have?
  - e.g. 3 of a kind, in poker could be $\{J\clubsuit, J\spadesuit, J\heartsuit\}$ or $\{A\clubsuit, A\spadesuit, A\heartsuit\}$, etc.
- Finally, make sure you're accounting for *all* of the variation in your process (e.g. suit *and* rank, regular lotto numbers *and* powerball, etc.)

Counting

Important tips to consider when applying these formulas:

- You'll almost never be asked to count outcomes in such certain terms (e.g. how many ways can I choose 5 objects from 60 objects without replacement?) So you'll have to think about how those formulas fit into the design of your process
- You'll often have to apply combinations of these formulas (see lottery example)
- It's helpful to think about your possible outcomes as vectors, or sequences of events. What actually constitutes a unique sequence? How many elements can they all have?
  - e.g. 3 of a kind, in poker could be $\{J\clubsuit, J\spadesuit, J\heartsuit\}$ or $\{A\clubsuit, A\spadesuit, A\heartsuit\}$, etc.
- Finally, make sure you're accounting for *all* of the variation in your process (e.g. suit *and* rank, regular lotto numbers *and* powerball, etc.)

Counting

Important tips to consider when applying these formulas:

- You'll almost never be asked to count outcomes in such certain terms (e.g. how many ways can I choose 5 objects from 60 objects without replacement?) So you'll have to think about how those formulas fit into the design of your process
- You'll often have to apply combinations of these formulas (see lottery example)
- It's helpful to think about your possible outcomes as vectors, or sequences of events. What actually constitutes a unique sequence? How many elements can they all have?
  - e.g. 3 of a kind, in poker could be $\{J\clubsuit, J\spadesuit, J\heartsuit\}$ or $\{A\clubsuit, A\spadesuit, A\heartsuit\}$, etc.
- Finally, make sure you're accounting for *all* of the variation in your process (e.g. suit *and* rank, regular lotto numbers *and* powerball, etc.)

Counting

Important tips to consider when applying these formulas:

- You'll almost never be asked to count outcomes in such certain terms (e.g. how many ways can I choose 5 objects from 60 objects without replacement?) So you'll have to think about how those formulas fit into the design of your process
- You'll often have to apply combinations of these formulas (see lottery example)
- It's helpful to think about your possible outcomes as vectors, or sequences of events. What actually constitutes a unique sequence? How many elements can they all have?
  - e.g. 3 of a kind, in poker could be $\{J\clubsuit, J\spadesuit, J\heartsuit\}$ or $\{A\clubsuit, A\spadesuit, A\heartsuit\}$, etc.
- Finally, make sure you're accounting for *all* of the variation in your process (e.g. suit *and* rank, regular lotto numbers *and* powerball, etc.)

Counting

Important tips to consider when applying these formulas:

- You'll almost never be asked to count outcomes in such certain terms (e.g. how many ways can I choose 5 objects from 60 objects without replacement?) So you'll have to think about how those formulas fit into the design of your process
- You'll often have to apply combinations of these formulas (see lottery example)
- It's helpful to think about your possible outcomes as vectors, or sequences of events. What actually constitutes a unique sequence? How many elements can they all have?
  - e.g. 3 of a kind, in poker could be $\{J\clubsuit, J\spadesuit, J\heartsuit\}$ or $\{A\clubsuit, A\spadesuit, A\heartsuit\}$, etc.
- Finally, make sure you're accounting for *all* of the variation in your process (e.g. suit *and* rank, regular lotto numbers *and* powerball, etc.)

Counting

Important tips to consider when applying these formulas:

- You'll almost never be asked to count outcomes in such certain terms (e.g. how many ways can I choose 5 objects from 60 objects without replacement?) So you'll have to think about how those formulas fit into the design of your process
- You'll often have to apply combinations of these formulas (see lottery example)
- It's helpful to think about your possible outcomes as vectors, or sequences of events. What actually constitutes a unique sequence? How many elements can they all have?
  - e.g. 3 of a kind, in poker could be $\{J\clubsuit, J\spadesuit, J\heartsuit\}$ or $\{A\clubsuit, A\spadesuit, A\heartsuit\}$, etc.
- Finally, make sure you're accounting for *all* of the variation in your process (e.g. suit *and* rank, regular lotto numbers *and* powerball, etc.)

Definition

- In practice, probability frequently concerns questions and data generating processes that are too complex even to express in our naïve ratio form.
- But we need a way to express those too! So let's begin defining them.
- Formally, probability is a mathematical model of uncertain outcomes in the real world
- Probability is a model that conforms to a particular set of rules (axioms) and is expressed in a specific type of language
- We'll discuss the axioms of probability at length throughout this section, but first let's define some specific concepts that will help us introduce those axioms in concise terms

Definition

- In practice, probability frequently concerns questions and data generating processes that are too complex even to express in our naïve ratio form.
- But we need a way to express those too! So let's begin defining them.
- Formally, probability is a mathematical model of uncertain outcomes in the real world
- Probability is a model that conforms to a particular set of rules (axioms) and is expressed in a specific type of language
- We'll discuss the axioms of probability at length throughout this section, but first let's define some specific concepts that will help us introduce those axioms in concise terms

Definition

- In practice, probability frequently concerns questions and data generating processes that are too complex even to express in our naïve ratio form.
- But we need a way to express those too! So let's begin defining them.
- Formally, probability is a mathematical model of uncertain outcomes in the real world
- Probability is a model that conforms to a particular set of rules (axioms) and is expressed in a specific type of language
- We'll discuss the axioms of probability at length throughout this section, but first let's define some specific concepts that will help us introduce those axioms in concise terms

