KOLYVAGIN'S CONJECTURE, BIPARTITE EULER SYSTEMS, AND HIGHER CONGRUENCES OF MODULAR FORMS

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Abstract. Let $E/\mathbb{Q}$ be an elliptic curve and let K be an imaginary quadratic field. Under a certain Heegner hypothesis, Kolyvagin constructed cohomology classes for E using K-CM points and conjectured they did not all vanish. Conditional on this conjecture, he described the Selmer rank of E using his system of classes. We extend work of Wei Zhang to prove new cases of Kolyvagin's conjecture by considering congruences of modular forms modulo large powers of p. Additionally, we prove an analogous result, and give a description of the Selmer rank, in a complementary "definite" case (using certain modified L-values rather than CM points). Similar methods are also used to improve known results on the Heegner point main conjecture of Perrin-Riou. One consequence of our results is a new converse theorem, that p-Selmer rank one implies analytic rank one, when the residual representation has dihedral image.

Contents

1. Introduction

Let $f : \mathbb{T}_N \to \mathcal{O}_f$ be a non-CM newform of level N, weight 2, and trivial nebentypus, where $\mathcal{O}_f$ is the ring of integers in an algebraic extension of $\mathbb{Q}$. The Birch and Swinnerton-Dyer Conjecture asserts the equality:

$$r(A/\mathbb{Q}) = \operatorname{ord}_{s=1} L(A, s), \tag{1}$$

where $A = A_f$ is the associated abelian variety to f and r is its Mordell-Weil rank. In pioneering works on this problem, Perrin-Riou [39] and Kolyvagin [30, 31] studied ranks of elliptic curves over an auxiliary imaginary quadratic field K through the theory of Heegner points on modular curves. We prove, in new cases, conjectures made by both authors.

Fix a quadratic imaginary field K, and a prime $\wp \subset \mathcal{O}_f$ of residue characteristic. We write T_f for (a lattice in) the $\wp$-adic Galois representation associated to f. Assume the following generalized Heegner hypothesis:

(Heeg) $\quad N = N^+N^-$, where all $\ell | N^+$ are split in K, all $\ell | N^-$ are inert in K, and N^- is squarefree,

as well as:

(unr) $\quad p \nmid 2N \operatorname{disc}(K).$

For purposes of exposition in this introduction, we also assume:

(sclr) $\quad$ The image of the G_K action on $\overline{T}_f$ contains a nonzero scalar.

To state Kolyvagin's conjecture, assume that the number of prime factors $\nu(N^-)$ is even. If m is a squarefree product of primes inert in K, one can use Heegner points of conductor m on the Shimura curve X_{N^+,N^-} to construct classes

$$c(m) \in H^1(K, T_f/I_m),$$

where I_m is the ideal of $\mathcal{O} = \mathcal{O}_{f,\wp}$ generated by $\ell + 1$ and a_ℓ for all $\ell | m$. (In the text, $c(m)$ is denoted $\overline{c}(m, 1)$.) These classes are a mild generalization of the ones constructed by Kolyvagin [31]. We are able to prove the following result towards Kolyvagin's conjecture on the nonvanishing of the system $\{c(m)\}$:

Theorem A. *[Corollary 8.3.7] Assume (Heeg), (unr), and (sclr) hold for $f, \wp$, and K, and $\nu(N^-)$ is even. Suppose the following conditions hold:*

$$(\diamondsuit) \quad \begin{cases} \bullet \text{ The modulo } \wp \text{ representation } \overline{T}_f \text{ associated to } f \text{ is absolutely irreducible; if } p = 3, \text{ then } \overline{T}_f \\ \text{is not induced from a character of } G_{\mathbb{Q}\sqrt{-3}}. \\ \bullet \text{ If } p \text{ is inert in } K \text{ or } a_p \text{ is not a } \wp\text{-adic unit, then there exists some prime } \ell || N. \\ \bullet \text{ If } a_p \text{ is not a } \wp\text{-adic unit, then either } \ell \text{ may be chosen above so that } A_f \text{ has non-split toric} \\ \text{reduction at } \ell, \text{ or the image of the Galois action on } T_f \text{ contains a conjugate of } SL_2(\mathbb{Z}_p). \end{cases}$$

Then there exists a nonzero Kolyvagin class

$$0 \neq c(m) \in H^1(K, T_f/I_m).$$

As Kolyvagin observed, Theorem A can be used to give a description of the Selmer ranks

$$r^\pm = \mathrm{rk}_{\mathcal{O}} \, \mathrm{Sel}(K, T_f)^\pm,$$

where superscripts refer to the action of complex conjugation. Indeed, define the vanishing order of the system $\{c(m)\}$ as

$$\nu := \min \{\nu(m) \, : \, c(m) \neq 0\} \tag{2}$$

where as before ν denotes the number of prime factors. Then we have:

Corollary B. *Under the assumptions of Theorem A,*

$$\max \left\{r^+, r^-\right\} = \nu + 1.$$

Moreover $r^+ + r^-$ is odd, and the larger eigenspace has sign $(-1)^{\nu+1}\epsilon_f$, where ϵ_f is the global root number of f.

Of course, the latter two assertions follow from the parity conjecture for f, already proven by Nekovar [36].

Since $c(1) \in \mathrm{Sel}(K, T_f)$ is the Kummer image of the classical Heegner point, the Gross-Zagier formula implies that $L'(f/K, 1) \neq 0$ if and only if $c(1) \neq 0$. Hence Corollary B yields a so-called p-converse theorem (in fact, under a slightly weaker hypothesis):

Corollary C. *Assume that (Heeg), (unr), and Condition $\diamondsuit$ hold for f, $\wp$, and K, and $\nu(N^-)$ is even. Then*

$$L'(f/K, 1) \neq 0 \iff \mathrm{rk}_{\mathcal{O}} \, \mathrm{Sel}(K, T_f) = 1 \iff \mathrm{rk}_{\mathbb{Z}} \, A_f(K) = [\mathcal{O}_f : \mathbb{Z}].$$

Now suppose instead that $\nu(N^-)$ is odd; it turns out that Kolyvagin's construction, suitably modified, may still be used to relate Selmer ranks and CM points. The Jacquet-Langlands correspondence associates to f a quaternionic modular form

$$\phi_f : X_{N^+,N^-} \to \mathcal{O}_f, \tag{3}$$

where X_{N^+,N^-} is a double coset space for a definite quaternion algebra, usually called a Shimura set. If m is a squarefree product of primes inert in K, there exist analogues of CM points of conductor m on the Shimura set. Using the values of ϕ_f at these points, we construct certain special elements (well-defined up to units)

$$\lambda(m) \in \mathcal{O}/I_m \tag{4}$$

($\lambda(m, 1)$ in the text). Here the ideal $I_m \subset \mathcal{O}$ is as before. The elements $\lambda(m)$ encode the same information about the Selmer ranks of A_f/K as Kolyvagin's classes $c(m)$.

Theorem D. *Suppose that (Heeg), (unr), (sclr), and Condition $\diamondsuit$ hold for $f, \wp$, and K, and that $\nu(N^-)$ is odd. Then the vanishing order*

$$\nu \coloneqq \min \{\nu(m) \,:\, \lambda(m) \neq 0\}$$

is finite and

$$\nu = \max \left\{r^+, r^-\right\}.$$

Moreover $(-1)^\nu = \epsilon_f$ and $r^+ + r^-$ is even.

As before, the final statement is a consequence of the parity conjecture; we include it only to emphasize that it follows from the non-vanishing of some $\lambda(m)$, in analogy to the indefinite case.

1.1. **Comparison to previous results.** In the indefinite case, the first results towards Kolyvagin's conjecture were obtained by Zhang [53], under a number of additional assumptions: that $p \geq 5$, that the Galois representation associated to $\overline{T}_f$ is surjective, and additional hypotheses on the residual ramification. In particular, under the hypotheses of [53], there exists a class $c(m)$ whose reduction in $H^1(K, \overline{T}_f)$ is nonzero; this is not the case in general. In the definite case, the classes $\lambda(m)$ are a novel feature of this work and were not considered in [53].

The converse theorem we obtain (Corollary C) is new in several cases, most notably when the image of the Galois action on $\overline{T}_f$ is dihedral, or when $p = 3$. Previous results, under various additional hypotheses, were obtained by Zhang as a corollary of his work on Kolyvagin's conjecture, and by Skinner [46] by a purely Iwasawa-theoretic method. For converse theorems in other settings, see Burungale [7] for the CM case, Castella-Grossi-Lee-Skinner [8] for the residually reducible case, Castella-Wan [9] for the supersingular case, and Skinner-Zhang [48] for the case of multiplicative reduction.

1.2. **Iwasawa theory.** Now suppose again that $\nu(N^-)$ is even. While the Kolyvagin classes are constructed by varying the conductor of CM points on X_{N^+,N^-} over squarefree integers, one may instead p-adically interpolate CM points of p-power conductor to obtain a class:

$$\boldsymbol{\kappa}_\infty \in H^1(K, T_f \otimes \Lambda(\Psi)), \tag{5}$$

where $\Lambda = \mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]$ is the anticyclotomic Iwasawa algebra, given G_K-action by the tautological character Ψ. (Note that the specialization of $\boldsymbol{\kappa}_\infty$ at the trivial character is a multiple of $c(1)$.) The methods used to prove Theorem A also yield the following result towards Perrin-Riou's Heegner point main conjecture.

Theorem E (Theorem 8.2.1)**.** *Suppose that (Heeg), (unr), and Condition $\diamondsuit$ hold for $f, \wp$, and K, and that $\nu(N^-)$ is even. Suppose further that a_p is a $\wp$-adic unit and p splits in K. Then there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}(K_\infty, A_f[\wp^\infty])^\vee \approx \Lambda \oplus M \oplus M$$

for some torsion Λ-module M, and

$$\mathrm{char}_\Lambda \left(\frac{\mathrm{Sel}(K, T_f \otimes \Lambda)}{\Lambda \cdot \boldsymbol{\kappa}_\infty} \right) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. If (sclr) holds, then the equality is true in Λ.

For precise definitions of these Selmer groups and of $\boldsymbol{\kappa}_\infty$, which is denoted $\boldsymbol{\kappa}(1)$ in the text, see §6.1.

Finally, we have the following result on the anticyclotomic main conjecture for f when $\nu(N^-)$ is odd. Evaluating the quaternionic modular form ϕ_f on CM points of p-power conductor on the Shimura set X_{N^+,N^-}, one constructs the algebraic p-adic L-function

$$\boldsymbol{\lambda}_\infty \in \Lambda, \tag{6}$$

denoted $\boldsymbol{\lambda}(1)$ in the text. The square of $\boldsymbol{\lambda}_\infty$ has an interpolation property for twisted L-values of f.

Theorem F (Theorem 3.4.6, Proposition 6.1.5)**.** *Suppose that (Heeg), (unr), and Condition $\diamondsuit$ hold for $f, \wp$, and K, and that $\nu(N^-)$ is odd. Suppose further that a_p is a $\wp$-adic unit and p splits in K. Then there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}(K_\infty, A_f[\wp^\infty])^\vee \approx M \oplus M$$

for some torsion Λ-module M, and

$$(\boldsymbol{\lambda}_\infty) \subset \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. If additionally (sclr) holds, then the inclusion is true in Λ.

The opposite inclusion of ideals may be deduced directly from Skinner-Urban's proof of one divisibility in the three-variable main conjecture [47]; indeed, this is an essential ingredient in all of our results, as explained below.

1.3. **Comparison to previous results.** The technical hypotheses in Zhang's proof of Kolyvagin's conjecture were carried over to Burungale, Castella, and Kim's proof [6] of the lower bound on the Selmer group in the Heegner point main conjecture, where it is also assumed that p is not anomalous. While the methods used in this paper build on those of [6], Castella and Wan [10] have given a completely independent proof of a three-variable main conjecture when $\nu(N^-)$ is even. Their result also requires some hypotheses on residual ramification avoided here, and that N be squarefree.

For upper bounds on the Selmer group in Theorem E and Theorem F, various technical assumptions on the residual representation and on the image of the Galois action were used in prior works by Howard [24, 25] and Chida-Hsieh [11].

1.4. **Overview of the proofs.** To prove Theorems A and D, we extend Kolyvagin's construction to a larger system of classes

$$c(m, Q_1) \in H^1(K, T_f/\wp^M), \quad \lambda(m, Q_2) \in \mathcal{O}/\wp^M, \tag{7}$$

where M is a fixed integer, and m, Q_1, Q_2 are squarefree product of auxiliary primes satisfying certain congruence conditions, such that $\nu(N^-Q_1)$ is even and $\nu(N^-Q_2)$ is odd. The classes (7) form a bipartite Euler system in the sense of Howard [25] for each fixed m and a Kolyvagin system for each fixed Q_1. If $\nu(N^-)$ itself is even, then the classes $c(m, 1)$ agree with Kolyvagin's original construction. The Euler system relations are of the form:

$$\operatorname{loc}_q c(m, Q_1) \sim \lambda(m, Q_1 q) \sim \partial_{q'} c(m, Q_1 q q'), \tag{8}$$

where q, q' are two additional auxiliary primes not dividing Q_1; and

$$\operatorname{loc}_\ell^{\pm} c(m, Q_1) \sim \partial_\ell^{\mp} c(m\ell, Q_1), \tag{9}$$

where ℓ is an additional auxiliary prime not dividing m. (Here loc_q, $\partial_{q'}$, $\operatorname{loc}_\ell^{\pm}$, $\partial_\ell^{\pm}$ are certain localization maps landing in subspaces of the local cohomology free of rank one over $\mathcal{O}/\wp^M$.) The classes $c(m, Q_1)$ were introduced by Zhang, although the $\lambda(m, Q_2)$ are only implicit in [53].

If $c(m, Q_1) \neq 0$, then one can use the Kolyvagin system relation to find an auxiliary ℓ — either prime or equal to 1— such that $\partial_q c(m\ell, Q_1) \neq 0$. By the bipartite Euler system relation, this implies $\lambda(m\ell, Q_1/q) \neq 0$. On the other hand if $\lambda(m, Q_2) \neq 0$ and $q | Q_2$, then $c(m, Q_2/q) \neq 0$. Combining these two observations, we reduce the non-vanishing of some class $c(m, 1)$ or $\lambda(m, 1)$ — depending on the parity of $\nu(N^-)$ — to exhibiting a single Q_2 such that $\lambda(1, Q_2) \neq 0$.

Now, if there exists a newform g of level NQ_2 with a congruence to f modulo $\wp^M$, then $\lambda(1, Q_2)$ is essentially the reduction of the algebraic L-value $L^{\mathrm{alg}}(g/K, 1)$ modulo $\wp^M$, which is related to the length of the Selmer group of g by the Iwasawa main conjecture [47, 51]. To complete the proof, it therefore suffices to choose a suitable Q_2 and construct such a g with a small Selmer group. We remark that our results can only be obtained by working modulo $\wp^M$ for a large M, since in general it will not be possible to choose g such that $L^{\mathrm{alg}}(g/K, 1)$ is a $\wp$-adic unit; in [53], $M = 1$ is fixed throughout, and the need to show that the L-value is a unit is responsible for most of the additional hypotheses.

To construct g, we use the deformation-theoretic techniques developed by Ramakrishna [41]. Standard level-raising methods work by producing a modulo $\wp$ eigenform of the desired level, and then using that all modulo $\wp$ eigenforms lift to characteristic zero, but this is not the case modulo $\wp^M$. Instead, we deform the representation $T_f/\wp^M$ to a $\wp$-adic Galois representation of a suitable auxiliary level, and then apply modularity lifting to ensure the resulting representation is modular. The auxiliary level Q_2 must be chosen to control two Selmer groups: the adjoint Selmer group governing the deformation problem, and the Selmer group $\mathrm{Sel}(K, A_g[\wp^\infty])$ that is related to the L-value.

We now make some remarks on the construction of the Euler system. The elements $c(m, Q_1)$ (resp. $\lambda(m, Q_2)$) are constructed from CM points of conductor m on the Shimura curve X_{N^+, N^-Q_1} (resp. Shimura set X_{N^+, N^-Q_2}). Similar Euler system constructions have been made by many authors, e.g. in [11, 3] as well as in [53], but all have relied on certain hypotheses ensuring an integral multiplicity one property for the space of algebraic modular forms on X_{N^+, N^-Q_i}, which we do not impose here. Instead, we obtain a control on the failure of multiplicity one, using the work of Helm [22] on maps between Jacobians of modular curves and Shimura curves. The construction of the Euler system is intimately related to level-raising, and so our method can also be viewed as improving results on level-raising of f to algebraic eigenforms modulo $\wp^M$ new at multiple auxiliary primes, which had previously been restricted to the multiplicity one case.

The proof of Theorem E is similar to that of Theorem A: the p-adically interpolated Heegner class $\boldsymbol{\kappa}_\infty$ is viewed as the bottom layer of an Euler system $\{\boldsymbol{\kappa}(Q_1), \boldsymbol{\lambda}(Q_2)\}$. (The squarefree conductor m no longer plays a role.) If g, as above, is a newform of level NQ_2 with a congruence to f, then $\boldsymbol{\lambda}(Q_2)$ is congruent to Bertolini and Darmon's anticyclotomic p-adic L-function of g [3]. Using this and an Euler system argument, we reduce the lower bound on the Selmer group in the Heegner point main conjecture to the anticyclotomic main conjecture for g, which was proven in [47]. Finally, the upper bound on the Selmer group in the Heegner point main conjecture, as well as Theorem F, follow by standard arguments from the construction of the Euler system.

In the text, the arguments described above are phrased in the language of ultrapatching, which amounts to a formalism for letting M tend to infinity; this also forces each prime factor of m, Q_1, Q_2 to tend to infinity in order to satisfy the congruence conditions. (The number of prime factors of m, Q_1, and Q_2 remain bounded.) This method was inspired by [45], where ultrapatching was applied to the Taylor-Wiles construction. Our setting is different in that we patch Galois cohomology groups and Selmer groups rather than geometric étale cohomology groups. The benefit of ultrapatching is that it allows us to consider the Euler system classes as characteristic zero objects in patched Selmer groups, significantly streamlining the Euler system arguments. For instance, with patching, we are able to make precise the heuristic that the non-vanishing of each Euler system class $c(m, Q_1)$ or $\lambda(m, Q_2)$ is equivalent to the (m, Q_i)-transverse Selmer group being rank one or zero, respectively, cf. Lemma 8.3.4.

Structure of the paper. In §2, we review basic properties of ultrafilters and introduce patched cohomology and Selmer groups. In §3, we present a simplified version of the theory of bipartite Euler systems that appeared in [25], using patched cohomology. In §4, we recall the geometric inputs that will be used to construct bipartite Euler systems: the work of Helm on maps between modular curves and Shimura curves, and the behavior of Heegner points on Shimura curves under reduction and specialization. In §5, we prove the modulo $\wp^M$ level-raising result and present a general framework for constructing bipartite Euler systems out of CM points, which we then specialize in §6 for our applications. In §7, we give the deformation-theoretic input to construct the newform g (in fact a sequence g_n satisfying increasingly deep congruence conditions). Finally, we prove the main results in §8. An additional calculation in cyclotomic Iwasawa theory is required for Kolyvagin's conjecture when p is non-ordinary or inert in K; this is done in the appendix.

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2. Ultrafilters and patching

2.1. Ultraproducts.

The facts recalled in this subsection are discussed in more detail in [32].

2.1.1. A (non-principal) ultrafilter $\mathfrak{F}$ for the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ is a collection of subsets of $\mathbb{N}$ satisfying the following properties:

(1) Every set $S \in \mathfrak{F}$ is infinite.
(2) For every $S \subset \mathbb{N}$, either $S \in \mathfrak{F}$ or $\mathbb{N} - S \in \mathfrak{F}$.
(3) If $S_1 \subset S_2 \subset \mathbb{N}$ and $S_1 \in \mathfrak{F}$, then $S_2 \in \mathfrak{F}$.
(4) If $S_1, S_2 \in \mathfrak{F}$, then $S_1 \cap S_2 \in \mathfrak{F}$.

Throughout this paper, we fix once and for all a non-principal ultrafilter $\mathfrak{F}$ on $\mathbb{N}$, which is possible assuming the axiom of choice. We will say that a statement P holds for $\mathfrak{F}$-many $n \in \mathbb{N}$ if the set S of n for which P holds lies in $\mathfrak{F}$.

Proposition 2.1.2. *Suppose that $\mathcal{C}$ is a finite set and $S \subset \mathbb{N}$ lies in $\mathfrak{F}$. Then for any function $t : S \to \mathcal{C}$, there is a unique $c \in \mathcal{C}$ such that $t(n) = c$ for $\mathfrak{F}$-many n.*

Proof. The function t defines a finite partition of $\mathbb{N}$:

$$\mathbb{N} = (\mathbb{N} - S) \sqcup \bigsqcup_{c \in \mathcal{C}} t^{-1}(c).$$

An easy induction argument shows that, for any partition of $\mathbb{N}$ into a finite number sets, exactly one of the sets lies in $\mathfrak{F}$. Since $\mathbb{N} - S \notin \mathfrak{F}$, the result follows. $\square$

2.1.3. If $\mathcal{M} = \{M_n\}_{n\in\mathbb{N}}$ is a sequence of sets indexed by $\mathbb{N}$, then $\mathfrak{F}$ defines an equivalence relation $\sim$ on $\prod M_n$:

$$(m_n)_{n\in\mathbb{N}} \sim (m'_n)_{n\in\mathbb{N}} \iff \{n : m_n = m'_n\} \in \mathfrak{F}.$$

The quotient $\prod M_n/\sim$ is called the **ultraproduct** of the sequence $\mathcal{M}$ and is denoted $\mathcal{U}(\mathcal{M})$. The ultraproduct is functorial: let $\mathcal{M}' = \{M'_n\}$ be another sequence of sets and suppose given, for $\mathfrak{F}$-many n, maps $\varphi_n : M_n \to M'_n$. Then there is a natural map $\varphi^{\mathcal{U}} : \mathcal{U}(\mathcal{M}) \to \mathcal{U}(\mathcal{M}')$. Similarly, if R is a fixed (topological) ring and each M_n is a (continuous) R-module, then $\mathcal{U}(\mathcal{M})$ may be naturally endowed with the structure of a (continuous) R-module; in particular, if each M_n is an abelian group, then $\mathcal{U}(\mathcal{M})$ has a natural abelian group structure as well.

The following basic properties are proven in [32, §I.1] using the functoriality of the ultraproduct.

Proposition 2.1.4. *Suppose that each M_n is a finite set, and that $\#M_n < C$ for some constant C and for $\mathfrak{F}$-many n. Then:*

1. *$\mathcal{U}(\mathcal{M})$ is finite and $\#\mathcal{U}(\mathcal{M}) = \#M_n$ for $\mathfrak{F}$-many n.*
2. *Suppose that each M_n is additionally endowed with the structure of a (continuous) R-module, and let A be another (continuous) R-module. Given a family of isomorphisms $\varphi_n : M_n \xrightarrow{\sim} A$ for $\mathfrak{F}$-many n, there is an induced isomorphism*

$$\varphi^{\mathcal{U}} : \mathcal{U}(\mathcal{M}) \xrightarrow{\sim} A.$$

Remark 2.1.5. If R is (topologically) finitely generated, then there are only finitely many isomorphism classes of (continuous) R-modules of a fixed cardinality. Hence, if $\mathcal{M}$ is a sequence of finite R-modules of bounded cardinality, Proposition 2.1.2 and Proposition 2.1.4 together imply that $\mathcal{U}(\mathcal{M})$ is non-canonically isomorphic to $\mathfrak{F}$-many M_n.

