Decoupling Coupled Constraints Through Utility Design

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***Abstract*—Several multiagent systems exemplify the need for establishing distributed control laws that ensure the resulting agents' collective behavior satisfies a given coupled constraint. This technical note focuses on the design of such control laws through a game-theoretic framework. In particular, this technical note provides two systematic methodologies for the design of local agent objective functions that guarantee all resulting Nash equilibria optimize the system level objective while also satisfying a given coupled constraint. Furthermore, the designed local agent objective functions fit into the framework of state based potential games. Consequently, one can appeal to existing results in game-theoretic learning to derive a distributed process that guarantees the agents will reach such an equilibrium.**

***Index Terms*—Game theory, multiagent systems, networked control systems.**

I. Introduction

The central goal in multiagent systems is to derive desirable collective behaviors through the design of individual agent control algorithms [1]–[8]. In many systems, the desired collective behavior must also satisfy a given coupled constraint on the agents' behavior [3]–[9]. One example is the problem of TCP control where the users' sending rates need to satisfy link capacity constraints [5]. An alternative example is the problem of economic dispatch in an electricity power system where the total power generation needs to satisfy the total power demands [9]. Regardless of the specific application domain, these coupled constraints bring additional complexity to the control algorithm design.

There are two main research directions aimed at designing distributed control algorithms to satisfy performance criteria involving coupled constraints. The first direction seeks to design algorithms that ensure the coupled constraint is always satisfied, e.g., the well-studied consensus algorithm [2], [4], [10], [11]. While theoretically appealing, such algorithms lack a robustness to environmental uncertainties, noisy measurements, and inconsistent clock rates amongst the agents. The second direction seeks to design algorithms that ensure the asymptotic behavior satisfies the coupled constraints, e.g., dual decomposition [5], [12]–[15] and subgradient methods [16], [17]. Such algorithms often require a two-time scale solution approach by introducing intermediate state variables, such as pricing terms or communication variables, to help coordinate behavior.[1] Depending on the application domain, these approaches may be prohibitive either by the informational dependence on the pricing terms or the rigidity of the update algorithm.

Recently, there has been a surge of research focused on deriving distributed control algorithms to satisfy performance criteria (with and without coupled constraints) under stringent informational constraints [15]–[17]. Here, each agents' control policy is only able to depend on information related to a limited subset of neighboring agents. A common facet associated with these algorithms is the introduction of a local estimate of the global state for each agent. The algorithms proceed as a repeated two-step process where each agent independently makes a decision using only information pertaining to his local estimate and the local estimate of his neighboring agents. Then, each agent updates his local estimate using only information regarding the decisions of neighboring agents, and the two-step process is repeated. While the above algorithms provide asymptotic guarantees for their respective problem domains, it remains an open question as to whether these guarantees are robust to delays in information, asynchronous clock rates, dynamically changing agent capabilities, or component failures.

This technical note focuses on dealing with the underlying control design through a complementary direction which involves assigning each agent a local objective function. More specifically, the underlying control policies are derived from the following two-step design approach:

1) Define a local objective function for each agent. This local objective may be distinct from the system-level objective.
2) Define an adaptation rule for each agent. This rule specifies how each agent utilizes his local objective function and available information to formulate a decision.

The goal is to complete this two-step design approach to ensure that the resulting distributed algorithm is efficient with regards to the system-level objective.

One of the main advantages associated with formally introducing intermediary agent objective functions into the underlying control design is to exploit the rich body of literature in game-theoretic learning to derive distributed algorithms with desirable performance guarantees. This literature provides a wide array of adaptation rules with strong theoretical guarantees pertaining to the emergent collective behavior under various structural assumptions on the agents' objective functions, e.g., potential games [18]. For example, if a system designer can design local agent objective functions such that i) the agents' objective functions constitute a potential game and ii) the resulting Nash equilibria are efficient with regards to the system-level objective function, then the underlying control design can be completed by appealing to existing adaptation rules that guarantee convergence to a Nash equilibrium in potential games. Furthermore, such an approach can also provide "automatic" robustness guarantees. More specifically, [19], [20] prove that the collective behavior will converge to a pure Nash equilibrium in potential games for virtually any adaptation rule where agents seek to optimize their individual objective functions. These guarantees hold true even when the underlying adaptation rule is corrupted with delays in information, component failures, or asynchronous clock rates.

In line with this theme, the main contribution of the technical note is the development of two systematic methodologies for the design of agent objective functions such that i) the agents' objective functions depend only on local information, ii) *all* resulting Nash equilibria satisfy the desired performance criterion which embodies coupled constraints, and iii) the resulting game is a close variant of a *potential game*. In Section III-B, we specify a methodology for designing local

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[1]It is important to highlight that there are special classes of objective functions, e.g.,[5], that do not require the use of a two-time scale algorithm to achieve the desired performance guarantees. However, such simplifications do not hold for the general class of system-level objective functions considered in this technical note.