Definition

- In practice, probability frequently concerns questions and data generating processes that are too complex even to express in our naïve ratio form.
- But we need a way to express those too! So let's begin defining them.
- Formally, probability is a mathematical model of uncertain outcomes in the real world
- Probability is a model that conforms to a particular set of rules (axioms) and is expressed in a specific type of language
- We'll discuss the axioms of probability at length throughout this section, but first let's define some specific concepts that will help us introduce those axioms in concise terms

Definition

- In practice, probability frequently concerns questions and data generating processes that are too complex even to express in our naïve ratio form.
- But we need a way to express those too! So let's begin defining them.
- Formally, probability is a mathematical model of uncertain outcomes in the real world
- Probability is a model that conforms to a particular set of rules (axioms) and is expressed in a specific type of language
- We'll discuss the axioms of probability at length throughout this section, but first let's define some specific concepts that will help us introduce those axioms in concise terms

Definition

- In practice, probability frequently concerns questions and data generating processes that are too complex even to express in our naïve ratio form.
- But we need a way to express those too! So let's begin defining them.
- Formally, probability is a mathematical model of uncertain outcomes in the real world
- Probability is a model that conforms to a particular set of rules (axioms) and is expressed in a specific type of language
- We'll discuss the axioms of probability at length throughout this section, but first let's define some specific concepts that will help us introduce those axioms in concise terms

Definition

- In practice, probability frequently concerns questions and data generating processes that are too complex even to express in our naïve ratio form.
- But we need a way to express those too! So let's begin defining them.
- Formally, probability is a mathematical model of uncertain outcomes in the real world
- Probability is a model that conforms to a particular set of rules (axioms) and is expressed in a specific type of language
- We'll discuss the axioms of probability at length throughout this section, but first let's define some specific concepts that will help us introduce those axioms in concise terms

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{\ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{ \}$ or as $S = \emptyset$

- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process

- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{\ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{\ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{\ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{\ \}$ or as $S = \emptyset$

- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process

- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Set Theory Review

- **Set:**
  - Notation: S (or any other letter of the alphabet, typically)
  - Defined as: any well defined collection of elements. If some element x is an element of a set S, we can argue $x \in S$ (read as: x is in S)
  - Note that the "null set", or set containing no elements, can be written as $S = \{\ \}$ or as $S = \emptyset$
- **Sample Space:**
  - Notation: Ω
  - Defined as: set of *all* possible outcomes from some process
- **Event:**
  - Notation: A (these are typically defined in single letters or phrases)
  - Defined as: Any collection of possible outcomes from some process. Any subset of Ω, including Ω itself. Given event A in Ω, we can say $A \subset \Omega$ (read as: A is a subset of Ω)

Example: Pick A Card, Any Card

- **The real-world process:** Suppose you're drawing cards from a standard, 52-card deck of playing cards
- **Sample Space:** $\Omega = 13$ rank cards in each of 4 suits

  $2\clubsuit$ $3\clubsuit$ $4\clubsuit$ $5\clubsuit$ $6\clubsuit$ $7\clubsuit$ $8\clubsuit$ $9\clubsuit$ $10\clubsuit$ $J\clubsuit$ $Q\clubsuit$ $K\clubsuit$ $A\clubsuit$
  $2\spadesuit$ $3\spadesuit$ $4\spadesuit$ $5\spadesuit$ $6\spadesuit$ $7\spadesuit$ $8\spadesuit$ $9\spadesuit$ $10\spadesuit$ $J\spadesuit$ $Q\spadesuit$ $K\spadesuit$ $A\spadesuit$
  $2\diamondsuit$ $3\diamondsuit$ $4\diamondsuit$ $5\diamondsuit$ $6\diamondsuit$ $7\diamondsuit$ $8\diamondsuit$ $9\diamondsuit$ $10\diamondsuit$ $J\diamondsuit$ $Q\diamondsuit$ $K\diamondsuit$ $A\diamondsuit$
  $2\heartsuit$ $3\heartsuit$ $4\heartsuit$ $5\heartsuit$ $6\heartsuit$ $7\heartsuit$ $8\heartsuit$ $9\heartsuit$ $10\heartsuit$ $J\heartsuit$ $Q\heartsuit$ $K\heartsuit$ $A\heartsuit$
- **Event:** Drawing a queen. $A = \{Q\clubsuit\ Q\spadesuit\ Q\diamondsuit\ Q\heartsuit\}$

Example: Pick A Card, Any Card

- **The real-world process:** Suppose you're drawing cards from a standard, 52-card deck of playing cards

- **Sample Space:** Ω = 13 rank cards in each of 4 suits

  2♣ 3♣ 4♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣ A♣
  2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠ A♠
  2♢ 3♢ 4♢ 5♢ 6♢ 7♢ 8♢ 9♢ 10♢ J♢ Q♢ K♢ A♢
  2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡ A♡

- **Event:** Drawing a queen. $A = \{$Q♣ Q♠ Q♢ Q♡$\}$

Example: Pick A Card, Any Card

- **The real-world process:** Suppose you're drawing cards from a standard, 52-card deck of playing cards
- **Sample Space:** Ω = 13 rank cards in each of 4 suits

  2♣ 3♣ 4♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣ A♣
  2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠ A♠
  2♢ 3♢ 4♢ 5♢ 6♢ 7♢ 8♢ 9♢ 10♢ J♢ Q♢ K♢ A♢
  2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡ A♡

- **Event:** Drawing a queen. $A = \{$Q♣ Q♠ Q♢ Q♡$\}$

Example: Pick A Card, Any Card

- **The real-world process:** Suppose you're drawing cards from a standard, 52-card deck of playing cards