Proposition 2.1.6. *Let $\mathcal{C}$ be the category of sequences of (continuous) R-modules of uniformly bounded cardinality. Then $\mathcal{U}$ is exact as a functor from $\mathcal{C}$ to the category of (continuous) R-modules.*

Proof. We wish to show that $\mathcal{U}$ preserves finite limits and colimits. Since any given finite limit or colimit in $\mathcal{C}$ involves only sequences of N-torsion R-modules, for some integer N, the limit or colimit may be computed in the category of $\mathbb{Z}/N\mathbb{Z}$-modules, in which case [32, Proposition I.2.2] applies. $\square$

2.2. **Ultraprimes.**

2.2.1. Fix a number field L and let M_L be its set of places; for each $v \in M_L$, fix as well an embedding $\overline{L} \hookrightarrow \overline{L_v}$. If $\mathcal{M}_L$ is the constant sequence of sets $\{M_L\}_{n\in\mathbb{N}}$, then we define the set of ultraprimes of L as

$$\mathsf{M}_L = \mathcal{U}(\mathcal{M}_L).$$

By definition, an ultraprime $\mathsf{v} \in \mathsf{M}_L$ is an equivalence class of sequences $(v_n)_{n\in\mathbb{N}}$, where each v_n is a place of L; note that $\mathrm{Gal}(L/\mathbb{Q})$ acts by set automorphisms on M_L, compatibly with the natural projection $\mathsf{M}_E \to \mathsf{M}_L$ for a finite extension E/L. The map $v \mapsto (v, v, \ldots)$ induces an embedding $M_L \hookrightarrow \mathsf{M}_L$, written $v \mapsto \underline{v}$, and we say an ultraprime is constant if it lies in the image of this embedding.

Proposition 2.2.2. *Let v be a non-constant ultraprime. Then there exists a unique Frobenius element* $\mathrm{Frob}_{\mathsf{v}} \in \mathrm{Gal}(\overline{L}/L)$ *with the following property: for each finite Galois extension $L \subset E \subset \overline{L}$, and for any representative (v_n) of v, there are $\mathfrak{F}$-many n such that v_n is unramified in E/L and the Frobenius of v_n in* $\mathrm{Gal}(E/L)$ *is the natural image of* $\mathrm{Frob}_{\mathsf{v}}$.

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a representative of v, and fix for the time being a finite extension E/L. If v_n is archimedean or ramified in E for $\mathfrak{F}$-many n, then Proposition 2.1.2 implies that v is constant. Thus the map that sends n to the Frobenius of v_n in $\mathrm{Gal}(E/L)$ is defined for $\mathfrak{F}$-many n; by Proposition 2.1.2, it sends $\mathfrak{F}$-many n to a (unique) common value $g_E \in \mathrm{Gal}(E/L)$. Note that g_E does not depend on the representative

(v_n). By the uniqueness of g_E, the association $E \mapsto g_E$ is compatible with restriction to subextensions $E' \subset E$, hence defines an element of the absolute Galois group. $\square$

2.2.3. Let v be an ultraprime. We define its abstract Galois group G_v as $\mathrm{Gal}(\overline{L_v}/L_v)$ if $\mathsf{v} = \underline{v}$ is constant, and as the semi direct product

$$\widehat{\mathbb{Z}}(1) \rtimes \langle \mathrm{Frob}_\mathsf{v} \rangle$$

otherwise. Here, $\langle \mathrm{Frob}_\mathsf{v} \rangle$ denotes the free profinite group on one generator, acting on $\widehat{\mathbb{Z}}(1)$ by Frob_v. We define the inertia group $\mathsf{I}_\mathsf{v} \subset \mathsf{G}_\mathsf{v}$ of v to be the usual inertia group if v is constant, and the normal subgroup $\widehat{\mathbb{Z}}(1) \subset \mathsf{G}_\mathsf{v}$ otherwise.

2.3. **Local cohomology.**

2.3.1. For any (continuous) Galois module A defined over L, and for any $\mathsf{v} \in \mathsf{M}_L$, there is a natural action of G_v on A (factoring through the quotient $\mathsf{G}_\mathsf{v} \to \mathrm{Frob}_\mathsf{v}$ if v is nonconstant). We define local cohomology groups by:

$$\begin{aligned} \mathsf{H}^i(L_\mathsf{v}, A) &\coloneqq H^i_{cts}(\mathsf{G}_\mathsf{v}, A), \\ \mathsf{H}^i(L_\mathsf{v}^{nr}, A) &\coloneqq H^i_{cts}(\mathsf{I}_\mathsf{v}, A), \quad i \geq 0. \end{aligned}$$

Note that the local cohomology commutes with direct limits and countable inverse limits of finite, discrete Galois modules; the former is essentially by definition of continuous cohomology and the latter is by [38, Corollary 2.6.7] applied to $\mathsf{G}_\mathsf{v}, \mathsf{I}_\mathsf{v}$.

Proposition 2.3.2. *Let $\mathsf{v} \in \mathsf{M}_L$ be an ultraprime represented by a sequence $(v_n)_{n\in\mathbb{N}}$. If A is a finite, discrete Galois module over L, then for $\mathfrak{F}$-many n there are natural isomorphisms (compatible with the restriction maps and with the cup product):*

$$\begin{aligned} H^i(L_{v_n}, A) &\simeq \mathsf{H}^i(L_\mathsf{v}, A), \\ H^i(L_{v_n}^{nr}, A) &\simeq \mathsf{H}^i(L_\mathsf{v}^{nr}, A), \quad i \geq 0. \end{aligned}$$

Proof. If v is the constant ultraprime $\underline{v}$, then $v_n = v$ for $\mathfrak{F}$-many n, and the desired isomorphisms are given by the identity maps; so suppose v is nonconstant. For $\mathfrak{F}$-many n, the action of the decomposition group G_{v_n} at v_n on A is unramified and the Frobenius of v_n acts by Frob_v. Let ℓ_n be the prime of $\mathbb{Q}$ lying under v_n; since $L/\mathbb{Q}$ is a finite extension and A is a finite Galois module, for $\mathfrak{F}$-many n we have $\ell_n \nmid |A|$. Restricting to these n, the inflation map induces isomorphisms:

$$H^i(G_{v_n}^t, A) \simeq H^i(L_{v_n}, A), \;\; H^i(I_{v_n}^t, A) \simeq H^i(L_{v_n}^{nr}, A),$$

where $G_{v_n}^t$ and $I_{v_n}^t$ denote the tame quotients. The tame Galois group $G_{v_n}^t$ is identified with the semi direct product:

$$I_{v_n}^t \rtimes \langle \mathrm{Frob}_{v_n} \rangle \simeq \widehat{\mathbb{Z}}^{(\ell_n)}(1) \rtimes \langle \mathrm{Frob}_{v_n} \rangle.$$

Since Frob_{v_n} and Frob_v may act differently on the Tate twist, $G_{v_n}^t$ and G_v cannot be compared directly; we wish to show that the cohomologies are nonetheless canonically isomorphic for $\mathfrak{F}$-many n.

Let $G = I \rtimes \langle F \rangle$ be an abstract group, where I is abelian and profinite, and $\langle F \rangle$ denotes the free profinite group on one generator, acting on I by an automorphism. If A is a $\mathbb{Z}[F]$-module, then the Galois cohomology groups $H^i(G, A)$ and $H^i(I, A)$ depend only on A and $\mathrm{Hom}(I, A)$ as $\mathbb{Z}[F]$-modules; in particular, the cohomology groups for G are canonically isomorphic to the cohomology groups for its quotient $I/|A| \rtimes \langle F \rangle$, and similarly for I and $I/|A|$. Applying this to $G_{v_n}^t$ and G_v completes the proof, since Frob_{v_n} and Frob_v have the same action on the finite Tate twist $\mathbb{Z}/|A|(1)$ for $\mathfrak{F}$-many n. $\square$

2.4. **Patched cohomology.**

2.4.1. Let $\mathsf{S} \subset \mathsf{M}_L$ be a finite set of ultraprimes $\{\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_r\}$. A **representative** of S is a sequence of sets $S^n \subset M_L$ such that $S^n = \{s_1^n, \cdots, s_r^n\}$ for some sequences $(s_i^n)_{n\in\mathbb{N}}$ representing s_i. If A is a $\mathrm{Gal}(\overline{L}/L)$ module, we say A is unramified outside $\mathsf{S} \subset \mathsf{M}_L$ if it is unramified outside $\mathsf{S} \cap M_L$.

Definition 2.4.2. Let A be a topological $\mathrm{Gal}(\overline{L}/L)$-module unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$, represented by a sequence $S^n \subset M_L$. If A is profinite, then we define the ith unramified-outside-S patched cohomology, for all $i \geq 0$, by:

$$\mathsf{H}^i(L^{\mathsf{S}}/S, A) = \varprojlim_{A \twoheadrightarrow A'} \mathcal{U}\left(\left\{H^i(L^{S^n}/L, A')\right\}\right)_{n\in\mathbb{N}},$$

where the inverse limit runs over continuous finite quotients of A. If A is ind-finite, then its unramified-outside-S patched cohomology is defined as:

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \varinjlim_{A' \subset A} \mathcal{U}\left(\left\{H^i(L^{S^n}/L, A')\right\}_{n\in\mathbb{N}}\right),$$

where the direct limit runs over finite submodules. If A is either profinite or ind-finite, then the totally patched cohomology is defined as

$$\mathsf{H}^i(L, A) = \varinjlim_{\mathsf{S}\subset \mathsf{M}_L} \mathsf{H}^i(L^{\mathsf{S}}/L, A),$$

where the direct limit is over finite subsets and the transition maps are induced by the functoriality of the ultraproduct.

Remark 2.4.3. (1) To see that these cohomology groups are well-defined, first note that they are independent of the choice of S^n since any two representatives of a finite set $\mathsf{S} \subset \mathsf{M}_L$ agree for $\mathfrak{F}$-many n. Moreover, if A is both profinite and ind-finite, then it is finite, and it is clear that either definition gives the same cohomology groups.

(2) There is a canonical isomorphism $\mathsf{H}^0(L^{\mathsf{S}}/L, A) = H^0(L, A)$ for all finite $\mathsf{S} \subset \mathsf{M}_L$ and all profinite or ind-finite A.

(3) The assignments

$$A \mapsto \mathsf{H}^i(L^{\mathsf{S}}/L, A), \quad A \mapsto \mathsf{H}^i(L, A)$$

are functorial in A. If A is an R-module for some ring R, then each patched cohomology group $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$, $\mathsf{H}^i(L, A)$ has a natural R-module structure.

(4) In practice, we will want our profinite Galois modules to be **countably profinite**, i.e. to have a presentation as a countable inverse limit of finite, discrete topological Galois modules. The significance of this technical hypothesis is that countable inverse limits of finite abelian groups are exact. For example, see [38, Corollary 2.7.6].

(5) Suppose A is ind-finite or countably profinite. If every ultraprime in S is constant, and $S \subset M_L$ is the corresponding finite set of places, then $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ is canonically isomorphic to $H^i(L^S/L, A)$.

(6) Suppose A is ind-finite or countably profinite. For each ultraprime v, there are natural localization maps

$$\mathrm{Res}_{\mathsf{v}} : \mathsf{H}^i(L, A) \to \mathsf{H}^i(L_{\mathsf{v}}, A)$$

deduced from Proposition 2.3.2 (and from [38, Corollary 2.7.6] applied to G_{v} in the profinite case).

(7) If the Galois action on A is the restriction of an action of G_K, where L/K is a Galois extension, then $\mathrm{Gal}(L/K)$ acts naturally on $\mathsf{H}^i(L, A)$, again by functoriality of ultraproducts; this is compatible with the localization maps in the obvious way.

Lemma 2.4.4. *For any finite set of primes $S \subset M_L$, and any finite Galois module A over L, the cardinality of $H^i(L^S/L, A)$ is uniformly bounded, with a bound depending only on A, L, and $|S|$. In particular, if $\mathsf{S} \subset \mathsf{M}_L$ is finite, then the patched cohomology groups $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ are finite for each finite Galois module A and each $i \geq 0$.*

Proof. The first claim is easily seen from [35, Theorem 4.10]; the second follows by Proposition 2.1.4. □

Proposition 2.4.5. *If A is either countably profinite or ind-finite, then, for all i, the natural map induces an isomorphism*

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) \simeq \ker\left(\mathsf{H}^i(L, A) \to \prod_{\mathsf{v}\in \mathsf{M}_L - \mathsf{S}} \mathsf{H}^i(L_{\mathsf{v}}^{nr}, A)\right).$$

Proof. It suffices to show that, for all finite sets $\mathsf{T} \subset \mathsf{M}_L - \mathsf{S}$,

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) \simeq \ker\left(\mathsf{H}^i(L^{\mathsf{S}\cup\mathsf{T}}, A) \to \prod_{\mathsf{t}\in\mathsf{T}} \mathsf{H}^i(L_{\mathsf{t}}^{nr}, A)\right).$$

This holds when A is finite by Lemma 2.4.4 and Proposition 2.1.6; the general case follows by taking limits. □

Lemma 2.4.6. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of either countably profinite or ind-finite Galois modules unramified outside S*. Then there is an induced long exact sequence beginning:*

$$\begin{aligned}0 \to \mathsf{H}^0(L^{\mathsf{S}}/L, A) \to \mathsf{H}^0(L^{\mathsf{S}}/L, B) \to \mathsf{H}^0(L^{\mathsf{S}}/L, C) \to \\ \to \mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \mathsf{H}^1(L^{\mathsf{S}}/L, B) \to \cdots\end{aligned}$$

Proof. If A, B, and C are all finite, then this follows from Proposition 2.1.6 and Lemma 2.4.4.

Now suppose that A, B, and C are all profinite. Let I, J, and K be directed sets indexing the finite quotients $A \twoheadrightarrow A_i$, $B \twoheadrightarrow B_j$, and $C \twoheadrightarrow C_k$, respectively. We define morphisms of directed sets $t : J \to I$ and $s : J \to K$ by

$$A_{t(j)} = \operatorname{im}(A \to B_j), \;\; C_{s(j)} = B_j/A_{t(j)}.$$

Because the subgroup and quotient topologies on A and C agree with the profinite topologies, the images of t and s are cofinal in I and K, respectively. We therefore have:

$$\mathsf{H}^*(L^{\mathsf{S}}/L, A) = \varprojlim_{j\in J} \mathsf{H}^*(L^{\mathsf{S}}/L, A_{t(j)}), \;\; \mathsf{H}^*(L^{\mathsf{S}}/L, C) = \varprojlim_{j\in J} \mathsf{H}^*(L^{\mathsf{S}}/L, C_{s(j)}).$$

For each j, we have a long exact sequence associated to the short exact sequence of finite Galois modules

$$0 \to A_{t(j)} \to B_j \to C_{s(j)} \to 0;$$

by Lemma 2.4.4, each term in the long exact sequence is finite. Since countable inverse limits of finite abelian groups are exact, taking limits completes the proof. The ind-finite case is completely analogous. □

2.5. **Selmer structures and patched Selmer groups.**

Definition 2.5.1. Let A be a countably profinite or ind-finite $\mathbb{Z}_p[G_L]$-module. A **generalized Selmer structure** $(\mathcal{F}, \mathsf{S})$ for A consists of:

- a finite set $\mathsf{S} \subset \mathsf{M}_L$ containing all Archimedean places, all places over p, and all ramified places for A;
- for each $\mathsf{v} \in \mathsf{M}_L$, a closed $\mathbb{Z}_p$-submodule (the **local condition**)

$$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) \subset \mathsf{H}^1(L_{\mathsf{v}}, A)$$

such that

$$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) = \mathsf{H}^1_{\mathrm{unr}}(L_{\mathsf{s}}, A) := \ker\left(\mathsf{H}^1(L_{\mathsf{s}}, A) \to \mathsf{H}^1(L_{\mathsf{s}}^{nr}, A)\right)$$

for all $\mathsf{v} \notin \mathsf{S}$.

If A is an R-module for some ring R and G_L acts on A by R-module automorphisms, a Selmer structure for A **over** R is a Selmer structure such that every local condition is an R-submodule.

2.5.2. If $B \subset A$ is any closed Galois-stable submodule, then a Selmer structure $(\mathcal{F}, \mathsf{S})$ for A induces Selmer structures on B and A/B defined in the usual way:

$$\begin{aligned}\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, B) &= \ker\left(\mathsf{H}^1(L_{\mathsf{s}}, B) \to \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}\right), \\ \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A/B) &= \operatorname{im}\left(\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) \to \mathsf{H}^1(L_{\mathsf{s}}, A/B)\right).\end{aligned}$$

2.5.3. To a generalized Selmer structure we associate the **patched Selmer group**, defined by the exact sequence:

$$0 \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}, \tag{10}$$

or equivalently (by Proposition 2.4.5):

$$0 \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathsf{H}^1(L, A) \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)} \times \prod_{\mathsf{s}\notin\mathsf{S}} \mathsf{H}^1(L_{\mathsf{s}}^{nr}, A). \tag{11}$$

(Note that the Selmer group attached to a Selmer structure does not depend on the choice of set S but only on the local conditions; we will therefore sometimes omit S from the notation when there is no risk of confusion.)

2.5.4. If $B \subset A$ is Galois-stable, and $B, A/B$ are equipped with the induced Selmer structures, then by definition there are natural maps:

$$\mathrm{Sel}_{\mathcal{F}}(B) \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathrm{Sel}_{\mathcal{F}}(A/B).$$

Proposition 2.5.5. *Let $(\mathcal{F}, \mathsf{S})$ be a generalized Selmer structure for A. If A is countably profinite and each continuous finite quotient $A \twoheadrightarrow A'$ is equipped with the Selmer structure induced by $\mathcal{F}$, then:*

$$\varprojlim \mathrm{Sel}_{\mathcal{F}}(A') \simeq \mathrm{Sel}_{\mathcal{F}}(A).$$

If instead A is ind-finite and each finite submodule $A' \subset A$ is given its induced Selmer structure, then:

$$\varinjlim \mathrm{Sel}_{\mathcal{F}}(A') \simeq \mathrm{Sel}_{\mathcal{F}}(A).$$

Proof. We show the countably profinite case; the ind-finite case is similar. By definition, $\mathrm{Sel}_{\mathcal{F}}(A)$ is the kernel of

$$\varprojlim \mathsf{H}^1(L^{\mathsf{S}}/L, A') \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)},$$

whereas

$$\varprojlim \mathrm{Sel}_{\mathcal{F}}(A') = \varprojlim \ker \left(\mathsf{H}^1(L^{\mathsf{S}}/L, A') \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A')}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A')} \right) = \ker \left(\mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \varprojlim \frac{\mathsf{H}^1(L_{\mathsf{s}}, A')}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A')} \right).$$

Since $\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)$ is closed, we have

$$\varprojlim \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A') = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A),$$

which implies the result. □

2.5.6. Given two Selmer structures $(\mathcal{F}, \mathsf{S})$ and $(\mathcal{G}, \mathsf{T})$ for A, we may define Selmer structures $(\mathcal{F} + \mathcal{G}, \mathsf{S} \cup \mathsf{T})$ and $(\mathcal{F} \cap \mathcal{G}, \mathsf{S} \cup \mathsf{T})$ by the local conditions:

$$\mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_{\mathsf{v}}, A) = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) + \mathsf{H}^1_{\mathcal{G}}(L_{\mathsf{v}}, A), \quad \mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_{\mathsf{v}}, A) = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) \cap \mathsf{H}^1_{\mathcal{G}}(L_{\mathsf{v}}, A). \tag{12}$$

2.6. Dual Selmer groups.

2.6.1. Fix an ultraprime $\mathsf{v} \in \mathsf{M}_L$. If A is a countably profinite $\mathbb{Z}_p$-Galois module and A^* denotes the Cartier dual, then the cup product induces pairings:

$$\langle \cdot, \cdot \rangle_{\mathsf{v}} : \mathsf{H}^i(L_{\mathsf{v}}, A) \times \mathsf{H}^{2-i}(L_{\mathsf{v}}, A^*) \to \mathbb{Q}_p/\mathbb{Z}_p, \quad i = 0, 1, 2. \tag{13}$$

Proposition 2.6.2. *The pairing (13) is perfect if v is non-Archimedean. Moreover, the induced pairing*

$$\mathsf{H}^1(L^{\mathsf{S}}/L, A) \times \mathsf{H}^1(L^{\mathsf{S}}/L, A^*) \to \prod_{\mathsf{s}\in\mathsf{S}} \mathsf{H}^1(L_{\mathsf{s}}, A) \times \mathsf{H}^1(L_{\mathsf{s}}, A^*) \xrightarrow{\Sigma\langle\cdot,\cdot\rangle_{\mathsf{s}}} \mathbb{Q}_p/\mathbb{Z}_p$$

is identically zero.

Proof. For the perfectness of (13), the usual proof of Poitou-Tate duality applies equally well to G_{v}; alternatively, one may take limits using Proposition 2.3.2. The second claim is clear when A is finite by functoriality of the ultraproduct, and the general case follows by taking limits. □

2.6.3. Suppose that A is either countably profinite or countably ind-finite, i.e. the Pontryagin dual of a countably profinite Galois module. If $(\mathcal{F}, \mathsf{S})$ is any Selmer structure for A, then we define the dual Selmer structure $(\mathcal{F}^*, \mathsf{S})$ for A^* by:

$$\mathsf{H}^1_{\mathcal{F}^*}(L_{\mathsf{s}}, A^*) = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)^{\perp}.$$

Here $\perp$ denotes the orthogonal complement under either the pairing of (2.6.1), or the usual modified Tate pairing of [14, Theorem 2.17] at Archimedian places. We observe that the dual Selmer structure to $(\mathcal{F}^*, \mathsf{S})$ is again $(\mathcal{F}, \mathsf{S})$. When$A$ is finite, the dual Selmer groups are related by the Greenberg-Wiles formula:

Proposition 2.6.4. *Let $(\mathcal{F}, \mathsf{S})$ be a Selmer structure for a finite $\mathbb{Z}_p[G_L]$-module A. We have:*

$$\frac{\#\,\mathrm{Sel}_{\mathcal{F}}(A)}{\#\,\mathrm{Sel}_{\mathcal{F}^*}(A^*)} = \frac{\#\mathsf{H}^0(L^{\mathsf{S}}/L, A)}{\#\mathsf{H}^0(L^{\mathsf{S}}/L, A^*)} \prod_{\mathsf{s} \in \mathsf{S}} \frac{\#\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}{\#\mathsf{H}^0(L_{\mathsf{s}}, A)}.$$

Proof. This follows from [14, Theorem 2.19] by the exactness of ultraproducts and Proposition 2.1.4(1). □

2.7. **Selmer groups over discrete valuation rings.**

2.7.1. Let R be a discrete valuation ring with uniformizer π which is a finite, flat extension of $\mathbb{Z}_p$, and suppose that $A = T$ is a free R-module of finite rank, with G_L action through R-module automorphisms. In particular, T is countably profinite. Suppose $\mathsf{S} \subset \mathsf{M}_L$ is a finite set containing all Archimedian places and all places over p, such that T is unramified outside S. If $T^\dagger = \mathrm{Hom}_R(T, R(1))$ is the dual, then the cup product induces a local Tate pairing

$$\langle \cdot, \cdot \rangle_{\mathsf{v}} : \mathsf{H}^1(L_{\mathsf{v}}, T) \times \mathsf{H}^1(L_{\mathsf{v}}, T^\dagger) \to R. \tag{14}$$

Proposition 2.7.2. *The kernels on the left and right of (14) are the R-torsion submodules; moreover, the induced pairing*

$$\mathsf{H}^1(L^{\mathsf{S}}/L, T) \times \mathsf{H}^1(L^{\mathsf{S}}/L, T^\dagger) \to \prod_{\mathsf{v} \in \mathsf{S}} \mathsf{H}^1(L_{\mathsf{v}}, T) \times \mathsf{H}^1(L_{\mathsf{v}}, T^\dagger) \xrightarrow{\sum \langle \cdot, \cdot \rangle_{\mathsf{v}}} R$$

is identically zero.

