An optimization-based demand response in radial distribution networks

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***Abstract*—Demand response has recently become a topic of active research. Most of work however considers only the balance between aggregate load and supply, and abstracts away the underlying power network and the associated power flow constraints and operating constraints. In this paper, we study demand response in a radial distribution network, by formulating it as an optimal power flow problem that maximizes the aggregate user utilities and minimizes the power line losses, subject to the power flow constraints and operating constraints. As the resulting problem is non-convex and difficult to solve, we propose a convex relaxation that is usually exact for the real-world distribution circuits. We then propose a distributed algorithm for the load-serving entity to set the price signal to coordinate the users' demand response so as to achieve the optimum. Numerical examples show that the proposed algorithm converges fast for real-world distribution systems.**

I. INTRODUCTION

The need for load management increases as we need to improve power system efficiency and integrate ever-increasing renewable generation [1], [2], [3]. It is expected that large-scale fast demand response, though not currently existing, will be enabled in the near future by the large-scale deployment of a sensing, control, and two-way communication infrastructure, including the flexible AC transmission systems, the GPS-synchronized phaser measurement units, and the advanced metering infrastructure, that is currently underway around the world [4].

There exists a large literature on demand response; see, e.g., [5], [6], [7], [8], [9], [10], [11]. Most of work however considers only the balance between aggregate load and supply, and abstracts away the underlying power network and the associated power flow constraints and operating constraints. As a result, the schemes proposed may end up with an electricity consumption/shedding decision that would violate those network and operating constraints. There are some recent work on demand management that take into consideration the physical network constraints; see, e.g., [12], [13], [14], [15], [16]. But they usually use the bus injection model for the electricity networks and propose location-based marginal pricing schemes for demand management, which is more suitable for the transmission systems.

In this paper, we study optimal demand response in the presence of the network and operating constraints for radial distribution networks, using the branch flow model [17], [18] for the electricity network rather than the bus injection model. The branch flow model focuses on currents and powers on the branches. It has been used mainly for modeling distribution circuits that are usually radial. We consider a radial distribution network that is served by a single load-serving entity (LSE), which may represent a regulated monopoly like utility companies, or a non-profit cooperative that serves a community of end users. The objective for the LSE is to choose a proper price signal, which can be the actual price of electricity or just a control signal, to coordinate individual users' decisions to promote the overall system welfare and efficiency.

Specifically, we formulate demand response problem as an optimal power flow problem whose objective is to maximize the aggregate user utilizes and minimize the power line losses, subject to the power flow constraints and operating constraints such as the voltage regulation constraint. Since the resulting optimization problem is non-convex and thus difficult to solve, we propose a convex relaxation of the optimization problem, and discuss whether the relaxation can be exact and under what conditions. We then show that there exists an optimal price under which, if each user maximizes its net utility, the global welfare, i.e., the aggregate utilities minus the power losses, turns out to be maximized, provided that an exact convex relaxation of the optimal power flow problem is admitted. We next develop a distributed algorithm to iteratively calculate the optimal price, where i) the LSE does not need to know users' information such as the utility functions or consumption constraints, and ii) each user makes demand response decision based only on the price and its own utility function and consumption constrains. Numerical examples show that the proposed algorithm converges fast for real-world distribution systems. We also extend the result and algorithm to load management over multiple instances.

The rest of the paper is organized as follows. We first formulate the optimization problem for designing the one-instance demand response in a radial network in Section II. We then discuss convex relaxation of the resulting optimization problem in Section III, and develop a pricing scheme and a distributed algorithm for the relaxed problem in Section IV. We next extend the result and algorithm to demand response over multiple instances in Section V. Finally, in Section VI, we provide numerical examples to complement the theoretical analysis, using a real-word distribution circuit.

Notations:

- V_i, v_i: complex voltage on bus i with $v_i = |V_i|^2$;
- $s_i = p_i + \mathbf{i}q_i$: complex net load on bus i;
- I_{ij}, ℓ_{ij}: complex current from buses i to j with $\ell_{ij} =$

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$|I_{ij}|^2$;

- $S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$: complex power flowing out from buses i to bus j;
- $z_{ij} = r_{ij} + \mathbf{i}x_{ij}$: impedance on line (i,j).

