Demand Response in Radial Distribution Networks: Distributed Algorithm

(Invited Paper)

Na Li, Lijun Chen and Steven H. Low

***Abstract*—Demand response has recently become a topic of active research. Most of work however considers only the balance between aggregate load and supply, and abstracts away the underlying power network. In this paper, we study demand response in a radial distribution network, by formulating it as an optimal power flow problem that maximizes the aggregate user utilities and minimizes the supply cost and the power line losses, subject to the power flow constraints and operating constraints. We propose a fully distributed algorithm for the users to coordinate their demand response decisions through local communication with their neighbors so as to achieve the optimum. Numerical examples with the real-world distribution circuits are provided to complement our theoretical analysis.**

***Index Terms*—Demand response, Distributed algorithm, Branch flow model, Optimal power flow problem, Distribution networks**

I. Introduction

Load management is increasingly needed to improve power system efficiency and integrate ever-increasing renewable generation [1], [2]. A large literature have developed different schemes for load management; see, e.g., [3]–[7]. Most of work however considers only the balance between aggregate load and supply, and abstracts away the underlying power network and the associated power flow constraints and operating constraints. As a result, the schemes proposed may end up with an electricity consumption/shedding decision that would violate those network and operating constraints. There are some recent work on load management that take into consideration the physical network constraints; see, e.g., [8]–[11]. But they usually use the bus injection model for the electricity network and propose location-based marginal pricing schemes for load management, which is more suitable for the transmission system.

In this paper, we study optimal load management in the presence of the network and operating constraints for the radial distribution networks, using the branch flow model [12], [13] for the electricity network rather than the bus injection model. The branch flow model focuses on currents and powers on the branches. It has been used mainly for modeling distribution circuits that are usually radial. Specifically, we formulate load management problem as an optimal power flow (OPF) problem whose objective is to maximize the aggregate user utilizes and minimize the supply cost and the power line losses, subject to the power flow constraints and operating constraints such as the voltage regulation constraint and power injection constraints. Since the resulting OPF problem is non-convex and thus difficult to solve, we propose a convex relaxation of the optimization problem, and discuss whether the relaxation can be exact and under what conditions. Convexity of the optimization problem is usually required for the development of computationally-efficient and distributed algorithms for system operations.

In a previous paper [14], we consider a scenario where the radial distribution network is served by a single load-serving entity (LSE), which coordinates the end users' demand response decisions by setting the right prices, and propose a distributed algorithm for the LSE to find such price signals. This algorithm requires two-way communication between the LSE and each user, and at each iteration, the LSE is required to solve a large OPF problem. In this paper, we instead develop a fully distributed OPF algorithm for demand response, where the end users make and coordinate their local demand response decisions through local communication with their neighbors. This demand response scheme requires two-way communication only between the end users that are directly connected in the distribution network, and each user only needs to solve a small optimization problem. We develop this demand response algorithm based on a well-known distributed algorithm, Predictor Corrector Proximal Multiplier (PCPM) [15]. Provided that the convex relaxation of the OPF problem for demand response is exact, the algorithm is guaranteed to converge to the global optimum of the problem.

The rest of the paper is organized as follows. In Section II we formulate the optimal demand response problem, introduce the PCPM algorithm, and discuss convex relaxation of the optimization problem. In Section III, we develop a fully decentralized algorithm for demand response. In Section IV, we provide numerical examples to complement the theoretical analysis, using a real-word distribution circuit.