- **Sample Space:** Ω = 13 rank cards in each of 4 suits

  2♣ 3♣ 4♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣ A♣
  2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠ A♠
  2♢ 3♢ 4♢ 5♢ 6♢ 7♢ 8♢ 9♢ 10♢ J♢ Q♢ K♢ A♢
  2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡ A♡

- **Event:** Drawing a queen. $A = \{$Q♣ Q♠ Q♢ Q♡$\}$

Example: Pick A Card, Any Card

- **The real-world process:** Suppose you're drawing cards from a standard, 52-card deck of playing cards
- **Sample Space:** Ω = 13 rank cards in each of 4 suits

  2♣ 3♣ 4♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣ A♣
  2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠ A♠
  2♢ 3♢ 4♢ 5♢ 6♢ 7♢ 8♢ 9♢ 10♢ J♢ Q♢ K♢ A♢
  2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡ A♡

- **Event:** Drawing a queen. $A = \{Q\clubsuit\ Q\spadesuit\ Q\diamondsuit\ Q\heartsuit\}$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

Let A and B be two events in some sample space Ω, then:

- **Complements:**
  - Notation: A^C
  - Definition: The complement of some event or set is everything in Ω that isn't in that event or set. Note that $\Omega^C = \emptyset$
    - e.g. in our previous example, the complement of picking a **red card** is picking a **black card**
- **Union:**
  - Notation: $A \cup B$
  - Definition: The union of two events A and B is the event that A or B occurs. In other words, its the event that you observe any of the elements in A or B
    - e.g. If A is the event that you draw a club and B is the event that you draw a queen, then $A \cup B$ is the event that you draw a queen or a club: $Q \cup \clubsuit$

Set Operations

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

Set Operations

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

Set Operations

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$

- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

Set Operations

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

Set Operations

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

Set Operations

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

Set Operations

- **Intersection:**
  - Notation: $A \cap B$
  - Definition: The intersection of two events A and B is the event that *both* A and B occur, or the event that you observe only elements contained in *both* A and B.
    - e.g. If A is the event that you draw a club in our previous example and B is the event that you draw a queen, then $A \cap B$ is the event that you draw the queen of clubs: $Q \cap \clubsuit = Q\clubsuit$
- **Disjoint Events:**
  - Notation: $A \cap B = \emptyset$
  - Definition: Two events A and B are said to be disjoint if they are mutually exclusive. In other words, if A occurs then we know that B cannot have occurred and vice-versa.
    - e.g. If A is the event that you draw a club and B is the event that you draw a spade, then $A \cap B = \emptyset$ because no single card can belong to multiple suits.

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Properties of Set Operations

- **Commutative:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative:**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:**
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Laws:**
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Axioms of Probability

Probability just quantifies how likely or unlikely it is that the events we've been defining in this section will come to pass. Before we start thinking about how to do that math, let's define three basic axioms of probability:

- **Nonnegativity:**
  - For any event A, $P(A) \geq 0$
- **Normalization:**
  - $P(\Omega) = 1$
- **Additivity:**
  - For any sequence of disjoint events $A_1, A_2, A_3, A_4, ...A_k$:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

A Quick Note on Notation

- Given *any* events $A_1, A_2, A_3, A_4, ...A_k$:
  - $\bigcup_{i=1}^{k} A_i = A_1 \cup A_2 \cup A_3 \cup ...A_k$
  - $P\left(\bigcup_{i=1}^{k} A_i\right) = P(A_1 \cup A_2 \cup A_3 \cup ...A_k)$
  - $\bigcap_{i=1}^{k} A_i = A_1 \cap A_2 \cap A_3 \cap ...A_k$
  - $P\left(\bigcap_{i=1}^{k} A_i\right) = P(A_1 \cap A_2 \cap A_3 \cap ...A_k)$
- Just as we use summation notation to summarize a series, we can use the large union or intersection signs you see above to indicate that we want the union or intersection of a long sequence of events!

A Quick Note on Notation

- Given *any* events $A_1, A_2, A_3, A_4, ...A_k$:
  - $\bigcup_{i=1}^{k} A_i = A_1 \cup A_2 \cup A_3 \cup ...A_k$
  - $P\left(\bigcup_{i=1}^{k} A_i\right) = P(A_1 \cup A_2 \cup A_3 \cup ...A_k)$
  - $\bigcap_{i=1}^{k} A_i = A_1 \cap A_2 \cap A_3 \cap ...A_k$
  - $P\left(\bigcap_{i=1}^{k} A_i\right) = P(A_1 \cap A_2 \cap A_3 \cap ...A_k)$
- Just as we use summation notation to summarize a series, we can use the large union or intersection signs you see above to indicate that we want the union or intersection of a long sequence of events!

A Quick Note on Notation

- Given *any* events $A_1, A_2, A_3, A_4, ...A_k$:
  - $\bigcup_{i=1}^{k} A_i = A_1 \cup A_2 \cup A_3 \cup ...A_k$
  - $P\left(\bigcup_{i=1}^{k} A_i\right) = P(A_1 \cup A_2 \cup A_3 \cup ...A_k)$
  - $\bigcap_{i=1}^{k} A_i = A_1 \cap A_2 \cap A_3 \cap ...A_k$
  - $P\left(\bigcap_{i=1}^{k} A_i\right) = P(A_1 \cap A_2 \cap A_3 \cap ...A_k)$
- Just as we use summation notation to summarize a series, we can use the large union or intersection signs you see above to indicate that we want the union or intersection of a long sequence of events!