II. Problem formulation

Consider a radial distribution circuit that consists of a set N of buses and a set E of distribution lines connecting these buses. We index the buses in N by $i = 0, 1, \ldots, n$, and denote a line in E by the pair (i,j) of buses it connects and the index i denotes the bus that is closer to the feeder. Bus 0 denotes the feeder, which has fixed voltage but flexible power injection to balance the loads; each of the other buses $i \in N\backslash\{0\}$ represents an aggregator that can participate in demand response. For convenience we call aggregator i as user i, which actually represents a or a group of customers that are connected to bus i and join the demand response system as a single entity.

For each link $(i,j) \in E$, let $z_{ij} = r_{ij} + \mathbf{i}x_{ij}$ be the impedance on line (i,j), and $S_{i,j} = P_{i,j} + \mathrm{i}Q_{i,j}$ and $I_{i,j}$ the complex power and current flowing from bus i to bus j. At each bus $i \in N$, let $s_i = p_i + \mathbf{i}q_i$ be the complex load and V_i the complex voltage. The branch flow model, first proposed in [17], models power flows in a steady state in a radial distribution network: for each $(i,j) \in E$,

$$\frac{P_{i,j}^2 + Q_{i,j}^2}{v_i} = \ell_{i,j}, \quad (1)$$

$$P_{i,j} = \sum_{h:(j,h)\in E} P_{j,h} + r_{i,j}\ell_{i,j} + p_j, \quad (2)$$

$$Q_{i,j} = \sum_{h:(j,h)\in E} Q_{j,h} + x_{i,j}\ell_{i,j} + q_j, \quad (3)$$

$$v_i - v_j = 2(r_{i,j}P_{i,j} + x_{i,j}Q_{i,j}) - (r_{i,j}^2 + x_{i,j}^2)\ell_{i,j}, \quad (4)$$

where $\ell_{i,j} := |I_{i,j}|^2$, $v_i := |V_i|^2$. Each user $i \in N\backslash\{0\}$ achieves certain utility $f_i(p_i)$ when its (real) power consumption is p_i. The utility function $f_i(\cdot)$ is usually assumed to be continuous, nondecreasing, and concave. Furthermore, there are the following operating constraints for each $i \in N\backslash\{0\}$:

$$\underline{v}_i \le v_i \le \bar{v}_i, i = 1, \cdots, n, \quad (5)$$

$$\underline{q}_i \le q_i \le \bar{q}_i, i = 1, \cdots, n, \quad (6)$$

$$\underline{p}_i \le p_i \le \bar{p}_i, i = 1, \cdots, n. \quad (7)$$

The radial distribution network is owned and controlled by a load-serving entity (LSE), which is typically a utility company. To serve the electricity demand, the LSE procures electricity from the electricity markets and delivers it to the radial distribution network through the feeder (i.e., the bus 0). The total (real) power supply P_0 is given by $P_0 := \sum_{j:(0,j)\in E} P_{0,j}$.

We consider a situation where the power supply P_0 is constrained by an upper bound $\bar{P}_0$, i.e.,

$$P_0 = \sum_{j:(0,j)\in E} P_{0,j} \le \bar{P}_0. \quad (8)$$

Under such a situation, the LSE would like to design certain mechanism to guide each user i to choose a proper load p_i, so as to i) meet the supply constraint as well as the power flow constraints and operating constraints listed in (1) - (7) and ii) maximize the aggregate user utilities and minimize the power line losses. This load side management is formulated as the following optimal power flow problem (OPF):

$$\textbf{OPF:} \max_{P,Q,l,v,p,q} \quad \sum_{i=1}^n f_i(p_i) - \rho \sum_{(i,j)\in E} r_{i,j}\ell_{i,j}$$
$$\text{s.t.} \quad (1) - (8),$$

where ρ is a parameter used to trade off between the user utility maximization and the power line loss minimization.

In this paper, we focus on developing decentralized mechanisms, where the utility functions and constraints (6)-(7) are private information of the users, while the LSE has the network information, i.e., power loss $\sum_{(i,j)\in E} r_{i,j}\ell_{i,j}$ and the constraints (1)-(5) and (8). Each user i chooses power consumption according to certain price signal μ_i sent by LSE, and the LSE adapts the price signal $\mu := (\mu_1, \ldots, \mu_n)$ to coordinate users' consumptions. The price signal μ_i can be implemented as the actual price of electricity usage or just a control signal that is used to coordinate users' decisions. Each user $i \in N\backslash\{0\}$ is assumed to choose p_i to maximize its net utility, i.e., user utility minus payment:

$$\textbf{DR-User:} \max_{p_i} \quad f_i(p_i) - \mu_i p_i$$
$$\text{s.t.} \quad \underline{p}_i \le p_i \le \bar{p}_i.$$

Since the reactive power q_i is not directly involved in the net utility of user i, we assume that user i is willing to report the feasible range $[\underline{q}_i, \bar{q}_i]$ for q_i to the LSE.[1] Hence, the LSE has the following information, the power loss $\sum_{(i,j)\in E} r_{i,j}\ell_{i,j}$ and the constraints (1)-(6) and (8). In the rest of the paper, we show that how the LSE chooses the price signal μ to coordinate the users' demand response decisions so as to solve Problem OPF.