Notations:

- V_i, v_i: complex voltage on bus i with $v_i = |V_i|^2$;
- $s_i = p_i + \mathbf{i}q_i$: complex net load on bus i;
- I_{ij}, ℓ_{ij}: complex current from buses i to j with $\ell_{ij} = |I_{ij}|^2$;
- $S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$: complex power flowing out from buses i to bus j;
- $z_{ij} = r_{ij} + \mathbf{i}x_{ij}$: impedance on line (i, j);

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II. Problem formulation & Preliminary

A. Problem formulation

Consider a radial distribution circuit that consists of a set N of buses and a set E of distribution lines connecting these buses. We index the buses in N by $i = 0, 1, \ldots, n$, and denote a line in E by the pair (i, j) of buses it connects and the index i denotes the bus that is closer to the feeder. Bus 0 denotes the feeder, which has fixed voltage but flexible power injection to balance the loads; each of the other buses $i \in N\backslash\{0\}$ represents an aggregator that can participate in demand response. For convenience we call aggregator i as user i, which actually represents a or a group of customers that are connected to bus i and join the demand response system as a single entity.

For each link $(i, j) \in E$, let $z_{ij} = r_{ij} + \mathbf{i}x_{ij}$ be the impedance on line (i, j), and $S_{i,j} = P_{i,j} + \mathrm{i}Q_{i,j}$ and $I_{i,j}$ the complex power and current flowing from bus i to bus j. At each bus $i \in N$, let $s_i = p_i + \mathbf{i}q_i$ be the complex load and V_i the complex voltage. As customary, we assume that the complex voltage V_0 on the feeder is given and fixed. The branch flow model, first proposed in [12], models power flows in a steady state in a radial distribution network: for each $(i, j) \in E$,

$$\frac{P_{i,j}^2 + Q_{i,j}^2}{v_i} = \ell_{i,j}, \quad (1)$$

$$P_{i,j} = \sum_{h:(j,h)\in E} P_{j,h} + r_{i,j}\ell_{i,j} + p_j, \quad (2)$$

$$Q_{i,j} = \sum_{h:(j,h)\in E} Q_{j,h} + x_{i,j}\ell_{i,j} + q_j, \quad (3)$$

$$v_i - v_j = 2(r_{i,j}P_{i,j} + x_{i,j}Q_{i,j}) - (r_{i,j}^2 + x_{i,j}^2)\ell_{i,j}, \quad (4)$$

where $\ell_{i,j} := |I_{i,j}|^2$, $v_i := |V_i|^2$. Each user $i \in N\backslash\{0\}$ achieves certain utility $f_i(p_i)$ when its (real) power consumption is p_i. The utility function $f_i(\cdot)$ is usually assumed to be continuous, nondecreasing, and concave. Furthermore, there are the following operating constraints for each $i \in N\backslash\{0\}$:

$$\underline{v}_i \le v_i \le \bar{v}_i, i = 1, \cdots, n, \quad (5)$$

$$\underline{q}_i \le q_i \le \bar{q}_i, i = 1, \cdots, n, \quad (6)$$

$$\underline{p}_i \le p_i \le \bar{p}_i, i = 1, \cdots, n. \quad (7)$$

The electricity is delivered from the main grid to the radial distribution network through the feeder (i.e., the bus 0). The total (real) power supply P_0 is given by $P_0 := \sum_{j:(0,j)\in E} P_{0,j}$.

We consider a situation where the power supply P_0 is constrained by an upper bound $\bar{P}_0$, i.e.,

$$P_0 = \sum_{j:(0,j)\in E} P_{0,j} \le \bar{P}_0. \quad (8)$$

Under such a situation, we would like to design a distributed mechanism to guide each user i to choose a proper load p_i, so as to i) meet the supply constraint (8) as well as the power flow constraints and operating constraints listed in (1) - (7) and ii) maximize the aggregate user utilities and minimize the power supply costs and power line losses. This demand response problem is formulated as the following optimal power flow problem (OPF):

$$\textbf{OPF:} \max_{P,Q,l,v,p,q} \quad \sum_{i=1}^{n} f_i(p_i) - C_0(P_0) - \rho \sum_{(i,j)\in E} r_{i,j}\ell_{i,j}$$
$$\text{s.t.} \quad (1) - (8),$$

where ρ is a trade off parameter. Throughout the paper, we assume that the feasible set of this problem is nonempty. In the following, we will develop a fully distributed OPF algorithm for demand response, where the end users make and coordinate their local demand response decisions through local communication with their neighbors.