Proof. This follows from Proposition 2.6.2. □

Given a Selmer structure $(\mathcal{F}, \mathsf{S})$ for T over R, taking the orthogonal complement of each local condition under (14) yields a Selmer structure $(\mathcal{F}^\dagger, \mathsf{S})$ for $T^\dagger$. Note that $\mathcal{F}^{\dagger\dagger} \neq \mathcal{F}$ in general, but we always have $\mathcal{F}^{\dagger\dagger\dagger} = \mathcal{F}^\dagger$.

Proposition 2.7.3. *Let $(\mathcal{F}, \mathsf{S})$ be a Selmer structure such that*

$$\frac{\mathsf{H}^1(L_{\mathsf{v}}, T)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, T)}$$

is torsion-free. Then, for all j and all $\mathsf{v} \in \mathsf{M}_L$,

$$\mathsf{H}^1_{\mathcal{F}^*}(L_{\mathsf{v}}, T^*[\pi^j]) = \mathsf{H}^1_{\mathcal{F}^\dagger}(L_{\mathsf{v}}, T^\dagger/\pi^j)$$

under an identification $T^*[\pi^j] \simeq T^\dagger/\pi^j$, and in particular

$$\mathrm{Sel}_{\mathcal{F}^*}(T^*[\pi^j]) = \mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger/\pi^j).$$

Proof. Although this fact is presumably standard, we give a proof for lack of a reference. For ease of notation, we abbreviate $\mathsf{H}^i(T^\dagger) = \mathsf{H}^i(L_{\mathsf{v}}, T^\dagger)$, etc. and $R_j = R/\pi^j$. The choice of uniformizer induces an identification $T^\dagger \otimes_R (R[1/\pi]/R) \simeq T^*$ and an embedding $T^\dagger/\pi^j \hookrightarrow T^*$; let $\mathsf{H}^1_{\mathcal{F}^*}(T^\dagger/\pi^j)$ be the induced local condition from this embedding. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{H}^0(T^*)_{/\,\mathrm{div}} \arrow[r] \arrow[d, "\alpha"] & \mathsf{H}^1_{\mathcal{F}^\dagger}(T^\dagger) \arrow[r] \arrow[d, "\beta"] & \mathrm{Hom}(\mathsf{H}^1(T), R) \arrow[r] \arrow[d, "\gamma"] & \mathrm{Hom}(\mathsf{H}^1_{\mathcal{F}}(T), R) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathsf{H}^0(T^*)/\pi^j \arrow[r] & \mathsf{H}^1_{\mathcal{F}^*}(T^\dagger/\pi^j) \arrow[r] & \mathrm{Hom}(\mathsf{H}^1(T), R_j) \arrow[r] & \mathrm{Hom}(\mathsf{H}^1_{\mathcal{F}}(T), R_j) \arrow[r] & 0
\end{tikzcd}$$

Here, the first horizontal map on each row is the Kummer map, and the subscript $/\operatorname{div}$ refers to the quotient by the maximal divisible submodule. By the hypothesis on $\mathsf{H}^1_{\mathcal{F}}(T)$, the maps $\operatorname{coker}\gamma \to \operatorname{coker}\delta$ and $\ker\gamma \to \ker\delta$ are injective and surjective, respectively. Also, α is clearly surjective. Breaking the diagram into two and applying the snake lemma, it follows that β is surjective. □

Proposition 2.7.4. *Let $(\mathcal{F}, \mathsf{S})$ be a Selmer structure for T over R. Then:*

$$\operatorname{rk}_R \operatorname{Sel}_{\mathcal{F}}(T) - \operatorname{rk}_R \operatorname{Sel}_{\mathcal{F}^\dagger}(T^\dagger) = \operatorname{rk}_R H^0(L,T) - \operatorname{rk}_R H^0(L,T^\dagger) + \sum_{\mathsf{s}\in\mathsf{S}} \left(\operatorname{rk}_R \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}},T) - \operatorname{rk}_R \mathsf{H}^0(L_{\mathsf{s}},T)\right).$$

Proof. Without loss of generality, we may assume that

$$\frac{\mathsf{H}^1(L_{\mathsf{v}},T)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}},T)}$$

is torsion-free for all v. By Propositions 2.6.4 and 2.7.3, we then have for each j:

$$\begin{aligned}\lg \operatorname{Sel}_{\mathcal{F}}(T/\pi^j) - \lg \operatorname{Sel}_{\mathcal{F}^\dagger}(T^\dagger/\pi^j) &= \lg \mathsf{H}^0(L,T/\pi^j) - \lg \mathsf{H}^0(L,T^\dagger/\pi^j)\\ &\quad + \sum_{\mathsf{v}\in\mathsf{S}} \left(\lg \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}},T/\pi^j) - \lg \mathsf{H}^0(L_{\mathsf{v}},T/\pi^j)\right).\end{aligned}$$

Since $\operatorname{Sel}_{\mathcal{F}}(T)$ is a finitely generated R-module, it follows from [33, Lemma 3.7.1] that

$$\lg \operatorname{Sel}_{\mathcal{F}}(T/\pi^j) = (\operatorname{rk}_R \operatorname{Sel}_{\mathcal{F}}(T)) \cdot \lg R/\pi^j + O(1)$$

as j varies, and likewise for $\operatorname{Sel}_{\mathcal{F}^\dagger}(T^\dagger)$ and each term on the right-hand side; the proposition follows. □

3. Bipartite Euler systems

3.1. **Admissible primes.**

3.1.1. Let f be a modular form of weight two, trivial character, and level N, and let $\wp \subset \mathcal{O}_f$ be a prime ideal of the ring of integers of its field of coefficients. We assume the rational prime p lying under $\wp$ is odd, and write $\mathcal{O}$ for the completion of $\mathcal{O}_f$ at $\wp$. Fix a Galois-stable $\mathcal{O}$-lattice T_f in the $\wp$-adic Galois representation associated to f, and let $\overline{T}_f$ be the residual representation $T_f/\wp$; we also write W_f for $T_f \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Also let $K/\mathbb{Q}$ be an imaginary quadratic field. We assume throughout this section that $\overline{T}_f$ is absolutely irreducible as a G_K-module. We will sometimes use the condition:

(sclr) The image of the G_K action on $\overline{T}_f$ contains a nonzero scalar.

Definition 3.1.2. A nonconstant ultraprime $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$ is said to be **admissible** with sign $\epsilon_{\mathsf{q}} = \pm 1$ for f if $\operatorname{Frob}_{\mathsf{q}}$ has nonzero image in $\operatorname{Gal}(K/\mathbb{Q})$, $\chi(\operatorname{Frob}_{\mathsf{q}}) \not\equiv 1 \pmod p$, and there is a rank-one direct summand $\operatorname{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f \subset T_f$ on which $\operatorname{Frob}_{\mathsf{q}}$ acts as $\chi(\operatorname{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$. (Equivalently, $\chi(\operatorname{Frob}_{\mathsf{q}}) \not\equiv 1 \pmod p$ and T_f admits a basis of eigenvectors for $\operatorname{Frob}_{\mathsf{q}}$ with eigenvalues ϵ_{q} and $\chi(\operatorname{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$.)

For example, if $\operatorname{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ is a complex conjugation, then q is admissible with either choice of ϵ_{q}. We abusively write q for the unique ultraprime in M_K lying over $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$, whose Frobenius is $\operatorname{Frob}_{\mathsf{q}}^2$.

Definition 3.1.3. If q is admissible with sign ϵ_{q} for f, then we define the **ordinary** local condition (with sign ϵ_{q}) as:

$$\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}},T_f) = \operatorname{im}\left(\mathsf{H}^1(K_{\mathsf{q}},\operatorname{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f) \to \mathsf{H}^1(K_{\mathsf{q}},T_f)\right).$$

The subscript ϵ_q will often be omitted (from this and future notation) when there is no risk of confusion.

3.1.4. Note that the ordinary local condition is self-annihilating under the local Tate pairing

$$\mathsf{H}^1(K_{\mathsf{q}},T_f) \times \mathsf{H}^1(K_{\mathsf{q}},T_f) \to \mathcal{O}$$

induced by (14) and the Weil pairing.

3.1.5. For any finite set $\mathsf{S} \subset \mathsf{M}_K$ such that T_f is unramified outside S, and any admissible $\mathsf{q} \notin \mathsf{S}$ with sign ϵ_{q}, define a localization map

$$\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}} : \mathsf{H}^1(K^{\mathsf{S}}/K, T_f) \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \to \frac{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f)}{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \cap \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)} \approx \mathcal{O}. \tag{15}$$

Define as well a residue map

$$\partial_{\mathsf{q},\epsilon_{\mathsf{q}}} : \mathsf{H}^1(K, T_f) \to \mathsf{H}^1(K_{\mathsf{q}}, T_f) \to \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f) \to \frac{\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)}{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \cap \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)} \approx \mathcal{O}, \tag{16}$$

where the second map is given by the projection $T_f \twoheadrightarrow (\mathrm{Frob}_{\mathsf{q}} - \epsilon_{\mathsf{q}})T_f \simeq \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f$. The maps $\mathrm{loc}_{\mathsf{q},\epsilon_q}$ and $\partial_{\mathsf{q},\epsilon_{\mathsf{q}}}$ may be extended in the obvious way to the patched cohomology for W_f and all $T_f/\wp^j$.

3.2. **Euler systems for anticyclotomic twists.**

3.2.1. Let R be a complete flat Noetherian local $\mathcal{O}$-algebra with finite residue field, equipped with an anticyclotomic character $\varphi : G_K \to R^\times$ which is trivial modulo the maximal ideal of R. We write T_φ for the anticyclotomic twist $T_f \otimes_{\mathcal{O}} R(\varphi)$, which is a countably profinite Galois module. If q is admissible with sign ϵ_{q}, then $\varphi(\mathrm{Frob}^2_{\mathsf{q}}) = 1$, so

$$\mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) = \mathsf{H}^1(K_{\mathsf{q}}, T_f) \otimes_{\mathcal{O}} R.$$

We extend the ordinary local condition of the previous subsection by linearity to define $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_\varphi)$, and likewise the maps $\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}}, \partial_{\mathsf{q},\epsilon_{\mathsf{q}}}$.

3.2.2. Suppose given a finite set $\mathsf{S} \subset \mathsf{M}_K$ and a generalized Selmer structure $(\mathcal{F}, \mathsf{S})$ for T_φ. Let $\mathsf{N} = \mathsf{N}_{\mathsf{S}}$ be the set of pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ where $\mathsf{Q} \subset \mathsf{M}_K - \mathsf{S}$ is a finite set of ultraprimes and $\epsilon_{\mathsf{Q}} : \mathsf{Q} \to \{\pm 1\}$ is a function such that q is admissible with sign $\epsilon_{\mathsf{Q}}(\mathsf{q})$ for all $\mathsf{q} \in \mathsf{Q}$. (We will drop the subscript S when it is clear from context, or when S contains only constant ultraprimes.) Given a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, define a generalized Selmer structure $(\mathcal{F}(\mathsf{Q}, \epsilon_{\mathsf{Q}}), \mathsf{S} \cup \mathsf{Q})$ for T_φ by the local conditions:

$$\mathsf{H}^1_{\mathcal{F}(\mathsf{Q},\epsilon_{\mathsf{Q}})}(K_{\mathsf{v}}, T_\varphi) = \begin{cases} \mathsf{H}^1_{\mathcal{F}}(K_{\mathsf{v}}, T_\varphi), & \mathsf{v} \notin \mathsf{Q} \\ \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{Q}}(\mathsf{q})}(K_{\mathsf{q}}, T_\varphi), & \mathsf{v} = \mathsf{q} \in \mathsf{Q}. \end{cases} \tag{17}$$

For $\delta \in \mathbb{Z}/2\mathbb{Z}$, let $\mathsf{N}^\delta \subset \mathsf{N}$ be the collection of pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that $|\mathsf{Q}| \equiv \delta \pmod 2$. Also, given two pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$ and $\{\mathsf{Q}', \epsilon_{\mathsf{Q}'}\} \in \mathsf{N}^{\delta'}$ such that $\mathsf{Q} \cap \mathsf{Q}' = \emptyset$, write

$$\{\mathsf{Q}\mathsf{Q}', \epsilon_{\mathsf{Q}\mathsf{Q}'}\} \in \mathsf{N}^{\delta+\delta'}$$

for the pair formed in the obvious way from $\mathsf{Q} \cup \mathsf{Q}'$ and the sign functions $\epsilon_{\mathsf{Q}}, \epsilon_{\mathsf{Q}'}$. The pair $\{\emptyset, \emptyset\} \in \mathsf{N}$ will be abbreviated as 1.

Definition 3.2.3. A **bipartite system** (κ, λ) for $(T_\varphi, \mathcal{F}, \mathsf{S})$ of parity $\delta \in \mathbb{Z}/2\mathbb{Z}$ consists of the following data:

(1) for each pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, a principal submodule
$$(\kappa(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi);$$

(2) for each pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, a principal ideal
$$(\lambda(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \subset R.$$

A **bipartite Euler system** is a bipartite system satisfying the "reciprocity laws":

(1) For each $\{\mathsf{Q}\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}}\} \in \mathsf{N}^{\delta+1}$,
$$\mathrm{loc}_{\mathsf{q}}((\kappa(\mathsf{Q})) = (\lambda(\mathsf{Q}\mathsf{q})) \subset R.$$

(2) For each $\{\mathsf{Q}\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}}\} \in \mathsf{N}^\delta$,
$$\partial_{\mathsf{q}}((\kappa(\mathsf{Q}\mathsf{q})) = (\lambda(\mathsf{Q})) \subset R.$$

We say (κ, λ) is **nontrivial** if there exists some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that either $\lambda(\mathsf{Q}, \epsilon_{\mathsf{Q}}) \neq 0$ or $\kappa(\mathsf{Q}, \epsilon_{\mathsf{Q}}) \neq 0$ depending on the parity of $|\mathsf{Q}| + \delta$.

3.3. **Euler systems over discrete valuation rings.**

3.3.1. Suppose that R is a discrete valuation ring with uniformizer π, and let $W_\varphi = T_\varphi \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Exactly as in [24], there is a perfect pairing $T_\varphi \times T_\varphi \to R(1)$, G_K-equivariant up to a twist, which induces local pairings:

$$\begin{gathered}\mathsf{H}^1(K_{\mathsf{v}}, T_\varphi) \times \mathsf{H}^1(K_{\bar{\mathsf{v}}}, W_\varphi) \to R \otimes \mathbb{Q}_p/\mathbb{Z}_p,\\ \mathsf{H}^1(K_{\mathsf{v}}, T_\varphi/\pi^j) \times \mathsf{H}^1(K_{\bar{\mathsf{v}}}, W_\varphi[\pi^j]) \to R/\pi^j,\\ \mathsf{H}^1(K_{\mathsf{v}}, T_\varphi) \times \mathsf{H}^1(K_{\bar{\mathsf{v}}}, T_\varphi) \to R.\end{gathered}$$

Here $\bar{\mathsf{v}} \in \mathsf{M}_K$ is the complex conjugate of v; the first two pairings are perfect. A Selmer structure $(\mathcal{F}, \mathsf{S})$ for T_φ induces a Selmer structure for W_φ, denoted the same way, by taking orthogonal complement local conditions.

Definition 3.3.2. We say $(\mathcal{F}, \mathsf{S})$ is **self-dual** if, for all $\mathsf{v} \in \mathsf{M}_K$, $\mathsf{H}^1_{\mathcal{F}}(K_{\mathsf{v}}, T_\varphi)$ and $\mathsf{H}^1_{\mathcal{F}}(K_{\bar{\mathsf{v}}}, T_\varphi)$ are exact annihilators under the local pairing.

Proposition 3.3.3. *Suppose that $(\mathcal{F}, \mathsf{S})$ is a self-dual Selmer structure for T_φ. Then, for each $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$:*

(1) $(\mathcal{F}(\mathsf{Q}), \mathsf{S})$ *is self-dual and*

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi) \approx (R \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{r_{\mathsf{Q}}} \oplus M_{\mathsf{Q}} \oplus M_{\mathsf{Q}}$$

for some torsion R-module M_{Q} and an integer r_{Q}.

(2) $r_{\mathsf{Q}} = \mathrm{rk}_R \, \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)$.

(3) For any $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\} \in \mathsf{N}$, one of the following holds:

(a) $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)) = 0$, $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) \neq 0$, $r_{\mathsf{Qq}} = r_{\mathsf{Q}} + 1$, *and there exists an exact sequence of R-modules:*

$$0 \to M_{\mathsf{Qq}} \to M_{\mathsf{Q}} \to \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})})(W_\varphi) \to 0.$$

Moreover,

$$\lg \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})})(W_\varphi) = \lg \mathrm{coker}\, \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)).$$

(b) $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)) \neq 0$, $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) = 0$, $r_{\mathsf{Qq}} = r_{\mathsf{Q}} - 1$, *and there exists an exact sequence of R-modules:*

$$0 \to M_{\mathsf{Q}} \to M_{\mathsf{Qq}} \to \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)) \to 0.$$

Moreover,

$$\lg \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)) = \lg \mathrm{coker}\, \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)).$$

Proof. (1) The self-duality claim is clear since $H^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_f)$ is self-dual. Now, for all $j \geq 0$,

$$(*) \qquad \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi[\pi^j]) \simeq \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\pi^j]$$

by Lemma 2.4.6 and the definition of the induced Selmer structure on $W_\varphi[\pi^j]$. (Note $H^0(K, \overline{T}_f) = 0$ since we have assumed $\overline{T}_f$ is an absolutely irreducible G_K-module.) Since $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)$ is co-finitely generated, we may conclude by [24, Theorem 1.4.2] (or its proof).

(2) As explained in [24, 25], the cohomological pairings deduced from $T_\varphi \times T_\varphi \to R(1)$ behave "exactly like" the Tate pairing; in particular, by the self-duality of the local conditions and Proposition 2.7.3, $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi/\pi^j) = \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi[\pi^j])$ and the result follows as in the proof of Proposition 2.7.4.

(3) Consider the Selmer structures $\mathcal{F}^{\mathsf{q}}(\mathsf{Q}) = \mathcal{F}(\mathsf{Q}) + \mathcal{F}(\mathsf{Qq})$ and $\mathcal{F}_{\mathsf{q}}(\mathsf{Q}) = \mathcal{F}(\mathsf{Q}) \cap \mathcal{F}(\mathsf{Qq})$. By Proposition 2.7.4,

$$\mathrm{rk}_R \, \mathrm{Sel}_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(T_\varphi) = \mathrm{rk}_R \, \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(T_\varphi) + 1.$$

Moreover, because $\mathcal{F}(\mathsf{Q})$ is self-dual, Proposition 2.7.2 implies that the image of

$$\frac{\mathrm{Sel}_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(T_\varphi)}{\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(T_\varphi)} \hookrightarrow \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} = \frac{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \oplus \frac{\mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \approx R^2$$

is self-annihilating under the induced local pairing, hence is contained either in the ordinary or unramified part.

For the relation between M_{Q} and M_{Qq}, we suppose we are in case (a), because the two arguments are identical. Using the perfect pairing between W_φ and T_φ, we see by Proposition 2.6.2 that

$\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)) \oplus \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi))$ is the exact annihilator of $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi))$ under the perfect induced local pairing

$$\frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \times \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(K_{\mathsf{q}},\mathsf{Q})}(W_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)} \to R \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

This implies that $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})})(W_\varphi)$ is divisible and

$$\lg \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})})(W_\varphi) = \lg \mathrm{coker}\, \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)).$$

Furthermore, the direct sum decomposition of $\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)$ may be chosen so that $\partial_{\mathsf{q}}(M_{\mathsf{Qq}} \oplus M_{\mathsf{Qq}}) = 0$, and in particular

$$M_{\mathsf{Qq}} \oplus M_{\mathsf{Qq}} \simeq (M_{\mathsf{Q}} \oplus M_{\mathsf{Q}}) \cap \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})} \subset (\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})})(W_\varphi).$$

Since $\mathrm{loc}_{\mathsf{q}}(M_{\mathsf{Q}} \oplus M_{\mathsf{Q}})$ must generate the image of $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})})(W_\varphi)$, the desired exact sequence follows.

□

The following result will allow us to control the alternative in Proposition 3.3.3(3).

Theorem 3.3.4. *Let $c \in \mathsf{H}^1(K^{\mathsf{T}}/K, T_\varphi)$ be any nonzero element, where $\mathsf{T} \supset \mathsf{S}$ is a finite set. Then there are infinitely many admissible ultraprimes $\mathsf{q} \notin \mathsf{T}$, with associated signs ϵ_{q}, such that $\mathrm{loc}_{\mathsf{q}}\, c \neq 0$.*

The proof is via a series of lemmas.

Lemma 3.3.5. *There is an integer j such that, for all $n \geq 0$,*

$$\pi^j H^1(K(T_\varphi)/K, T_\varphi/\pi^n) = 0.$$

If (sclr) holds, then we may take $j = 0$.

Proof. Let $G = \mathrm{Gal}(K(T_\varphi)/K)$, and let $Z \subset G$ be its center; since T_f is absolutely irreducible over K, Z acts on T_φ by scalars. We claim:

$$Z \neq \{1\}. \tag{18}$$

Assuming (18), the lemma follows from the inflation-restriction exact sequence

$$H^1(G/Z, H^0(Z, T_\varphi/\pi^n)) \hookrightarrow H^1(G, T_\varphi/\pi^n) \to H^1(Z, T_\varphi/\pi^n).$$

Let us now prove (18). Let $G' = \mathrm{Gal}(K(T_f)/K)$, and let L/K be the Galois subfield of $K(T_f)$ cut out by the center $Z' = Z(G') \subset G'$. By a result of Momose [43], Z' is nontrivial. Let E/K be the Galois extension determined by the kernel of φ; then it suffices to show that EL/L and $K(T_f)/L$ are linearly disjoint. Both EL and $K(T_f)$ are Galois over $\mathbb{Q}$, so $G_{\mathbb{Q}}$ acts on $\mathrm{Gal}(EL/L)$ and $\mathrm{Gal}(K(T_f)/L)$ by conjugation. If $\tau \in G_{\mathbb{Q}}$ is a complex conjugation, then τ acts trivially on $\mathrm{Gal}(K(T_f)/L)$ but nontrivially on $\mathrm{Gal}(EL/L)$, so the two groups have no nontrivial common quotient compatible with the $G_{\mathbb{Q}}$-action; hence $EL \cap K(T_f) = L$. □

Lemma 3.3.6. *Suppose given a cocycle*

$$c \in H^1(K, T_\varphi/\pi^n)$$

such that $\pi^j c \neq 0$, where j is as in Lemma 3.3.5. Then, for any integer $N \geq n$, there exists a sign $\epsilon = \pm 1$ and infinitely many rational primes q such that:

(1) q is inert in K and unramified in the splitting field $\mathbb{Q}(T_f, c)$.
(2) $\mathrm{Frob}_q \in \mathrm{Gal}(\mathbb{Q}(T_f)/\mathbb{Q})$ has distinct eigenvalues ± 1 on $T_f \otimes R/\pi^N$ (where R has trivial Galois action).
(3) For any cocycle representative, $c(\mathrm{Frob}_q^2)$ has nonzero component in the ϵ eigenspace for Frob_q.