III. Convexification of Problem OPF

OPF is non-convex due to the quadratic equality constraints in (1) and thus difficult to solve. We therefore consider the following convex relaxation of OPF:

$$\textbf{ROPF:} \max_{P,Q,l,v,p,q} \quad \sum_{i=1}^n f_i(p_i) - \rho \sum_{(i,j)\in E} r_{i,j}l_{i,j}$$
$$s.t. \quad (2) - (7)$$
$$\frac{P_{i,j}^2 + Q_{i,j}^2}{v_i} \le l_{i,j}, \ (i,j) \in E, \quad (9)$$

where the equality constraints (1) are relaxed to the inequality constraints (9). Obviously, ROPF provides a lower bound on OPF. After obtaining a solution $X^* := (P^*, Q^*, \ell^*, v^*, p^*, q^*)$ to ROPF, if the equality in (9) is attained at X^*, then X^* is also a solution to OPF. We call ROPF an *exact relaxation* of OPF if every solution to ROPF is also a solution to OPF, and vice versa.

[1]Note that in practice, VAR control is usually carried out by the LSE. So, it is reasonable to assume that the LSE knows the feasible range of reactive power.

In previous work [18], [19], [20], we have studied whether and when ROPF is an exact relaxation of OPF for the radial networks. It is shown in [18] that the relaxation is exact when there are no upper bounds on the loads. However, removing upper bounds on the loads may be unrealistic, especially in the context of demand response. It is shown in [19], [20] that the relaxation is exact provided that instead there are no upper bounds on the voltage magnitudes and certain other conditions hold, which are verified to hold for many real-world distribution systems. Moreover, the upper bounds on the voltage magnitudes for the relaxation solution are characterized [20].

When ROPF is an exact relaxation of OPF, we can just focus on solving the convex optimization problem ROPF. Given the strict feasibility of ROPF under those cases in [18], [19], [20] that guarantee exact relaxation, there is no duality gap between ROPF and its Lagrangian dual [21]. As we will see later, the Lagrangian multipliers are the prices to coordinate the users' demand response decisions. When the problem is nonconvex, those multipliers can not be interpreted or implemented as prices. Hence the conditions for exact relaxation of OPF to ROPF specified in [19], [20] is important for our demand response design. In the rest of the paper, we will assume that ROPF is an exact relaxation of OPF and strong duality holds for ROPF.

IV. Demand Management

In this section, we investigate how the LSE sets price signal μ to coordinate the users' demand response decisions.

Recall that the LSE has only information on the power loss function $\sum_{(i,j)\in E} r_{i,j} l_{i,j}$ and the constraints (1)-(6) and (8). Given price μ, the LSE maximizes its net benefit, i.e., the total payment received minus the power loss:

$$\textbf{DR-LSE:} \max_{P,Q,l,v,p,q} \quad \sum_{i=1}^{n} \mu_i p_i - \rho \sum_{(i,j)\in E} r_{i,j}\ell_{i,j}$$
$$\text{s.t.} \quad (2)-(6),(8),(9).$$

Definition 1. *The price $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ and the variable $(P^*, Q^*, \ell^*, v^*, p^*, q^*)$ are in equilibrium if i) p_i^* is an optimal solution of DR-User for each user i given the price μ_i^*, and ii) $(P^*, Q^*, \ell^*, v^*, p^*, q^*)$ is an optimal solution of DR-LSE for the LSE given the price μ^*.*

The above definition implies that if such an equilibrium $(\mu^*; P^*, Q^*, \ell^*, v^*, p^*, q^*)$ exists, μ^* can serve as the price signal for the LSE to guide users' decisions. The following result establishes the existence of the equilibrium $(\mu^*; P^*, Q^*, \ell^*, v^*, p^*, q^*)$ and characterizes its properties. Let $\lambda_{i,j}$, $\underline{\xi}_i$, $\bar{\xi}_i$ denote the corresponding Lagrangian dual variables of ROPF for the constraint (2), $P_{i,j} = \sum_{h:(j,h)\in E} P_{j,h} + r_{i,j}\ell_{i,j} + p_j$, and the constraint (7), $\underline{p}_i \le p_i$, $p_i \le \bar{p}_i$, respectively.