B. A decentralized optimization algorithm: predictor corrector proximal multiplier (PCPM)

This paper we focus on using the decentralized algorithm, predictor corrector proximal multiplier (PCPM) [15] to develop a distributed demand response scheme. Consider the following convex problem:

$$\min_{x\in X, y\in Y} \quad f(x) + g(y) \quad (9a)$$

$$\text{s.t.} \quad Ax + By = C \quad (9b)$$

Introduce the Lagrangian variable z for constraint (9b).
The algorithm PCPM is given as follows:

1) Initially set $k \leftarrow 0$ and randomly choose initial (x^0, y^0, z^0).
2) For each $k \ge 0$, update a virtual variable $\hat{z}^k := z^k + \gamma(Ax^k - By^k - C)$ Here $\gamma > 0$ is a constant parameter.
3) Based on the virtual variable $\hat{z}^k$, update x, y according to:
$$x^{k+1} = \arg\min_{x\in X}\{f(x) + (\hat{z}^k)^T Ax + (1/(2\gamma))||x - x^k||^2\},$$
$$y^{k+1} = \arg\min_{y\in Y}\{g(y) + (\hat{z}^k)^T By + (1/(2\gamma))||y - y^k||^2\}.$$
4) z is updated according to $z^{k+1} = z^k + \gamma(Ax^{k+1} + By^{k+1} - C)$.
5) $k \leftarrow k + 1$, and go to step 2).

From the algorithm, we see that PCPM is highly decomposable. In terms of convergence, it has been shown in [15] that as long as strong duality holds for the convex problem (9), the algorithm will converge to a primal-dual optimal solution (x^*, y^*, z^*) for sufficient small positive γ.

C. Convexification of Problem OPF

OPF is non-convex due to the quadratic equality constraints in (1) and thus difficult to solve. Moreover, most decentralized algorithms require convexity to ensure convergence, e.g., PCPM as described in II-B. We therefore consider the following convex relaxation of OPF:

$$\textbf{ROPF:} \max_{P,Q,l,v,p,q} \quad \sum_{i=1}^{n} f_i(p_i) - C_0(P_0) - \rho \sum_{(i,j)\in E} r_{i,j}l_{i,j}$$
$$s.t. \quad (2) - (7)$$
$$\frac{P_{i,j}^2 + Q_{i,j}^2}{v_i} \le l_{i,j}, \ (i,j) \in E, \quad (10)$$

where the equality constraints (1) are relaxed to the inequality constraints (10). ROPF provides a lower bound on OPF. For an optimal solution $X^* := (P^*, Q^*, \ell^*, v^*, p^*, q^*)$ of ROPF, if the equality in (10) is attained at X^*, then X^* is also a solution to OPF. We call ROPF an *exact relaxation* of OPF if every solution to ROPF is also a solution to OPF, and vice versa. In previous work [13], [16], we have studied whether and when ROPF is an exact relaxation of OPF for the radial networks. It is shown in [13] that the relaxation is exact when there are no upper bounds on the loads. However, removing upper bounds on the loads may be unrealistic, especially in the context of demand response. It is shown in [16] that the relaxation is exact provided that instead there are no upper bounds on the voltage magnitudes and certain other conditions hold, which are verified to hold for many real-world distribution systems. Moreover, the upper bounds on the voltage magnitudes for the relaxation solution are characterized [16].

The benefit of convexity is that convexity does not only facilitates the design of efficient pricing schemes for power market and demand response, but it also facilitates the development of tractable, scalable and distributed algorithms for system operations. Hence the conditions for exact relaxation of OPF to ROPF specified in [16] is important for our demand response design. In the rest of the paper, we will assume that ROPF is an exact relaxation of OPF and strong duality holds for ROPF. When ROPF is an exact relaxation of OPF, we can just focus on solving the convex optimization problem ROPF.