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agent objective functions using exterior penalty functions while in Section III-C we specify a similar methodology for designing local agent objective functions using barrier functions. Both methodologies achieve our three design objectives as shown in Theorem 1 and Section III-C, respectively. The main difference between the derived methodologies is that the design using barrier functions can also be used to ensure that the coupled constraint is satisfied dynamically in addition to asymptotically as highlighted in Section V. This represents a significant extension over the guarantees associated with the previous algorithms in [15]–[17] which only ensure that the coupled constraint is satisfied asymptotically.

II. Problem Formulation

We consider a multiagent system consisting of n agents denoted by the set $N = \{1, \cdots, n\}$. Each agent $i \in N$ is endowed with a set of possible decisions (or values), denoted by $\mathcal{V}_i$, which we assume is a convex subset of $\mathbb{R}^{d_i}$, i.e., $\mathcal{V}_i \subseteq \mathbb{R}^{d_i}$ for some integer $d_i \geq 1$. We denote a joint decision by the tuple $v = (v_1, \ldots, v_n) \in \mathcal{V} = \prod_{i \in N} \mathcal{V}_i$ where $\mathcal{V}$ is referred to as the set of joint decisions. The goal of this technical note is to establish a methodology for attaining a distributed solution to the following optimization problem

$$\min_{v_i \in \mathcal{V}_i, i \in N} \quad \phi(v) = \sum_{i \in N} C_i(v_i)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} A_i^k v_i - B^k \leq 0, \ k \in M \tag{1}$$

where $C_i : \mathcal{V}_i \to \mathbb{R}$ represents a local cost function for agent i, which is assumed to be a differentiable convex function, and the linear inequalities $\{\sum_{i=1}^{n} A_i^k v_i - B^k \leq 0\}_{k \in M}$ where $M = \{1, \ldots, m\}$ characterize the coupled constraints on the agents' decisions. The distributed algorithm will produce a sequence of decision profiles $v(1), v(2), \ldots$, where the decision of each agent $i \in N$ at each iteration $t \in \{1, 2, \ldots\}$ is selected according to a control law of the form

$$v_i(t) = \Pi_i\left(\{\text{Information about agent } j \text{ at time } t\}_{j \in N_i}\right). \tag{2}$$

where $N_i \subseteq N$ identifies the neighbor set (or information set) of agent i. The neighbor sets $\{N_i\}_{i \in N}$, which we will refer to as the *communication graph*, capture the locality of the distributed algorithm. By convention, we assume $i \in N_i$ for each $i \in N$.[2]

III. A Methodology for Objective Function Design

In this section we present two methodologies for the design of local agent objective functions. Both designs, which incorporate exterior penalty functions and barrier functions respectively, guarantee that i) the resulting game is a state based potential game and ii) all resulting equilibria correspond to optimal solutions of (1).

A. Preliminaries: State Based Potential Games

We consider the class of state based potential games introduced in [20]. A (deterministic) state based game consists of the following elements: i) an agent set N, ii) a state space X, iii) state dependent action sets of the form $\mathcal{A}_i(x)$ for each agent $i \in N$ and state $x \in X$, iv) state dependent cost functions of the form $J_i(x, a) \in \mathbb{R}$ for each agent $i \in N$, $x \in X$, and $a \in \mathcal{A}(x) = \prod_{i \in N} \mathcal{A}_i(x)$, and v) a deterministic state transition function $f(x, a) \in X$ for $x \in X$ and $a \in \mathcal{A}(x)$.[3] Furthermore, we focus on state based games where for any $x \in X$ there exists a null action $\mathbf{0} \in \mathcal{A}(x)$ such that $x = f(x, \mathbf{0})$. This means that the state will not change if all of the agents take the null action.

State based potential games represents an extension of potential games [18] to the framework of state based games as shown in the following definition [20], [21].

Definition 1: (State Based Potential Game) A (deterministic) state based game G with a null action $\mathbf{0}$ is a (deterministic) state based potential game if there exists a potential function $\Phi : X \times \mathcal{A} \to \mathbb{R}$ satisfying the following two properties for every state $x \in X$: i) for every agent $i \in N$, action profile $a \in \mathcal{A}(x)$ and action $a_i' \in \mathcal{A}_i(x)$

$$J_i\left(x, a_i', a_{-i}\right) - J_i(x, a) = \Phi\left(x, a_i', a_{-i}\right) - \Phi(x, a) \tag{3}$$

and ii) for every action profile $a \in \mathcal{A}(x)$ and ensuing state $\tilde{x} = f(x, a)$, the potential function satisfies $\Phi(x, a) = \Phi(\tilde{x}, \mathbf{0})$.