Theorem 1. *There exists at lease one equilibrium $(\mu^*; P^*, Q^*, \ell^*, v^*, p^*, q^*)$. Moreover, a tuple $(\mu^*; P^*, Q^*, \ell^*, v^*, p^*, q^*)$ is an equilibrium if and only if $(P^*, Q^*, \ell^*, v^*, p^*, q^*)$ is an optimal solution of ROPF and for each $i > 0$, $\mu_i^* = f_i'(p_i^*) - \xi_i^* = \lambda^*_{\pi(i),i}$, where $\pi(i)$ is the parent of bus i.*

Proof: First note that problems ROPF, DR-User, and DR-LSE are convex problems and strong duality holds for all of them. The main idea of the proof is to compare the KKT optimality conditions for these convex problems.

Let $\alpha = (\lambda_{i,j}, \theta_{i,j}, \omega_{i,j}, \underline{\gamma}_i, \bar{\gamma}_i, \underline{\eta}_i, \bar{\eta}_i, \underline{\xi}_i, \underline{\xi}_i, \kappa_0, \mu_{i,j})$ be the Lagrangian dual variables of ROPF corresponding to the constraints (2) – (9) respectively. Given an optimal primal-dual pair $(P^*, Q^*, \ell^*, v^*, p^*, q^*; \alpha^*)$ of ROPF, $(P^*, Q^*, \ell^*, v^*, p^*, q^*; \alpha^*)$ satisfies the KKT condition of ROPF. This implies that $f_i'(p_i^*) - \underline{\xi}_i^* + \bar{\xi}_i^* = \lambda_{\pi(i),i}$. Let $\mu_i^* = f_i'(p_i^*) - \underline{\xi}_i^* + \bar{\xi}_i^* = \lambda^*_{\pi(i),i}$ for all $i = 1, \cdots, n$. Then the KKT condition for ROPF implies that $(p_i^*, \underline{\xi}_i^*, \bar{\xi}_i^*)$ satisfies the KKT condition for problem DR-User for each $i = 1, \cdots, n$; and $(P^*, Q^*, \ell^*, v^*, p^*, q^*, \beta^*)$ satisfies the KKT condition for DR-LSE where $\beta^* = (\lambda^*_{i,j}, \theta^*_{i,j}, \omega^*_{i,j}, \kappa_0^*, \underline{\gamma^*}_i, \bar{\gamma^*}_i, \underline{\eta^*}_i, \bar{\eta^*}_i, \mu^*_{i,j})$. Therefore, i) p_i^* is an optimal solution of DR-User for each user i given the price μ_i^*, and ii) $(P^*, Q^*, \ell^*, v^*, p^*, q^*)$ is an optimal solution of DR-LSE for the LSE given the price μ^*.

On the other hand, suppose $(\mu^*; P^*, Q^*, \ell^*, v^*, p^*, q^*)$ is an equilibrium. The KKT conditions of DR-LSE and DR-User imply that there exists a dual variable α^* such that $(P^*, Q^*, \ell^*, v^*, p^*, q^*; \alpha^*)$ satisfies the KKT condition of ROPF. Thus $(P^*, Q^*, \ell^*, v^*, p^*, q^*)$ is an optimal solution of problem ROPF. ■

A. Distributed Algorithm

Following the algorithm *predictor corrector proximal multiplier (PCPM)* [22], we propose a distributed learning algorithm to achieve an equilibrium $(\mu^*; P^*, Q^*, \ell^*, v^*, p^*, q^*)$:

1) Intially set $k \leftarrow 0$. The LSE randomly chooses initial price μ_i^k and inital p_i^k for each bus i. Each user i randomly chooses initial $\hat{p}_i^k$ and returns $\hat{p}_i^k$ to the LSE.
2) For each $k \ge 0$, the LSE sends a virtual price signal $\hat{\mu}_i^k := \mu_i^k + \gamma(\hat{p}^k - p^k)$ to each bus i. Here $\gamma > 0$ is a constant parameter.
3) Based on the virtual price $\hat{\mu}_i^k$, each bus $i \in N\backslash\{0\}$ solves the following problem:
$$\max_{\hat{p}_i} \quad f_i(\hat{p}_i) - \hat{\mu}_i^k \hat{p}_i - \frac{1}{2\gamma}||\hat{p}_i - \hat{p}_i^k||^2$$
$$\text{s.t.} \quad \underline{p}_i \le \hat{p}_i \le \bar{p}_i.$$
The optimal $\hat{p}_i$ is set as $\hat{p}_i^{k+1}$.
4) The LSE solves the following problem:
$$\max_{P,Q,l,v,p,q} (\hat{\mu}^k)^T p - \rho \sum_{(i,j)\in E} r_{i,j}\ell_{i,j} - \frac{1}{2\gamma}||p - p^k||_2^2$$
$$\text{s.t.} (2)-(6),(8),(9).$$
The optimal p is set as p_i^{k+1}.
5) Each bus i returns $\hat{p}_i^{k+1}$ to the LSE and the LSE updates the price μ as $\mu^{k+1} = \mu^k + \gamma(\hat{p}^{k+1} - p^{k+1})$ with an appropriate stepsize $\epsilon > 0$.

6) $k \leftarrow k+1$, and go to step 2).

For sufficiently small γ, $(\mu^k; P^k, Q^k, \ell^k, v^k, p^k, q^k)$ will converge to an equilibrium, and $\hat{p}^k - p^k$ and $\hat{\mu}^k - \mu^k$ will become zero [22]. Numerical experiments show that this algorithm converges to the optimum of problem ROPF(OPF) very fast.

V. Demand management over Multi-instances

In the previous sections, we have studied demand management at one instance. The method and results can be easily extended to demand management over multiple instances. The LSE may need to schedule supply to meet the demand for each time period of the next day, represented as a set of time slots $\mathcal{T} = \{1, 2, \cdots, T\}$, and its objective is to maximize the aggregate user utilities minus the power line losses over the whole period of $\mathcal{T}$. Let $\mathbf{P} = (P(1), \ldots, P(T))$, $\mathbf{Q} = (Q(1), \ldots, Q(T))$, $\mathbf{l} = (\ell(1), \ldots, \ell(T))$,$\mathbf{v} = (v(1), \ldots, v(T))$, $\mathbf{p} = (p(1), \ldots, p(T))$, and $\mathbf{q} = (q(1), \ldots, q(T))$ be the corresponding variables of the power network at different times. Mathematically, the load management problem over multiple instances can be formulated as the following optimization problem:

$$\textbf{MOPF:} \max_{\mathbf{P},\mathbf{Q},\mathbf{l},\mathbf{v},\mathbf{p},\mathbf{q}} \quad \sum_{i=1}^{n} f_i(\mathbf{p}_i) - \rho \sum_{t\in\mathcal{T},(i,j)\in E} r_{i,j}\ell_{i,j}(t) - \sum_{t\in\mathcal{T}} C_t\left(\sum_{j:(0,j)\in E} P_{0,j}(t)\right)$$

$$\text{s.t.} \quad (1)-(5), \forall t \in \mathcal{T}$$

$$\underline{q}_i(t) \le q_i(t) \le \bar{q}_i(t), \forall t \in \mathcal{T}, \quad (10)$$

$$\underline{p}_i(t) \le p_i(t) \le \bar{p}_i(t), \forall t \in \mathcal{T}, \quad (11)$$

$$\sum_{t\in\mathcal{T}} p_i(t) \ge \underline{d}_i, \ \forall i \in N\backslash\{0\}. \quad (12)$$

Here $C_t(\cdot)$ is a cost function of the total real power injected to the network through the feeder at time t. It can be interpreted as the cost in power provisioning for the LSE. The cost function $C_t(\cdot)$ is assumed to be nondecreasing and convex. Compared with OPF, MOPF has the following differences:

1) Instead of constraining the power supply P_0 at the feeder as in (8), MOPF allows greater elasticity in power supply subject to a cost $C_t(P_0(t))$;
2) The utility function $f_i(p_i)$ in OPF for one instance is replaced by the utility function $f_i(\mathbf{p}_i) = f_i(p_i(1), \cdots, p_i(T))$ which characterizes user i's utility over the whole period of $\mathcal{T}$ given the demand profile $(p_i(1), \cdots, p_i(T))$;
3) The constraints (6), (7) that bound power consumption for each user i become time-dependent constraints (10), (11).
4) There is a constraint (12) on the total real power consumption over the whole period of $\mathcal{T}$, corresponding to a minimum demand that is required to power basic daily routines for user i.