III. A FULLY DECENTRALIZED ALGORITHM

In this paper, we develop a fully distributed OPF algorithm for demand response, where the end users make and coordinate their local demand response decisions through local communication with their neighbors. Specifically, we assume that each user has certain computation ability to decide a set of local variables of the OPF. The composition of those variables determines the global status of the power flow over the distribution network. We also assume that there is two way communication available between any two users that are directly connected in the distribution network. In the decentralized OPF algorithm, at each iteration each user makes decisions about the local variables, communicate those decisions with neighbors, and then update their local variables and repeat the process.

Before establishing the algorithm, let us define the local decision variables for each user. Let $\pi(i)$ be the parent of bus i and $\delta(i)$ be the direct children of bus i. The local decision variables for each buses are:

- For bus 0, P_0, v_0, where v_0 is fixed by convention.
- For bus $i > 0$, $P_{\pi(i),i}$, $Q_{\pi(i),i}$, $\ell_{\pi(i),i}$, p_i,q_i,v_i,$\hat{v}_i$. Here $\hat{v}_i$ is bus i's estimation about its parent's voltage $v_{\pi(i)}$. To simplify the notations, we denote $P_{\pi(i),i}$, $Q_{\pi(i),i}$, $\ell_{\pi(i),i}$ as P_i, Q_i, ℓ_i; and $r_{\pi(i),i}$, $x_{\pi(i),i}$ as r_i, x_i.

With the new notations, OPF can be rewritten as:

$$\max_{P,Q,l,v,p,q} \quad \sum_{i=1}^{n} f_i(p_i) - C_0(P_0) - \sum_{i=1}^{n} r_i l_i \tag{11a}$$

$$\text{s.t.} \quad P_0 = \sum_{j:(0,j)\in E} P_j, \tag{11b}$$

$$P_i = \sum_{j\in\delta(i)} P_j + r_i l_i + p_i, i \in N\backslash\{0\} \tag{11c}$$

$$Q_i = \sum_{j\in\delta(i)} Q_j + x_i l_i + q_i, i \in N\backslash\{0\} \tag{11d}$$

$$\hat{v}_i = v_{\pi(i)}, i \in N\backslash\{0\} \tag{11e}$$

$$P_0 \le \bar{P}, \tag{11f}$$

$$\underline{v}_i \le v_i \le \bar{v}_i, i \in N\backslash\{0\} \tag{11g}$$

$$\underline{p}_i \le p_i \le \bar{p}_i, i \in N\backslash\{0\} \tag{11h}$$

$$\underline{q}_i \le q_i \le \bar{q}_i, i \in N\backslash\{0\} \tag{11i}$$

$$\frac{P_i^2 + Q_i^2}{\hat{v}_i} \le l_i, i \in N\backslash\{0\} \tag{11j}$$

$$\hat{v}_i - v_i = 2(r_i P_i + x_i Q_i) - (r_i^2 + x_i^2) l_i, \quad i \in N\backslash\{0\}, \tag{11k}$$

The new formulation has the following properties that can be utilized for the design of distributed algorithms:

- The objective function (11a) is fully decomposable.
- Constraints (11b-11e) are linear coupled constraints but each constraint only constrains "local" information, namely that each constraint is defined over the local variables of one node and its direct neighbors over the radial network.
- Constraints (11f-11k) are just local constraints that are defined over bus i's local decision variables.

Then we can apply algorithm *PCPM* to define a decentralized algorithm. We will use PCPM to decouple those linear coupled constraints (11b-11e). Let the Lagrangian dual variable corresponding to constraint (11b) be λ_0 and dual variables corresponding to constraints (11c-11e) be $\lambda_i, \theta_i, \omega_i$ for each $i \in N\backslash\{0\}$. In the following distributed algorithm, node 0 takes charge of updating λ_0 and node $i \in N\backslash\{0\}$ takes charge of updating $\lambda_i, \theta_i, \omega_i$. Now let us introduce the distributed demand response algorithm which converges to a global optimal solution of the OPF.