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, ...A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, \ldots A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, ...A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, ...A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, ...A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, ...A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, ...A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Additional Properties of Probability

We can derive all the properties of probability from the three axioms we just discussed:

- For any event $A : 0 \leq P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(\emptyset) = 0$
- If $A \subset B$, $P(A) \leq P(B)$
- For *any* two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For *any* sequence of k events $A_1, A_2, A_3, ...A_k$ (which need not be disjoint):

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Conditional Probability

- Notation: $P(A \mid B)$ (read as: probability of A *given* B)
- Definition: Given two events A and B, the **conditional probability** of A given B refers to the probability that A occurs given that we know B has already occurred. If $P(B) > 0$, we can express this as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- In this example, we are conditioning our analysis on the assumption that B has occurred. In practice, all probability is conditional probability; you're almost always making implicit assumptions about the process that generates your data.

Conditional Probability

- Notation: $P(A \mid B)$ (read as: probability of A *given* B)
- Definition: Given two events A and B, the **conditional probability** of A given B refers to the probability that A occurs given that we know B has already occurred. If $P(B) > 0$, we can express this as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- In this example, we are conditioning our analysis on the assumption that B has occurred. In practice, all probability is conditional probability; you're almost always making implicit assumptions about the process that generates your data.

Conditional Probability

- Notation: $P(A \mid B)$ (read as: probability of A *given* B)
- Definition: Given two events A and B, the **conditional probability** of A given B refers to the probability that A occurs given that we know B has already occurred. If $P(B) > 0$, we can express this as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- In this example, we are conditioning our analysis on the assumption that B has occurred. In practice, all probability is conditional probability; you're almost always making implicit assumptions about the process that generates your data.

Conditional Probability

- Notation: $P(A \mid B)$ (read as: probability of A *given* B)
- Definition: Given two events A and B, the **conditional probability** of A given B refers to the probability that A occurs given that we know B has already occurred. If $P(B) > 0$, we can express this as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- In this example, we are conditioning our analysis on the assumption that B has occurred. In practice, all probability is conditional probability; you're almost always making implicit assumptions about the process that generates your data.

Conditional Probability

- Notation: $P(A \mid B)$ (read as: probability of A *given* B)
- Definition: Given two events A and B, the **conditional probability** of A given B refers to the probability that A occurs given that we know B has already occurred. If $P(B) > 0$, we can express this as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- In this example, we are conditioning our analysis on the assumption that B has occurred. In practice, all probability is conditional probability; you're almost always making implicit assumptions about the process that generates your data.

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Example: Conditional Probability

Suppose the table below represents the composition of the U.S. Senate, and you're surveying senators at random:

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
Total	51	47	2	100

- What's the probability of choosing a female senator?
  - $P(\text{Female}) = \frac{17}{100} = 0.17$
- What's the probability of choosing a female, Republican senator?
  - $P(\text{Female} \cap \text{Republican}) = \frac{5}{100} = 0.05$
- What's the probability that a randomly selected Republican senator is female?
  - $P(\text{Female} \mid \text{Republican}) = \frac{P(\text{Female} \cap \text{Republican})}{P(\text{Republican})} = \frac{5}{47} = 0.106$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Properties of Conditional Probability

- First, a conditional probability PSA:
  - The rules of probability apply to the **left-hand** side of the conditioning bar in a probability expression.
  - Everything on the right hand side represents what you already know about the world. You can't do mathematical operations on that.
  - So, even if two events B and C are disjoint:

$$P(A \mid B \cup C) \neq P(A \mid B) + P(A \mid C)$$

- You can, however, perform mathematical operations on the left hand side.
  - If two events A_1 and A_2 are disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal} \\ \text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional} \\ \text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal}\\\text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional}\\\text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal}\\\text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional}\\\text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal} \\ \text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional} \\ \text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal}\\\text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional}\\\text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal}\\\text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional}\\\text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal}\\\text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional}\\\text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

- Definition for **two events**:
  - Given *any* two events, A_1 and A_2:

$$\underbrace{P(A_1 \cap A_2)}_{\text{Joint Distribution}} = \underbrace{P(A_1)}_{\substack{\text{Marginal}\\ \text{Distribution}}} \underbrace{P(A_2 \mid A_1)}_{\substack{\text{Conditional}\\ \text{Distribution}}}$$

  - We can derive the joint distribution of two (or more) events using the marginal and conditional distributions for those events!
- Definition for **three events**:
  - Given *any* three events, A_1,A_2, and A_3:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1)P(A_2 \mid A_1)P(A_1)$$

Multiplication Rule

We can generalize the multiplication rule to k events:

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3 \cap ...A_k) = &P(A_k \mid A_{k-1} \cap A_{k-2} \cap ...A_1) \\
&P(A_{k-1} \mid A_{k-2} \cap A_{k-3} \cap ...A_1) \\
&P(A_{k-2} \mid A_{k-3} \cap A_{k-4} \cap ...A_1)... \\
&P(A_2 \mid A_1)P(A_1)
\end{aligned}
$$

Multiplication Rule

We can generalize the multiplication rule to k events:

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3 \cap ...A_k) = & P(A_k \mid A_{k-1} \cap A_{k-2} \cap ...A_1) \\ & P(A_{k-1} \mid A_{k-2} \cap A_{k-3} \cap ...A_1) \\ & P(A_{k-2} \mid A_{k-3} \cap A_{k-4} \cap ...A_1)... \\ & P(A_2 \mid A_1)P(A_1) \end{aligned}$$

Multiplication Rule

We can generalize the multiplication rule to k events:

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3 \cap ...A_k) = & P(A_k \mid A_{k-1} \cap A_{k-2} \cap ...A_1) \\ & P(A_{k-1} \mid A_{k-2} \cap A_{k-3} \cap ...A_1) \\ & P(A_{k-2} \mid A_{k-3} \cap A_{k-4} \cap ...A_1)... \\ & P(A_2 \mid A_1)P(A_1) \end{aligned}$$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) =$??
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \ ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) P(\text{Ace}_2 \mid \text{Ace}_1) P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) =$??
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) P(\text{Ace}_2 \mid \text{Ace}_1) P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) =$??
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \; ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) =\ ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) P(\text{Ace}_2 \mid \text{Ace}_1) P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \; ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) P(\text{Ace}_2 \mid \text{Ace}_1) P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \ ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) P(\text{Ace}_2 \mid \text{Ace}_1) P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) =$??
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \ ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \ ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) =$??
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Example: Multiplication Rule

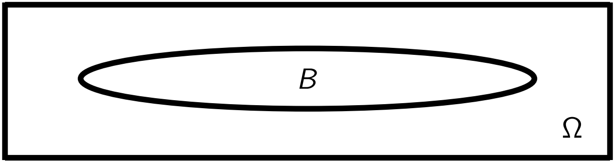
Suppose we draw three cards at random from a standard 52-card deck of playing cards. What is the probability that we draw three aces?

- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \; ??$
- Apply the multiplication rule!
  - $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1)P(\text{Ace}_2 \mid \text{Ace}_1)P(\text{Ace}_1)$
- What do these individual probabilities actually correspond to?
  - $P(\text{Ace}_1) = \frac{4}{52}$ (4 aces in the full deck)
  - $P(\text{Ace}_2 \mid \text{Ace}_1) = \frac{3}{51}$ (3 remaining aces in 51 remaining cards)
  - $P(\text{Ace}_3 \mid \text{Ace}_2 \cap \text{Ace}_1) = \frac{2}{50}$ (2 remaining aces in 50 remaining cards)
- $P(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = 0.00018$

Law of Total Probability

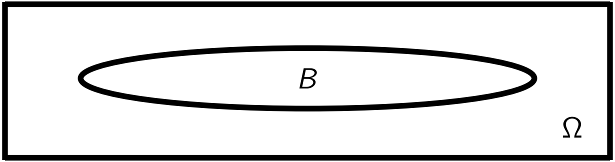
- In the set above, how do we calculate the probability of B?
- Parition Ω into disjoint events:

Law of Total Probability



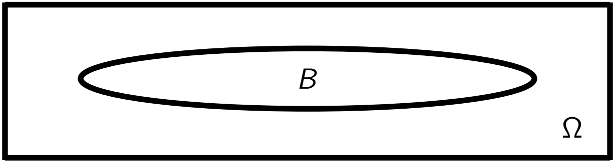
- In the set above, how do we calculate the probability of B?
- Parition Ω into disjoint events:

Law of Total Probability



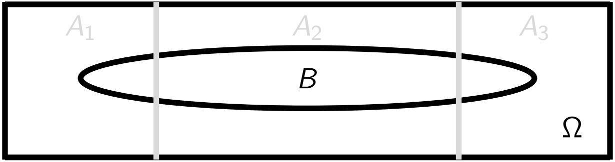
- In the set above, how do we calculate the probability of B?
- Parition Ω into disjoint events:

Law of Total Probability



- In the set above, how do we calculate the probability of B?
- Parition Ω into disjoint events:

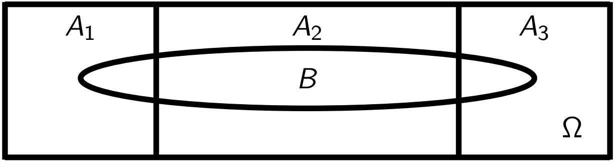
Law of Total Probability



- **Law of Total Probability**
  - Partition Ω into a set of disjoint events, $A_1, A_2, A_3, ...A_k$ such that $\bigcup_{i=1}^{k} A_i = \Omega$
  - Given the partition above, the Law of Total Probability (LOTP), states:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

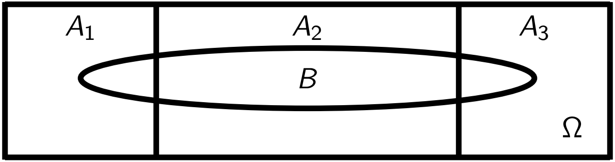
Law of Total Probability



- **Law of Total Probability**
  - Partition Ω into a set of disjoint events, $A_1, A_2, A_3, ...A_k$ such that $\bigcup_{i=1}^{k} A_i = \Omega$
  - Given the partition above, the Law of Total Probability (LOTP), states:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

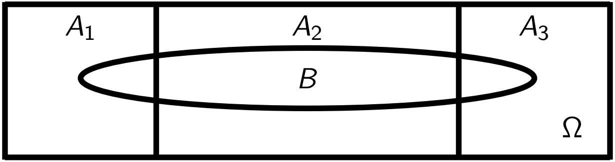
Law of Total Probability



- **Law of Total Probability**
  - Partition Ω into a set of disjoint events, $A_1, A_2, A_3, ...A_k$ such that $\bigcup_{i=1}^{k} A_i = \Omega$
  - Given the partition above, the Law of Total Probability (LOTP), states:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

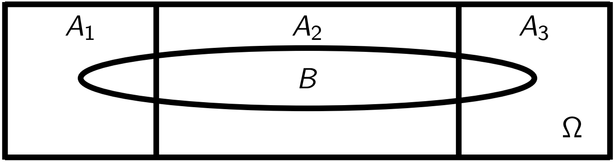
Law of Total Probability



- **Law of Total Probability**
  - Partition Ω into a set of disjoint events, $A_1, A_2, A_3, \ldots A_k$ such that $\bigcup_{i=1}^{k} A_i = \Omega$
  - Given the partition above, the Law of Total Probability (LOTP), states:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