Proof. Abbreviate $L = K(T_\varphi/\pi^N)$, and let $\phi \in \mathrm{Hom}_{G_K}(G_L, T_\varphi/\pi^n)$ be the image of c under restriction; by hypothesis $\phi \neq 0$. Without loss of generality, we may suppose that the image of ϕ is contained in $T_\varphi/\pi^n[\pi] \simeq T_\varphi/\pi$, which, since φ is residually trivial, is an extension of scalars $\overline{T}_f \otimes_{\mathcal{O}/\wp} k$. Now,

$$\mathrm{Hom}_{G_K}(G_L, \overline{T}_f \otimes k)$$

has a natural action of $\mathrm{Gal}(K/\mathbb{Q})$, and we may assume without loss of generality that ϕ lies in the ϵ eigenspace for some $\epsilon \in \{\pm 1\}$. Fix a complex conjugation $\tau \in G_{\mathbb{Q}}$. Since $\overline{T}_f$ is absolutely irreducible over G_K, there exists $g \in G_L$ such that $\phi(g)$ has nonzero component in the ϵ eigenspace of τ. Then

$$\phi(\tau g \tau g) = \epsilon\tau\phi(g) + \phi(g)$$

has nonzero component in the ϵ eigenspace as well. Any q with Frobenius τg in $L(\phi)$ satisfies the desired conditions. □

Proof of Theorem 3.3.4. Since $H^0(K, T_\varphi/\pi) = 0$, Lemma 2.4.6 implies that

$$\mathsf{H}^1(K^{\mathsf{T}}/K, T_\varphi)[\pi] = 0.$$

Thus there exists some n such that the image $\overline{c}$ of c in $\mathsf{H}^1(K^{\mathsf{T}}/K, T_\varphi/\pi^n)$ satisfies $\pi^j\overline{c} \neq 0$, for some j as in Lemma 3.3.5. By definition, $\overline{c}$ is represented by a sequence of classes $c_m \in H^1(K^{T_m}/K, T_\varphi/\pi^n)$ such that $\pi^j c_m \neq 0$ for $\mathfrak{F}$-many m, where $\{T_m\}_{m\in\mathbb{N}}$ represents T. For each m, apply Lemma 3.3.6 (with $N = m$) to obtain a prime $q_m \notin T_m$ and a sign ϵ_m. If $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$ is the equivalence class of the sequence $\{q_m\}_{m\in\mathbb{N}}$, and $\epsilon \in \mathcal{U}(\{\pm 1\}_{m\in\mathbb{N}}) \simeq \{\pm 1\}$ is the equivalence class of the sequence $\{\epsilon_m\}_{m\in\mathbb{N}}$, then the pair $\{\mathsf{q}, \epsilon\}$ has the desired properties. Since there are infinitely many choices for each q_m, there are also infinitely many choices for q. □

Corollary 3.3.7. *For any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, there exists some $\{\mathsf{Q}\mathsf{Q}', \epsilon_{\mathsf{Q}\mathsf{Q}'}\} \in \mathsf{N}$ such that $r_{\mathsf{Q}\mathsf{Q}'} = 0$.*

Proof. This is an obvious induction argument using Theorem 3.3.4 and Proposition 3.3.3. □

Combining Proposition 3.3.3 and Theorem 3.3.4 allows us to prove the main result of this subsection.

Theorem 3.3.8. *Suppose that $(\mathcal{F}, \mathsf{S})$ is self-dual and that (κ, λ) is a nontrivial bipartite Euler system with sign δ for $(T_\varphi, \mathcal{F}, \mathsf{S})$. Then there exists a nonzero fractional ideal I of R such that:*

(1) For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, r_{Q} is odd, $r_{\mathsf{Q}} = 1$ if and only if $\kappa(\mathsf{Q}) \neq 0$, and in that case

$$\mathrm{char}_R(M_{\mathsf{Q}}) \cdot I = \mathrm{char}_R\left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))}\right).$$

(2) For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, r_{Q} is even, $r_{\mathsf{Q}} = 0$ if and only if $\lambda(\mathsf{Q}) \neq 0$, and in that case

$$\mathrm{char}_R(M_{\mathsf{Q}}) \cdot I = (\lambda(\mathsf{Q})).$$

In particular,

$$\delta = \mathrm{rk}_R \,\mathrm{Sel}_{\mathcal{F}}(T_\varphi) + 1 \pmod 2.$$

Proof. The proof will be in several steps.

Step 1. *If $\lambda(\mathsf{Q}) \neq 0$ for some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, then $r_{\mathsf{Q}} = 0$.*

Proof. If $0 \neq c \in \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)$, then by Theorem 3.3.4, there exists an admissible ultraprime q with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}}\, c \neq 0$. By Proposition 3.3.3, $\partial_{\mathsf{q}}(\kappa(\mathsf{Q}\mathsf{q})) = 0$, which contradicts the reciprocity laws. □

Step 2. *If $\kappa(\mathsf{Q}) \neq 0$ for some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, then $r_{\mathsf{Q}} = 1$.*

Proof. Choose an admissible ultraprime q with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}}\, \kappa(\mathsf{Q}) \neq 0$. Then by the reciprocity laws, $\lambda(\mathsf{Q}\mathsf{q}) \neq 0$, so by Step 1 $r_{\mathsf{Q}\mathsf{q}} = 0$. Proposition 3.3.3 implies $r_{\mathsf{Q}} = 1$. □

Step 3. *For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, $r_{\mathsf{Q}} \equiv \delta + |\mathsf{Q}| + 1 \pmod 2$.*

Proof. If $\{\mathsf{Q}\mathsf{Q}', \epsilon_{\mathsf{Q}\mathsf{Q}'}\} \in \mathsf{N}$, then by Proposition 3.3.3

$$r_{\mathsf{Q}} - r_{\mathsf{Q}\mathsf{Q}'} \equiv |\mathsf{Q}'| \pmod 2.$$

So Steps 1 and 2 imply Step 3. □

Step 4. *Suppose $r_{\mathsf{Q}} = 0$ for some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$. Then, for all admissible ultraprimes $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{S}$ with sign ϵ_{q}, $r_{\mathsf{Q}\mathsf{q}} = 1$ and*

$$\mathrm{char}_R(M_{\mathsf{Q}\mathsf{q}}) \cdot (\lambda(\mathsf{Q})) = \mathrm{char}_R(M_{\mathsf{Q}}) \cdot \mathrm{char}_R\left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}\mathsf{q})}(T_\varphi)}{(\kappa(\mathsf{Q}\mathsf{q}))}\right).$$

Proof. By Step 3, $\lambda(\mathsf{Q})$ and $\kappa(\mathsf{Qq})$ are well-defined. Then Step 4 follows from Proposition 3.3.3, since

$$\mathrm{char}_R\left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)}{(\kappa(\mathsf{Qq}))}\right)\cdot(\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) = (\lambda(\mathsf{Q}))\subset R.$$

□

The exact same reasoning implies:

Step 5. *Suppose that $r_{\mathsf{Q}}=1$ and $\mathsf{q}\notin \mathsf{Q}\cup\mathsf{S}$ is an admissible ultraprime with sign ϵ_{q} such that $r_{\mathsf{Qq}}=0$. Then*

$$\mathrm{char}_R(M_{\mathsf{Qq}})\cdot\left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))}\right) = \mathrm{char}_R(M_{\mathsf{Q}})\cdot(\lambda(\mathsf{Qq})).$$

Now consider the graph $\mathcal{X}$ [25] whose vertices are the elements of N, and where the edges are between vertices of the form $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}$ and $\{\mathsf{Qq},\epsilon_{\mathsf{Qq}}\}$, for some admissible ultraprime q with sign ϵ_{q}. We say $\{\mathsf{Q},\epsilon_Q T\}$ is a **core vertex** if $r_{\mathsf{Q}}\le 1$. The **core subgraph** $\mathcal{X}_0$ of $\mathcal{X}$ is the full subgraph on core vertices. Applying Steps 4 and 5, it suffices to show that $\mathcal{X}_0$ is path-connected to complete the proof of the theorem.

Step 6. *If $v=\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}$ and $v'=\{\mathsf{QQ}',\epsilon_{\mathsf{QQ}'}\}$ are core vertices, then they are connected by a path in $\mathcal{X}_0$.*

Proof. We proceed by induction on $|\mathsf{Q}'|$, where the base case is trivial. If $r_{\mathsf{QQ}'/\mathsf{q}}\le 1$ for any $\mathsf{q}\in\mathsf{Q}'$, then we may apply the inductive hypothesis, so assume otherwise. By Proposition 3.3.3, $r_{\mathsf{QQ}'}=1$ and $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{QQ}')}(T_\varphi))=0$ for all $\mathsf{q}\in\mathsf{Q}'$. Hence

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{QQ}')}(T_\varphi)\subset \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi).$$

Then, by Theorem 3.3.4 and Proposition 3.3.3, there exists an admissible ultraprime $\mathsf{q}\notin\mathsf{Q}\cup\mathsf{Q}'\cup\mathsf{S}$ with sign ϵ_{q} such that $r_{\mathsf{Qq}}=r_{\mathsf{QQ'q}}=0$. If $\mathsf{q}'\in\mathsf{Q}'$ is any factor, then $\left\{\mathsf{QQ'q/q'},\epsilon_{\mathsf{QQ'q/q'}}\right\}\in\mathsf{N}$ is a core vertex, which is connected to v' in $\mathcal{X}_0$. By the inductive hypothesis, $\left\{\mathsf{QQ'q/q'},\epsilon_{\mathsf{QQ'q/q'}}\right\}$ is also connected to the core vertex $\{\mathsf{Qq},\epsilon_{\mathsf{Qq}}\}$, hence to v, by a path in $\mathcal{X}_0$. This completes the inductive step. □

Step 7. *If $v=\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}$ is a core vertex and $\mathsf{T}\subset\mathsf{M}_{\mathbb{Q}}$ is any finite set, then there exists a core vertex $v'=\{\mathsf{Q}',\epsilon_{\mathsf{Q}'}\}$ such that v and v' are connected by a path in $\mathcal{X}_0$ and $\mathsf{Q}'\cap\mathsf{T}=\emptyset$.*

Proof. By iterating, it suffices to assume that $\mathsf{Q}\cap\mathsf{T}$ consists of exactly one ultraprime $\mathsf{q}\in\mathsf{Q}$. If $r_{\mathsf{Q/q}}\le 1$, then the conclusion is obvious, so suppose otherwise. As in the proof of Step 6, choose an admissible ultraprime $\mathsf{q}'\notin\mathsf{Q}\cup\mathsf{S}\cup\mathsf{T}$ with associated sign $\epsilon_{\mathsf{q}'}$ such that $r_{\mathsf{Qq}'}=0$, which implies $r_{\mathsf{Qq'/q}}=1$. The core vertex $v'=\left\{\mathsf{Qq'/q},\epsilon_{\mathsf{Qq'/q}}\right\}$ has the desired properties. □

Finally, we have:

Step 8. *The core subgraph $\mathcal{X}_0$ is path-connected.*

Proof. Let $\{\mathsf{Q}_1,\epsilon_{\mathsf{Q}_1}\}$ and $\{\mathsf{Q}_2,\epsilon_{\mathsf{Q}_2}\}$ be two core vertices. Without loss of generality, by Step 7, we may assume $\mathsf{Q}_1\cap\mathsf{Q}_2=\emptyset$. (This step is necessary because the sign functions ϵ_{Q_1} and ϵ_{Q_2} need not agree on $\mathsf{Q}_1\cap\mathsf{Q}_2$.) Consider $\{\mathsf{Q}_1\mathsf{Q}_2,\epsilon_{\mathsf{Q}_1\mathsf{Q}_2}\}\in\mathsf{N}$. This may not be a core vertex, but, by repeatedly applying Theorem 3.3.4 and Proposition 3.3.3, there exists $\{\mathsf{Q}_3,\epsilon_{\mathsf{Q}_3}\}\in\mathsf{N}$ such that $\{\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3,\epsilon_{\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3}\}$ is a core vertex. We may then conclude by Step 6. □

□

Proposition 3.3.9. *Under the hypotheses of Theorem 3.3.8, there exists a constant C depending on $|\mathsf{S}|$, T_f, and the ramification index of $R/\mathcal{O}$, but not on φ, such that $I\pi^C\subset R$. If (sclr) holds, then we may take $C=0$.*

Proof. By Theorem 3.3.8, it suffices to show that there exists a constant with the desired dependencies and a pair $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}\in\mathsf{N}^\delta$ such that $\pi^C\in\mathrm{char}_R(M_{\mathsf{Q}})$. We first note that the constant j in Lemma 3.3.6 depends only on T_f and the ramification index of $R/\mathcal{O}$, and can be taken to be 0 under (sclr).

Moreover, if k is the residue field of R, then $d=\dim_k \mathsf{H}^1(K^{\mathsf{S}}/K,W_\varphi[\pi])$ is also bounded with the desired uniformity. We now construct a sequence $\{\mathsf{Q}_i,\epsilon_{\mathsf{Q}_i}\}$ recursively (starting from $\mathsf{Q}_1=1$) by the following rules:

- If $r_{\mathsf{Q}_i}>0$, then choose any $\mathsf{q}_{i+1}\notin\mathsf{Q}_i$ with sign $\epsilon_{\mathsf{q}_{i+1}}$ such that

$$\mathrm{lg}\,\mathrm{coker}(\mathrm{loc}_{\mathsf{q}}\,\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(T_\varphi))\le j.$$

- If $r_{\mathsf{Q}_i} = 0$ and the exponent of $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi) \neq 0$ is $n_i > i \cdot j$, then choose any $\mathsf{q}_{i+1} \notin \mathsf{Q}_i$ with sign $\epsilon_{\mathsf{q}_{i+1}}$ such that the exponent of $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi))$ is at least $n_i - j$.

These choices are possible by Lemma 3.3.6. In either of the above two cases, set

$$\left\{\mathsf{Q}_{i+1}, \epsilon_{\mathsf{Q}_{i+1}}\right\} = \left\{\mathsf{Q}_i\mathsf{q}_{i+1}, \epsilon_{\mathsf{Q}_i\mathsf{q}_{i+1}}\right\};$$

if neither holds, then end the construction. For each i, let r'_{Q_i} be the minimal number of generators of the R-module $\pi^{i\cdot j}M_{\mathsf{Q}_i}$. In the first case of the construction, $r'_{\mathsf{Q}_{i+1}} \leq r'_{\mathsf{Q}_i}$; in the second case, $r'_{\mathsf{Q}_{i+1}} < r'_{\mathsf{Q}_i}$ (by Proposition 3.3.3(3b,a) respectively). After $r_1 \leq d$ steps, we alternate between the two cases of the construction, taking at most $2r'_1 \leq 2d$ more steps. Hence for some $i \leq 3d$, $r'_{\mathsf{Q}_i} = 0$ and $r_{\mathsf{Q}_i} = 0$, and the construction halts. For this i,

$$\lg M_{\mathsf{Q}_i} \leq ij \dim_k \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)[\pi] \leq 3dj(d+3d),$$

the last inequality by the reasoning of [25, Corollary 2.2.10]. (Less precisely, we could deduce the bound $3dj(d+6d)$ directly from Proposition 3.3.3(3).)

Since d and j have bounds of the desired sort, the claim follows. □

3.4. **Euler systems over Λ.** Let Λ be the anticyclotomic Iwasawa algebra $\mathcal{O}[\![T]\!]$ with canonical character

$$\Psi : G_K \to \Lambda^\times.$$

For each height-one prime $\mathfrak{P} \subset \Lambda$, let $S_\mathfrak{P}$ be the integral closure of $\Lambda/\mathfrak{P}$ in its field of fractions, so that Ψ induces a character $G_K \to \Lambda^\times \to S_\mathfrak{P}^\times$. We write $T_\mathfrak{P}$ for the twist $T \otimes_\mathcal{O} S_\mathfrak{P}(\Psi)$ and $\mathbf{T}$ for the interpolated twist $T \otimes_\mathcal{O} \Lambda(\Psi)$. Also let $\mathbf{W} = \mathbf{T}^*$ be the Cartier dual with Λ action twisted by the canonical involution ι, so that for each $\mathfrak{P}$ there is a natural map

$$W_\mathfrak{P} \to \mathbf{W}$$

of $\Lambda[G_K]$-modules (see, e.g., [24]). The following definition is motivated by [33, Lemma 5.3.13] and its applications in [24, 25].

Definition 3.4.1. An **interpolated self-dual** Selmer structure

$$(\mathsf{S}, \mathcal{F}_\Lambda, \mathcal{F}_\mathfrak{P}, \Sigma_\Lambda)$$

for $\mathbf{T}$ consists of the following data:

- A finite set $\mathsf{S} \subset \mathsf{M}_K$.
- For each height-one prime $\mathfrak{P} \subset \Lambda$, a self-dual Selmer structure $(\mathcal{F}_\mathfrak{P}, \mathsf{S})$ for $T_\mathfrak{P}$.
- A finite set Σ_Λ of height-one primes $\mathfrak{P} \subset \Lambda$.
- A Selmer structure $(\mathcal{F}_\Lambda, \mathsf{S})$ for $\mathbf{T}$ such that, for all $\mathsf{v} \in \mathsf{M}_K$ and all $\mathfrak{P} \subset \Lambda$, there are well-defined maps induced in the obvious way:

$$\begin{aligned} \mathsf{H}^1_{\mathcal{F}_\Lambda}(K_\mathsf{v}, \mathbf{T}/\mathfrak{P}) &\to \mathsf{H}^1_{\mathcal{F}_\mathfrak{P}}(K_\mathsf{v}, T_\mathfrak{P}), \\ \mathsf{H}^1_{\mathcal{F}_\mathfrak{P}}(K_\mathsf{v}, W_\mathfrak{P}) &\to \mathsf{H}^1_{\mathcal{F}_\Lambda}(K_\mathsf{v}, \mathbf{W}[\mathfrak{P}]). \end{aligned} \tag{19}$$

  Moreover, for all $\mathfrak{P} \notin \Sigma_\Lambda$, the maps (19) have finite kernel and cokernel with order bounded by a constant depending only on $[S_\mathfrak{P} : \Lambda/\mathfrak{P}]$ as $\mathfrak{P}$ varies.

Proposition 3.4.2 ([33], Proposition 5.3.14)**.** *Suppose* $(\mathsf{S}, \mathcal{F}_\mathfrak{P}, \mathcal{F}_\Lambda, \Sigma_\Lambda)$ *is an interpolated self-dual Selmer structure for* $\mathbf{T}$*. Then for all* $\mathfrak{P} \subset \Lambda$*, there are well-defined maps induced in the obvious way:*

$$\begin{aligned} \mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T})/\mathfrak{P} &\to \mathrm{Sel}_{\mathcal{F}_\mathfrak{P}}(T_\wp) \\ \mathrm{Sel}_{\mathcal{F}_\mathfrak{P}}(W_\mathfrak{P}) &\to \mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{W})[\mathfrak{P}]. \end{aligned}$$

For all $\mathfrak{P} \notin \Sigma_\Lambda$*, these maps have finite kernel and cokernel with a bound depending on* $\mathcal{F}$ *and on* $[S_\mathfrak{P} : \Lambda/\mathfrak{P}]$*, but not on* $\mathfrak{P}$ *itself.*

3.4.3. Recall that, for any finitely generated Λ-module M, there exists a unique Λ-module N of the form $\Lambda^r \oplus \bigoplus \Lambda/\mathfrak{P}_i^{e_i}$ such that M admits a map to N with finite kernel and cokernel; we denote this relationship by $M \sim N$. The characteristic ideal $\mathrm{char}_\Lambda(M)$ is zero if $r \geq 1$, and equal to $\prod \mathfrak{P}_i^{e_i}$ otherwise. The following easy lemma is implicit in [33, p. 66].

Lemma 3.4.4. *Let $\mathfrak{P} \subset \Lambda$ be a height-one prime. Then there exists an integer d and a sequence of height-one primes $\mathfrak{P}_m$ such that, for all finitely generated torsion Λ-modules M,*

$$\mathrm{lg}_{\mathcal{O}}(M/\mathfrak{P}_m) = md\, \mathrm{ord}_{\mathfrak{P}}\, \mathrm{char}_\Lambda(M) + O(1)$$

as m varies (holding M fixed). Moreover $[S_{\mathfrak{P}_m} : \Lambda/\mathfrak{P}_m]$ is constant for large enough m, and if $\mathfrak{P} \neq (\wp)$, then the rings $\Lambda/\mathfrak{P}_m$ are abstractly isomorphic.

Proof. If $\mathfrak{P} \neq (\wp)$ is generated by a distinguished polynomial $f \in \Lambda$, and π is a uniformizer for $\mathcal{O}$, then we may take $\mathfrak{P}_m = f + \pi^m$ (for sufficiently large m) and $d = [S_{\mathfrak{P}} : \mathcal{O}]$. If $\mathfrak{P} = (\wp) = (\pi)$, then we may take $\mathfrak{P}_m = T^m + \pi$ and $d = 1$. □

Proposition 3.4.5. *Suppose that $(\mathsf{S}, \mathcal{F}_{\mathfrak{P}}, \mathcal{F}_\Lambda, \Sigma_\Lambda)$ is an interpolated self-dual Selmer structure for $\mathbf{T}$. Then for all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$:*

(1) *$(\mathsf{S} \cup \mathsf{Q}, \mathcal{F}_{\mathfrak{P}}(\mathsf{Q}), \mathcal{F}_\Lambda(\mathsf{Q}), \Sigma_\Lambda)$ is an interpolated self-dual Selmer structure for $\mathbf{T}$ and*

$$\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W})^\vee \sim \Lambda^{r_{\mathsf{Q}}} \oplus M_{\mathsf{Q}} \oplus M_{\mathsf{Q}}$$

for some torsion Λ-module M_{Q} and an integer r_{Q}.

(2) *$r_{\mathsf{Q}} = \mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})$.*

Proof. At places $\mathsf{q} \in \mathsf{Q}$,

$$\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_{\mathsf{q}}, \mathbf{T}) = \mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_f) \otimes \Lambda$$

and

$$\mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(K_{\mathsf{q}}, \mathbf{T}) = \mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_f) \otimes S_{\mathfrak{P}},$$

so we clearly have local maps with kernel and cokernels bounded as desired (and similarly for $\mathbf{W}_f$ and $W_{\mathfrak{P}}$); so indeed $(\mathsf{S} \cup \mathsf{Q}, \mathcal{F}_{\mathfrak{P}}(\mathsf{Q}), \mathcal{F}_\Lambda(\mathsf{Q}), \Sigma_\Lambda)$ is an interpolated self-dual Selmer structure. The rest of the claims are deduced from Proposition 3.4.2 and Proposition 3.3.3(1,2) exactly as in [24, Theorem 2.2.10]. □

Theorem 3.4.6. *Suppose that $(\mathsf{S}, \mathcal{F}_{\mathfrak{P}}, \mathcal{F}_\Lambda, \Sigma_\Lambda)$ is an interpolated self-dual Selmer structure for $\mathbf{T}$ and $\{\boldsymbol{\kappa}, \boldsymbol{\lambda}\}$ is a nontrivial bipartite Euler system with parity δ for the triple $(\mathbf{T}, \mathcal{F}_\Lambda, \mathsf{S})$. Then there exists a nonzero fractional ideal $I \subset \Lambda \otimes \mathbb{Q}_p$ such that:*

(1) *For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, r_{Q} is odd, $r_{\mathsf{Q}} = 1$ if and only if $\boldsymbol{\kappa}(\mathsf{Q}) \neq 0$, and in that case*

$$\mathrm{char}_\Lambda(M_{\mathsf{Q}}) \cdot I = \mathrm{char}_\Lambda \left(\frac{\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})}{(\boldsymbol{\kappa}(\mathsf{Q}))} \right).$$

(2) *For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, r_{Q} is even, $r_{\mathsf{Q}} = 0$ if and only if $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$, and in that case*

$$\mathrm{char}_\Lambda(M_{\mathsf{Q}}) \cdot I = (\boldsymbol{\lambda}(\mathsf{Q})).$$

In particular,

$$\delta = \mathrm{rk}_R \mathrm{Sel}_{\mathcal{F}}(\mathbf{T}) + 1 \pmod 2.$$

If (sclr) holds, then $I \subset \Lambda$.