Our motivation for considering state based potential games is the availability of distributed learning algorithms that converge to the following class of equilibria [21].

Definition 2: (Stationary State Nash Equilibrium) A state action pair $[x^*, a^*]$ is a stationary state Nash equilibrium if

(D-1): For any agent $i \in N$, $a_i^* \in \arg\min_{a_i \in \mathcal{A}_i(x^*)} J_i(x^*, a_i, a_{-i}^*)$.

(D-2): The state x^* is a fixed point of the state transition function, i.e., $x^* = f(x^*, a^*)$.

Proposition 1 in [21] proves the existence of a stationary state Nash equilibrium in any state based potential game.

B. Design Using Exterior Penalty Functions

Our first design methodology integrates *exterior penalty functions* into the agents' cost functions. The forthcoming design embodies the following four properties:

(i) The state represents a compilation of local state variables, i.e., the state x can be represented as $x = (x_1, \ldots, x_n)$ where each x_i represents the state of agent i. Furthermore, the state transition function depends only on local information.

(ii) The objective function for each agent i is local and of the form $J_i(\{x_j, a_j\}_{j \in N_i}) \in \mathbb{R}$.

(iii) The resulting game is a state based potential game.

(iv) The stationary state Nash equilibria are optimal in the sense that they represent solutions to the optimization problem in (1).

State Space: The starting point of our design is an underlying state space X where each state $x \in X$ is defined as a tuple $x = (v, e)$, where $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is the profile of values and $e = \{e_i^k\}_{k \in M, i \in N}$ is the profile of estimation terms. The term e_i^k represents agent i's estimate for the k-th constraint, i.e., $e_i^k \sim \sum_{j=1}^{n} A_j^k v_j - B^k$.

[2] We now provide a few remarks regarding the optimization problem presented in (1). First, we do not explicitly highlight the equality constraint, $\sum_{i=1}^{n} A_i^k v_i - B^k = 0$, since this can be handled by two inequalities of the form $\sum_{i=1}^{n} A_i^k v_i - B^k \leq 0$ and $-(\sum_{i=1}^{n} A_i^k v_i - B^k) \leq 0$. Second, for ease of exposition we focus purely on the case $d_i = 1$ for all $i \in N$. However, the forthcoming results also hold for both higher dimensions, i.e, $\mathcal{V}_i \subseteq \mathbb{R}^{d_i}$ where $d_i > 1$, and heterogenous dimensions, i.e., d_i need not equal d_j for $i \neq j$. Lastly, since the focus of this technical note is about decoupling coupled constraints, we focus purely on the case when the objective function ϕ is decomposable but the constraints are coupled. By combining the forthcoming design with our previous work in [21], which focuses on optimization problems with coupled objective functions but decoupled constraints, we can also deal with coupled objective functions in (1) in a similar fashion.

[3] We will frequent express an action profile $a = (a_1, \ldots, a_n)$ as (a_i, a_{-i}) where $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ denotes the collection of actions other than agent i in the action profile a.

Note that each agent possesses an estimation term for each constraint $k \in M$.

Actions: Each agent i is assigned a state dependent action set $\mathcal{A}_i(x)$ that permits the agent to change its value and constraint estimation through communication with neighboring agents. Specifically, an action a_i is defined as a tuple $a_i = (\hat{v}_i, \{\hat{e}_i^1, \ldots, \hat{e}_i^m\})$ where $\hat{v}_i$ indicates a change in the agent's value and $\hat{e}_i^k$ indicates a change in the agent's estimate of the k-th constraint. Here, the change in estimation terms for agent i pertaining to constraint k is represented by a tuple $\hat{e}_i^k = \{\hat{e}_{i\to j}^k\}_{j\in N_i}$. The term $\hat{e}_{i\to j}^k$ indicates the estimation value that agent i exchanges (or passes) to agent $j \in N_i$ regarding the k-th constraint.

State Dynamics: For any state $x = (v, e)$ and action $a = (\hat{v}, \hat{e})$, the ensuing state $(\tilde{v}, \tilde{e}) = f(x, a)$ is chosen according to the state transition function is of the form

$$\begin{aligned}\tilde{v}_i &= v_i + \hat{v}_i,\\ \tilde{e}_i &= \left\{e_i^k + A_i^k \hat{v}_i + \hat{e}_{i\leftarrow \text{in}}^k - \hat{e}_{i\to \text{out}}^k\right\}_{k\in M}\end{aligned} \tag{4}$$

where $\hat{e}_{i\leftarrow \text{in}}^k = \sum_{j\in N_i} \hat{e}_{j\to i}^k$ and $\hat{e}_{i\to \text{out}}^k = \sum_{j\in N_i} \hat{e}_{i\to j}^k$. For each state $x \in X$, define

$$\mathcal{A}_i^{\text{PF}}(x) = \{(\hat{v}, \hat{e}) : v_i + \hat{v}_i \in \mathcal{V}_i\} \tag{5}$$

as the admissible action set of agent i. The null action, $\mathbf{0}$, takes on the form $\hat{v}_i = 0$ and $\hat{e}_{i\to j}^k = 0$ for all $i, k \in N$ and $j \in N_i$. Note that $\mathbf{0} \in \mathcal{A}_i(x)$ for any $i \in N$, $x \in X$.