1) Initially set $k \leftarrow 0$. Node 0 randomly chooses P_0^k and λ_0^k and node $i \in N\backslash\{0\}$ randomly chooses P_i^k, Q_i^k, ℓ_i^k, p_i^k,q_i^k,v_i^k,$\hat{v}_i^k$ and the dual variables $\lambda_i^k, \theta_i^k, \omega_i^k$. Each node $i \in N\backslash\{0\}$ send the primal variables P_i^k, Q_i^k, ℓ_i^k to its parent $\pi(i)$, and each node $i \in N$ except leaves in the network send v_i^k to its children. Note that v_0^k is fixed for any k.
2) For each $k \ge 0$, node 0 send a virtual dual signal $\hat{\lambda}_0^k := \lambda_0^k + \gamma(P_0^k - \sum_{j:(0,j)\in E} P_j^k)$ to its children; and each node $i \in N\backslash\{0\}$ except the leaves send the following

virtual signals to its children:

$$\hat{\lambda}_i^k = \lambda_i^k + \gamma \left(P_i^k - \left(\sum_{j\in\delta(i)} P_j^k + r_i l_i^k + p_i^k \right) \right),$$

$$\hat{\theta}_i^k = \theta_i^k + \gamma \left(Q_i^k - \left(\sum_{j\in\delta(i)} Q_j^k + x_i l_i^k + q_i^k \right) \right);$$

and each node $i \in N\backslash\{0\}$ send the following virtual signals to its parent:

$$\hat{\omega}_i^k = \omega_i^k + \gamma(\hat{v}_i^k - v_{\pi(i)}^k).$$

Here $\gamma > 0$ is a constant parameter.

3) Each node update their local primal variables according to the following rules.

Case 1: Node 0 solves the following problem:

$$\begin{aligned} \min_{P_0} \quad & C_0(P_0) + \hat{\lambda}_0^k P_0 + \frac{1}{2\gamma}||P_0 - P_0^k||^2 \\ \text{s.t.} \quad & P_0 \le \bar{P}. \end{aligned}$$

The optimal P_0 is set as P_0^{k+1}.

Case 2: Each node i such that $(0,i) \in E$, solves the following problem:

$$\begin{aligned} \min \quad & -f_i(p_i) + r_i\ell_i - \hat{\lambda}_0^k P_i + \hat{\lambda}_i(P_i - r_i\ell_i - p_i) \\ & +\hat{\theta}_i(Q_i - x_i\ell_i - q_i) + \hat{\omega}_i\hat{v}_i - \sum_{j:(i,j)\in E} \hat{\omega}_j v_i \\ & +\frac{1}{2\gamma}\big((P_i - P_i^k)^2 + (Q_i - Q_i^k)^2 \\ & +(\ell_i - \ell_i^k)^2 + (p_i - p_i^k)^2 + (q_i - q_i^k)^2 \\ & +(v_i - v_i^k)^2 + (\hat{v}_i - \hat{v}_i^k)^2\big) \\ \text{over} \quad & P_i, Q_i, \ell_i, p_i, q_i, v_i, \hat{v}_i \\ \text{s.t.} \quad & (11g - 11k) \end{aligned}$$

The optimal $P_i, Q_i, \ell_i, p_i, q_i, v_i, \hat{v}_i$ is set as $P_i^{k+1}, Q_i^{k+1}, \ell_i^{k+1}, p_i^{k+1}, q_i^{k+1}, v_i^{k+1}, \hat{v}_i^{k+1}$.

