Optimal Power Flow in Distribution Networks

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Abstract— **The optimal power flow (OPF) problem seeks to control the generation/consumption of generators/loads to optimize certain objectives such as to minimize the generation cost or power loss in the network. It is becoming increasingly important for distribution networks due to the emerging distributed generation and controllable loads. In this paper, we study the OPF problem in distribution networks. In particular, OPF is nonconvex and we study solving it through convex relaxations. We prove that after a "small" modification to the OPF problem, the solution to its second-order-cone programming (SOCP) relaxation is the global optimum of OPF, under a "mild" condition that can be checked prior to solving the relaxation. Empirical studies justify that the modification to OPF is "small" and that the "mild" condition holds for all test distribution networks, including the IEEE 13-bus test distribution network and practical distribution networks with high penetration of distributed generation.**

I. Introduction

The optimal power flow (OPF) problem seeks to control the generation/consumption of generators/loads to optimize certain objectives such as to minimize the generation cost or power loss in the network. It is first proposed by Carpentier in 1962, and has been one of the fundamental problems in power system operation ever since.

The OPF problem is becoming increasingly important for distribution networks due to the advent of high penetration of distributed generation and controllable loads such as electric vehicles. Distributed generation is difficult to predict, calling the traditional control strategy of "generation follows demand" into question. Meanwhile, controllable loads provide significant potential to compensate for the randomness in distributed generation. To incorporate distributed generation and realize the potential of controllable loads, solving the OPF problem for distribution networks is inevitable.

The OPF problem is difficult to solve due to the nonconvex power flow constraints, and there are in general three ways to deal with this challenge: (i) linearize the power flow constraints; (ii) look for local optima; and (iii) convexify power flow constraints, which are described in turn.

The power flow constraints can be approximated by some linear constraints in transmission networks, and then the OPF problem reduces to a linear program [1]. This method is widely used in practice for transmission networks, but does not apply to distribution networks.

Various algorithms have been proposed to find local optima of OPF, e.g., successive linear/quadratic programming [2], trust-region based methods [3], Lagrangian Newton method [4], and interior-point methods [5]. However, not only is convergence of these algorithms not guaranteed, but also a local optimum can be highly suboptimal.

Convexification methods are the focus of this paper. It is proposed in [6]–[8] to transform the nonconvex power flow constraints into linear constraints on a rank-one matrix, and then replace the rank-one constraint by a semidefinite constraint, to obtain a semidefinite programming (SDP) relaxation of OPF. If the solution one obtains by solving the SDP relaxation is feasible for the OPF problem, then the global optima of OPF can be found by solving the convex SDP relaxation. In this case, we say that the SDP relaxation is *exact*. Strikingly, the SDP relaxation is exact for the IEEE 14-, 30-, 57-, and 118-bus test transmission networks [8], highlighting the potential of convexification methods.

There is another type of convex relaxations for OPF, i.e., second-order-cone programming (SOCP) relaxations [9]–[11]. While having a lower computational complexity than the SDP relaxation, SOCP relaxations are exact if and only if the SDP relaxation is exact, for tree networks [12]. Since distribution networks are tree networks, we focus on the SOCP relaxations and do not distinguish the SDP or SOCP relaxations being exact. In particular, we focus on the SOCP relaxation proposed in [9].

Up to date, sufficient conditions that have been derived for the exactness of the SOCP relaxation do not hold in practice, and whether the SOCP relaxation is exact can only be checked after solving it. More specifically, these conditions require some bus to be able to draw infinite power, which is referred to as *load over-satisfaction* in the literature. For example, the conditions given in [9], [13], [14] require load over-satisfaction on some/all of the buses.

Summary of contributions

The goal of this paper is to provide a prior guarantee that the SOCP relaxation be exact for a modified OPF problem. The modified OPF problem has the same objective function as the OPF problem but a slightly smaller feasible set. After modifying OPF, its SOCP relaxation is exact under a condition that can be checked in priori and holds for all test distribution networks considered in this paper. In particular, contributions of this paper are threefold.

First, we prove that *under Condition C1, the SOCP relaxation is exact if its optimal power injections lie in some region $\mathcal{S}$*. Condition C1 can be checked prior to solving the SOCP relaxation, and holds for all test distribution networks considered in this paper, including the IEEE 13-bus test distribution network and practical distribution networks with high penetration of distributed generation.

Second, we *modify the OPF problem by restricting the power injections to $\mathcal{S}$*. This modification is necessary for an exact SOCP relaxation since otherwise practical examples exist where the SOCP relaxation is not exact. Remarkably, if we restrict the power injections to $\mathcal{S}$, then only feasible points that are "close" to the voltage regulation upper bounds are eliminated, and then the SOCP relaxation is exact under Condition C1. Empirical studies justify that the modification to the OPF problem is "small" for all test distribution networks considered in this paper.

Third, we prove that *the SOCP relaxation has a unique solution if it is exact*. In this case, any convex programming solver gives the same solution.

II. The optimal power flow problem

This paper studies the optimal power flow (OPF) problem in distribution networks, which includes Volt/VAR control and demand response. In the following we present a model of this scenario that serves as the basis for our analysis. The model incorporates nonlinear power flow physical laws, considers a variety of controllable devices including distributed generators, inverters, controllable loads, and shunt capacitors, and allows for a wide range of control objectives such as minimizing the power loss or generation cost, which are described in turn.

A. Power flow model

A distribution network is composed of buses and lines connecting these buses, and has a tree topology.