Note that the requirements and constraints on demand (10-12) and the utility function f_i can be modeled in a more complicated form if we consider every appliance for each user; see previous work [23] for the detailed user models.

All the results in Section III and Section IV can be readily extended to MOPF. The convex relaxation of MOPF is given as follows:

$$\textbf{RMOPF:} \max_{\mathbf{P},\mathbf{Q},\mathbf{l},\mathbf{v},\mathbf{p},\mathbf{q}} \quad \sum_{i=1}^{n} f_i(\mathbf{p}_i) - \rho \sum_{t\in\mathcal{T},(i,j)\in E} r_{i,j}\ell_{i,j}(t) - \sum_{t\in\mathcal{T}} C_t\left(\sum_{j:(0,j)\in E} P_{0,j}(t)\right)$$

$$\text{s.t.} \quad (2)-(5), (9-12).$$

Provided that the sufficient conditions for exact relaxation in [18], [19], [20] are satisfied, RMOPF is also an exact relaxation of MOPF. Similarly, in the rest of this paper, we assume that RMOPF is an exact relaxation and strong duality holds for RMOPF.

In the demand response setting, the utility functions f_i and the constraints (10-12) are private information of the users, while the LSE has the network information. Each user i chooses power consumption according to certain price signal $\{\mu_i(t)\}_{t\in\mathcal{T}}$ sent by LSE, and the LSE adjusts the price signal $\{\mu_i(t)\}_{i=1,\cdots,n}^{t=1,\cdots,T}$ to coordinate the users' consumption decisions. Specifically, given prices $\{\mu_i(t)\}_{t\in\mathcal{T}}$, each user $i \in N\backslash\{0\}$ solves the following problem:

$$\textbf{MDR-User:} \max_{\mathbf{p}_i} \quad f_i(\mathbf{p}_i) - \textstyle\sum_{t\in T} \mu_i(t)p_i(t)$$

$$\text{s.t.} \quad \underline{p}_i(t) \le p_i(t) \le \bar{p}_i(t), \quad \textstyle\sum_{t\in\mathcal{T}} p_i(t) \ge \underline{d}_i.$$

We also assume that user i is willing to report the feasible range $[\underline{q}_i(t), \bar{q}_i(t)]$ for $q_i(t)$ to the LSE. The LSE solves the following problem for each $t \in \mathcal{T}$:

MDR-LSE:

$$\max_{\mathbf{P},\mathbf{Q},\mathbf{l},\mathbf{v},\mathbf{p},\mathbf{q}} \quad \sum_{t\in T} \mu_i(t)p_i(t) - \rho \sum_{t\in\mathcal{T},(i,j)\in E} r_{i,j}\ell_{i,j}(t) - \sum_{t\in\mathcal{T}} C_t\left(\sum_{j:(0,j)\in E} P_{0,j}(t)\right)$$

$$\text{s.t.} \quad (2-5), (9), (10).$$

Definition 2. *The price $\mu^* = \{\mu_i^*(t)\}_{i=1,\cdots,n}^{t=1,\cdots,T}$ and the variable $(\mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$ are in equilibrium if i) $\mathbf{p}_i^*$ is an optimal solution of MDR-User for each user i given the price $\{\mu_i^*(t)\}_{t\in\mathcal{T}}$, and ii) $(\mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$ is an optimal solution of MDR-LSE for the LSE given the price $\{\mu_i^*(t)\}_{i=1,\cdots,n}$ for each $t \in \mathcal{T}$.*

The above definition implies that if such an equilibrium $(\mu^*; \mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$ exists, μ^* can serve as the price signal for the LSE to guide users' decisions. The following result establishes the existence of the equilibrium and characterizes its properties.

Theorem 2. *There exists at lease one equilibrium $(\mu^*; \mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$. Moreover, if a tuple*

$(\mu^*; \mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$ *is an equilibrium, then* $(\mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$ *is an optimal solution of ROPF; and if* $(\mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$ *is an optimal solution of ROPF, then there exists* μ^* *such that* $(\mu^*; \mathbf{P}^*, \mathbf{Q}^*, \mathbf{l}^*, \mathbf{v}^*, \mathbf{p}^*, \mathbf{q}^*)$ *is an equilibrium. The equilibrium* μ^* *is determined by the optimal Lagrangian dual variables of RMOPF corresponding to the constraints (2), (11).*

Proof: The proof is similar to that for Theorem 1. ■

A. Distributed Algorithm

Similarly, we have the following distributed learning algorithm to achieve an equilibrium of demand response over multiple instances.