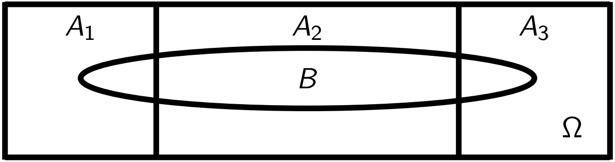
Law of Total Probability



- **Law of Total Probability**
  - Partition Ω into a set of disjoint events, $A_1, A_2, A_3, \ldots A_k$ such that $\bigcup_{i=1}^{k} A_i = \Omega$
  - Given the partition above, the Law of Total Probability (LOTP), states:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

Law of Total Probability



- **Law of Total Probability**
  - Partition Ω into a set of disjoint events, $A_1, A_2, A_3, \ldots A_k$ such that $\bigcup_{i=1}^{k} A_i = \Omega$
  - Given the partition above, the Law of Total Probability (LOTP), states:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i) P(A_i)$$

Bayes' Rule

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Bayes' Rule

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Bayes' Rule

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Bayes' Rule

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Bayes' Rule

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Bayes' Rule

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- $P(\text{Being smart} \mid \text{Enrolling in Gov2k}) =$ High!
- $P(\text{Enrolling in Gov2k} \mid \text{Being smart}) =$ Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Before we get into this, another important probability PSA:

$$P(A \mid B) \neq P(B \mid A)$$

Seriously, that's not a thing. Think about it:

- P(Being smart $\mid$ Enrolling in Gov2k) = High!
- P(Enrolling in Gov2k $\mid$ Being smart) = Pretty low!
  - We don't have room for all of the smart people in the world. We barely had room for all of you in K354. Plus, lots of smart people who aren't you are too far away and too busy for Gov2k. (Ain't nobody got time for that)
- Well, that's good to know. But what if we have $P(A \mid B)$, but we really want to know $P(B \mid A)$? What do we do?

Bayes' Rule

We talk to the reverend:

- **Bayes' Rule**
  - If $P(B) > 0$, then:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Bayes' Rule

We talk to the reverend:

- **Bayes' Rule**
  - If $P(B) > 0$, then:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

We talk to the reverend:

- **Bayes' Rule**
  - If $P(B) > 0$, then:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Bayes' Rule

We talk to the reverend:

- **Bayes' Rule**
  - If $P(B) > 0$, then:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Bayes' Rule

We talk to the reverend:

- **Bayes' Rule**
  - If $P(B) > 0$, then:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Proof: Bayes' Rule

- To prove Bayes' Rule (or Bayes' Theorem), let's begin with the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Let's rearrange, multiplying both sides by $P(A)$:

$$P(B \mid A)P(A) = P(A \cap B)$$

- Ok, but what's $P(A \cap B)$? If we recall the definition of conditional probability again, this time for $P(A \mid B)$, we get:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proof: Bayes' Rule

- To prove Bayes' Rule (or Bayes' Theorem), let's begin with the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Let's rearrange, multiplying both sides by $P(A)$:

$$P(B \mid A)P(A) = P(A \cap B)$$

- Ok, but what's $P(A \cap B)$? If we recall the definition of conditional probability again, this time for $P(A \mid B)$, we get:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proof: Bayes' Rule

- To prove Bayes' Rule (or Bayes' Theorem), let's begin with the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Let's rearrange, multiplying both sides by $P(A)$:

$$P(B \mid A)P(A) = P(A \cap B)$$

- Ok, but what's $P(A \cap B)$? If we recall the definition of conditional probability again, this time for $P(A \mid B)$, we get:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proof: Bayes' Rule

- To prove Bayes' Rule (or Bayes' Theorem), let's begin with the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Let's rearrange, multiplying both sides by $P(A)$:

$$P(B \mid A)P(A) = P(A \cap B)$$

- Ok, but what's $P(A \cap B)$? If we recall the definition of conditional probability again, this time for $P(A \mid B)$, we get:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proof: Bayes' Rule

- To prove Bayes' Rule (or Bayes' Theorem), let's begin with the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Let's rearrange, multiplying both sides by $P(A)$:

$$P(B \mid A)P(A) = P(A \cap B)$$

- Ok, but what's $P(A \cap B)$? If we recall the definition of conditional probability again, this time for $P(A \mid B)$, we get:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proof: Bayes' Rule

- To prove Bayes' Rule (or Bayes' Theorem), let's begin with the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Let's rearrange, multiplying both sides by $P(A)$:

$$P(B \mid A)P(A) = P(A \cap B)$$

- Ok, but what's $P(A \cap B)$? If we recall the definition of conditional probability again, this time for $P(A \mid B)$, we get:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proof: Bayes' Rule

- To prove Bayes' Rule (or Bayes' Theorem), let's begin with the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Let's rearrange, multiplying both sides by $P(A)$:

$$P(B \mid A)P(A) = P(A \cap B)$$

- Ok, but what's $P(A \cap B)$? If we recall the definition of conditional probability again, this time for $P(A \mid B)$, we get:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proof: Bayes' Rule

- Now we can plug in our first expression for $P(A \cap B)$:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- And that's Bayes' rule!

Proof: Bayes' Rule

- Now we can plug in our first expression for $P(A \cap B)$:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- And that's Bayes' rule!

Proof: Bayes' Rule

- Now we can plug in our first expression for $P(A \cap B)$:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- And that's Bayes' rule!

Proof: Bayes' Rule

- Now we can plug in our first expression for $P(A \cap B)$:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- And that's Bayes' rule!