Proof. Let $\mathfrak{P} \subset \Lambda$ be any height-one prime; via the natural maps $\mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T}) \to \mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}}(T_{\mathfrak{P}})$ and $\Lambda \to S_{\mathfrak{P}}$, the Euler system $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ defines an Euler system $(\kappa_{\mathfrak{P}}, \lambda_{\mathfrak{P}})$ of parity δ for the triple $(T_{\mathfrak{P}}, \mathcal{F}_{\mathfrak{P}}, \mathsf{S})$. In particular, Theorem 3.3.8 applies.

If $\boldsymbol{\kappa}(\mathsf{Q}) \neq 0$, then by Proposition 3.4.2 $\kappa_{\mathfrak{P}}(\mathsf{Q}) \neq 0$ for all but finitely many $\mathfrak{P}$ (since $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})$ is torsion-free), and similarly for $\boldsymbol{\lambda}(\mathsf{Q})$. Because

$$\mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T}) \leq \mathrm{rk}_{S_{\mathfrak{P}}} \mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}}(T_{\mathfrak{P}})$$

with equality for all but finitely many $\mathfrak{P}$, the claims about r_{Q} follow from Theorem 3.3.8.

For any $\mathfrak{P}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$ such that $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$, by Proposition 3.4.5 and Lemma 3.4.4 we have

$$\begin{aligned} e_{\mathfrak{P}}(\mathsf{Q}) &\coloneqq \operatorname{ord}_{\mathfrak{P}}(\boldsymbol{\lambda}(\mathsf{Q})) - \operatorname{ord}_{\mathfrak{P}} \operatorname{char}_\Lambda(M_{\mathsf{Q}}) \\ &= \lim_{m\to\infty} \frac{\lg_{\mathcal{O}}(S_{\mathfrak{P}_m}/\lambda_{\mathfrak{P}_m}(\mathsf{Q})) - \lg_{\mathcal{O}} M_{\mathsf{Q},\mathfrak{P}_m}}{md}. \end{aligned}$$

Applying Theorem 3.3.8, this quantity does not depend on $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ (as long as $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$); it is also clearly zero for almost all $\mathfrak{P}$, so that $\prod_{\mathfrak{P}} \mathfrak{P}^{e_{\mathfrak{P}}}$ defines a fractional ideal I of Λ satisfying (2). The same calculation shows that I satisfies (1) as well, and the integrality properties follow from Proposition 3.3.9. □

4. Geometry of modular Jacobians

4.1. **Multiplicity one.**

4.1.1. Let N_1 and N_2 be coprime positive integers. Consider the Hecke algebra $\mathbb{T} = \mathbb{T}_{N_1,N_2}$ generated over $\mathbb{Z}$ by operators T_ℓ for all primes $\ell \nmid N = N_1N_2$ and U_ℓ for all $\ell | N$, acting on the modular forms of weight two and level $\Gamma_0(N)$ which are new at all factors $\ell | N_2$. If I is the kernel of the projection $\mathbb{T}_{N_1N_2,1} \to \mathbb{T}$, then we set

$$J_{\min}^{N_1,N_2} \coloneqq J_0(N)/IJ_0(N), \tag{20}$$

an abelian variety with a (faithful) action of $\mathbb{T}$. If N_1, N_2 are clear from context, we will omit the superscript.

For any abelian variety A with an action of $\mathbb{T}$, and any maximal ideal $\mathfrak{m} \subset \mathbb{T}$, the $\mathfrak{m}$-adic Tate module is defined to be the localization

$$T_{\mathfrak{m}}A \coloneqq T_pA \otimes_{\mathbb{T}} \mathbb{T}_{\mathfrak{m}}, \tag{21}$$

where p is the residue characteristic of $\mathfrak{m}$. (Note that this is dual to the notation of [22].) For any $\mathfrak{m}$ which is non-Eisenstein with odd residue characteristic $p \nmid N$, it follows from [49] that $T_{\mathfrak{m}}J_{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$; by [22, Corollary 4.7], the natural map then induces an isomorphism

$$\mathbb{T}_{\mathfrak{m}} \xrightarrow{\sim} \operatorname{End}_{\mathbb{T}}(J_{\min})_{\mathfrak{m}}. \tag{22}$$

4.1.2. Now suppose that A is an abelian variety with a Hecke-equivariant isogeny to $J_{\min}$. For any $\ell || N_2$, let $\mathcal{A}_{/\mathbb{Z}_\ell}$ be the Néron model of A. The connected component $\mathcal{A}^0_{\mathbb{F}_\ell}$ of the special fiber of $\mathcal{A}$ is a torus, and we write $\mathcal{X}_\ell(A) = \operatorname{Hom}(\mathcal{A}^0_{\mathbb{F}_\ell}, \mathbb{G}_m)$ for its character group. The association $A \mapsto \mathcal{X}_\ell(A)$ is contravariantly functorial.

Proposition 4.1.3 (Helm)**.** *Let $\mathfrak{m} \subset \mathbb{T}$ be non-Eisenstein of residue characteristic $p \nmid 2N$. Then the natural maps induce $\mathbb{T}_{\mathfrak{m}}$-module isomorphisms:*

$$T_{\mathfrak{m}}J_{\min} \otimes \operatorname{Hom}(J_{\min}, A)_{\mathfrak{m}} \xrightarrow{\sim} T_{\mathfrak{m}}A,$$

$$\mathcal{X}_\ell(J_{\min}^\vee) \otimes \operatorname{Hom}(J_{\min}, A)_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{X}_\ell(A^\vee),$$

$$\operatorname{Hom}(A, J_{\min})_{\mathfrak{m}} \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{T}_{\mathfrak{m}}}\left(\operatorname{Hom}(J_{\min}, A)_{\mathfrak{m}}, \operatorname{End}(J_{\min})_{\mathfrak{m}}\right).$$

Here, all Hom-sets are understood to be $\mathbb{T}$-equivariant morphisms, and tensor products are taken modulo $\mathbb{Z}$-torsion.

Proof. This follows by duality from [22, Corollary 4.1, Theorem 4.11, Proposition 4.14]. □

We record the following elementary lemma for later use.

Lemma 4.1.4. *Let $\mathcal{X} = \mathcal{X}_\ell(J_{\min}^\vee)_{\mathfrak{m}}$ for some $\ell | N_2$ and $\mathfrak{m} \subset \mathbb{T}$, where $\mathfrak{m}$ is non-Eisenstein of odd residue characteristic p. If the associated residual representation $\overline{\rho}_{\mathfrak{m}}$ is ramified at ℓ, then $\mathcal{X}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$. In general, there exist $\mathbb{T}_{\mathfrak{m}}$-module maps*

$$\phi_i : \mathcal{X} \to \mathbb{T}_{\mathfrak{m}}, \;\; \psi_i : \mathbb{T}_{\mathfrak{m}} \to \mathcal{X}, \;\; i = 1, 2$$

such that

$$\phi_i \circ \psi_i = \psi_i \circ \phi_i = t_i \in \mathbb{T}_{\mathfrak{m}} \subset \operatorname{End}(\mathcal{X})$$

and

$$t_1 + t_2 = \ell - 1 \in \mathbb{T}_{\mathfrak{m}}.$$

Proof. If $\ell - 1$ is a p-adic unit, or if $\overline{\rho}_{\mathfrak{m}}$ is ramified, then this follows from [22, Lemma 6.5]. In general, we have

$$\mathcal{X} = \mathrm{Hom}((\mathcal{J}_{\min}^{\vee})^0_{\mathbb{F}_\ell}[\mathfrak{m}^\infty], \mu_{p^\infty}) \tag{23}$$

so that $\mathcal{X}$ may be identified with a $\mathbb{T}_{\mathfrak{m}}[G_{\mathbb{Q}_\ell}]$-module quotient

$$\pi : T_{\mathfrak{m}} J_{\min} \to \mathcal{X};$$

the Galois action on $\mathcal{X}$ is unramified and Frobenius acts as U_ℓ, which is a constant ± 1 because the residue characteristic of $\mathfrak{m}$ is $p > 2$.

Because $T_{\mathfrak{m}} J_{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$, it may be equipped with a basis $\{e_1, e_2\}$, and moreover an alternating $\mathbb{T}_{\mathfrak{m}}$-module pairing

$$\langle \cdot, \cdot \rangle : T_{\mathfrak{m}} J_{\min} \times T_{\mathfrak{m}} J_{\min} \to \mathbb{T}_{\mathfrak{m}} \tag{24}$$

such that

$$y = \langle e_1, y\rangle e_2 - \langle e_2, y\rangle e_1 \tag{25}$$

for all $y \in T_{\mathfrak{m}} J_{\min}$. Define maps

$$\begin{aligned} \phi_i : T_{\mathfrak{m}} J_{\min} &\to \mathbb{T}_{\mathfrak{m}}, \quad i = 1, 2 \\ \phi_1 : y &\mapsto \langle y, (F - U_\ell) e_2 \rangle \\ \phi_2 : y &\mapsto \langle y, (F - U_\ell) e_1 \rangle, \end{aligned}$$

where $F \in G_{\mathbb{Q}_\ell}$ is any lift of Frobenius. We first claim that the maps ϕ_i factor through π. Since $\mathbb{T}_{\mathfrak{m}}$ is p-torsion-free, it suffices to check this after inverting p. On $T_{\mathfrak{m}} J_{\min} \otimes \mathbb{Q}_p$, F acts with distinct eigenvalues U_ℓ and ℓU_ℓ, and $\pi \otimes \mathbb{Q}_p : T_{\mathfrak{m}} J_{\min} \otimes \mathbb{Q}_p \to \mathcal{X} \otimes \mathbb{Q}_p$ coincides with the projection onto the U_ℓ-eigenspace. Since $\langle \cdot, \cdot \rangle$ is alternating and $\mathbb{T}_{\mathfrak{m}}$-linear, it follows that each ϕ_i does indeed descend to a $\mathbb{T}_{\mathfrak{m}}$-module map $\mathcal{X} \to \mathbb{T}_{\mathfrak{m}}$. Now define maps

$$\begin{aligned} \psi_i : \mathbb{T}_{\mathfrak{m}} &\to \mathcal{X}, \quad i = 1, 2 \\ \psi_1 : 1 &\mapsto U_\ell \pi(e_1) \\ \psi_2 : 1 &\mapsto -U_\ell \pi(e_2). \end{aligned}$$

We claim that ψ_i and ϕ_i satisfy the conclusion of the lemma. One readily calculates:

$$\begin{aligned} \phi_1 \circ \psi_1(1) &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \\ \psi_1 \circ \phi_1(e_1) &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \pi(e_1) \\ \psi_1 \circ \phi_1(e_2) &= U_\ell \langle e_2, (F - U_\ell) e_2 \rangle \pi(e_1) \\ &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \pi(e_2) - U_\ell (F - U_\ell) \pi(e_2) \\ &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \pi(e_2), \end{aligned}$$

where in the last two steps we have used (25) and the fact that $F = U_\ell$ on $\mathcal{X}$. Similarly,

$$\phi_2 \circ \psi_2 = \psi_2 \circ \phi_2 = -U_\ell \langle e_2, (F - U_\ell) e_1 \rangle,$$

and

$$U_\ell \langle e_1, (F - U_\ell) e_2 \rangle - U_\ell \langle e_2, (F - U_\ell) e_1 \rangle = \mathrm{tr}_{T_{\mathfrak{m}} J_{\min}} U_\ell (F - U_\ell) = \ell - 1.$$

□

4.2. **Shimura curves.**

4.2.1. If $\nu(N_2)$ is even, then there exists a Shimura curve X_{N_1,N_2}, with $\Gamma_0(N_1)$ level structure, associated to the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant N_2. Let

$$J^{N_1,N_2} := J(X_{N_1,N_2}),$$

an abelian variety with a natural action of $\mathbb{T}$ by correspondences (induced by Picard functoriality). When N_1 and N_2 are understood, we abbreviate $J = J^{N_1,N_2}$. There is a noncanonical Hecke-equivariant isogeny $J \to J_{\min}$. Consider the following technical hypothesis on the residual representation $\overline{\rho}_{\mathfrak{m}} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}/\mathfrak{m})$ associated to $\mathfrak{m}$:

$$(*)\qquad \begin{array}{l}\text{If } p = 3 \text{ and } \overline{\rho}_{\mathfrak{m}} \text{ is induced from a character of } G_{\mathbb{Q}(\sqrt{-3})},\ \exists\, \ell || N_2 \\ \text{such that either } \ell \equiv -1 \pmod 3 \text{ or } \overline{\rho}_{\mathfrak{m}} \text{ is ramified at } \ell.\end{array}$$

Theorem 4.2.2 (Helm)**.** *Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2N$ satisfying $(*)$. Then there is an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules:*

$$\mathrm{Hom}(J_{\min}, J) \simeq \otimes_{\ell|N_2} \mathcal{X}_\ell(J_{\min}^\vee)_{\mathfrak{m}},$$

modulo $\mathbb{Z}$-torsion on the right-hand side.

Proof. This is essentially [22, Theorem 8.7]; to complete the case $p = 3$, by [22, Remark 8.12] one only needs a level-raising input that is provided by [16]. $\square$

4.3. **Shimura sets.** Now suppose that $\nu(N_2)$ is odd, and consider the finite double coset space (often called a **Shimura set**):

$$X_{N_1,N_2} := R(\mathbb{A}_{\mathbb{Q}})^\times \backslash B(\mathbb{A}_{\mathbb{Q}})^\times / B(\mathbb{Q})^\times, \tag{26}$$

where B is a definite quaternion algebra over $\mathbb{Q}$ ramified at N_2 and ∞, and R is an Eichler order in B of level $\Gamma_0(N_1)$. When N_1 and N_2 are clear from context, the subscripts may be omitted.

4.3.1. The $\mathbb{Z}$-module $\mathbb{Z}[X]^0$ of formal degree-zero divisors in X has two natural actions of $\mathbb{T} = \mathbb{T}_{N_1,N_2}$ by correspondences: an "Albanese" action induced by viewing an element of $\mathbb{Z}[X]^0$ as a formal sum of points in a double coset space, and a "Picard" action induced by identifying $\mathbb{Z}[X] = \mathrm{Hom}_{\mathrm{Set}}(X, \mathbb{Z})$. We will consider $\mathbb{Z}[X]^0$ as a $\mathbb{T}$-module through the latter action. The analogue of Theorem 4.2.2 is:

Theorem 4.3.2. *Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2N$ satisfying $(*)$. Then there is an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules:*

$$\mathbb{Z}[X]^0 \simeq \otimes_{\ell|N_2} \mathcal{X}_\ell(J_{\min}^\vee)_{\mathfrak{m}},$$

modulo $\mathbb{Z}$-torsion on the right-hand side.

Proof. Choose any prime $q|N_2$, so that $\nu(N_2/q)$ is even. Let $\mathbb{T}' = \mathbb{T}_{N_1q,N_2/q}$, and write $\mathfrak{m}$ as well for the maximal ideal of $\mathbb{T}'$ induced by the map $\mathbb{T}' \to \mathbb{T}$.

Applying Theorem 4.2.2 to the pair $N_1q, N_2/q$, we obtain an isomorphism of $\mathbb{T}'_{\mathfrak{m}}$-modules (modulo $\mathbb{Z}$-torsion)

$$\mathrm{Hom}(J_{\min}^{N_1q,N_2/q}, J^{N_1q,N_2/q})_{\mathfrak{m}} \simeq \otimes_{\ell|N_2/q} \mathcal{X}_\ell(J_{\min}^{N_1q,N_2/q,\vee}). \tag{27}$$

By [22, Corollary 5.3, Lemma 8.2], this implies an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules

$$\mathrm{Hom}(J_{\min}^{N_1,N_2}, J_{q\text{-new}}^{N_1,N_2/q})_{\mathfrak{m}} \simeq \otimes_{\ell|N_2/q} \mathcal{X}_\ell(J_{\min}^{N_1,N_2,\vee}), \tag{28}$$

where $J_{q\text{-new}}^{N_1q,N_2/q}$ is the q-new quotient of $J^{N_1q,N_2/q}$. Then, by Proposition 4.1.3, we have

$$\mathcal{X}_q(J_{q\text{-new}}^{qN_1,N_2/q,\vee})_{\mathfrak{m}} \simeq \mathcal{X}_q(J_{\min}^{N_1,N_2,\vee})_{\mathfrak{m}} \otimes_{\ell|N_2/q} \mathcal{X}_\ell(J_{\min}^{N_1,N_2,\vee})_{\mathfrak{m}}. \tag{29}$$

By [3, Proposition 5.3], $\mathcal{X}_q(J^{qN_1,N_2/q,\vee})$ is canonically identified with $\mathbb{Z}[X_{N_1,N_2}]^0$. It remains to show that the inclusion $J_{q\text{-new}}^{N_1q,N_2/q,\vee} \hookrightarrow J^{N_1q,N_2/q,\vee}$ induces an isomorphism on character groups at q. Indeed, since $J_{q\text{-new}}^{N_1q,N_2/q,\vee}$ has purely toric reduction at q, there is a surjection of character groups $\mathcal{X}_q(J^{N_1q,N_2/q,\vee}) \twoheadrightarrow \mathcal{X}_q(J_{q\text{-new}}^{N_1q,N_2/q,\vee})$, which is clearly an isomorphism after tensoring both sides with $\mathbb{Q}$, hence also before. $\square$

4.4. **CM points.**

4.4.1. Let us now fix an imaginary quadratic field $K/\mathbb{Q}$, and, once and for all, an embedding $K \hookrightarrow GL_2(\mathbb{Q})$ such that $K \cap M_2(\mathbb{Z}) = \mathcal{O}_K$.

For any integer $N = N^+N^-$ such that every prime factor of N^+ is (unramified and) split in K, and N^- is a squarefree product of primes (unramified and) inert in K, let $B = B_{N^-}$ be the quaternion algebra over $\mathbb{Q}$ ramified exactly at the factors of N^- (and possibly ∞), and $R \subset B$ an Eichler order of level N^+. For each such B, we fix an optimal embedding $K \hookrightarrow B$ such that $K \cap R = \mathcal{O}_K$. Then we may define the space of K-CM points:

$$\mathscr{C}_{N^+,N^-} = K^\times \backslash B(\mathbb{A}_f)/\widehat{R}^\times.$$

The Galois group G_K^{ab} acts on $\mathscr{C}_{N^+,N^-}$ through the reciprocity map

$$\mathrm{rec} : G_K^{\mathrm{ab}} \xrightarrow{\sim} K^\times \backslash \widehat{K}^\times, \tag{30}$$

i.e. $\sigma[b] = [\mathrm{rec}(\sigma)b]$.

4.4.2. If $S \subset M_{\mathbb{Q}}$ is a finite set of primes, consider the set of S-CM points:

$$\mathscr{C}^S = K^\times \backslash \widehat{K}^\times GL_2(\mathbb{A}_S)/\widehat{O}_K^\times GL_2(\widehat{\mathbb{Z}}_S).$$

We again have an action of G_K^{ab}, and by [37, Proposition 2.5] the field of definition $K[y]$ of any $y \in \mathscr{C}^S$ is contained in the compositum $K[S]$ of all ring class fields unramified outside S. If S is disjoint from the factors of N, then there is a unique map $GL_2(\mathbb{A}_S) \hookrightarrow B(\mathbb{A}_S)$ so that the composite

$$K \otimes_{\mathbb{Q}} \mathbb{A}_S \hookrightarrow GL_2(\mathbb{A}_S) \hookrightarrow B(\mathbb{A}_S)$$

agrees with the embedding deduced from $K \hookrightarrow B$. This embedding identifies $\mathscr{C}^S$ with a Galois-stable subset of $\mathscr{C}_{N^+,N^-}$.

4.4.3. If $\nu(N^-)$ is **even**, then the Shimura curve $X = X_{N^+,N^-}$ of (4.2.1) admits a complex uniformization:

$$X(\mathbb{C}) = B^\times \backslash \mathcal{H}^\pm \times B(\mathbb{A}_f)^\times/\widehat{R}^\times, \tag{31}$$

where $\mathcal{H}^\pm = \mathbb{C} \setminus \mathbb{R}$. We have an injective map:

$$\begin{aligned} \mathrm{CM}_{N^+,N^-} : \mathscr{C}_{N^+,N^-} &\to X(\mathbb{C}) \\ y &\mapsto [(h_0, y)], \end{aligned} \tag{32}$$

where h_0 is the unique fixed point on $\mathcal{H}^+$ of the action of K through the embedding $K \hookrightarrow B$. By Shimura's reciprocity law, the image of CM_{N^+,N^-} is contained in $X(K^{\mathrm{ab}})$, and CM_{N^+,N^-} is Galois-equivariant for the action on $\mathscr{C}_{N^+,N^-}$ defined above.

4.4.4. If instead $\nu(N^-)$ is **odd**, then there is a natural projection

$$\mathrm{CM}_{N^+,N^-} : \mathscr{C}_{N^+,N^-} \to X_{N^+,N^-},$$

where X_{N^+,N^-} is the Shimura set of (4.3).

4.4.5. With notation as above in (4.4.1), let $q \nmid N$ be a prime inert in K and not in S; note that the unique prime of K above q splits completely in $K[S]/K$. We fix a prime $\mathfrak{q}$ of $K[S]$ lying above q according to a fixed embedding $G_{\mathbb{Q}_q} \hookrightarrow G_{\mathbb{Q}}$. If $\nu(N^-)$ is **even**, then all points of $\mathrm{CM}_{N^+,N^-}(\mathscr{C}^S)$ have supersingular reduction modulo $\mathfrak{q}$, and the set of supersingular points $X(\mathbb{F}_{q^2})^{ss}$ may be identified with the Shimura set $X_q = X_{N^+,N^-q}$ associated to the definite quaternion algebra B_q ramified at $N^-q\infty$. Consider the following hypothesis on the residual representation $\overline{\rho}_{\mathfrak{m}}$ associated to a non-Eisenstein maximal ideal $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ of residue characteristic p (for any Hecke algebra $\mathbb{T}_{N_1,N_2}$):

$$\text{if } p = 3, \text{ then } \overline{\rho}_{\mathfrak{m}} \text{ is absolutely irreducible over } \mathbb{Q}\sqrt{-3}. \tag{TW}$$

Note that this is strictly stronger than condition $(*)$ above.

Proposition 4.4.6. *With identifications chosen compatibly, there is a commutative diagram:*

$$\begin{tikzcd} \mathbb{Z}\left[\mathscr{C}^S\right]^0 \arrow[r] \arrow[d, equal] & \mathbb{Z}\left[\mathrm{CM}_{N^+,N^-}(\mathscr{C}^S)\right]^0 \arrow[r] \arrow[d, "\mathrm{Red}_{\mathfrak{q}}"] & J(K[S]) \arrow[d, "\mathrm{Red}_{\mathfrak{q}}"] \\ \mathbb{Z}\left[\mathscr{C}^S\right]^0 \arrow[r] & \mathbb{Z}[X_q]^0 \arrow[r] & J(\mathbb{F}_{q^2}) \end{tikzcd}$$

Moreover, the map $\mathbb{Z}[X_q]^0 \to J(\mathbb{F}_{q^2})$ *is compatible with the action of* $\mathbb{T}_{N^+q,N^-Q}$ *where* U_q *acts on* $J(\mathbb{F}_{q^2})$ *through* Frob_q*, and is surjective after localizing at a non-Eisenstein maximal ideal* $\mathfrak{m} \subset \mathbb{T}_{N^+q,N^-Q}$ *satisfying condition (TW).*

Proof. See [52, Lemma 5.4.3] for the commutativity of the diagram and [42] for the U_q action. The surjectivity is an application of Ihara's Lemma which can be deduced from the argument in [3, Proposition 9.2]: we add auxiliary level of the form $\Gamma_1(\ell)$, where $\ell \nmid N$ is a prime such that $\ell - 1, T_\ell - \ell - 1 \notin \mathfrak{m}$. That such a prime exists follows from condition (TW) by [16, Lemma 3]. ☐

4.4.7. Now suppose instead that $\nu(N^-)$ is **odd**, and let $q \nmid N^-$ be a prime inert in K and not lying in S. The Shimura curve $X_q = X_{N^+,N^-q}$ has a canonical, semistable integral model over $\mathbb{Z}_q$, whose irreducible components are identified with two copies of $X = X_{N^+,N^-}$. We denote this set by $X^\pm$, where the positive copy is the one containing the reduction of the point $[(h_0, 1)]$ in the uniformization of (31). We define a map $\mathscr{C}^S \to X^\pm$ by the composition $\mathscr{C}^S \xrightarrow{\mathrm{CM}_{N^+,N^-}} X \simeq X^+ \subset X^\pm$.