There is a substation in the network, which has fixed voltage and flexible power injection for power balance. Index the substation bus by 0 and the other buses by $1,\ldots,n$. Let $N := \{0,\ldots,n\}$ denote the set of all buses and $N^+ := \{1,\ldots,n\}$ denote the set of all non-substation buses. Each line connects an ordered pair (i,j) of buses where bus j is in the middle of bus i and bus 0. Let E denote the set of all lines and abbreviate $(i,j)\in E$ by $i\to j$. If $i\to j$ or $j\to i$, denote $i\sim j$; otherwise denote $i\nsim j$.

For each bus $i\in N$, let V_i denote its voltage and I_i denote its current injection. Specifically, the substation voltage, V_0, is given and fixed. Let $s_i = p_i + \mathbf{i}q_i$ denote the power injection of bus i where p_i and q_i denote its real and reactive power injections respectively. Specifically, s_0 is the power that the substation draws from the transmission network for power balance. Let $\mathcal{P}_i$ denote the path (a collection of buses in N and lines in E) from bus i to bus 0.

For each line $i\sim j$, let $y_{ij} = g_{ij} - \mathbf{i}b_{ij}$ denote its admittance and $z_{ij} = r_{ij} + \mathbf{i}x_{ij}$ denote its impedance, then $y_{ij}z_{ij} = 1$.

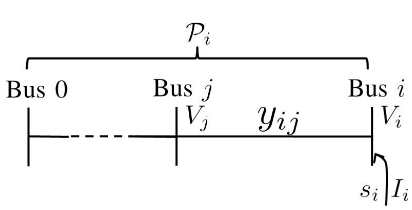

Fig. 1. Some of the notations.

Some of the notations are summarized in Fig. 1. Further, we use a letter without subscript to denote a vector of the corresponding quantity, e.g., $V = (V_1,\ldots,V_n)$, $y = (y_{ij}, i\sim j)$. Note that subscript 0 is not included in nodal variables.

Given the network graph (N,E), the admittance y, and the substation voltage V_0, then the other variables (s,V,I,s_0) are described by the following physical laws.

- Current balance and Ohm's law:
$$I_i = \sum_{j:\, j\sim i} y_{ij}(V_i - V_j), \qquad i\in N;$$
- Power balance:
$$s_i = V_i I_i^*, \qquad i\in N.$$

If we are only interested in voltages and power, then the two sets of equations can be combined into a single one

$$s_i = V_i \sum_{j:\, j\sim i} (V_i^* - V_j^*)y_{ij}^*, \qquad i\in N. \tag{1}$$

In this paper, we use (1) to model the power flow.

B. Controllable devices and control objective

Controllable devices in a distribution network include distributed generators, inverters that connect distributed generators to the grid, controllable loads like electric vehicles and smart appliances, and shunt capacitors.

Real and reactive power generation/consumption of these devices can be controlled to achieve certain objectives. For example, in Volt/VAR control, reactive power injection of the inverters and shunt capacitors are controlled to regulate the voltages; in demand response, real power consumption of controllable loads are reduced or shifted in response to power supply conditions. Mathematically, power injection s is the control variable, after specifying which the other variables V and s_0 are determined by (1).

Constraint on the power injection s_i of a bus $i\in N^+$ is captured by some feasible power injection set $\mathcal{S}_i$, i.e.,

$$s_i \in \mathcal{S}_i, \qquad i\in N^+. \tag{2}$$

The set $\mathcal{S}_i$ for typical control devices are summarized below.

- If bus i has only shunt capacitor with nameplate capacity $\overline{q}_i$, then
$$\mathcal{S}_i = \{s \mid \mathbf{Re}(s)=0,\ \mathbf{Im}(s) = 0 \text{ or } \overline{q}_i\}.$$
- If bus i has a solar photovoltaic panel with real power generation capacity $\overline{p}_i$, and an inverter with nameplate capacity $\overline{|s_i|}$, then
$$\mathcal{S}_i = \{s \mid 0\le \mathbf{Re}(s)\le \overline{p}_i,\ |s|\le \overline{|s_i|}\}.$$
- If bus i only has a controllable load with constant power factor η, whose real power consumption can vary continuously from $-\overline{p}_i$ to $-\underline{p}_i$, then
$$\mathcal{S}_i = \{s \mid \underline{p}_i \le \mathbf{Re}(s)\le \overline{p}_i,\ \mathbf{Im}(s) = \sqrt{1-\eta^2}\,\mathbf{Re}(s)/\eta\}.$$

For ease of presentation, we focus on the case where

$$\mathcal{S}_i = \{s \mid \underline{p}_i \le \mathbf{Re}(s)\le \overline{p}_i,\ \underline{q}_i \le \mathbf{Im}(s)\le \overline{q}_i\} \tag{3}$$

for $i \in N^+$ in this paper, but the results hold more generally.

The control objective in a distribution network is twofold. The first one is regulating the voltages within a specified range since voltages deviate significantly from their nominal values in distribution networks. This is captured by externally specified voltage lower and upper bounds $\underline{|V_i|}$ and $\overline{|V_i|}$, i.e.,

$$\underline{|V_i|} \le |V_i| \le \overline{|V_i|}, \qquad i \in N^+. \tag{4}$$

For example, if 5% voltage deviation from the nominal value is allowed, then $0.95 \le |V_i| \le 1.05$ [15].