Cancer Testing

Suppose that a certain rare cancer is present in 1 in every 10,000 people (0.0001). A test for the cancer exists, and the test correctly classifies people with 99% accuracy. (That is, if you have the cancer, the test comes up positive correctly 99% of the time, and if you don't have the cancer, the test will suggest you're cancer-free correctly 99% of the time). Suppose you get tested and it comes up positive. What's the probability that you actually have the cancer?

Cancer Testing

Suppose that a certain rare cancer is present in 1 in every 10,000 people (0.0001). A test for the cancer exists, and the test correctly classifies people with 99% accuracy. (That is, if you have the cancer, the test comes up positive correctly 99% of the time, and if you don't have the cancer, the test will suggest you're cancer-free correctly 99% of the time). Suppose you get tested and it comes up positive. What's the probability that you actually have the cancer?

Cancer Testing

Suppose that a certain rare cancer is present in 1 in every 10,000 people (0.0001). A test for the cancer exists, and the test correctly classifies people with 99% accuracy. (That is, if you have the cancer, the test comes up positive correctly 99% of the time, and if you don't have the cancer, the test will suggest you're cancer-free correctly 99% of the time). Suppose you get tested and it comes up positive. What's the probability that you actually have the cancer?

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid + \text{test}) = \frac{P(+\text{ test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{ test})}$$

Example: Bayes' Rule

- Let's begin by defining some events:
  - Let A = the event that you actually have the cancer
  - Let B = the event that you test positive
- Now let's plug these into Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- In words, that's:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

Example: Bayes' Rule

- So let's plug in what we know:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

$$P(\text{Cancer} \mid +\text{test}) = \frac{(0.99)(0.0001)}{P(+\text{test})}$$

- Hmm, what do we do with this $P(+\text{test})$? Do we have the unconditional probability of testing positive? No. We know the conditional probabilities of both testing positive when you have cancer and testing negative when you don't have cancer.

- So let's plug in what we know:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

$$P(\text{Cancer} \mid +\text{test}) = \frac{(0.99)(0.0001)}{P(+\text{test})}$$

- Hmm, what do we do with this $P(+\text{test})$? Do we have the unconditional probability of testing positive? No. We know the conditional probabilities of both testing positive when you have cancer and testing negative when you don't have cancer.

Example: Bayes' Rule

- So let's plug in what we know:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

$$P(\text{Cancer} \mid +\text{test}) = \frac{(0.99)(0.0001)}{P(+\text{test})}$$

- Hmm, what do we do with this $P(+\text{test})$? Do we have the unconditional probability of testing positive? No. We know the conditional probabilities of both testing positive when you have cancer and testing negative when you don't have cancer.

Example: Bayes' Rule

- So let's plug in what we know:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

$$P(\text{Cancer} \mid +\text{test}) = \frac{(0.99)(0.0001)}{P(+\text{test})}$$

- Hmm, what do we do with this $P(+\text{test})$? Do we have the unconditional probability of testing positive? No. We know the conditional probabilities of both testing positive when you have cancer and testing negative when you don't have cancer.

Example: Bayes' Rule

- So let's plug in what we know:

$$P(\text{Cancer} \mid +\text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

$$P(\text{Cancer} \mid +\text{test}) = \frac{(0.99)(0.0001)}{P(+\text{test})}$$

- Hmm, what do we do with this $P(+\text{test})$? Do we have the unconditional probability of testing positive? No. We know the conditional probabilities of both testing positive when you have cancer and testing negative when you don't have cancer.

Example: Bayes' Rule

- So let's plug in what we know:

$$P(\text{Cancer} \mid + \text{test}) = \frac{P(+\text{test} \mid \text{Cancer})P(\text{Cancer})}{P(+\text{test})}$$

$$P(\text{Cancer} \mid + \text{test}) = \frac{(0.99)(0.0001)}{P(+\text{test})}$$

- Hmm, what do we do with this $P(+\text{test})$? Do we have the unconditional probability of testing positive? No. We know the conditional probabilities of both testing positive when you have cancer and testing negative when you don't have cancer.

Example: Bayes' Rule

- So what do we apply there? The LOTP! Here's how it works:
- Let's partition $P(+\text{ test})$ into all of the disjoint possibilities:

$$P(+\text{ test}) = \underbrace{P(+\text{ test} \mid \text{Cancer})}_{\text{We know this, it's 0.99}} \underbrace{P(\text{Cancer})}_{0.0001} +$$

$$\underbrace{P(+\text{ test} \mid \text{No Cancer})}_{\text{This is a false positive, so 1 - 0.99}} \underbrace{P(\text{No Cancer})}_{(1-0.0001)=0.9999}$$

- Plugging back into our Bayes' rule equation, this becomes:

$$P(\text{Cancer} \mid +\text{ test}) = \frac{(0.99)(0.0001)}{0.99 \cdot 0.0001 + (1-0.99) \cdot 0.9999} = 0.0098$$

Example: Bayes' Rule

- So what do we apply there? The LOTP! Here's how it works:
- Let's partition $P(+\text{ test})$ into all of the disjoint possibilities:

$$P(+\text{ test}) = \underbrace{P(+\text{ test} \mid \text{Cancer})}_{\text{We know this, it's 0.99}} \underbrace{P(\text{Cancer})}_{0.0001} +$$

$$\underbrace{P(+\text{ test} \mid \text{No Cancer})}_{\text{This is a false positive, so 1 - 0.99}} \underbrace{P(\text{No Cancer})}_{(1-0.0001)=0.9999}$$

- Plugging back into our Bayes' rule equation, this becomes:

$$P(\text{Cancer} \mid +\text{ test}) = \frac{(0.99)(0.0001)}{0.99 \cdot 0.0001 + (1 - 0.99) \cdot 0.9999} = 0.0098$$