4.4.8. The Néron model $\mathcal{J}_q$ of the Jacobian $J_q = J^{N^+,N^-q}$ has purely toric reduction, and we write $\mathcal{X}$ and Φ for the character group and the group of connected components, respectively, of its special fiber. Recall the rigid-analytic uniformization of J_q, which gives rise to an exact sequence:

$$0 \to \mathcal{X} \to \mathcal{X}^\dagger \otimes \overline{\mathbb{Q}}_q \to J_q(\overline{\mathbb{Q}}_q) \to 0. \tag{33}$$

Here, $\mathcal{X}^\dagger = \mathrm{Hom}(\mathcal{X}, \mathbb{Z})$, and the maps are Hecke-equivariant if $\mathcal{X}$ is given Hecke action through Albanese functoriality, and the actions on $\mathcal{X}^\dagger$ and $J_q(\overline{\mathbb{Q}}_q)$ are induced by Picard functoriality. Importantly for our later applications, (33) is compatible with the Galois action of $G_{\mathbb{Q}_q}$ [5], where the action on $\mathcal{X}$ is unramified and Frobenius acts through U_q [42, Proposition 3.8]. The rigid analytic uniformization is related to the monodromy pairing $j : \mathcal{X} \to \mathcal{X}^\dagger$ of Grothendieck [21] by the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{X} \arrow[r] \arrow[d, equal] & \mathcal{X}^\dagger \otimes \mathbb{Q}_{q^2} \arrow[r] \arrow[d, "\mathrm{ord}"] & J_q(\mathbb{Q}_{q^2}) \arrow[r] \arrow[d, "\mathrm{Sp}_q"] & 0 \\ 0 \arrow[r] & \mathcal{X} \arrow[r, "j"] & \mathcal{X}^\dagger \arrow[r] & \Phi \arrow[r] & 0 \end{tikzcd}$$

In particular, the specialization map is well-defined on $J_q(K[S])$.

Proposition 4.4.9. *With identifications chosen compatibly, there is a commutative diagram:*

$$\begin{tikzcd} \mathbb{Z}\left[\mathscr{C}^S\right]^0 \arrow[r] \arrow[d, equal] & \mathbb{Z}\left[\mathrm{CM}_{N^+,N^-q}(\mathscr{C}^S)\right]^0 \arrow[r] \arrow[d, "\mathrm{Sp_q}"] & J_q(K[S]) \arrow[d, "\mathrm{Sp_q}"] \\ \mathbb{Z}\left[\mathscr{C}^S\right]^0 \arrow[r] & \mathbb{Z}[X^\pm]^0 \arrow[r] & \Phi \end{tikzcd}$$

Moreover, the map $\mathbb{Z}[X^\pm]^0 \to \Phi$ *is compatible with the action of* $\mathbb{T}_q = \mathbb{T}_{N^+q,N^-}$*, where* U_q *acts on* $\mathbb{Z}[X^\pm]^0$ *by the matrix*

$$\begin{pmatrix} T_q & q \\ -1 & 0 \end{pmatrix}.$$

After localizing at a non-Eisenstein maximal ideal $\mathfrak{m} \subset \mathbb{T}_q$*, the map* $\mathbb{Z}[X^\pm]^0 \to \Phi$ *induces an isomorphism*

$$\mathbb{Z}[X^\pm]^0_{\mathfrak{m}} \otimes_{\mathbb{T}_q} \mathbb{T}_q/(U_q^2 - 1) \xrightarrow{\sim} \Phi_{\mathfrak{m}}.$$

Proof. For the commutativity of the diagram, see [52, Lemma 5.4.6]; for the rest, see [3, Proposition 5.5]. ☐

5. CM classes in cohomology

5.1. **Level raising.**

5.1.1. Fix notation as in (3.1.1), and suppose that $N = N^+N^-$ is coprime to pD_K, where all factors of N^+ are split in K and N^- is a squarefree product of primes inert in K. (In particular, $H^0(G_K, \overline{T}_f) = 0$.) We shall denote by π a uniformizer of $\mathcal{O}$. From now on, we additionally assume that the maximal ideal $\mathfrak{m}$ associated to T_f satisfies (TW) above, i.e.:

$$\text{if } p = 3, \text{ then } \overline{T}_f \text{ is absolutely irreducible over } \mathbb{Q}\sqrt{-3}. \tag{TW}$$

5.1.2. We say a prime $q \nmid N$ is weakly admissible with sign $\epsilon_q = \pm 1$ if q is inert in K, $a_q \equiv \epsilon_q(q+1) \pmod{\wp}$, and $q \not\equiv 1 \pmod{p}$. A weakly admissible pair $\{Q, \epsilon_Q\}$ is an ordered pair of a squarefree number Q and a function $\epsilon_Q : \{q|Q\} \to \{\pm 1\}$ such that q is weakly admissible with sign $\epsilon_Q(q)$ for all $q|Q$. If $\{Q, \epsilon_Q\}$ is a weakly admissible pair, then for all $q|Q$, there is a unique root $u_q \in \mathcal{O}$ of the polynomial $y^2 - ya_q + q$ such that $u_q \equiv \epsilon_Q(q) \pmod{\wp}$. We may view $\mathcal{O}$ as a $\mathbb{T}_{N^+Q,N^-}$-algebra by letting U_q act through u_q; let $\mathfrak{m}_Q^{\epsilon_Q}$ be the associated maximal ideal (we will usually drop the superscript).

If $Q = Q'Q''$, then we abbreviate $\mathbb{T}_{Q''}^{Q'} = \mathbb{T}_{N^+Q',N^-Q''}$, omitting any superscript or subscript which is equal to 1.

5.1.3. In light of the structural similarity of Theorems 4.2.2 and 4.3.2, let

$$M_Q = \begin{cases} \mathrm{Hom}(J_{\min}^{N^+,N^-Q}, J^{N^+,N^-Q}), & \nu(N^-Q) \text{ even,} \\ \mathbb{Z}[X_{N^+,N^-Q}]^0, & \nu(N^-Q) \text{ odd.} \end{cases} \tag{34}$$

It is well-known that M_Q is a faithful $\mathbb{T}_Q$-module, and indeed $M_Q \otimes \mathbb{Q}$ is free of rank one over $\mathbb{T}_Q \otimes \mathbb{Q}$.

Lemma 5.1.4. *Suppose $\{Q, \epsilon_Q\}$ is a weakly admissible pair, and let*

$$C = \sum_{\substack{\ell | N^- \\ \overline{T}_f \text{ unram at } \ell}} \mathrm{ord}_\pi(\ell - 1).$$

Then there exists an $\mathcal{O}$-module map

$$M_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \to \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$$

with kernel and cokernel annihilated by π^C; in particular, $\pi^C(M_Q \otimes \mathcal{O})$ is principal of length at least $\lg(\mathbb{T}_Q \otimes \mathcal{O}) - 2C$.

Proof. We may assume that $\mathfrak{m}_Q \subset \mathbb{T}^Q$ descends to $\mathbb{T}_Q$. Now, by Theorems 4.2.2 and 4.3.2, we have

$$M_{Q,\mathfrak{m}_Q} \simeq \otimes_{\ell | N^-Q} \mathcal{X}_\ell \left(J_{\min}^{N^+,N^-Q,\vee} \right)_{\mathfrak{m}_Q},$$

modulo $\mathbb{Z}$-torsion on the right. Lemma 4.1.4 implies that there exist a collection of $\mathbb{T}_Q$-module maps

$$\phi_i : M_{Q,\mathfrak{m}_Q} \to \mathbb{T}_{Q,\mathfrak{m}_Q}, \;\; \psi_i : \mathbb{T}_{Q,\mathfrak{m}_Q} \to M_{Q_n,\mathfrak{m}_{Q_n}}, \;\; i = 1, \ldots, r$$

such that

$$\phi_i \circ \psi_i = \psi_i \circ \phi_i = t_i \in \mathbb{T}_{Q,\mathfrak{m}_Q} \subset \mathrm{End}(M_{Q_n,\mathfrak{m}_{Q_n}})$$

and

$$t_1 + \ldots + t_r = \prod_{\substack{\ell | N^- \\ \overline{T}_f \text{ unram at } \ell}} (\ell - 1) \in \mathbb{T}_{Q,\mathfrak{m}_Q}.$$

Since $\mathcal{O}$ is principal, we may choose some i such that the image of t_i in $\mathbb{T}_Q \otimes \mathcal{O}$ divides π^C. Then ϕ_i and ψ_i induce $\mathcal{O}$-module maps

$$M_Q \otimes \mathcal{O} \to \mathbb{T}_Q \otimes \mathcal{O} \to M_Q \otimes \mathcal{O}$$

whose composition in either direction is multiplication by a divisor of π^C, which implies the result. □

Theorem 5.1.5. *If $\{Qq, \epsilon_{Qq}\}$ is a weakly admissible pair, then*

$$\lg(\mathbb{T}_{Qq} \otimes_{\mathbb{T}^{Qq}} \mathcal{O}) \geq \lg\left(\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{a_q - \epsilon_q(q+1)} \right) - C,$$

where C is the number of Lemma 5.1.4.

Proof. The proof depends on the parity of $\nu(N^-Q)$.

Case 1. *$\nu(N^-Q)$ is even.*

Let us abbreviate $J^Q = J^{N^+,N^-Q}$ and $J^Q_{\min} = J^{N^+,N^-Q}_{\min}$. Consider the composite

$$M_{Qq} \to J^Q(\mathbb{F}_{q^2}) \to H^1\left(\mathbb{F}_q^2, T_{\mathfrak{m}_Q} J^Q\right)_{\mathfrak{m}_{Qq}} \simeq M_Q \otimes \frac{T_{\mathfrak{m}_Q} J^Q_{\min}}{(U_q - \epsilon_q)}$$

induced from the diagram of Proposition 4.4.6, the Kummer map, and Proposition 4.1.3. These are surjective maps of $\mathbb{T}^q_Q$-modules, where U_q acts on the three latter modules through Frob_q. Since $T_{\mathfrak{m}_Q} J^Q_{\min}$ is free of rank two over $\mathbb{T}_{Q,\mathfrak{m}_Q}$ and Frob_q acts with the characteristic polynomial $\mathrm{Frob}_q^2 - T_q \mathrm{Frob}_q + q$ (whose roots are distinct modulo $\mathfrak{m}_Q$), we may fix an identification

$$\frac{T_{\mathfrak{m}_Q} J^Q_{\min}}{U_q - \epsilon_q} \simeq \frac{\mathbb{T}_{Q,\mathfrak{m}_Q}}{T_q - \epsilon_q(q+1)},$$

considered as a $\mathbb{T}^q_{Q,\mathfrak{m}_{Qq}}$-module again through U_q acting by ϵ_q. Tensoring with $\mathcal{O}$, we obtain a surjective map

$$M_{Qq} \to M_Q \otimes_{\mathbb{T}^{Qq}} \frac{\mathcal{O}}{a_q - \epsilon_q(q+1)},$$

hence (by Lemma 5.1.4) a map of $\mathbb{T}^{Qq}$-modules $M_{Qq} \to \frac{\mathbb{T}_Q \otimes \mathcal{O}}{a_q - \epsilon_q(q+1)}$ with cokernel annihilated by π^C. Since the action of $\mathbb{T}^{Qq}$ on M_{Qq} factors through $\mathbb{T}_{Qq}$, we obtain, by taking eigenvalues, a surjection $\mathbb{T}_{Qq} \to \mathcal{O}/\pi^j$ for some

$$j \geq \lg\left(\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{a_q - \epsilon_q(q+1)}\right) - C.$$

Case 2. *$\nu(N^-Q)$ is odd.*

By Proposition 4.4.9, the action of $\mathbb{T}^q_{Q,\mathfrak{m}_{Qq}}$ on

$$M_{Q,\mathfrak{m}_Q} \otimes_{\mathbb{T}_Q} \left(\mathbb{T}_Q^2 / \operatorname{im}\begin{pmatrix} T_q - \epsilon_q & q \\ -1 & -\epsilon_q \end{pmatrix}\right),$$

with U_q acting by ϵ_q, factors through $\mathbb{T}_{Qq,\mathfrak{m}_{Qq}}$. Hence the action of $\mathbb{T}^q_Q$ on

$$A = M_Q \otimes_{\mathbb{T}^Q} \frac{\mathcal{O}}{a_q - \epsilon_q(q+1)}$$

likewise factors through $\mathbb{T}_{Qq}$ (again with U_q acting by ϵ_q). The conclusion of Lemma 5.1.4 implies that A has a $\mathbb{T}^q_Q$-module map to

$$\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{a_q - \epsilon_q(q)}$$

with cokernel annihilated by π^C, from which the result follows. $\square$

Remark 5.1.6. If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, then for $\mathfrak{F}$-many n there is a corresponding weakly admissible pair $\{Q_n, \epsilon_{Q_n}\}$, where Q_n is a sequence representing Q. To be precise, if $\mathsf{Q} = \{\mathsf{q}_1, \ldots, \mathsf{q}_r\}$, we choose sequences q_i^n representing each q_i; for $\mathfrak{F}$-many n, the product $Q_n = q_1^n \cdots q_r^n$, equipped with sign function $\epsilon_{Q_n}(q_i^n) = \epsilon_{\mathsf{Q}}(\mathsf{q}_i)$, forms a weakly admissible pair $\{Q_n, \epsilon_{Q_n}\}$. It follows from the definition of N and from the theorem that, for any $j \geq 0$, there exist $\mathfrak{F}$-many n such that $\mathbb{T}^{Q_n} \to \mathcal{O} \to \mathcal{O}/\pi^j$ factors through $\mathbb{T}_{Q_n}$. We say that a sequence of weakly admissible pairs $\{Q_n, \epsilon_{Q_n}\}$ (defined for $\mathfrak{F}$-many n) represents the pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ if it is obtained from this construction for some choice of representatives q_i^n.

5.2. **The CM class construction.**

5.2.1. Let $S \subset M_{\mathbb{Q}}$ be a finite set of primes, and fix an element

$$y \in \mathbb{Z}[\mathscr{C}^S)]^0;$$

let $K[y] \subset K[S]$ be its field of definition and $G_y = \mathrm{Gal}(K[y]/K)$ the corresponding Galois group. A weakly admissible prime q with sign ϵ_q is called j-admissible if $a_q \equiv \epsilon_q(q+1) \pmod{\pi^j}$; in this case, $T_j := T_f/\pi^j$ has a unique subspace $\mathrm{Fil}^+_{q,\epsilon_q} T_j$, free of rank one over $\mathcal{O}_j := \mathcal{O}/\pi^j$, on which Frob_q acts as $q\epsilon_q$. We will omit the subscript ϵ_q when there is no risk of confusion. A weakly admissible set $\{Q, \epsilon_Q\}$ is called j-admissible if $\lg(\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}) \geq j + 2C$; note that each $q|Q$ is then necessarily j-admissible (since $U_q = \epsilon_q$ in $\mathbb{T}_Q \otimes \mathcal{O}$). Let N_j be the collection of j-admissible sets.

For any j-admissible prime with sign ϵ_q, we define the ordinary subspace:

$$H^1_{\mathrm{ord},\epsilon_q}(K_q, T_j) = \mathrm{im}\left(H^1(K_q, \mathrm{Fil}^+_{q,\epsilon_q} T_j) \to H^1(K_q, T_j)\right). \tag{35}$$

Using the map obtained from Schapiro's Lemma (e.g. [47, §3.1.2])

$$\mathrm{Res}_q : H^1(K[y], T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_y, H^1(K_q, T_j)), \tag{36}$$

we also have maps:

$$\partial_{q,\epsilon_q} : H^1(K[y], T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_y, H^1(I_q, \mathrm{Fil}^+_q T_j)) \approx \mathcal{O}_j[G_y],$$
$$\mathrm{loc}_{q,\epsilon_q} : H^1(K[y]^\Sigma/K[y], T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_y, T_j/\mathrm{Fil}^+_q T_j), \approx \mathcal{O}_j[G_y], \quad q \notin \Sigma,$$

defined as in (15,16).

Construction 5.2.2. *If $\Sigma \subset M_{\mathbb{Q}}$ is the set of places dividing $Np\infty$, then for all $\{Q, \epsilon_Q\} \in N_j$, there exist principal sub-$\mathcal{O}_j$-modules:*

$$(\kappa_j(y, Q, \epsilon_Q)) \subset H^1(K[y]^{\Sigma\cup Q}/K[y], T_j), \qquad \nu(N^-Q) \text{ even},$$
$$(\lambda_j(y, Q, \epsilon_Q)) \subset \mathcal{O}_j[G_y], \qquad \nu(N^-Q) \text{ odd},$$

compatible under the natural reduction maps for $j' \le j$, and satisfying the following properties.

(1) If $\{Qq, \epsilon_Q\} \in N_j$ where $\nu(N^-Q)$ is even, then for all $q|Q$ and all $g \in G_y$,

$$\mathrm{Res}_q(\kappa_j(y, Q, \epsilon_Q))(g) \subset H^1_{\mathrm{ord},\epsilon_Q(q)}(K_{\mathfrak{q}}, T_j).$$

(2) If $\{Qq, \epsilon_{Qq}\}, \{Q, \epsilon_Q\} \in N_j$ where $\epsilon_Q = \epsilon_{Qq}|_Q$ and $\nu(N^-Qq)$ is even, then

$$\partial_{q,\epsilon_{Qq}(q)}(\kappa_j(y, Qq, \epsilon_{Qq})) = (\lambda_j(y, Q, \epsilon_Q)) \subset \mathcal{O}_j[G_y].$$

(3) If $\{Qq, \epsilon_{Qq}\}, \{Q, \epsilon_Q\} \in N_j$ where $\epsilon_Q = \epsilon_{Qq}|_Q$ and $\nu(N^-Qq)$ is odd, then

$$\mathrm{loc}_{q,\epsilon_{Qq}(q)}(\kappa_j(y, Q, \epsilon_Q)) = (\lambda_j(y, Qq, \epsilon_{Qq})) \subset \mathcal{O}_j[G_y].$$

Proof. The specifications ϵ_Q will be dropped to ease notation. Suppose first that $\nu(N^-Q)$ is odd. By Lemma 5.1.4, there is a unique map (up to scalars) $M_Q \to \mathcal{O}_j$ of $\mathbb{T}^Q$-modules that factors through multiplication by π^C and is surjective after $\mathcal{O}$-linearization. We define $\lambda_j(y, Q)(g)$ to be the image of $gy \in \mathbb{Z}[\mathscr{C}^S]^0$ by the composite $\mathbb{Z}[\mathscr{C}^S]^0 \to M_Q \to \mathcal{O}_j$.

Now suppose that $\nu(N^-Q)$ is even. Adopting the abbreviations of Case 1 of Theorem 5.1.5, $CM_{N^+,N^-Q}(y)$ is a formal divisor on $X_Q = X_{N^+,N^-Q}$ for each n, and its image in J^Q is defined over $K[y]$. Let

$$d(y, Q) \in H^1(K[y]^{\Sigma\cup Q}/K[y], T_{\mathfrak{m}_Q} J^Q)$$

be the Kummer image. By Lemma 5.1.4 and Proposition 4.1.3, there is a unique (up to scalars) map of $\mathbb{T}^Q[G_{\mathbb{Q}}]$-modules $T_{\mathfrak{m}_Q} J^Q \to T_j$ that is surjective after $\mathcal{O}$-linearization. We define $\kappa_j(y, Q)$ to be the image of $d(y, Q)$ under the induced map

$$H^1(K[y]^{\Sigma\cup Q}/K[y], T_{\mathfrak{m}_Q} J^Q) \to H^1(K[y]^{\Sigma\cup Q}/K[y], T_j).$$

Note that, for each $g \in G_y$, $\mathrm{Res}_q\, \kappa_j(y, Q)(g)$ is the local Kummer image of $g \cdot \mathrm{CM}_{N^+,N^-}(y) = \mathrm{CM}_{N^+,N^-}(gy)$ by Shimura's reciprocity law.

(1) This follows from the rigid analytic uniformization (33) by the argument of [20, p. 15]. Indeed, the argument there shows that the image of the Kummer map

$$J^{Qq}(\mathbb{Q}_{q^2}) \to H^1(\mathbb{Q}_{q^2}, T_{\mathfrak{m}_Q} J^Q)$$

agrees with the image of the map

$$H^1(\mathbb{Q}_{q^2}, \mathcal{X}(J^Q)(1)) \to H^1(\mathbb{Q}_{q_n^2}, T_{\mathfrak{m}_Q} J^Q),$$

where $\mathcal{X}(J^Q)(1) \to (T_{\mathfrak{m}_Q} J^Q)[\mathrm{Frob}_q - U_q q]$ is a canonical isomorphism. Since $q \nmid p - 1$, the surjection

$$T_{\mathfrak{m}_Q} J^Q \twoheadrightarrow T_j$$

induces

$$\mathcal{X}(J^Q)(1) \twoheadrightarrow \mathrm{Fil}^+_q T_j,$$

and the claim follows.

(2) We claim that the composite

$$J^{Qq}(\mathbb{Q}_{q^2}) \to H^1(\mathbb{Q}_{q^2}, T_{\mathfrak{m}_Q} J^Q) \to H^1(I_q, T_j)$$

factors through

$$\mathrm{Sp}_q : J^{Qq}(\mathbb{Q}_{q^2}) \to \Phi_{Q,\mathfrak{m}_Q}.$$

Indeed, the target of the composite map has Frobenius eigenvalue U_q, and, because $p \nmid q-1$, a diagram chase using (33) shows that the pro-p part of the kernel of Sp_q has Frobenius eigenvalue $-U_q$ (if it is nontrivial at all).

Using the commutativity of the diagram in Proposition 4.4.9 (and the fixed embedding $K[S] \hookrightarrow \mathbb{Q}_{q^2}$), $\partial_q(\kappa_j(y, \mathsf{Q}))(g)$ is therefore the image of gy under the composite of the canonical map

$$\mathbb{Z}[\mathscr{C}^S]^0 \to M_{Qq}$$

with *some* surjective map of Hecke modules $M_{Qq} \to \mathcal{O}_j$, which factors through multiplication by π^C by the choice of map $T_{\mathfrak{m}_Q} J^Q \to T_j$. We may conclude by Lemma 5.1.4.

(3) The proof is very similar to (2), invoking instead Proposition 4.4.6.

□

6. From CM classes to bipartite Euler systems

6.1. p-adic interpolation.

6.1.1. Suppose for this subsection that:

(spl) $$p \text{ splits in } K$$

and

(ord) $$a_p \notin \wp.$$

6.1.2. Fix an auxiliary prime $\ell_0 \nmid Np$ such that $a_{\ell_0} - \ell_0 - 1 \in \mathcal{O}^\times$ (one exists by the irreducibility of $\overline{T}_f$), and set $S = \{\ell_0, p\}$. For each $m \geq 0$, consider the point

$$y_{m,0} = \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q}_p) \subset \mathscr{C}^S;$$

note that $y_{m,0}$ is defined over $K[p^m]$. If T_{ℓ_0} is the usual adelic Hecke operator, then set $y_m = (T_{\ell_0} - \ell_0 - 1) \operatorname{Tr}_{K[p^m]/K_m} y_{m,0} \in \mathbb{Z}[\mathscr{C}^S]^0$, where K_m is the mth layer of the anticyclotomic $\mathbb{Z}_p$-extension.