The second objective is minimizing the power loss since distribution networks have significant power loss due to their low voltages and high line resistances. Power loss on line $i \to j$ is $\mathbf{Re}(S_{ij} + S_{ji}) = g_{ij}|V_i - V_j|^2$, therefore power loss in the network is

$$\sum_{i \to j} g_{ij}|V_i - V_j|^2. \tag{5}$$

C. The OPF problem

We can now formally state the OPF problem that we seek to solve: minimize the power loss (5), subject to power flow constraints (1), power injection constraints (2), and voltage regulation constraints (4).

$$\begin{aligned}
\textbf{OPF:}\ \min \quad & \sum_{i \to j} g_{ij}|V_i - V_j|^2 \\
\text{over} \quad & s, V, s_0 \\
\text{s.t.} \quad & s_i = V_i \sum_{j:\, j \sim i} (V_i^* - V_j^*) y_{ij}^*, && i \in N; \\
& s_i \in \mathcal{S}_i, && i \in N^+; \\
& \underline{|V_i|} \le |V_i| \le \overline{|V_i|}, && i \in N^+.
\end{aligned}$$

The results in this paper generalize to a much broader class of OPF problems with different objective functions and different power injection constraints, as discussed after each theorem.

The challenge in solving the OPF problem comes from the nonconvex quadratic equality constraints in (1). To overcome this challenge, we enlarge the feasible set of OPF to a convex set. To state the convex relaxation, define

$$W_{ij} := V_i V_j^*, \qquad i \sim j \text{ or } i = j \tag{6}$$

and

$$W\{i,j\} := \begin{pmatrix} W_{ii} & W_{ij} \\ W_{ji} & W_{jj} \end{pmatrix}, \qquad i \sim j.$$

Then the OPF problem can be equivalently formulated as

$$\begin{aligned}
\textbf{OPF':}\ \min \quad & \sum_{i \to j} g_{ij}(W_{ii} - W_{ij} - W_{ji} + W_{jj}) \\
\text{over} \quad & s, W, s_0 \\
\text{s.t.} \quad & s_i = \sum_{j:\, j \sim i} (W_{ii} - W_{ij}) y_{ij}^*, && i \in N; \quad (7) \\
& s_i \in \mathcal{S}_i, && i \in N^+; \\
& \underline{|V_i|}^2 \le W_{ii} \le \overline{|V_i|}^2, && i \in N^+; \quad (8) \\
& \mathbf{Rank}(W\{i,j\}) = 1, && i \to j \quad (9)
\end{aligned}$$

for tree networks according to Theorem 1, which is proved in the extended version of this paper [16]. Theorem 1 establishes a bijective map between the feasible sets of OPF and OPF', that preserves the objective value. Hence, we can solve OPF' to obtain the solution of OPF. To state the theorem, let $\mathcal{F}_{\text{OPF}}$ and $\mathcal{F}_{\text{OPF'}}$ denote the feasible sets of OPF and OPF' respectively, and let L and L' denote the objective functions of OPF and OPF' respectively. Besides, for any feasible point $x = (s, V, s_0)$ of OPF, define a map $\phi(x) := (s, W, s_0)$ where W is defined according to (6).

Theorem 1 *For any $x \in \mathcal{F}_{OPF}$, the point $\phi(x) \in \mathcal{F}_{OPF'}$ and satisfies $L(x) = L'(\phi(x))$. Furthermore, the map $\phi : \mathcal{F}_{OPF} \to \mathcal{F}_{OPF'}$ is bijective for tree networks.*

After transforming OPF to OPF', we can relax OPF' to a convex problem, by relaxing the rank constraints in (9) to

$$W\{i,j\} \succeq 0, \qquad i \to j, \tag{10}$$

i.e., matrices $W\{i,j\}$ being positive semidefinite, as in the following second-order-cone programming (SOCP) relaxation [17].

$$\begin{aligned}
\textbf{SOCP:}\ \min \quad & \sum_{i \to j} g_{ij}(W_{ii} - W_{ij} - W_{ji} + W_{jj}) \\
\text{over} \quad & w = (s, W, s_0) \\
\text{s.t.} \quad & s_i = \sum_{j:\, j \sim i} (W_{ii} - W_{ij}) y_{ij}^*, && i \in N; \\
& s_i \in \mathcal{S}_i, && i \in N^+; \\
& \underline{|V_i|}^2 \le W_{ii} \le \overline{|V_i|}^2, && i \in N^+; \\
& W\{i,j\} \succeq 0, && i \to j.
\end{aligned}$$

If the solution w one obtains by solving the SOCP relaxation is feasible for OPF', i.e., w satisfies (9), then w is the global optima of OPF'. This motivates the definition of "exactness" for the SOCP relaxation.

Definition 1 *The SOCP relaxation is exact if every of its solutions satisfies* (9).

If the SOCP relaxation is exact, then a global optimum of OPF can be found by solving a convex problem—the SOCP relaxation.

D. Related work

This paper studies exactness of the SOCP relaxation. Before this paper, it has been proved that the SOCP relaxation is exact under certain conditions [9], [13], [14], [18].

It is proved in [9] that the SOCP relaxation is exact if there are no lower bounds on the power injections. The results in [13], [14] slightly improve over this condition.

Proposition 1 ([9]) *The SOCP relaxation is exact if $\underline{p}_i = -\infty$ and $\underline{q}_i = -\infty$ for $i \in N^+$.*

In practice, $\underline{p}_i$ and $\underline{q}_i$ are finite for every $i \in N^+$.

In contrast, reference [18] considers lower bounds on the power injections, but ignores upper bounds on the voltages. To state the result, for every $i \to j$, let

$$S_{ij}^{\text{lin}}(p + \mathbf{i}q) = P_{ij}^{\text{lin}}(p) + \mathbf{i}Q_{ij}^{\text{lin}}(q) := \sum_{k:\, i\in\mathcal{P}_k} p_k + \mathbf{i}\sum_{k:\, i\in\mathcal{P}_k} q_k \tag{11}$$

denote the downstream total power injection.