Invariance Property: If the initial estimation terms $e(0)$ satisfy $\sum_{i\in N} e_i^k(0) = \sum_{i\in N} A_i^k v_i(0) - B^k, \ \forall k \in M$, then for any sequence of action profiles $a(0), a(1), \ldots$, the resulting state trajectory generated according to process $x(t+1) = f(x(t), a(t))$ satisfies

$$\sum_{i\in N} e_i^k(t) = \sum_{i\in N} A_i^k v_i(t) - B^k \tag{6}$$

for all constraints $k \in M$ and $t \geq 0$. Hence, for any constraint $k \in M$ we have

$$\sum_{i\in N} e_i^k(t) \leq 0 \Leftrightarrow \sum_{i\in N} A_i^k v_i(t) - B^k \leq 0. \tag{7}$$

Accordingly, the estimation terms encode information pertaining to constraint violations. Note that if the initial value profile $v(0)$ satisfies the constraints $k \in M$, then assigning $e_i^k(0) = A_i^k v_i(0) - (1/n)B^k$ ensures that $e(0)$ satisfies the above condition. We will assume throughout that the initial value and estimation profiles satisfy these initial conditions.

Agent Cost Functions: For any state $x \in X$ and admissible action profile $a \in \prod_{i\in N} \mathcal{A}_i(x)$, the cost function of agent i is defined as

$$J_i^{\text{PF}}(x, a) = C_i(\tilde{v}_i) + \mu \sum_{j\in N_i} \sum_{k=1}^{m} \left[\max\left(0, \tilde{e}_j^k\right)\right]^2 \tag{8}$$

where $(\tilde{v}, \tilde{e}) = f(x, a)$ is the ensuing state and $\mu > 0$ is a trade-off parameter. The first term captures agent i's local cost function while the second term introduces a penalty on inconsistencies in estimation terms between neighboring agents.

We now provide the main result of this technical note.

Theorem 1: Model the constrained optimization problem in (1) as a state based game with a fixed trade-off parameter $\mu > 0$ as depicted in Section III-B. The state based game is a state based potential game with potential function

$$\Phi^{\text{PF}}(x, a) = \phi(\tilde{v}) + \mu \sum_{i\in N} \sum_{k=1}^{m} \left[\max\left(0, \tilde{e}_i^k\right)\right]^2 \tag{9}$$

where $(\tilde{v}, \tilde{e}) = f(x, a)$ represents the ensuing state. Furthermore, if the objective function $\phi : \mathcal{V} \to \mathbb{R}$ is convex and differentiable and the communication graph is undirected and connected, then a state action pair $[x, a] = [(v, e), (\hat{v}, \hat{e})]$ is a stationary state Nash equilibrium if and only if the following four conditions are satisfied:

i) The value profile v is an optimal point of the uncoupled constrained optimization problem

$$\min_{v\in\mathcal{V}} \phi(v) + \frac{\mu}{n} \sum_{k\in M} \left[\max\left(0, \sum_{i\in N} A_i^k v_i - B^k\right)\right]^2. \tag{10}$$

ii) The estimation profile e satisfies that for all $i \in N$, $k \in M$

$$\max\left(0, e_i^k\right) = \frac{1}{n} \max\left(0, \sum_{i\in N} A_i^k v_i - B^k\right)$$

iii) The change in value profile satisfies $\hat{v}_i = 0$ for all agents $i \in N$.

iv) The net change in estimation profile is 0, i.e., $\hat{e}_{i\leftarrow \text{in}}^k - \hat{e}_{i\to \text{out}}^k = 0$ for all agents $i \in N$ and constraints $k \in M$.

This characterization proves the equivalence between the stationary state Nash equilibria of the designed game and solutions to the uncoupled constrained optimization problem in (10). Therefore, as $\mu \to \infty$, all equilibria of our designed game are solutions to (1) [22].

Proof: It is straightforward to show that the potential function in (9) satisfies the conditions of state based potential games given in Definition 1. Hence, we will focus purely on the presented characterization. Throughout, we will use the notation $(\tilde{v}, \tilde{e})$ to represent the ensuing state for a state action pair $[x, a]$, i.e., $(\tilde{v}, \tilde{e}) = f(x, a)$.