6.1.3. Now suppose given any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$, and let $\{Q_n, \epsilon_{Q_n}\}$ be a representative sequence of weakly admissible pairs as in Remark 5.1.6. Since $T_p \notin \mathfrak{m}$, Hensel's Lemma implies that the Hecke algebras $\mathbb{T}_{Q_n, \mathfrak{m}_{Q_n}}$ contain a (unique) element $u \notin \mathfrak{m}_{Q_n}$ such that $u^2 - ua_p + p = 0$. Suppose first that $|\mathsf{Q}| + \nu(N^-)$ is even. By the usual Heegner point norm relations (cf. e.g. [13, Proposition 3.10]), the classes

$$d(y_m, Q_n)' \coloneqq u^{-m+1} d(y_m, Q_n) - u^{-m} \operatorname{Res}_{K_m/K_{m-1}} d_n(y_{m-1}, Q_n)$$

of Construction 5.2.2 are compatible under the corestriction maps

$$H^1(K_m, T_{\mathfrak{m}_{Q_n}} J^{Q_n}) \to H^1(K_{m-1}, T_{\mathfrak{m}_{Q_n}} J^{Q_n}).$$

If we replace $d(y_m, Q_n)$ by $d(y_m, Q_n)'$, then, for any j and for $\mathfrak{F}$-many n, we obtain classes $\kappa_j(y_m, Q_n)' \in \mathsf{H}^1(K_m^{\Sigma \cup Q_n}/K_m, T_j)$ that are compatible under corestriction (as long the choices in the construction are made compatibly as m varies, which is clearly possible). We let

$$\boldsymbol{\kappa}(\mathsf{Q}) \in \varprojlim_{m,j} \mathsf{H}^1(K_m^{\Sigma \cup \mathsf{Q}}/K_m, T_j) \simeq \mathsf{H}^1(K, T_f \otimes \Lambda(\Psi))$$

be the class represented by the family $\kappa_j(y_m, Q_n)'$.

6.1.4. Similarly, if $|\mathsf{Q}| + \nu(N^-)$ is odd, the elements

$$\lambda_j(y_m, Q_n)' \coloneqq \alpha_p^{-m+1}\lambda(y_m, Q_n) - \alpha_p^{-m+1}\lambda(y_{m-1}, Q_n) \in \mathcal{O}_j[\mathrm{Gal}(K_m/K)]$$

are compatible under the natural projection maps

$$\mathcal{O}_j[\mathrm{Gal}(K_m/K)] \to \mathcal{O}_j[\mathrm{Gal}(K_{m-1}/K)].$$

We then obtain an element

$$\boldsymbol{\lambda}(\mathsf{Q}) \in \varprojlim_{m,j} \mathcal{U}\left(\{\mathcal{O}_j[\mathrm{Gal}(K_m/K)]\}_{n\in\mathbb{N}}\right) \simeq \mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!] \simeq \Lambda.$$

Let $\mathsf{S} \subset \mathsf{M}_K$ be the set of constant ultraprimes $\underline{v}$ such that $v|Np\infty$. We define a Selmer structure $(\mathcal{F}_\Lambda, \mathsf{S})$ for $\mathbf{T} \coloneqq T_f \otimes \Lambda$ in the usual way (see e.g. [24, 6]):

$$\mathsf{H}^1_{\mathcal{F}_\Lambda}(K_\mathsf{v}, \mathbf{T}) = \begin{cases} \mathrm{im}\left(H^1(K_v, \mathrm{Fil}_v^+\mathbf{T}) \to H^1(K_v, \mathbf{T})\right), & \mathsf{v} = \underline{v}, v|p, \\ H^1(K_v, \mathbf{T}), & \mathsf{v} = \underline{v}, v\nmid p, \\ \mathsf{H}^1_{\mathrm{unr}}(K_\mathsf{v}, \mathbf{T}), & \text{otherwise.} \end{cases}$$

Here $\mathrm{Fil}_v^+\mathbf{T} \subset \mathbf{T}$ is the unique free, rank-one direct summand on which I_v acts by the cyclotomic character. By [25, proposition 3.3.1], $\mathcal{F}_\Lambda$ extends naturally to an interpolated self-dual Selmer structure $(\mathsf{S}, \mathcal{F}_\Lambda, \mathcal{F}_\mathfrak{P}, \Sigma_\Lambda)$ for $\mathbf{T}$.

Proposition 6.1.5. *The pair $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ is a nontrivial bipartite Euler system for the triple $(\mathbf{T}, \mathcal{F}_\Lambda, \mathsf{S})$.*

Proof. We first show that $\boldsymbol{\kappa}(\mathsf{Q})$ lies in $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})$ for all $\{\mathsf{Q}, \epsilon_\mathsf{Q}\} \in \mathsf{N}^{\nu(N^-)}$. The only local conditions to verify are those at $v|p$; the local conditions for $\mathsf{q} \in \mathsf{Q}$ follow from Construction 5.2.2(1), and the rest are trivial. If Q is represented by the sequence Q_n, let $\mathrm{Fil}_v^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}$ be the maximal $\mathbb{T}_{\mathfrak{m}_{Q_n}}$ submodule on which I_v acts by the cyclotomic character (adopting the notation of Construction 5.2.2 and if necessary restricting our attention to $\mathfrak{F}$-many n). As in [11, Proposition 4.7], it suffices to show that, for all m and n and a fixed extension of v to K_∞, the image $d_{n,m}$ of the class $d(y_m, Q_n)'$ under the composite

$$H^1(K_m, T_{\mathfrak{m}_{Q_n}} J^{Q_n}) \to H^1(K_{m,v}, T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n})$$

is trivial. Since $d_n(y_m)'$ is a $\mathbb{T}_\mathfrak{m}$-linear combination of Kummer images over K_m, by [4, Example 3.11] and [36, Proposition 12.5.8] $d_{n,m}$ lies in the kernel of

$$H^1(K_{m,v}, T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}) \to H^1(K_{m,v}, \mathbb{Q}_p \otimes T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}).$$

Since the classes $d_{n,m}$ are corestriction-compatible as m varies, the argument of [26, Proposition 2.4.5] shows that indeed $d_{n,m} = 0$ for all n, m.

The explicit reciprocity laws are a consequence of Construction 5.2.2(2,3), and the nonvanishing of either $\boldsymbol{\kappa}(1)$ or $\boldsymbol{\lambda}(1)$ (according to the parity of $\nu(N^-)$) is due to the work of Cornut [12] and Vatsal [50]. □

6.2. **Kolyvagin classes.**

6.2.1. Before defining the Kolyvagin classes in patched cohomology, we begin by recalling a calculation explained in [19].

Let m be a squarefree product of primes ℓ inert in K, and let

$$y(m)_0 \in \prod_{\ell|m} GL_2(\mathbb{Q}_\ell)$$

be the element with ℓth component $\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}$. If ℓ_0 is the auxiliary prime of the previous subsection, and $S\,\{\ell_0\} \cup \{\ell|m\}$, then we define

$$y(m) = (T_{\ell_0} - (\ell_0 + 1))y(m)_0 \in \mathbb{Z}[\mathscr{C}^S]^0.$$

6.2.2. Note that $K[y(m)] = K[m]$ and that $\mathrm{Gal}(K[m]/K[1]) \simeq \prod_{\ell|m} \mathrm{Gal}(K[\ell]/K[1])$; each $\mathrm{Gal}(K[\ell]/K[1])$ is cyclic of order $\ell+1$. For any place λ of $\overline{\mathbb{Q}}$ over ℓ, fixed for the time being, let $\mathrm{Frob}_\lambda \in G_{\mathbb{Q}}$ be a lift of absolute Frobenius, and $\sigma_\lambda \in I_\lambda \subset G_K$ a generator of $\mathrm{Gal}(K[\ell]/K[1])$. Recall the Kolyvagin derivative operators [19]:

$$D_\ell = \sum_{i=1}^{\ell} i\sigma_\lambda^i \in \mathbb{Z}[\mathrm{Gal}(K[\ell]/K[1])], \quad D_m = \prod_{\ell|m} D_\ell.$$

Let

$$P(m) = D_m y(m) \in \mathbb{Z}[\mathscr{C}^S]^0.$$

Finally, let Q be a set of primes inert in K that is disjoint from S, p, and the factors of N, and let $P(m,Q) = \mathrm{CM}_{N^+,N^-Q}(P(m))$, $y(m,Q) = \mathrm{CM}_{N^+,N^-Q}(y(m))$.

Proposition 6.2.3 ([19], Proposition 3.7)**.** *For all $\ell|m$, we have:*

(1) $(\sigma_\ell - 1)P(m) = (\ell+1)D_{m/\ell}y(m) - T_\ell P(m/\ell)$.

(2) Suppose $\nu(N^-Q)$ is even. If λ lies over ℓ, then

$$D_{m/\ell}y(m,Q) \equiv \mathrm{Frob}_\lambda P(m/\ell,Q) \pmod{\lambda}.$$

Proposition 6.2.4. *Suppose $\mathfrak{m}_Q \subset \mathbb{T}_Q = \mathbb{T}_{N^+,N^-Q}$ is a maximal ideal whose associated residual representation has no $G_{K[m]}$-fixed points, and let $I_m \subset \mathbb{T}_Q$ be the ideal generated by $\ell+1$ and T_ℓ for all $\ell|m$. Then if $\nu(N^-Q)$ is even:*

(1) Restriction induces an isomorphism

$$\mathrm{Res}_m : H^1(K[1], T_{\mathfrak{m}_Q}J^Q/I_m) \xrightarrow{\sim} H^1(K[m], T_{\mathfrak{m}_Q}J^Q/I_m)^{\mathrm{Gal}(K[m]/K[1])}.$$

(2) The Kummer image $d'(m,Q)$ of $P(m,Q)$ in $H^1(K[m], T_{\mathfrak{m}_Q}J^Q/I_m)$ lies in the image of Res_m.

(3) If $c(m,Q) = \mathrm{Cores}_{K[1]/K}\mathrm{Res}_m^{-1} d'(m,Q)$, then for all $\ell|m$ and any choices of representatives,

$$c(m,Q)(\sigma_\lambda) = \mathrm{Frob}_\lambda^{-1} d'(m/\ell,Q)(\mathrm{Frob}_\lambda^2) \pmod{I}_m.$$

(4) The class $c(m,Q)$ is unramified at any place $v \nmid NpmQ\infty$.

Proof. (1) follows from the inflation restriction exact sequence as in [19], and (2) is immediate from Proposition 6.2.3. Also (4) is clear from the construction. For (3), it suffices to check the corresponding statement for $c'(m,Q) = \mathrm{Res}_m^{-1} d'(m,Q)$. The proof is a modification of the argument in [34]. Fix division points $\frac{P(m,Q)}{\ell+1}$ and $\frac{P(m/\ell,Q)}{\ell+1}$; one may verify that $c'(m,Q)(\sigma_\ell)$ is the unique element $A \in T_{\mathfrak{m}_Q}J^Q/I_m$ such that, for all $g \in G_{K[m/\ell]}$,

$$(g-1)A \equiv (g-1)(\widetilde{\sigma}_\ell - 1)\frac{P(m,Q)}{\ell+1} \in T_{\mathfrak{m}_Q}J^Q/I_m.$$

By Proposition 6.2.3(1), A is also the image of the (unique) point $T \in J^Q[\ell+1]$ such that

$$T \equiv D_{m/\ell}\,\mathrm{CM}_{N^+,N^-Q}(P(m,Q)) - T_\ell\frac{P(m/\ell,Q)}{\ell+1} \pmod{\lambda}.$$

But by Proposition 6.2.3(2), this is equivalent to

$$T \equiv \mathrm{Frob}_\lambda^{-1} P(m/\ell,Q) - T_\ell\frac{P(m/\ell,Q)}{\ell+1} \pmod{\lambda}. \tag{37}$$

By the Eichler-Shimura relation, and the fact that ℓ splits completely in $K[m/\ell]$, the image of T in $T_{\mathfrak{m}_Q}J^Q/I_m$ is precisely

$$\mathrm{Frob}_\lambda^{-1} d'(m/\ell,Q)(\mathrm{Frob}_\lambda^2).$$

□

Definition 6.2.5. For a squarefree product m of primes inert in K, let $I_m(f) \subset \mathcal{O}$ be the ideal generated by $a_\ell(f)$ and $\ell+1$ for all $\ell|m$. Suppose $\{Q,\epsilon_Q\} \in \mathcal{N}_j$ is j-admissible and $j \geq v_\wp(I_m(f))$. If $\nu(N^-Q)$ is even, then the Kolyvagin class

$$\overline{c}(m,Q) \in H^1(K^{\Sigma\cup Q\cup m}/K, T_f/I_m(f)) \tag{38}$$

is defined to be the image of $c(m,Q)$. If $\nu(N^-Q)$ is odd, then the reduction of $\lambda(P(m),Q)$ in

$$(\mathcal{O}/I_m)[\mathrm{Gal}(K[m]/K)]$$

is constant on cosets of $\mathrm{Gal}(K[m]/K[1])$ by Proposition 6.2.3(1) and therefore descends to

$$\lambda'(m,Q) \in (\mathcal{O}/I_m)[\mathrm{Gal}(K[1]/K)]. \tag{39}$$

The Kolyvagin element is then defined as:

$$\lambda(m,Q) = \mathrm{tr}_{K[1]/K}\, \lambda'(P(m),Q) \in \mathcal{O}/I_m. \tag{40}$$

Remark 6.2.6. When $Q = 1$ and $\nu(N^-)$ is even, this agrees with Kolyvagin's construction [31].

For applications to the parity conjecture for f, we will require the following:

Proposition 6.2.7. *If $\nu(N^-)$ is even, then $\bar{c}(m,1)$ lies in the $\epsilon_f \cdot (-1)^{\nu(m)+1}$-eigenspace for the action of τ. If $\nu(N^-)$ is odd and $\lambda(m,1) \neq 0$, then $\epsilon_f = (-1)^{\nu(m)}$.*

Proof. The maps $T_{\mathfrak{m}} J^{N^+,N^-} \to T_f/\pi^j$ or $\mathbb{Z}[X_{N^+,N^-}]^0 \to \mathcal{O}/\pi^j$ used in Construction 5.2.2 are equivariant for the action of the Atkin-Lehner involution because of the uniqueness property derived from Lemma 5.1.4. (Note this is not necessarily true of the corresponding maps at level N^+N^-Q, which are not necessarily reductions of genuine modular parameterizations.) Since the Atkin-Lehner eigenvalue of f is $-\epsilon_f$, the proposition follows exactly as in [19, Proposition 5.4]. □

Definition 6.2.8. An ultraprime l is called Kolyvagin-admissible if

$$\mathrm{Frob}_{\mathsf{l}} \in \mathrm{Gal}(K(T_f)/\mathbb{Q})$$

is a complex conjugation. A Kolyvagin-admissible set is a finite set of Kolyvagin-admissible ultraprimes, and the collection of all Kolyvagin-admissible sets is denoted K.

6.2.9. If l is Kolyvagin-admissible, then the local cohomology

$$\mathsf{H}^1(K_{\mathsf{l}}, T_f)$$

is free of rank four over $\mathcal{O}$, and carries a natural action of the complex conjugation $\tau \in \mathrm{Gal}(K/\mathbb{Q})$. It has a canonical splitting of the finite-singular exact sequence:

$$\mathsf{H}^1(K_{\mathsf{l}}, T_f) = \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f) \oplus \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f),$$

defined as follows. If the sequence ℓ_n represents l, then for any j and for $\mathfrak{F}$-many n, Frob_{ℓ_n} acts as complex conjugation on T_f/π^j, and

$$H^1_{\mathrm{tr}}(K_{\ell_n}, T_f/\pi^j) = \ker\left(H^1(K_{\ell_n}, T_f/\pi^j) \to H^1(K[\ell_n]_{\lambda_n}, T_f/\pi^j)\right)$$

is isomorphic to $H^1(I_{\ell_n}, T_f/\pi^j)^{\mathrm{Frob}_\ell^2 = 1}$, where λ_n is a prime of $K[\ell_n]$ over ℓ_n. Then

$$H^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f) = \varprojlim \mathcal{U}\left(\left\{H^1_{\mathrm{tr}}(K_{\ell_n}, T_f/\pi^j)\right\}_{n\in\mathbb{N}}\right) \subset \mathsf{H}^1(K_{\mathsf{l}}, T_f)$$

is our transverse subspace. We denote by $\mathrm{loc}_{\mathsf{l}}^{\pm}$ and $\partial_{\mathsf{l}}^{\pm}$ the composites $\mathsf{H}^1(K, T_f) \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f)^{\pm}$ and $\mathsf{H}^1(K, T_f) \to \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f)^{\pm}$, respectively, where $\pm$ is the Frobenius eigenvalue. The codomain of each is free of rank one over $\mathcal{O}$.

Let $\mathsf{S} \subset \mathsf{M}_K$ be the set of constant ultraprimes $\underline{v}$ such that $v \nmid Np\infty$. We will consider the Kolyvagin-transverse Selmer structure $(\mathcal{F}(\mathsf{m}), \mathsf{S} \cup \mathsf{m})$ on T_f, for any $\mathsf{m} \in \mathsf{K}$:

$$\mathsf{H}^1_{\mathcal{F}(\mathsf{m})}(K_{\mathsf{v}}, T_f) = \begin{cases} \mathrm{im}\left(A(K_v) \otimes_{\mathcal{O}_f} \mathcal{O} \to H^1(K_v, T_f)\right), & \mathsf{v} = \underline{v}, \\ \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f), & \mathsf{v} = \mathsf{l} \in \mathsf{m}, \\ \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{v}}, T_f), & \text{otherwise.} \end{cases} \tag{41}$$

Here A is an (optimal) abelian variety with real multiplication by $\mathcal{O}_f$ associated to f. If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}$, then we denote by $(\mathcal{F}(\mathsf{m}, \mathsf{Q}), \mathsf{S} \cup \mathsf{m} \cup \mathsf{Q})$ the modified Selmer structure of (3.2.2).

6.2.10. Let $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}^{\nu(N^-)}$, and fix representatives Q_n and m_n, which we may assume to be disjoint. Our patched Kolyvagin class is the element

$$\kappa(\mathsf{m}, \mathsf{Q}) \in \mathsf{H}^1(K^{\mathsf{S}\cup\mathsf{m}\cup\mathsf{Q}}/K, T_f)$$

whose image in T_j is represented by the sequence of images of the classes $\overline{c}(m_n, Q_n)$, well-defined for $\mathfrak{F}$-many n.

If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$, then we similarly set

$$\lambda(\mathsf{m}, \mathsf{Q}) \in \mathcal{O} \simeq \varprojlim \mathcal{U}\left(\{\mathcal{O}/\pi^j\}\right)$$

to be the element whose image in $\mathcal{O}/\pi^j$ is represented by the sequence $\lambda(m_n, Q_n)$.

Proposition 6.2.11. *For any* $\mathsf{m} \in \mathsf{K}$ *and* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}^{\nu(N^-)}$*,*

$$(\kappa(\mathsf{m}, \mathsf{Q})) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f).$$

Moreover:

(1) *For all* $\mathsf{l} \in \mathsf{m}$*,*

$$(\mathrm{loc}_{\mathsf{l}}^{\pm}(\kappa(\mathsf{m}/\mathsf{l}, \mathsf{Q}))) = (\partial_{\mathsf{l}}^{\mp}(\kappa(\mathsf{m}, \mathsf{Q}))$$

as submodules of $\mathcal{O}$*.*

(2) *For all* $\mathsf{q} \in \mathsf{Q}$*,*

$$(\partial_{\mathsf{q}}(\kappa(\mathsf{m}, \mathsf{Q}))) = (\lambda(\mathsf{m}, \mathsf{Q}/\mathsf{q}))$$

as submodules of $\mathcal{O}$*.*

(3) *For all* $\mathsf{q} \notin \mathsf{Q}$*, admissible with sign* ϵ_{q}*,*

$$(\mathrm{loc}_{\mathsf{q}}(\kappa(\mathsf{m}, \mathsf{Q}))) = (\lambda(\mathsf{m}, \mathsf{Q}\mathsf{q}))$$

as submodules of $\mathcal{O}$*.*

In particular, for any fixed m*,* $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ *forms a bipartite Euler system with sign* $\nu(N^-)$ *for the triple* $(T_f, \mathcal{F}(\mathsf{m}), \mathsf{S} \cup \mathsf{m})$*.*

Proof. We verify the local conditions for each $\mathsf{v} \in \mathsf{S} \cup \mathsf{m} \cup \mathsf{Q}$. If $\mathsf{v} = \underline{v}$ for a prime $v | N\infty$, then the local condition is all of $H^1(K_v, T_f)$, so there is nothing to show. If $v|p$, then we show that, for all j, the image c_j of $\mathrm{Res}_v\, \kappa(\mathsf{m}, \mathsf{Q})$ in $H^1(K_v, T_f/\pi^j)$ is a Kummer image. Recalling the notation used to construct $\kappa(\mathsf{m}, \mathsf{Q})$, the proof of [20, Lemma 7] implies that:

$$\delta_v(A(K_v)) = H^1_{\mathrm{fl}}(K_v, T_f/\pi^j) \subset H^1(K_v, T_f/\pi^j)$$
$$\kappa_v(J^{Q_n}(K[m_n]_v)) = H^1_{\mathrm{fl}}(K[m_n]_v, J^{Q_n}[p^M]) \subset H^1(K[m_n]_v, J^{Q_n}[p^M])$$

where we have extended v to a place of $K[n]$. For $\mathfrak{F}$-many n, the restriction of c_j to $K[m_n]_v$ is the image of a Kummer class in $H^1(K[m_n]_v, J^{Q_n}[p^M])$ by a map of Galois representations $J^{Q_n}[p^M] \to T_f/\pi^j \simeq A[\pi^j]$, which extends to a map of finite flat group schemes. As a consequence, the image of c_j in $H^1(K_v, A)$ is inflated from a class in $H^1(K[m_n]_v/K_v, A)$, which is trivial by [35, Proposition I.3.8]. (This argument is essentially [19, Proposition 6.2(1)].)

If $\mathsf{v} = \mathsf{l} | \mathsf{m}$, then, adopting as well the notation of (6.2.1), the class $c(P(m_n), Q_n)$ is zero when restricted to $K[m_n]_{\lambda_n}$ because $D_{\ell_n} = \ell_n(\ell_n + 1)$ on $\mathbb{F}_{\lambda_n}$; hence $\mathrm{Res}_{\mathsf{v}}\, \kappa(\mathsf{m}, \mathsf{Q}) \in \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f)$. The local conditions at $\mathsf{q} \in \mathsf{Q}$ are satisfied because every factor of Q_n splits completely in $K[m_n]$; for the same reason, (2, 3) follow from Construction 5.2.2(2, 3). (Note that the projection step in Definition 6.2.5 makes no difference to these identities.) Moreover (1) is clear from Proposition 6.2.4(3). □

Remark 6.2.12. The Euler system $(\kappa(1, \cdot), \lambda(1, \cdot))$ may be viewed as a specialization of $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$. Indeed, by the usual Heegner point norm relations [13, Proposition 3.10], if p splits in K, $\mathbb{1}(\boldsymbol{\lambda}(\mathsf{Q})) = (\alpha_p - 1)^2(\lambda(1, \mathsf{Q}))$ and $\mathbb{1}(\boldsymbol{\kappa}(\mathsf{Q})) = (\alpha_p - 1)^2(\kappa(1, \mathsf{Q}))$ when $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$, respectively. (Here $\mathbb{1} : \Lambda \to \mathcal{O}$ is specialization at the trivial character.)