Proposition 2 ([18]) *The SOCP relaxation is exact if* $\overline{|V_i|} = \infty$ *for* $i \in N^+$*, and any one of the following conditions hold:*

(i) $P_{ij}^{\text{lin}}(\overline{p}) \le 0$ *and* $Q_{ij}^{\text{lin}}(\overline{q}) \le 0$ *for all* $i \to j$.
(ii) $r_{ij}/x_{ij} = r_{jk}/x_{jk}$ *for all* $i \to j$, $j \to k$.
(iii) $r_{ij}/x_{ij} \ge r_{jk}/x_{jk}$ *for all* $i \to j$, $j \to k$, *and* $P_{ij}^{\text{lin}}(\overline{p}) \le 0$ *for all* $i \to j$.
(iv) $r_{ij}/x_{ij} \le r_{jk}/x_{jk}$ *for all* $i \to j$, $j \to k$, *and* $Q_{ij}^{\text{lin}}(\overline{q}) \le 0$ *for all* $i \to j$.

In distribution networks, the constraints $|V_i| \le \overline{|V_i|}$ cannot be ignored, especially with distributed generators making the voltages likely to exceed $\overline{|V|}$.

To summarize, all sufficient conditions in the literature that guarantee the exactness of the SOCP relaxation require removing some of the constraints in OPF, and do not hold in practice. In fact, the SOCP relaxation is in general not exact, and a 2-bus example is provided in the extended version of this paper [16].

III. A modified OPF problem

We answer the following two questions in this section.

- Under what conditions is the SOCP relaxation exact?
- How can we modify OPF to satisfy these conditions?

A. A sufficient condition

We start with providing a sufficient condition under which the SOCP relaxation is exact. The condition builds on a linear approximation of the power flow in "the worst case".

To state the condition, we first define the linear approximation. Define

$$W_{ii}^{\text{lin}}(s) := W_{00} + 2\sum_{(j,k)\in\mathcal{P}_i} \mathbf{Re}\left(z_{jk}^* S_{jk}^{\text{lin}}(s)\right)$$

for every $i \in N^+$ and every power injection s, then $W_{ii}^{\text{lin}}(s)$ is a linear (linear in s) approximation of the voltage $W_{ii} = |V_i|^2$. Also define

$$S_{ij} = P_{ij} + \mathbf{i}Q_{ij} := (W_{ii} - W_{ij})y_{ij}^* \tag{12}$$

as the sending-end power flow from bus i to bus j for $i \to j$, then $S_{ij}^{\text{lin}}(s)$ defined in (11) is a linear (linear in s) approximation of S_{ij}.

The linear approximations $W_{ii}^{\text{lin}}(s)$ and $S_{ij}^{\text{lin}}(s)$ are upper bounds on W_{ii} and S_{ij}, as stated in Lemma 1, which is proved in the extended version of this paper [16]. To state the lemma, let $S := (S_{ij}, i \to j)$ denote the collection of power flow on all lines. For two complex numbers $a, b \in \mathbb{C}$, define the operator $\preceq$ by

$$a \preceq b \overset{\text{def}}{\iff} \mathbf{Re}(a) \le \mathbf{Re}(b) \text{ and } \mathbf{Im}(a) \le \mathbf{Im}(b).$$

Lemma 1 *If* (s, S, W, s_0) *satisfies* (7), (10) *and* (12)*, then* $S_{ij} \preceq S_{ij}^{\text{lin}}(s)$ *for* $i \to j$ *and* $W_{ii} \le W_{ii}^{\text{lin}}(s)$ *for* $i \in N^+$.

The linear approximations $S_{ij}^{\text{lin}}(s)$ and $W_{ii}^{\text{lin}}(s)$ are close to S_{ij} and W_{ii} in practice. It can be verified that $\{S_{ij}^{\text{lin}}\}_{i\to j}$, $\{W_{ii}^{\text{lin}}\}_{i\in N^+}$ solves

$$\begin{aligned} S_{jk} &= s_j + \sum_{i:\, i\to j} S_{ij}, && j \to k; \\ W_{jj} &= W_{ii} - 2\mathbf{Re}(z_{ij}^* S_{ij}), && i \to j, \end{aligned}$$

which is called *Linear DistFlow model* in the literature and known to approximate the exact power flow model (7) well.

The linear approximations $S_{ij}^{\text{lin}}(s)$ and $W_{ii}^{\text{lin}}(s)$ are widely used in the literature since they are close to the power flow S_{ij} and the voltage $W_{ii} = |V_i|^2$. In particular, they have been used to study the optimal placement and sizing of shunt capacitors [19], [20], to minimize power loss and balance load [21], and to control reactive power injections for voltage regulation [22].

The sufficient condition we derive for the exactness of the SOCP relaxation is based on the linear approximations $S_{ij}^{\text{lin}}(p + \mathbf{i}q) = P_{ij}^{\text{lin}}(p) + \mathbf{i}Q_{ij}^{\text{lin}}(q)$ and $W_{ii}^{\text{lin}}(s)$ of the power flow, in the case where $p = \overline{p}$ and $q = \overline{q}$, i.e., power injection is maximized. In this case, power flow in the network is approximated by $P_{ij}^{\text{lin}}(\overline{p})$ and $Q_{ij}^{\text{lin}}(\overline{q})$.