($\Leftarrow$) We start by proving that if a state action pair $[x, a]$ satisfies Conditions i)–iv) then $[x, a]$ is a stationary state Nash equilibrium. First, we know that if a satisfies Conditions iii)–iv) then $x = f(x, a)$. Hence, we only need to prove that $a \in \arg\min_{\check{a}\in\mathcal{A}(x)} \Phi(x, \check{a})$. Let $\check{a} := (\check{v}, \check{e}) \in \mathcal{A}(x)$. Since $\Phi(x, \check{a}) = \Phi(x, (\check{v}, \check{e}))$ is convex over $(\check{v}, \check{e})$, the necessary and sufficient conditions for $a = (\hat{v}, \hat{e})$ to be an optimal solution of the optimization problem $\min_{\check{a}\in\mathcal{A}(x)} \Phi(x, \check{a})$ are

$$\left.\frac{\partial \Phi(x, \check{a})}{\partial \check{e}_{i\to j}^k}\right|_a = 0, \ \forall i \in N, \ j \in N_i, \ k \in M, \tag{11}$$

$$\left.\frac{\partial \Phi(x, \check{a})}{\partial \check{v}_i}\right|_a \cdot (\hat{v}_i' - \hat{v}_i) \geq 0, \ \forall i \in N, \ \hat{v}_i' \in \mathcal{A}_i^v(x). \tag{12}$$

Since $\Phi(x, a) = \Phi(\tilde{x}, \mathbf{0}) = \Phi(f(x, a), \mathbf{0})$, we have $\partial\Phi(x, a)/\partial a = \partial\Phi(f(x, a), \mathbf{0})/\partial a$. Therefore (11) and (12) simplify to

$$\max\left(0, \tilde{e}_j^k\right) - \max\left(0, \tilde{e}_i^k\right) = 0, \quad \forall i \in N, \ j \in N_i, \ k \in M \tag{13}$$

$$\left[\left.\frac{\partial \phi}{\partial v_i}\right|_{\tilde{v}} + 2\mu \sum_{k\in M} A_i^k \max\left(0, \tilde{e}_i^k\right)\right] \cdot (\tilde{v}_i' - \tilde{v}_i) \geq 0, \quad \forall i \in N, \ \tilde{v}_i' \in \mathcal{V}_i. \tag{14}$$

To complete the proof of this direction we will actually prove the following stronger statement: if the ensuing state $\tilde{x}$ of a state action pair $[x, a]$ satisfies Conditions i)–ii), then $a \in \arg\min_{\check{a}\in\mathcal{A}(x)} \Phi(x, \check{a})$.

For such a state action pair $[x, a]$, it is straightforward to show that $\tilde{x}$ satisfies the following conditions:

$$\max\left(0, \tilde{e}_i^k\right) = \max\left(0, \tilde{e}_j^k\right) = \frac{1}{n}\max\left(0, \sum_{i=1}^{n} A_i^k \tilde{v}_i - B^k\right), \quad \forall i, j \in N,\ k \in M \tag{15}$$

$$\left[\left.\frac{\partial \phi}{\partial v_i}\right|_{\tilde{v}} + \frac{2\mu}{n}\sum_{k\in M} A_i^k \max\left(0, \sum_{i=1}^{n} A_i^k \tilde{v}_i - B^k\right)\right](\tilde{v}_i' - \tilde{v}_i) \geq 0, \quad \forall i \in N,\ \tilde{v}_i' \in \mathcal{V}_i. \tag{16}$$

Equation (15) is from Condition i) and equality (6). Equation (16) is the optimal condition of the optimization problem $\min_{v\in\mathcal{V}} \phi(v) + (\mu/n)\alpha(v)$. Substituting (15) into (16) proves that $\tilde{x}$ satisfies the two optimality conditions in (13) and (14). Hence, $a \in \arg\min_{\check{a}\in\mathcal{A}(x)} \Phi(x, \check{a})$. Therefore, we can conclude that such $[x, a]$ is a stationary state Nash equilibrium.

($\Rightarrow$) Now we prove the other direction of this theorem. First, notice that if $[x, a]$ is a stationary state Nash equilibrium, then the action profile $a = (\hat{v}, \hat{e})$ must satisfy Conditions iii)–iv). Otherwise, $x = (v, e) \neq f(x, a)$. Second, if $[x, a]$ is a stationary state Nash equilibrium, then $J_i(x, a_i, a_{-i}) = \min_{\check{a}_i\in\mathcal{A}_i(x)} J_i(x, \check{a}_i, a_{-i})$ for each $i \in N$. Since $J_i(x, \check{a}_i, a_{-i})$ is a convex function on $\check{a}_i := (\check{v}_i, \check{e}_i) \in \mathcal{A}_i(x)$, we know that