Case 3: Each node i such that $(0,i) \notin E$ solves the following problem:

$$\begin{aligned} \min \quad & -f_i(p_i) + r_i\ell_i - \lambda_{\pi(i)}P_i - \theta_{\pi(i)}Q_i \\ & +\hat{\lambda}_i(P_i - r_i\ell_i - p_i) + \hat{\theta}_i(Q_i - x_i\ell_i - q_i) \\ & +\hat{\omega}_i\hat{v}_i - \sum_{j:(i,j)\in E} \hat{\omega}_j v_i \\ & +\frac{1}{2\gamma}\big((P_i - P_i^k)^2 + (Q_i - Q_i^k)^2 \\ & +(\ell_i - \ell_i^k)^2 + (p_i - p_i^k)^2 + (q_i - q_i^k)^2 \\ & +(v_i - v_i^k)^2 + (\hat{v}_i - \hat{v}_i^k)^2\big) \\ \text{over} \quad & P_i, Q_i, \ell_i, p_i, q_i, v_i, \hat{v}_i \\ \text{s.t.} \quad & (11g - 11k) \end{aligned}$$

The optimal $P_i, Q_i, \ell_i, p_i, q_i, v_i, \hat{v}_i$ is set as $P_i^{k+1}, Q_i^{k+1}, \ell_i^{k+1}, p_i^{k+1}, q_i^{k+1}, v_i^{k+1}, \hat{v}_i^{k+1}$.

4) Each node $i \in N\backslash\{0\}$ send the primal variables P_i^{k+1}, Q_i^{k+1}, ℓ_i^{k+1} to its parent $\pi(i)$, and each node $i \in N$ except leaves in the network send v_i^{k+1}. Note that v_0^k is fixed as v^* for any k. Then node 0 update the dual signal $\lambda_0^{k+1} := \lambda_0^k + \gamma(P_0^{k+1} - \sum_{j:(0,j)\in E} P_j^{k+1})$ to its children; and each node $i \in N\backslash\{0\}$ except the leaves update the following variables:

$$\lambda_i^{k+1} = \lambda_i^k + \gamma \left(P_i^{k+1} - \left(\sum_{j\in\delta(i)} P_j^{k+1} + r_i l_i^{k+1} + p_i^{k+1} \right) \right),$$

$$\theta_i^{k+1} = \theta_i^k + \gamma \left(Q_i^{k+1} - \left(\sum_{j\in\delta(i)} Q_j^{k+1} + x_i l_i^{k+1} + q_i^{k+1} \right) \right),$$

$$\omega_i^{k+1} = \omega_i^k + \gamma(\hat{v}_i^{k+1} - v_{\pi(i)}^{k+1}).$$

5) $k \leftarrow k+1$, and go to step 2).

For sufficiently small γ, the algorithm will converge to the optimal solutions. Notice that in the distributed algorithm, each node only needs determine a few variables by solving a small optimization problem.

IV. Case study

This section provides numerical examples to complement the analysis in previous sections. We apply the algorithm developed in Section III to a practical distribution circuit of the Southern California Edison (SCE) with 56 buses, as shown in Fig. 1. The corresponding network data including the line impedances, the peak MVA demand of loads, and the nameplate capacity of the shunt capacitors and the photovoltaic generations can be found in [17]. Note that there is a photovoltaic (PV) generator located at bus 45. Since the focus of this paper is to study demand response in power networks, so in the simulation we remove the PV generator. Previous work [16] has shown that this 56-bus circuit satisfies the sufficient conditions for the exact relaxation of OPF to ROPF. Therefore, we can apply the proposed algorithm for the demand response in this circuit. In the simulation, the user utility function $f_i(p_i)$ is set to the quadratic form $f_i(p_i) = -a_i(p_i - \bar{p}_i)^2 + a_i(\bar{p}_i)^2$ where a_i is randomly drawn from $[2,5]$. For each bus i, set $\bar{p}_i$ and $\bar{q}_i$ to the peak demand and $\underline{p}_i$ to the half of the peak demand. If there is no shunt capacitor attached to bus i, we set $\underline{q}_i$ to the half of the peak demand as well, and if there is shunt capacitor attached, we set $\underline{q}_i$ to the negative of the nameplate capacity. We set $\gamma = 0.01$, and $\bar{P}_0 = 2.5$MVA.