Now we can formally specify the condition under which the SOCP relaxation is exact, as in Lemma 2, which is proved in the extended version of this paper [16]. To state the lemma, define $a^+ := \max\{a, 0\}$ for $a \in \mathbb{R}$, let $a_0^1 = 1$, $a_0^2 = 0$, $a_0^3 = 0$, $a_0^4 = 1$, and define

$$\begin{aligned} a_i^1 &:= \prod_{(j,k)\in\mathcal{P}_i} \left(1 - \frac{2r_{jk}\left[P_{jk}^{\text{lin}}(\overline{p})\right]^+}{\underline{|V_j|}^2}\right), \\ a_i^2 &:= \sum_{(j,k)\in\mathcal{P}_i} \frac{2r_{jk}\left[Q_{jk}^{\text{lin}}(\overline{q})\right]^+}{\underline{|V_j|}^2}, \\ a_i^3 &:= \sum_{(j,k)\in\mathcal{P}_i} \frac{2x_{jk}\left[P_{jk}^{\text{lin}}(\overline{p})\right]^+}{\underline{|V_j|}^2}, \\ a_i^4 &:= \prod_{(j,k)\in\mathcal{P}_i} \left(1 - \frac{2x_{jk}\left[Q_{jk}^{\text{lin}}(\overline{q})\right]^+}{\underline{|V_j|}^2}\right) \end{aligned}$$

for $i \in N^+$.

Lemma 2 *If every optimal solution* $w = (s, W, s_0)$ *to the SOCP relaxation satisfies* $W_{ii}^{\text{lin}}(s) \le \overline{|V_i|}^2$ *for* $i \in N^+$*, and the condition*

C1 $a_j^1 r_{ij} > a_j^2 x_{ij}$, $a_j^3 r_{ij} < a_j^4 x_{ij}$ *for all* $i \to j$

holds, then the SOCP relaxation is exact.

The condition $W_{ii}^{\text{lin}}(s) \le \overline{|V_i|}^2$ depends on solutions of the SOCP relaxation, and cannot be checked before solving the relaxed problem. This shortcoming motivates us to modify the OPF problem in Section III-B.

B. A modified OPF problem

We modify OPF by imposing additional constraints

$$W_{ii}^{\text{lin}}(s) \le \overline{|V_i|}^2, \quad i \in N^+ \tag{13}$$

on the power injection s, so that the condition $W_{ii}^{\text{lin}}(s) \le \overline{|V_i|}^2$ in Lemma 2 always holds. Note that the constraints in (8) and (13) can be combined as

$$\underline{|V_i|}^2 \le W_{ii},\ W_{ii}^{\text{lin}}(s) \le \overline{|V_i|}^2, \qquad i \in N^+$$

since it is proved in Lemma 1 that $W_{ii} \le W_{ii}^{\text{lin}}(s)$.

To summarize, the modified OPF problem is

$$\begin{aligned}
\textbf{OPF-m:}\ \min\quad & \sum_{i\to j} g_{ij}(W_{ii} - W_{ij} - W_{ji} + W_{jj}) \\
\text{over}\quad & s, W, s_0 \\
\text{s.t.}\quad & s_i = \sum_{j:\, j\sim i} (W_{ii} - W_{ij}) y_{ij}^*, && i \in N; \\
& s_i \in \mathcal{S}_i, && i \in N^+; \\
& \underline{|V_i|}^2 \le W_{ii},\, W_{ii}^{\text{lin}}(s) \le \overline{|V_i|}^2, && i \in N^+; \\
& \textbf{Rank}(W\{i,j\}) = 1, && i \to j.
\end{aligned}$$

Note that modifying OPF is necessary for an exact SOCP relaxation, since the SOCP relaxation is in general not exact.

Remarkably, the feasible sets of OPF-m and OPF are close since $W_{ii}^{\text{lin}}(s)$ is close to W_{ii} in practice. This is justified by the empirical studies in Section IV-B.

The SOCP relaxation for the modified OPF problem is as follows, which we call SOCP-m.

$$\begin{aligned}
\textbf{SOCP-m:}\ \min\quad & \sum_{i\to j} g_{ij}(W_{ii} - W_{ij} - W_{ji} + W_{jj}) \\
\text{over}\quad & w = (s, W, s_0) \\
\text{s.t.}\quad & s_i = \sum_{j:\, j\sim i} (W_{ii} - W_{ij}) y_{ij}^*, && i \in N; \\
& s_i \in \mathcal{S}_i, && i \in N^+; \\
& \underline{|V_i|}^2 \le W_{ii},\, W_{ii}^{\text{lin}}(s) \le \overline{|V_i|}^2, && i \in N^+; \\
& W\{i,j\} \succeq 0, && i \to j.
\end{aligned}$$

The main contribution of this paper is to provide a sufficient condition for the exactness of SOCP-m, that can be checked in priori and holds in practice. In particular, the condition is given in Theorem 2, which follows from Lemma 2.

Theorem 2 *The SOCP-m relaxation is exact if C1 holds.*

C1 can be checked in priori since it does not depend on the solutions to SOCP-m. In fact, $\{a_j^k, j \in N, k = 1, 2, 3, 4\}$ are functions of $(r, x, \overline{p}, \overline{q}, \overline{|V|})$ that can be computed efficiently in $O(n)$ time, therefore the complexity of checking C1 is $O(n)$.

C1 requires $\overline{p}$ and $\overline{q}$ be "small". Fix $(r, x, \overline{|V|})$, then C1 is a condition on $(\overline{p}, \overline{q})$. It can be verified that if $(\overline{p}, \overline{q}) \preceq (\overline{p}', \overline{q}')$ where the operator $\preceq$ denotes componentwise $\le$, then

$$\text{C1 holds for } (\overline{p}', \overline{q}') \ \Rightarrow\ \text{C1 holds for } (\overline{p}, \overline{q}),$$

i.e., the smaller power injections, the more likely C1 holds. In particular, it can be verified that if $(\overline{p}, \overline{q}) \preceq (0, 0)$, i.e., there is no distributed generation, then C1 holds as long as $(r, x) \succ 0$ where the operator $\succ$ denotes componentwise $>$.