Example: Bayes' Rule

- So what do we apply there? The LOTP! Here's how it works:
- Let's partition $P(+\text{ test})$ into all of the disjoint possibilities:

$$P(+\text{ test}) = \underbrace{P(+\text{ test} \mid \text{Cancer})}_{\text{We know this, it's 0.99}} \underbrace{P(\text{Cancer})}_{0.0001} +$$

$$\underbrace{P(+\text{ test} \mid \text{No Cancer})}_{\text{This is a false positive, so 1 - 0.99}} \underbrace{P(\text{No Cancer})}_{(1-0.0001)=0.9999}$$

- Plugging back into our Bayes' rule equation, this becomes:

$$P(\text{Cancer} \mid +\text{ test}) = \frac{(0.99)(0.0001)}{0.99 \cdot 0.0001 + (1 - 0.99) \cdot 0.9999} = 0.0098$$

Example: Bayes' Rule

- So what do we apply there? The LOTP! Here's how it works:
- Let's partition $P(+\text{ test})$ into all of the disjoint possibilities:

$$P(+\text{ test}) = \underbrace{P(+\text{ test} \mid \text{Cancer})}_{\text{We know this, it's 0.99}} \underbrace{P(\text{Cancer})}_{0.0001} +$$

$$\underbrace{P(+\text{ test} \mid \text{No Cancer})}_{\text{This is a false positive, so 1 - 0.99}} \underbrace{P(\text{No Cancer})}_{(1-0.0001)=0.9999}$$

- Plugging back into our Bayes' rule equation, this becomes:

$$P(\text{Cancer} \mid +\text{ test}) = \frac{(0.99)(0.0001)}{0.99 \cdot 0.0001 + (1-0.99) \cdot 0.9999} = 0.0098$$

Example: Bayes' Rule

- So what do we apply there? The LOTP! Here's how it works:
- Let's partition $P(+\text{ test})$ into all of the disjoint possibilities:

$$P(+\text{ test}) = \underbrace{P(+\text{ test} \mid \text{Cancer})}_{\text{We know this, it's 0.99}} \underbrace{P(\text{Cancer})}_{0.0001} +$$

$$\underbrace{P(+\text{ test} \mid \text{No Cancer})}_{\text{This is a false positive, so 1 - 0.99}} \underbrace{P(\text{No Cancer})}_{(1-0.0001)=0.9999}$$

- Plugging back into our Bayes' rule equation, this becomes:

$$P(\text{Cancer} \mid +\text{ test}) = \frac{(0.99)(0.0001)}{0.99 \cdot 0.0001 + (1-0.99) \cdot 0.9999} = 0.0098$$

- So what do we apply there? The LOTP! Here's how it works:
- Let's partition $P(+\ \text{test})$ into all of the disjoint possibilities:

$$P(+\ \text{test}) = \underbrace{P(+\ \text{test} \mid \text{Cancer})}_{\text{We know this, it's 0.99}} \underbrace{P(\text{Cancer})}_{0.0001} +$$

$$\underbrace{P(+\ \text{test} \mid \text{No Cancer})}_{\text{This is a false positive, so 1 - 0.99}} \underbrace{P(\text{No Cancer})}_{(1-0.0001)=0.9999}$$

- Plugging back into our Bayes' rule equation, this becomes:

$$P(\text{Cancer} \mid +\ \text{test}) = \frac{(0.99)(0.0001)}{0.99 \cdot 0.0001 + (1-0.99) \cdot 0.9999} = 0.0098$$

Example: Bayes' Rule

- So what do we apply there? The LOTP! Here's how it works:
- Let's partition $P(+\text{ test})$ into all of the disjoint possibilities:

$$P(+\text{ test}) = \underbrace{P(+\text{ test} \mid \text{Cancer})}_{\text{We know this, it's 0.99}} \underbrace{P(\text{Cancer})}_{0.0001} +$$

$$\underbrace{P(+\text{ test} \mid \text{No Cancer})}_{\text{This is a false positive, so 1 - 0.99}} \underbrace{P(\text{No Cancer})}_{(1-0.0001)=0.9999}$$

- Plugging back into our Bayes' rule equation, this becomes:

$$P(\text{Cancer} \mid +\text{ test}) = \frac{(0.99)(0.0001)}{0.99 \cdot 0.0001 + (1-0.99) \cdot 0.9999} = 0.0098$$

Example: Bayes' Rule

- So what's the intuition here?
- We're talking about an incredibly rare cancer. Even though you've tested positive, your probability of having the disease is still extremely low!

Example: Bayes' Rule

- So what's the intuition here?
- We're talking about an incredibly rare cancer. Even though you've tested positive, your probability of having the disease is still extremely low!

- So what's the intuition here?
- We're talking about an incredibly rare cancer. Even though you've tested positive, your probability of having the disease is still extremely low!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

$$P(A \cap B) = P(A)P(B)$$

  and further:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

$$P(A \cap B) = P(A)P(B)$$

and further:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

$$P(A \cap B) = P(A)P(B)$$

  and further:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

$$P(A \cap B) = P(A)P(B)$$

and further:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

  $$P(A \cap B) = P(A)P(B)$$

  and further:

  $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

$$P(A \cap B) = P(A)P(B)$$

and further:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

$$P(A \cap B) = P(A)P(B)$$

and further:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!

Independence

Given two events, A and B:

- Definition: Two events are independent if the fact that one occurs has no impact on the probability of the other occuring. That is, if A and B are independent, knowing that A has already occurred gives us no information about whether or not B will occur - and vice-versa.
- Notation: $A \perp\!\!\!\perp B$
- If $A \perp\!\!\!\perp B$ holds, it follows that:

$$P(A \cap B) = P(A)P(B)$$

  and further:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

- B gives us no information about A!