$$\left.\frac{\partial J_i(x, \check{a}_i, a_{-i})}{\partial \check{e}_i}\right|_a = 0,\ \forall i \in N, k \in M \tag{17}$$

$$\left[\left.\frac{\partial J_i(x, \check{a}_i, a_{-i})}{\partial \check{v}_i}\right|_a\right] \cdot (\hat{v}_i' - \hat{v}_i) \geq 0,\ \forall i \in N, \hat{v}_i' \in \mathcal{A}_i^v(x) \tag{18}$$

which is equivalent to

$$2\mu\left(\max\left(0, \tilde{e}_l^k\right) - \max\left(0, \tilde{e}_i^k\right)\right) = 0, \quad \forall i \in N, j \in N_i, k \in M \tag{19}$$

$$\left[\left.\frac{\partial C_i}{\partial v_i}\right|_{\tilde{v}} + 2\mu\sum_{k\in M} A_i^k \max\left(0, \tilde{e}_i^k\right)\right] \cdot (\tilde{v}_i' - \tilde{v}_i) \geq 0, \quad \forall i \in N,\ \tilde{v}_i' \in \mathcal{V}_i. \tag{20}$$

Equation (19) implies that $\max(0, \tilde{e}_i^k) = \max(0, \tilde{e}_j^k)$ for all agents $i, j \in N$ and constraints $k \in M$ since the communication graph is connected. Applying the equality in (6), we have that for all agents $i \in N$ and constraints $k \in M$, $\max(0, \tilde{e}_i^k) = (1/n)\max(0, \sum_{i=1}^{n} A_i^k \tilde{v}_i - B^k)$. Substituting this equality into (20) gives us

$$\left[\left.\frac{\partial \phi}{\partial v_i}\right|_{\tilde{v}} + \frac{2\mu}{n}\sum_{k\in M} A_i^k \max\left(0, \sum_{i=1}^{n} A_i^k \tilde{v}_i - B^k\right)\right] \cdot (\tilde{v}_i' - \tilde{v}_i) \geq 0 \tag{21}$$

for all $\tilde{v}_i' \in \mathcal{V}_i$. Hence, $\tilde{v}$ the optimal solution to $\phi(v) + (\mu/n)\alpha(v)$. Combining with the fact that $x = \tilde{x} = f(x, a)$, we can conclude that $x = (v, e)$ satisfies Conditions i)–iv). ∎

C. Design Using Barrier Functions

In this section we introduce our second design which integrates *barrier functions*, as opposed to exterior penalty functions, into the design of the agents' cost functions. The key difference between the two approaches lies in the feasibility of both the intermediate and asymptotic solutions. In particular, barrier functions can be employed to ensure that both the intermediate and asymptotic solutions are in the interior feasible set. Accordingly, we assume that the interior feasible set of problem (1) is nonempty when implementing barrier function methods. Note that this implies that equality constraints are not permissible.

State Space, Actions, State Dynamics: These three parts are identical to those in Section III-B.

Admissible Action Sets: Let $x = (v, e)$ represent a *strictly feasible* state where the value profile v satisfies $\sum_{i=1}^{n} A_i^k v_i < B^k, v_i \in \mathcal{V}_i$ and the estimation profile e satisfies $e_i^k < 0$ for each $i \in N$ and $k \in M$. Define the admissible action set for each agent $i \in N$ as

$$\mathcal{A}_i^{BF}(x) = \left\{(\hat{v}_i, \hat{e}_i) : v_i + \hat{v}_i \in \mathcal{V}_i, \hat{e}_{i\to j}^k \leq 0,\ e_i^k + A_i^k \hat{v}_i - \hat{e}_{i\to\text{out}}^k < 0, \forall k \in M\right\}. \tag{22}$$

If the initial state $x(0)$ is strictly feasible and the initial estimation terms $e(0)$ satisfy $\sum_{i\in N} e_i^k(0) = \sum_{i\in N} A_i^k v_i(0) - B^k$, then the resulting state trajectory generated according to process $x(t+1) = f(x(t), a(t))$ where $a(t) \in \prod_{i\in N} \mathcal{A}_i^{BF}(x(t))$ for all $t \geq 0$ is also strictly feasible.

Agents' Cost Functions: For any state $x \in X$ and admissible action profile $a \in \prod_{i\in N} \mathcal{A}_i^{BF}(x)$, the cost function of agent i is defined as

$$J_i^{BF}(x, a) = C_i(\tilde{v}_i) - \mu\sum_{j\in N_i}\sum_{k=1}^{m} \log\left(-\tilde{e}_j^k\right) \tag{23}$$

where $\mu > 0$ is a trade-off parameter. Note that the sole difference between (23) and (8) rests on the penalty associated with being close to constraint violations.