As will be seen in the empirical studies in Section IV-C, C1 holds for all test networks, even those with high penetration of distributed generation, i.e., big $(\overline{p}, \overline{q})$. Hence, C1 should hold widely in practice.

Theorem 2 holds for more general objective functions and power injection constraints. In particular, the objective function in (5) can be generalized to $f(L(\ell), s)$ where the function $f(x, y) : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$ is strictly increasing in x. This includes generation costs of the form $\sum_{i\in N} f_i(\textbf{Re}(s_i))$ where f_0 is strictly increasing. The power injection constraints in (2) can be generalized to $s \in \mathcal{S}$ where $\mathcal{S}$ is an arbitrary set.

C. Uniqueness of solutions

If the solution to SOCP-m is unique, then any convex programming solver will obtain the same solution. Interestingly, this is the case if SOCP-m is exact, as stated in the following theorem, which is proved in the extended version of this paper [16].

Theorem 3 *SOCP-m has at most a unique solution if it is exact.*

Theorem 3 holds for more general objective functions and power injection constraints. In particular, the objective function in (5) can be generalized to any convex function $f(w)$, and the power injection constraints in (2) can be generalized to $s \in \mathcal{S}$ where $\mathcal{S}$ is an arbitrary convex set.

IV. Case Studies

In this section we use test distribution networks to demonstrate the following two arguments made in Section III: for a broad class of distribution networks,

1) The feasible sets of OPF and OPF-m are close;
2) Condition C1 holds.

In particular, we show that these two arguments hold for all test distribution networks considered in this paper.

A. Test distribution networks

We consider two types of test distribution networks: IEEE test distribution networks [23] and practical distribution networks in the service territory of Southern California Edison (SCE), a utility company in Southern California.

The IEEE 13-bus test network [23] is an unbalanced three-phase network with some circuit devices that are not modeled by (1). Hence, we adjust the network to demonstrate our arguments. The adjustment is detailed in the extended version of this paper [16]. We also consider two practical networks of SCE, a 47-bus network [9] and a 56-bus network [24].

The three test networks have increasing penetration level of distributed generation. While the IEEE network does not have distributed generation (0% penetration), the 47-bus network has 6.4MW nameplate distributed generation capacity (over 50% penetration in comparison with 11.3MVA peak spot load), and the 56-bus network has 5MW nameplate distributed generation capacity (over 100% penetration in comparison with 3.835MVA peak spot load).

B. Feasible sets of OPF-m and OPF are close.

We show that the feasible sets of OPF-m and OPF are empirically close for all three test networks in this section. Specifically, we show that OPF-m eliminates some feasible points of OPF that are close to the voltage upper bounds.

To state the results, we first define a measure that will be used to evaluate the difference between the feasible sets of OPF and OPF-m. It is claimed in [25] that given $s = p + \mathbf{i}q$, there exists a unique voltage $V(s)$ near the nominal value that satisfies the power flow constraint (1). Define

$$\varepsilon := \max\left\{ W_{ii}^{\text{lin}}(s) - |V_i(s)|^2 \mid s \text{ satisfies (2)},\ i \in N^+ \right\}$$

as the maximum deviation from $W_{ii}^{\text{lin}}(s)$ to $W_{ii}(s) = |V_i(s)|^2$. It follows from Lemma 1 that $W_{ii}^{\text{lin}}(s) \geq W_{ii}(s)$ for all s and all $i \in N^+$, therefore $\varepsilon \geq 0$.

The value "ε" serves as a measure for the difference between the feasible sets of OPF-m and OPF for the following reason. Consider the OPF problem with stricter voltage upper bound constraints $W_{ii} \leq \overline{|V_i|}^2 - \varepsilon$:

$$\begin{aligned} \textbf{OPF-}\boldsymbol{\varepsilon}\textbf{:}\ \min \quad & \sum_{i\to j} g_{ij}(W_{ii} - W_{ij} - W_{ji} + W_{jj}) \\ \text{over} \quad & s, W, s_0 \\ \text{s.t.} \quad & (1), (2), (9); \\ & \underline{|V_i|}^2 \leq W_{ii} \leq \overline{|V_i|}^2 - \varepsilon, \qquad i \in N^+. \end{aligned}$$

Then it follows from

$$W_{ii}(s) \leq \overline{|V_i|}^2 - \varepsilon \implies W_{ii}^{\text{lin}}(s) \leq \overline{|V_i|}^2, \qquad i \in N^+$$

that the feasible set $\mathcal{F}_{\text{OPF-}\varepsilon}$ of OPF-ε is contained in the feasible set $\mathcal{F}_{\text{OPF-m}}$ of OPF-m. Furthermore, we know that the feasible set of OPF-m is contained in the feasible set of OPF. Hence,

$$\mathcal{F}_{\text{OPF-}\varepsilon} \subseteq \mathcal{F}_{\text{OPF-m}} \subseteq \mathcal{F}_{\text{OPF}}.$$

To summarize, OPF-m is "sandwiched" between OPF and OPF-ε, which are "ε" apart, as illustrated in Fig. 2. If ε is small, then $\mathcal{F}_{\text{OPF-m}}$ is close to $\mathcal{F}_{\text{OPF}}$.