Fig. 2 shows the dynamics of the distributed algorithm proposed in Section III. We see that the algorithm converges fast for this distribution system. Notice that since at each iteration step, each node only needs to solve a small optimization problem and the algorithm is highly parallel, the total running time is very fast. We also solve problem ROPF by using CVX toolbox [18], which implements a centralized algorithm, and verify that it gives the same solution as our distributed algorithm. We further verify that the optimal solution of ROPF is a feasible point of OPF, i.e., ROPF is an exact relaxation of OPF.

V. Conclusion

In this paper, we have studied demand response in the radial distribution network with power flow constraints and

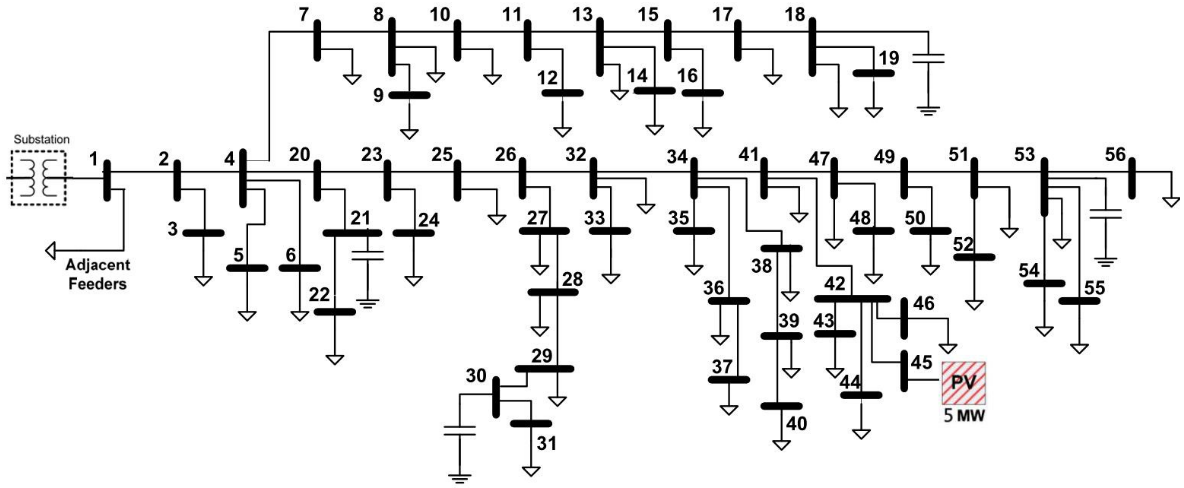

Fig. 1: Schematic diagram of the SCE distribution systems.

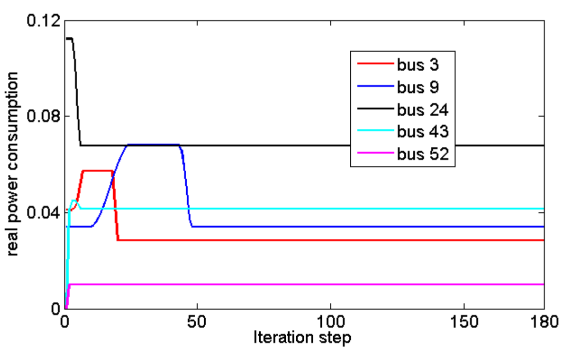

Fig. 2: Dynamics of the distributed demand response algorithm: Bus i's decision $\hat{p}_i$.

operating constraints, by formulating it as an optimal power flow problem. We discuss the exact convex relaxation of the OPF problem, based on which to propose a fully distributed algorithm where the end users make and coordinate their local demand response decisions through local communication with their direct neighbors in the distribution network. Numerical examples show that the proposed algorithm converges fast for the real-world distribution circuits.

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