The above methodology using barrier functions yields a game that possesses the same analytical properties of the designed game using exterior penalty function as given in Theorem 1 with two exceptions. First, the potential function of the state based game is now of the form

$$\Phi^{BF}(x, a) = \phi(\tilde{v}) - \mu\sum_{i\in N}\sum_{k=1}^{m} \log\left(-\tilde{e}_i^k\right). \tag{24}$$

Second, (10) is replaced with a new optimization problem of the form

$$\min_{v\in\mathcal{V}} \quad \phi(v) - n\mu\sum_{k\in M}\log\left(B^k - \sum_{i\in N} A_i^k v_i\right) \qquad \text{s.t.} \quad \sum_{i=1}^{m} A_i^k v_i - B^k < 0, k \in M. \tag{25}$$

We omit a formal statement of the theorem in addition to the proof for brevity as it is virtually identical to that of Theorem 1. Note that as $\mu \to 0$, all equilibria of our designed game are solutions to the constrained optimization problem in (1) [22].

IV. An Illustrative Example

Consider an economic dispatch problem in electricity power systems, introduced in [23], with N generators and a demand requirement $D \geq 0$. Each generator is capable of generating an amount of power $v_i \in \mathcal{V}_i = [\underline{v}_i, \bar{v}_i]$, where $\underline{v}_i$ and $\bar{v}_i$ denote the minimum and maximum generation levels respectively, subject to a cost $C_i(v_i)$. The system level objective is to meet the demand level D while minimizing the sum of the costs incurred by the generators. More specifically, the system level objective is of the form

$$\min_{v_i \in [\underline{v}_i, \bar{v}_i]} \quad \phi(v) = \sum_{i \in N} C_i(v_i)$$
$$\text{s.t.} \qquad \sum_{i \in N} v_i \geq D. \tag{26}$$

One of the central challenges associated with attaining generation levels $v \in \mathcal{V}$ to optimize (26) is that each individual generator selects its own generation level in response to incomplete information regarding the system as a whole.

Consider a simple economic dispatch problem where $N = \{1,2,3,4\}$, generation capabilities $\mathcal{V}_i = [0,5]$ for all $i \in N$, cost functions $C_i(v_i) = v_i^2 + v_i + 10$ for $i \in \{1,2\}$ and $C_i(v_i) = 0.5v_i^2 + v_i + 10$ for $i \in \{3,4\}$, a demand $D = 12$, and a communication graph of the form $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$. It is straightforward to verify that the optimal generation levels are (2,2,4,4).

The methodologies developed in this technical note can be used to attain a distributed solution to this economic dispatch problem that satisfies the given communication graph. The following highlights the specifics of our design, while focusing on generator 2, for the penalty function method given in Section III-B: i) state: $x_2 = (v_2, e_2)$; ii) action: $a_2 = (\hat{v}_2, \hat{e}_{2\to 1}, \hat{e}_{2\to 3})$; iii) admissible action set: $\mathcal{A}_2^{\text{PF}}(x_2) = \{(\hat{v}_2, \hat{e}_2) : v_2 + \hat{v}_2 \in [\underline{v}_2, \bar{v}_2]\}$; state dynamics: $(\tilde{v}, \tilde{e}) = f(x,a)$: $\tilde{v}_2 = v_2 + \hat{v}_2$ and $\tilde{e}_2 = e_2 - \hat{v}_2 + \hat{e}_{2\leftarrow \text{in}} - \hat{e}_{2\to\text{out}}$; and (v) cost function: $J_2^{\text{PF}}(x,a) = C_2(\tilde{v}_2) + \mu \sum_{j \in N_2} [\max(0, \tilde{e}_j)]^2$. The specifics for the alternative generators could be derived in a similar fashion. Likewise, integrating barrier functions as opposed to penalty functions would incorporate substituting $\mathcal{A}_i^{\text{BF}}$ for $\mathcal{A}_i^{\text{PF}}$ as defined in (22) and J_i^{BF} for J_i^{PF} as defined in (23).

Fig. 1 shows simulation results for both the penalty function method and barrier function method when employing the adaptation rule gradient play for state based potential games, defined in [21], and initializing the generation levels at $v(0) = (5,4,3,2)$ and the estimation $e_i(0) = v_i(0) - (1/4)D$. The learning algorithm gradient play guarantees convergence to a stationary state Nash equilibrium in any state based potential game [21], hence gradient play can be utilized to complete the control design. The gradient play algorithm takes on the following forms:

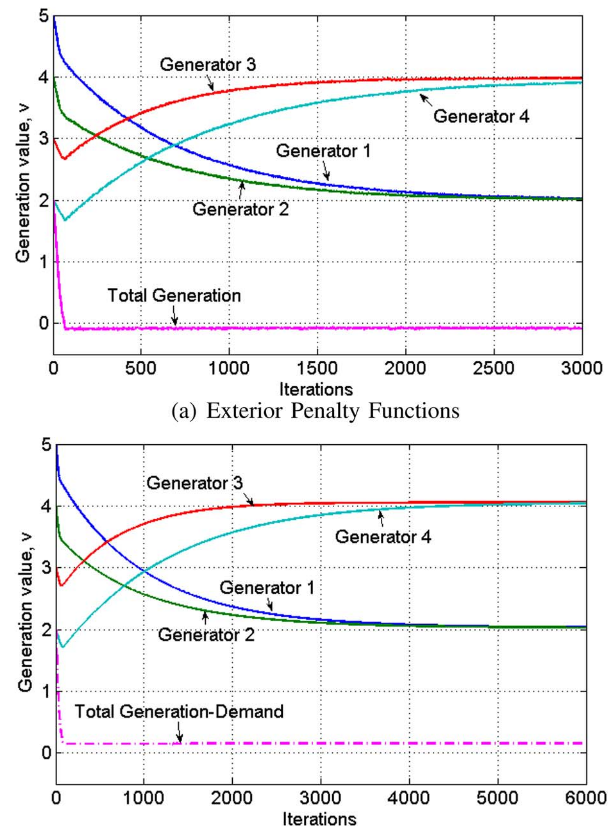
Penalty Function Method

$$\hat{v}_i(t) = \left[-\epsilon_i(t) \cdot \left. \frac{\partial J_i(x(t), a)}{\partial \hat{v}_i} \right|_{a=\mathbf{0}} \right]^+_{\mathcal{A}_i^{\text{PF}}(x(t))}$$
$$\hat{e}^k_{i\to j}(t) = -\epsilon_i(t) \cdot \left. \frac{\partial J_i(x(t), a)}{\partial \hat{e}^k_{i\to j}} \right|_{a=\mathbf{0}}$$

Barrier Function Method

$$\hat{v}_i(t) = \beta(t) \left(-\epsilon_i(t) \left. \frac{\partial J_i(x(t), a)}{\partial \hat{v}_i} \right|_{a=\mathbf{0}} \right)$$
$$\hat{e}^k_{i\to j}(t) = \beta(t) \min \left(0, -\epsilon_i(t) \left. \frac{\partial J_i(x(t), a)}{\partial \hat{e}^k_{i\to j}} \right|_{a=\mathbf{0}} \right)$$

where $\epsilon_i(t)$ is the step size of agent i at time t, $[\cdot]^+$ represents the projection onto the represented closed convex set, and $\beta(t) = (1/2)^{l(t)}$ where $l(t)$ is the smallest nonnegative integer l such that $(\hat{v}_i(t), \hat{e}_{i\to j}(t)) \in \mathcal{A}_i^{\text{BF}}(x(t))$. Note that computing such gradients only requires each agent $i \in N$ to have access to the state of neighboring agents $j \in N_i$, i.e., $\{x_j(t)\}_{j \in N_i}$. To illustrate the robustness of the algorithm, we consider the situation where the step size of each agent,



(a) Exterior Penalty Functions

(b) Barrier Functions

Fig. 1. Simulation results for the economic dispatch problem. Subfigure 1(a) shows the simulation results when using gradient play applied to the state based game with exterior penalty functions using a tradeoff parameter $\mu = 60$. The simulation demonstrates that the profile of generation levels quickly approaches (1.97, 1.97, 3.93, 3.93) which is close to optimal. However, the generation levels do necessarily satisfy the demand. Subfigure 1(b) shows the simulation results when using gradient play applied to the state based game with barrier functions using a tradeoff parameter $\mu = 0.2$. The simulation demonstrates that the profile of generation levels quickly approaches (2.03, 2.03, 4.02, 4.02) which is close to optimal. Furthermore, the generation levels always exceed the demand in this setting.

i.e., $\epsilon_i(t)$, is chosen uniformly at random from the interval [0,0.002] at each iteration t. The generation levels quickly converge close to the optimal generation levels for both approaches, as guaranteed by [21]. The barrier function approach takes longer to converge since this approach also ensures that the transient generation levels satisfy the given demand.

V. Conclusion

This technical note focuses on the general question of how to design local agent objective functions for distributed engineering systems. The influence and design of objective functions has been extensively studied in several branches of game theory ranging from cost sharing problems to mechanism design; however, as the role of game theory transitions from a descriptive tool for social systems to a prescriptive tool for engineering systems, several pertinent questions are unresolved. At the forefront is how do you design admissible agent objective functions such as to optimize the efficiency of the resulting equilibria? This manuscript provides some initial results along this direction but admittedly focuses on a special class of system level objective functions. Future work will investigate this question for broader problem settings.

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