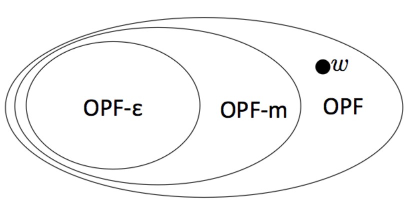

Fig. 2. Feasible sets of OPF-ε, OPF-m, and OPF. The point w is feasible for OPF but not for OPF-m.

Moreover, if ε is small, then any point w that is feasible for OPF but infeasible for OPF-m is close to the voltage upper bound since $W_{ii} > \overline{|V_i|}^2 - \varepsilon$ for some $i \in N^+$. Such points are perhaps undesirable for robust operation.

Now we show that ε takes relatively small values for the IEEE and SCE test networks. In all case studies, we assume that the substation voltage is fixed at the nominal value, i.e., $W_{00} = 1$, and that the voltage upper and lower bounds are $\overline{|V_i|} = 1.05$ and $\underline{|V_i|} = 0.95$ for $i \in N^+$.

To evaluate ε for the IEEE 13-bus test network, we further assume that $\overline{p} = \underline{p}$, $\overline{q} = \underline{q}$, and that they equal the values specified in the IEEE test cases documents. In this setup, $\varepsilon = 0.0043$. Therefore the voltage constraints are $0.9025 \leq W_{ii} \leq 1.1025$ for OPF and $0.9025 \leq W_{ii} \leq 1.0982$ for OPF-ε.

TABLE I

CLOSENESS OF OPF-M AND OPF

	ε
IEEE 13-bus	0.0043
SCE 47-bus	0.0036
SCE 56-bus	0.0106

To evaluate ε for the SCE test networks, we further assume that all loads draw peak spot apparent power at power factor 0.97, that all shunt capacitors are switched on, and that distributed generators generate real power at their nameplate capacities with zero reactive power (these assumptions enforce $\underline{p} = \overline{p}$, $\underline{q} = \overline{q}$, and simplify the calculation of ε). The values of ε are summarized in Table I. For example, the value ε is 0.0031 for the SCE 47-bus network. Therefore the voltage constraints are $0.9025 \leq W_{ii} \leq 1.1025$ for OPF and $0.9025 \leq W_{ii} \leq 1.0994$ for OPF-ε.

C. Condition C1 holds in all test networks.

We show that C1 holds for all test networks in this section.

To present the results, we transform C1 to a brief form. To state the brief form, first define[1]

$$\underline{b}_i := \frac{a_i^2}{a_i^1},\ \overline{b}_i := \frac{a_i^4}{a_i^3}$$

for $i \in N$, then C1 is equivalent to

$$\frac{r_{ij}}{x_{ij}} \in \left(\underline{b}_j, \overline{b}_j\right), \qquad i \to j. \tag{14}$$

We have checked that (14) holds for all three test networks.

To better present the result, note that (14) is implied by

$$\text{conv}\left(\left\{\frac{r_{ij}}{x_{ij}}, i \to j\right\}\right) \quad \subseteq \quad \bigcap_{j\in N} \left(\underline{b}_j, \overline{b}_j\right) \tag{15}$$

where $\text{conv}(A)$ denotes the convex hull of a set A. In the rest of this section, we focus on (15) since it only involves 2 intervals and is therefore easier to present.

We call the left hand side of (15) the *range of* r/x and the right hand side the *minimum interval*. The calculation of ranges of r/x is straightforward. To calculate the minimum intervals of the test networks, we consider two cases: a bad

[1]If $a_i^3 = 0$ for some $i \in N$, then set $\overline{b}_i = \infty$. In practice, $a_i^1 \approx 1$.

case and the worst case. In the bad case, we set the bounds $\overline{p}$ and $\overline{q}$ as follows:

- For a load bus i, we set $(\overline{p}_i, \overline{q}_i)$ to equal to the specified load data[2] because there is usually not much flexibility in controlling the loads.
- For a shunt capacitor bus i, we set $\overline{p}_i = 0$ and $\overline{q}_i$ to equal to its nameplate capacity.
- For a distributed generator bus i, we set $\overline{q}_i = 0$ and $\overline{p}_i$ to equal to its nameplate capacity. In practice, $\overline{p}_i$ is usually smaller.

In the bad case setup, $(\overline{p}_i, \overline{q}_i)$ is artificially enlarged except for load buses.

In the worst case, we further set $\overline{p}_i = 0$ and $\overline{q}_i = 0$ for load buses while they are negative in practice. Hence in the worst case setup, $(\overline{p}_i, \overline{q}_i)$ is artificially enlarged for all buses.

The minimum intervals for the three test networks in the bad case and in the worst case are calculated and summarized in Table II. Recall that C1 is more difficult to hold if $(\overline{p}, \overline{q})$ gets bigger, and that the test networks have increasing penetration of distributed generation, we expect C1 to be increasingly likely to be violated in the test networks.

TABLE II
THE RANGE OF r/x AND MINIMUM INTERVALS OF TEST NETWORKS

	range of r/x	minimum interval (worst case)	minimum interval (bad case)
IEEE 13-bus	$[0.331, 2.62]$	$(0.0175, \infty)$	$(0.0013, \infty)$
SCE 47-bus	$[0.321, 7.13]$	(0.0374,10.0)	(0.0187,995)
SCE 56-bus	$[0.414, 4.50]$	(0.0652,2.93)	(0.0528,5.85)

In the bad case, the minimum interval contains the range of r/x for all three networks with significant margins. In the worst case, the minimum interval covers the range of r/x for the first two networks, but not the third one. However, (14), which is equivalent to C1, still holds for the third network.