1) Initially set $k \leftarrow 0$. The LSE randomly chooses initial price $\mu_i^k(t)$ and inital $p_i^k(t)$ for each bus i at each time $t \in \mathcal{T}$. Each user i randomly chooses initial $\hat{p}_i^k(t)$ for each time $t \in \mathcal{T}$ with $\sum_{t\in\mathcal{T}} p_i^k(t) \geq \underline{d}_i$ and returns $\hat{p}_i^k(t)$ to the LSE.
2) For each $k \geq 0$, the LSE sends a virtual price signal $\hat{\mu}_i^k(t) := \mu_i^k(t) + \gamma(\hat{p}_i^k(t) - p_i^k(t))$ to bus i. Here $\gamma \geq 0$ is a constant parameter.
3) Based on the virtual price $\hat{\mu}_i^k(t)$, each bus $i \in N\backslash\{0\}$ solves the following problem:

$$\begin{aligned} \max_{\hat{\mathbf{p}}_i} \quad & f_i(\hat{\mathbf{p}}_i) - \sum_{t\in\mathcal{T}} \hat{\mu}_i^k(t)\hat{p}_i(t) - \frac{1}{2\gamma}\sum_{t\in\mathcal{T}} ||\hat{p}_i(t) - \hat{p}_i^k(t)||^2 \\ \text{s.t.} \quad & \underline{p}_i(t) \leq \hat{p}_i(t) \leq \bar{p}_i(t), \forall t \in \mathcal{T} \\ & \sum_{t\in\mathcal{T}} \hat{p}_i(t) \geq \underline{d}_i. \end{aligned}$$

   The optimal $\hat{p}_i(t)$ is set as $\hat{p}_i^{k+1}(t)$.
4) For each time $t \in \mathcal{T}$, the LSE solves the following problem:

$$\begin{aligned} \max \quad & (\hat{\mu}^k(t))^T p(t) - \rho \sum_{(i,j)\in E} r_{i,j}\ell_{i,j}(t) \\ & -C_t\left(\sum_{j:(0,j)\in E} P_{0,j}(t)\right) - \frac{1}{2\gamma}||p(t) - p^k(t)||_2^2 \\ \text{over:} \quad & P(t), Q(t), \ell(t), v(t), p(t), q(t) \\ \text{s.t.} \quad & (2-5), (9), (10). \end{aligned}$$

   The optimal $p_i(t)$ is set as $p_i^{k+1}(t)$.
5) Each user i return $\hat{p}_i^k(t)$ to the LSE and the LSE updates the price μ as $\mu^{k+1} = \mu^k + \gamma(\hat{p}^k(t) - p^k(t))$.
6) $k \leftarrow k+1$, and go to step 2).

VI. Case Study

This section provides numerical examples to complement the analysis in previous sections. We apply the algorithm developed in Section IV-A to a practical distribution circuit of the Southern California Edison (SCE) with 56 buses, as shown in Figure 1. The corresponding network data including the line impedances, the peak MVA demand of loads, and the nameplate capacity of the shunt capacitors and the photovoltaic generations can be found in [24]. Note that there is a photovoltaic (PV) generator located at bus 45. Since the focus of this paper is to study demand response in power networks, so in the simulation we remove the PV generator. Previous work [20] has shown that this 56-bus circuit satisfies the sufficient conditions for the exact relaxation of OPF to ROPF. Therefore, we can apply the algorithm proposed in Section IV-A for the demand response in this circuit.

In the simulation, the user utility function $f_i(p_i)$ is set to the quadratic form $f_i(p_i) = -a_i(p_i - \bar{p}_i)^2 + a_i(\bar{p}_i)^2$ where a_i is randomly drawn from $[2, 5]$. For each bus i, set $\bar{p}_i$ and $\bar{q}_i$ to the peak demand and $\underline{p}_i$ to the half of the peak demand. If there is no shunt capacitor attached to bus i, we set $\underline{q}_i$ to the half of the peak demand as well, and if there is shunt capacitor attached, we set $\underline{q}_i$ to the negative of the nameplate capacity. We set $\gamma = 0.02$, $\epsilon = 0.01$, and $\bar{P}_0 = 3.5$MV.