To summarize, C1 holds for all three test networks, even those with high penetration of distributed generation. Therefore, C1 should hold widely in practice.

V. Conclusion

We have proved that the SOCP relaxation for OPF is exact under a prior checkable condition C1, after imposing additional constraints on power injections. C1 holds for all test networks we use in the numerical experiments, including the IEEE 13-bus test network and practical networks with high penetration of distributed generation. The additional constraints on power injections eliminate feasible points of OPF that are close to the voltage upper bounds, which is justified using the same set of test networks.

[2] In the SCE networks, only apparent power is given. Therefore we assume a power factor of 0.97 to obtain the real and reactive power consumptions. For example, we set $\overline{p}_{22} = \underline{p}_{22} = -2.16$MW and $\overline{q}_{22} = \underline{q}_{22} = -0.54$MVAR for the load at bus 22 since it has 2.23MVA apparent power.

References

[1] B. Stott and O. Alsac, "Fast decoupled load flow," IEEE Transactions on Power Apparatus and Systems, vol. PAS-93, no. 3, pp. 859–869, 1974.

[2] G. C. Contaxis, C. Delkis, and G. Korres, "Decoupled optimal power flow using linear or quadratic programming," IEEE Transactions on Power Systems, vol. 1, no. 2, pp. 1–7, 1986.

[3] W. Min and L. Shengsong, "A trust region interior point algorithm for optimal power flow problems," International Journal on Electrical Power and Energy Systems, vol. 27, no. 4, pp. 293–300, 2005.

[4] E. C. Baptista, E. A. Belati, and G. R. M. da Costa, "Logarithmic barrier-augmented lagrangian function to the optimal power flow problem," International Journal on Electrical Power and Energy Systems, vol. 27, no. 7, pp. 528–532, 2005.

[5] G. L. Torres and V. H. Quintana, "An interior-point method for nonlinear optimal power flow using voltage rectangular coordinates," IEEE Transactions on Power Systems, vol. 13, no. 4, pp. 1211–1218, 1998.

[6] X. Bai, H. Wei, K. Fujisawa, and Y. Yang, "Semidefinite programming for optimal power flow problems," International Journal of Electric Power & Energy Systems, vol. 30, no. 6, pp. 383–392, 2008.

[7] X. Bai and H. Wei, "Semi-definite programming-based method for security-constrained unit commitment with operational and optimal power flow constraints," Generation, Transmission & Distribution, IET,, vol. 3, no. 2, pp. 182–197, 2009.

[8] J. Lavaei and S. H. Low, "Zero duality gap in optimal power flow problem," IEEE Transactions on Power Systems, vol. 27, no. 1, pp. 92–107, 2012.

[9] M. Farivar, C. R. Clarke, S. H. Low, and K. M. Chandy, "Inverter var control for distribution systems with renewables," in IEEE SmartGridComm, 2011, pp. 457–462.

[10] J. A. Taylor, "Conic optimization of electric power systems," Ph.D. dissertation, MIT, 2011.

[11] R. Jabr, "Radial distribution load flow using conic programming," IEEE Transactions on Power Systems, vol. 21, no. 3, pp. 1458–1459, 2006.

[12] S. Bose, S. Low, and K. Chandy, "Equivalence of branch flow and bus injection models," in submitted to Allerton, 2013.

[13] B. Zhang and D. Tse, "Geometry of injection regions of power networks," IEEE Transactions on Power Systems, to appear, 2012.

[14] S. Bose, D. Gayme, S. Low, and K. Chandy, "Quadratically constrained quadratic programs on acyclic graphs with application to power flow," arXiv:1203.5599, 2012.

[15] "Electric power systems and equipment—voltage ratings (60 hertz)," ANSI Standard Publication, no. ANSI C84.1, 1995.

[16] L. Gan, U. Topcu, N. Li, and S. Low, "Exact convex relaxation for optimal power flow in tree networks," arXiv preprint arXiv:1208.4076, 2012.

[17] S. Sojoudi and J. Lavaei, "Physics of power networks makes hard optimization problems easy to solve," in IEEE Power & Energy Society General Meeting, 2012, pp. 1–8.

[18] L. Gan, N. Li, U. Topcu, and S. H. Low, "On the exactness of convex relaxation for optimal power flow in tree networks," in IEEE CDC, 2012, pp. 465–471.

[19] M. E. Baran and F. F. Wu, "Optimal capacitor placement on radial distribution networks," IEEE Transactions on Power Delivery, vol. 4, no. 1, pp. 725–734, 1989.

[20] ——, "Optimal sizing of capacitors placed on a radial distribution systems," IEEE Transactions on Power Delivery, vol. 4, no. 1, pp. 735–743, 1989.

[21] ——, "Network reconfiguration in distribution systems for loss reduction and load balancing," IEEE Transactions on Power Delivery, vol. 4, no. 2, 1989.

[22] K. Turitsyn, P. Sulc, S. Backhaus, and M. Chertkov, "Local control of reactive power by distributed photovoltaic generators," in IEEE SmartGridComm, 2010, pp. 79–84.

[23] IEEE distribution test feeders, available at http://ewh.ieee.org/soc/pes/dsacom/testfeeders/.

[24] M. Farivar, R. Neal, C. Clarke, and S. Low, "Optimal inverter var control in distribution systems with high pv penetration," arXiv preprint arXiv:1112.5594, 2011.

[25] H.-D. Chiang and M. E. Baran, "On the existence and uniqueness of load flow solution for radial distribution power networks," IEEE Transactions on Circuits and Systems, vol. 37, no. 3, pp. 410–416, 1990.