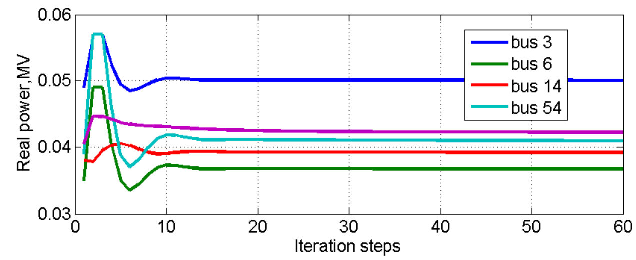

Fig. 2: Dynamics of the distributed demand response algorithm: Bus i's decision $\hat{p}_i$.

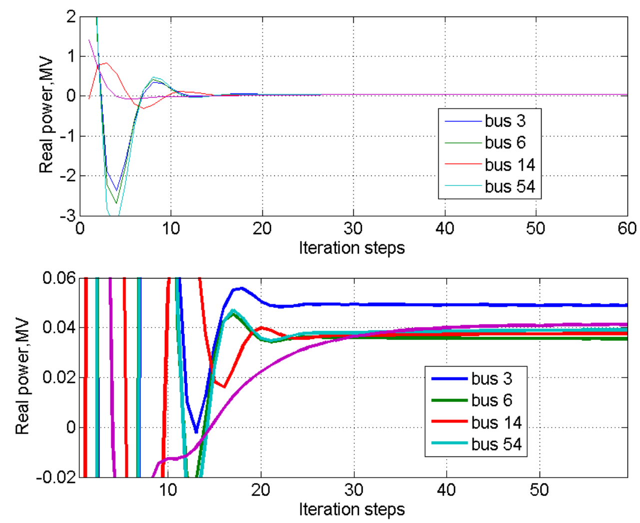

Fig. 3: Dynamics of the distributed demand response algorithm: LSE's decision p_i for each bus i. The bottom one is a zoomed figure of the top one.

Figures 2 and 3 show the dynamics of the distributed algorithm proposed in Section IV-A. We see that the algorithm converges very fast for this distribution system. We also solve problem ROPF by using CVX toolbox [25], which implements a centralized algorithm, and verify that it gives the same solution as our distributed algorithm. We further verify that the optimal solution of ROPF is a feasible point of OPF, i.e., ROPF is an exact relaxation of OPF.

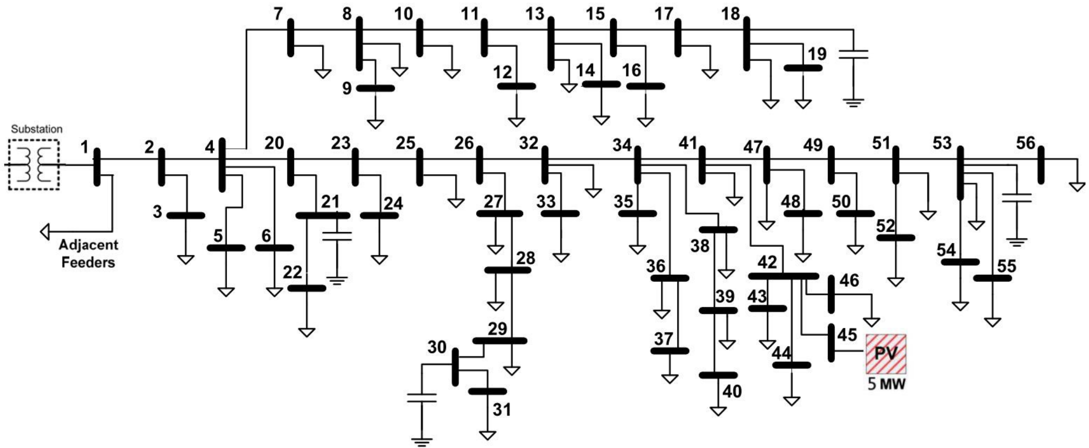

Fig. 1: Schematic diagram of the SCE distribution systems.

VII. Conclusion

In this paper, we have studied demand response in the radial distribution network with power flow constraints and operating constraints, by formulating it as an optimal power flow problem. We discuss the exact convex relaxation of the resulting optimal power flow problem, and accordingly propose a distributed demand response algorithm where the LSE sets a proper price signal to coordinate the users' demand response decisions. Numerical examples show that the proposed algorithm converges very fast for the real-world distribution circuits.

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