7. Deformation theory

Theorem 5.1.5 allows us to produce weak eigenforms (i.e. ring maps) $\mathbb{T}_{N^+,N^-Q} \to \mathcal{O}/\pi^j$ for arbitrarily large j, simply by requiring sufficiently deep congruence conditions on all $q|Q$. However, in general these maps do not lift to characteristic zero. To prove the main results, we also need to be able to $\wp$-adically approximate f by genuine level-raised newforms. In this section, we provide this input via the relative deformation theory of Fakhruddin-Khare-Patrikis [17].

7.1. **Patched adjoint Selmer groups.**

7.1.1. Continuing to fix notation as in (3.1.1) and (5.1.1), we now assume moreover that f is non-CM. Consider the (irreducible) adjoint representation

$$L = \mathrm{ad}^0 T_f$$

and its $\mathcal{O}$-dual, $L^\dagger \simeq L(1)$, and let $\overline{L}$ and $\overline{L}^* \simeq L^\dagger/\pi$ be the associated residual representations. For all $v|Np\infty$, a choice of framing for T_f defines a smooth point of the generic fiber of an appropriate framed universal deformation ring (of fixed determinant, and fixed Hodge type if $v|p$) by [1, Theorem D]. Taking this smooth point as the input, the construction of [17, Proposition 4.7] yields, for all j, certain orthogonal local conditions

$$H^1_{\mathcal{S}}(\mathbb{Q}_v, L/\pi^j) \subset H^1(\mathbb{Q}_v, L/\pi^j), \;\; H^1_{\mathcal{S}^*}(\mathbb{Q}_v, L^*[\pi^j]) \subset H^1(\mathbb{Q}_v, L^*[\pi^j]).$$

By [17, Lemma 6.1], taking inverse limits yields dual local conditions

$$H^1_{\mathcal{S}}(\mathbb{Q}_v, L) \subset H^1(\mathbb{Q}_v, L), \;\; H^1_{\mathcal{S}^\dagger}(\mathbb{Q}_v, L^\dagger) \subset H^1(\mathbb{Q}_v, L^\dagger).$$

We use these to define generalized Selmer structures $(\mathcal{S}, \mathsf{S})$ and $(\mathcal{S}^\dagger, \mathsf{S})$ for L and $L^\dagger$, where $\mathsf{S} \subset \mathsf{M}_{\mathbb{Q}}$ is the set of constant ultraprimes $\underline{v}$ for $v|Np\infty$.

7.1.2. Now suppose that q is an admissible ultraprime with sign ϵ_{q}. Using the exact sequence of $\mathcal{O}[G_{\mathsf{q}}]$-modules in Definition 3.1.2,

$$0 \to \mathrm{Fil}^+_{\mathsf{q}} T_f \to T_f \to T_f/\mathrm{Fil}^+_{\mathsf{q}} T_f \to 0,$$

we define

$$\mathrm{Fil}^+_{\mathsf{q}} L = \mathrm{Hom}(T_f/\mathrm{Fil}^+_{\mathsf{q}} T_f, \mathrm{Fil}^+_{\mathsf{q}} T_f) \subset L$$

and

$$\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) = \mathrm{im}\left(\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, \mathrm{Fil}^+_{\mathsf{q}} L) \to \mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)\right).$$

We also define $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)$ as the orthogonal complement of $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L)$ under the local Tate pairing; note that, since $\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)$ is torsion-free, $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)$ and $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L)$ are exact annihilators. We will require the restriction maps

$$\begin{aligned} \mathrm{loc}_{\mathsf{q}} : \mathsf{H}^1(\mathbb{Q}, L) &\to \frac{\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}, \\ \mathrm{loc}^\dagger_{\mathsf{q}} : \mathsf{H}^1(\mathbb{Q}, L^\dagger) &\to \frac{\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}. \end{aligned} \tag{42}$$

Analogously, if $q \in M_{\mathbb{Q}}$ is j-admissible with sign ϵ_q, then we may define $H^1_{\mathrm{ord}}(\mathbb{Q}_q, L/\pi^j)$, $H^1_{\mathrm{ord}}(\mathbb{Q}_q, L^\dagger/\pi^j)$, and the localization maps loc_q, $\mathrm{loc}^\dagger_q$.

7.1.3. For any $\{\mathsf{PQR}, \epsilon_{\mathsf{PQR}}\} \in \mathsf{N}$, define the modified Selmer structure $(\mathcal{S}(\mathsf{Q}), \mathsf{S} \cup \mathsf{Q})$ for L:

$$\mathsf{H}^1_{\mathcal{S}^{\mathsf{P}}_{\mathsf{R}}(\mathsf{Q})}(\mathbb{Q}_{\mathsf{v}}, L) = \begin{cases} \mathsf{H}^1_{\mathcal{S}}(\mathbb{Q}_{\mathsf{v}}, L), & \mathsf{v} \notin \mathsf{PQR} \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{Q}, \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) + \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{P}, \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{R}. \end{cases} \tag{43}$$

The corresponding dual Selmer structure for $L^\dagger$ will be written $\mathcal{S}_{\mathsf{P}}^{\mathsf{R},\dagger}(\mathsf{Q})$. Finally, define, for any finite set of places Σ containing all $v|Np\infty$:

$$\text{Ш}^1_\Sigma(\overline{L}^*) = \ker\left(H^1(\mathbb{Q}^\Sigma/\mathbb{Q},\overline{L}^*) \to \prod_{v\in\Sigma} H^1(\mathbb{Q}_v,\overline{L}^*)\right). \tag{44}$$

Proposition 7.1.4. *For all $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}\in\mathsf{N}$,*

$$d_{\mathsf{Q}} := \mathrm{rk}_{\mathcal{O}}\,\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L) = \mathrm{rk}_{\mathcal{O}}\,\mathrm{Sel}_{\mathcal{S}^\dagger(\mathsf{Q})}(L^\dagger).$$

Proof. It follows from the construction [17, Proposition 4.7] that

$$\mathrm{rk}_{\mathcal{O}}\,H^1_{\mathcal{S}}(\mathbb{Q}_v,L) = \mathrm{rk}_{\mathcal{O}}\,H^1_{\mathrm{unr}}(\mathbb{Q}_v,L)$$

for all $v|N$ and

$$\mathrm{rk}_{\mathcal{O}}\,H^1_{\mathcal{S}}(\mathbb{Q}_v,L) = \mathrm{rk}_{\mathcal{O}}\,H^0(\mathbb{Q}_v,L)+2$$

if $v|p$. Since

$$\mathrm{rk}_{\mathcal{O}}\,\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}},L) = \mathrm{rk}_{\mathcal{O}}\,\mathsf{H}^0(\mathbb{Q}_{\mathsf{q}},L) = 1$$

for all $\mathsf{q}\in\mathsf{Q}$, the claim results from Proposition 2.7.4. □

The "relative deformation theory" developed in [17] may be summarized (for our context) as follows.

Theorem 7.1.5 (Fakhruddin-Khare-Patrikis, Kisin)**.** *Suppose given $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}\in\mathsf{N}$ such that $d_{\mathsf{Q}}=0$, and a finite set of places Σ containing all $v|Np\infty$ such that $\text{Ш}^1_\Sigma(\mathbb{Q},\overline{L}^*)=0$. Fix a sequence $\{Q_n,\epsilon_{Q_n}\}$ of weakly admissible pairs representing $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}$ and an integer $j\geq 0$. Then there is a sequence (defined for $\mathfrak{F}$-many n) of newforms g_n of weight two, level NQ_n, and trivial nebentypus, with a prime $\wp_{g_n}$ of the ring of integers of its coefficient field $\mathcal{O}_{g_n}$, such that:*

- *The completion $\mathcal{O}_{g_n,\wp_{g_n}}$ is isomorphic to $\mathcal{O}$.*
- *The associated Galois representations satisfy $T_f|_{I_\ell}\simeq T_{g_n}|_{I_\ell}$ for all $\ell\nmid Q_n$, and $\rho_{g_n}|_{I_{q_n}}$ is a Steinberg representation twisted by the unramified character $\mathrm{Frob}_{q_n}\mapsto\epsilon_{Q_n}(q_n)$ for all $q_n|Q_n$.*
- *For any fixed j, there is a congruence of Galois representations (in some basis)*
$$T_f\equiv T_{g_n}\pmod{\pi^j}$$
for $\mathfrak{F}$-many n. In particular, the maps
$$\mathbb{T}_{Q_n}=\mathbb{T}_{N^+,N^-Q_n}\to\mathcal{O}/\pi^j$$
of Remark 5.1.6 admit $\mathcal{O}$-valued lifts for $\mathfrak{F}$-many n.

Proof. Since $d_{\mathsf{Q}}=0$, Proposition 2.5.5 implies that there exists some $k\geq 0$ such that the natural maps

$$\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L/\pi^k)\to\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(\overline{L}),\;\;\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L^\dagger/\pi^k)\to\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(\overline{L}^*) \tag{45}$$

are identically zero.

Now, for any j and for $\mathfrak{F}$-many n, we have the weakly admissible pair $\{Q_n,\epsilon_{Q_n}\}\in N_j$ of Remark 5.1.6. Consider the (non-patched) Selmer groups

$$\mathrm{Sel}_{Q_n}(L/\pi^k)=\ker\left(H^1(\mathbb{Q}^{S\cup Q_n},L/\pi^k)\to\prod_{v|Np\infty}\frac{H^1(\mathbb{Q}_v,L/\pi^k)}{H^1_{\mathcal{S}}(\mathbb{Q}_v,L/\pi^k)}\times\prod_{q_n|Q_n}\frac{H^1(\mathbb{Q}_{q_n},L/\pi^k)}{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_n},L/\pi^k)}\right),$$

where S is the set of places dividing $NpQ_n\infty$. For each $q_n|Q_n$, $H^1_{\mathrm{ord}}(\mathbb{Q}_{q_n},L/\pi^k)$ is exactly the local condition obtained from the smooth, Steinberg-with-sign-$\epsilon_{Q_n}(q_n)$ quotient of the framed local deformation ring at q_n via [17, Proposition 4.7] (as long as q_n is k-admissible). There are also dual Selmer groups $\mathrm{Sel}_{Q_n}(L^\dagger/\pi^k)=\mathrm{Sel}_{Q_n}(L^*[\pi^k])$ defined in the same way (see Proposition 2.7.3 for the equality). By (45), for $\mathfrak{F}$-many n, the maps

$$\mathrm{Sel}_{Q_n}(L/\pi^k)\to\mathrm{Sel}_{Q_n}(\overline{L}),\;\;\mathrm{Sel}_{Q_n}(L^*[\wp^k])\to\mathrm{Sel}_{Q_n}(\overline{L}^*)$$

are identically zero. Moreover, for all n,

$$\text{Ш}^1_{\Sigma\cup Q_n}\subset\text{Ш}^1_\Sigma=0.$$

The proof of [17, Claim 6.12] now implies that, for $j \geq 3k$ large enough in a manner depending on the local representations $\rho_f|_{G_{\mathbb{Q}_\ell}}$ for $\ell|Np$ and for $\mathfrak{F}$-many n, there exists a representation

$$\tau_n : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$$

such that:

- $\tau_n \equiv \rho_f \pmod{\pi^j}$ (for some choice of basis of T_f);
- $\det \tau_n = \chi$;
- the local representations $\tau_n|_{G_{\mathbb{Q}_\ell}}$ lie on the same irreducible component of the framed local deformation ring as $\rho_f|_{G_{\mathbb{Q}_\ell}}$ if $\ell \nmid Q_n$, and are Steinberg representations twisted by the unramified character $\mathrm{Frob}_{q_n} \mapsto \epsilon_{Q_n}(q_n)$ for all $q_n|Q_n$.

It remains to apply a modularity lifting theorem to conclude that τ_n arises from a suitable modular form g_n. We claim that $\overline{T}_f|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible: suppose otherwise for contradiction. Then the image G of the action of $G_{\mathbb{Q}}$ on $\overline{T}_f$ fixes a pair of lines. Since the image of the inertia group I_p contains a matrix whose square has distinct eigenvalues by (TW), the pair of lines must be the eigenspaces of I_p. But I_p surjects onto $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, so by assumption no element of $G_{\mathbb{Q}}$ interchanges the two eigenspaces of I_p, and $\overline{T}_f$ is reducible – a contradiction. Thus the Taylor-Wiles hypothesis in Kisin's result [29] is satisfied, and τ_n indeed arises from a newform g_n. □

7.2. **Annihilating two Selmer groups.**

7.2.1. In order to apply Theorem 7.1.5, we must make a suitable choice of $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$. In this subsection, we show that such a choice as possible.

Proposition 7.2.2. *There exists a finite set of places Σ, containing all $v|Np\infty$, such that*

$$Ш^1_\Sigma = 0.$$

Proof. It suffices to show that

$$H^1(\mathbb{Q}(\overline{L}^*)/\mathbb{Q}, \overline{L}^*) = 0, \tag{46}$$

for if so, for any $0 \neq c \in H^1(\mathbb{Q}, \overline{L}^*)$, c restricts to a nonzero homomorphism $G_{\mathbb{Q}(\overline{L}^*)} \to \overline{L}^*$, and there exist primes which are totally split in $\mathbb{Q}(\overline{L}^*)$ but not in the extension cut out by the restriction of c.

We now show (46). If $\mu_p \not\subset \mathbb{Q}(\overline{L})$, then the center of $\mathrm{Gal}(\mathbb{Q}(\overline{L}^*)/\mathbb{Q})$ contains elements that act by nontrivial scalars on $\overline{L}^*$, and (46) follows from inflation-restriction. So suppose that $\mu_p \subset \mathbb{Q}(\overline{L})$; then the projective image $\overline{G} = \mathrm{Gal}(\mathbb{Q}(\overline{L})/\mathbb{Q})$ of the (irreducible) Galois action on $\overline{T}_f$ has a cyclic quotient of order $p-1$, and a classical result of Dickson implies that $p = 3$ and $\overline{G}$ is either a dihedral group, or S_4. In the former case, the order of $\mathrm{Gal}(\mathbb{Q}(\overline{L}^*)/\mathbb{Q})$ is prime to p, so (46) still holds. We are left to consider the case $\overline{G} = S_4$ and $p = 3$. Let $G = \mathrm{Gal}(\mathbb{Q}(\overline{T}_f)/\mathbb{Q})$ be the image of the Galois action; since we have assumed that $\det : G \to \mathbb{F}_3^\times$ factors through $\overline{G}$, a complex conjugation c in G projects to a transposition in $\overline{G}$. Let $\overline{H} \subset \overline{G}$ be a copy of S_3 containing the image of c, and H the normalizer of its preimage in G, which is contained in a unique Borel subgroup B. Let N be the unipotent radical of $B \cap G$. To prove (46), it suffices to check that

$$H^1(H, \overline{L}^*) = H^1(N, \overline{L}^*)^{H/N} = 0.$$

This holds because $\mathrm{im}(N-1)$ is isomorphic to the subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \overline{L}^*$, while $c \in H$ acts on N by -1 and on $\overline{L}^*/\,\mathrm{im}(N-1)$ by 1. □

We now require a more elaborate version of Theorem 3.3.4; the proof is inspired by [8], and begins with a series of lemmas.

Lemma 7.2.3. *There exists an integer j that, for all $n \geq 0$,*

$$\pi^j H^1(K(T_f)/\mathbb{Q}, L/\pi^n) = \pi^j H^1(K(T_f)/\mathbb{Q}, L^\dagger/\pi^n) = 0.$$

Proof. Let $E = \mathbb{Q}(\mu_{p^\infty}) \subset K(T_f)$, and note that L and $L^\dagger$ are isomorphic G_E-modules. Since f is non-CM, $(L \otimes \mathbb{Q}_p)^{G_E} = 0$, and so $(L/\pi^n)^{G_E}$ is uniformly bounded in n.

The pro-p-Sylow subgroup of $\mathrm{Gal}(K(T_f)/E)$ is a compact p-adic Lie group with semisimple Lie algebra; hence, by [17, Lemma B.1], $H^1(K(T_f)/E, L/\pi^n)$ is uniformly bounded in n.

Now, by inflation-restriction, we have exact sequences

$$
\begin{aligned}
&0 \to H^1(E/\mathbb{Q}, (L/\pi^n)^{G_E}) \to H^1(K(T_f)/\mathbb{Q}, L/\pi^n) \to H^1(K(T_f)/E, L/\pi^n),\\
&0 \to H^1(E/\mathbb{Q}, (L^\dagger/\pi^n)^{G_E}) \to H^1(K(T_f)/\mathbb{Q}, L^\dagger/\pi^n) \to H^1(K(T_f)/E, L^\dagger/\pi^n),
\end{aligned} \tag{47}
$$

where the outer terms are isomorphic and uniformly bounded in n; the lemma follows. □

For the next lemma, we abbreviate $L_m \coloneqq L/\pi^m$, $L_m^\dagger \coloneqq L^\dagger/\pi^m$, and $T_m \coloneqq T_f/\pi^m$. Moreover, if $y \in M$ for any torsion $\mathcal{O}$-module M, let $\operatorname{ord}(y)$ be the smallest integer $t \geq 0$ such that $\pi^t y = 0$.

Lemma 7.2.4. *There is a global constant C, depending on T_f, with the following property. Given cocycles $\phi \in H^1(\mathbb{Q}, L_m)$, $\psi \in H^1(\mathbb{Q}, L_m^\dagger)$, and $c_1, c_2 \in H^1(K, T_m)^\delta$ for some $\delta = \pm 1$, there exist infinitely many primes $q \nmid Np$ such that all the cocycles are unramified at q and:*

- *The Frobenius of q in $\operatorname{Gal}(K(T_m)/\mathbb{Q})$ is a complex conjugation; in particular, q is m-admissible with sign δ.*
- $\operatorname{ord}\operatorname{loc}_q \phi \geq \operatorname{ord}\phi - C$.
- $\operatorname{ord}\operatorname{loc}_q^\dagger \psi \geq \operatorname{ord}\psi - C$, *or* $\operatorname{loc}_q^\dagger \psi = 0$, *as desired.*
- $\operatorname{ord}\operatorname{loc}_q c_i \geq \operatorname{ord} c_i - C$ *for* $i = 1, 2$.

Proof. Let us first fix a complex conjugation $c \in G_{\mathbb{Q}}$ and choose a basis for T_m in which c acts as $\begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix}$.

The restriction of the cocycles ϕ, ψ, c_i to $G_{K(T_m)}$ may be considered as a homomorphism

$$h : G_{K(T_m)} \to L_m \oplus L_m^\dagger \oplus (T_m)^2$$

compatible with the action of G_K; let H be the image of this homomorphism. Since there exists an element of $g_z \in G_K$ that acts by a scalar $z \neq \pm 1$ on T_f, we have:

$$
\begin{aligned}
H \supset &(g_z - z)(g_z - z^2)H + (g_z - z)(g_z - 1)H + (g_z - z^2)(g_z - 1)H\\
&\supset (z-1)(z^2-1)(z^2-z)\left(\pi_{L_m}(H) \oplus \pi_{L_m^\dagger}(H) \oplus \pi_{T_m^2}(H),\right)
\end{aligned}
$$

where $\pi_\bullet$ are the projection operators. Now, since L and $L^\dagger$ are absolutely irreducible, the natural maps $\mathbb{Q}_p[G_K] \to \operatorname{End}(L \otimes \mathbb{Q}_p)$ and $\mathbb{Q}_p[G_K] \to \operatorname{End}(L^\dagger \otimes \mathbb{Q}_p)$ are surjective. Combining these observations with Lemma 7.2.3, we see that, for some constant C depending only on T_f, there exists $\gamma \in G_{K(T_f)}$ satisfying:

- The $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ component of $\phi(\gamma)$ has order at least $\operatorname{ord}\phi - C$.
- The $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ component of $\psi(\gamma_\psi)$ has order at least $\operatorname{ord}\psi - C$, or is 0, as desired.
- The components of $c_i(\gamma)$ and $c_2(\gamma)$ in the δ eigenspace have order at least $\operatorname{ord} c_i - C$, where $i = 1, 2$.

For the final item, we are using the elementary fact that a group cannot be the union of two trivial subgroups, as well as the irreducibility of T_f.

Since $\phi(c^2) = c\phi(c) + \phi(c) = 0$, $\phi(c)$ lies in the -1 eigenspace for complex conjugation, whereas $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalue 1; hence the $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ component of $\phi(c\gamma)$ has order at least $\operatorname{ord}\phi - C$. Similarly, the $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ component of $\psi(c\gamma_\psi)$ has order at least $\operatorname{ord}\psi - C$, or is 0, as desired.

Any prime with Frobenius $c\gamma$ in $\ker h$ satisfies the conclusion of the lemma; cf. the proof of Lemma 3.3.6 for the assertions about c_i. □

Corollary 7.2.5. *Suppose given a finite set of ultraprimes T and non-torsion cocycles:*

- $\phi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{T}}/\mathbb{Q}, L)$;
- $\psi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{T}}/\mathbb{Q}, L^\dagger)$;
- $c_1, c_2 \in \mathsf{H}^1(K^{\mathsf{T}}/K, T_f)^\delta$ *for* $\delta = \pm 1$.

Then there exist infinitely many admissible ultraprimes $\mathsf{q} \notin \mathsf{T}$ with sign δ such that:

- $\operatorname{loc}_{\mathsf{q}} \phi \neq 0$.
- *Either* $\operatorname{loc}_{\mathsf{q}}^\dagger \psi \neq 0$ *or* $\operatorname{loc}_{\mathsf{q}}^\dagger \psi = 0$, *as desired.*
- $\operatorname{loc}_{\mathsf{q}} c_i \neq 0$.

Proof. Choose a sequence T_n representing T and sequences $\phi_n, \psi_n, c_n^1, c_n^2$ representing the respective cocycles in $H^1(\mathbb{Q}^{T_n}/\mathbb{Q}, L/\pi^n)$, etc. For each n, apply Lemma 7.2.4 with $m = n$ and the appropriate desideratum for ψ_n; by definition, any resulting admissible ultraprime q, represented by a sequence q_n, satisfies the desired conclusion. $\square$

Proposition 7.2.6. *Suppose given a self-dual Selmer structure $(\mathcal{F}, \mathsf{T})$ for T_f. Then there exists $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{T}}$ such that*

$$r_{\mathsf{Q}} = d_{\mathsf{Q}} = 0.$$

(Recall that $r_{\mathsf{Q}} = \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f)$.)

Proof. Without loss of generality, by Corollary 3.3.7 we may assume that $r_1 = 0$; for if not, choose any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{T}}$ with $r_{\mathsf{Q}} = 0$, and then relabel $\mathcal{F}(\mathsf{Q})$ as $\mathcal{F}$.

We will show that, if $d_1 > 0$, we may find $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ such that $r_{\mathsf{Q}} = 0$ and $d_{\mathsf{Q}} < d_1$; this clearly suffices by induction. By Proposition 7.1.4, there exist non-torsion elements $\phi \in \mathrm{Sel}_{\mathcal{S}}(L)$, $\psi \in \mathrm{Sel}_{\mathcal{S}^\dagger}(L^\dagger)$. We choose any admissible $\mathsf{q} \notin \mathsf{T}$ with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}} \phi \neq 0$, $\mathrm{loc}_{\mathsf{q}}^\dagger \psi \neq 0$. Then by Proposition 2.7.4,

$$\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}^{\dagger,\mathsf{q}}}(L^\dagger) + \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}^{\mathsf{q}}}(L) = 2 + \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}^\dagger_{\mathsf{q}}}(L^\dagger) + \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}_{\mathsf{q}}}(L)$$

(in the notation of (43)). The images of the localization maps

$$\mathrm{loc}_{\mathsf{q}} : \frac{\mathrm{Sel}_{\mathcal{S}^{\mathsf{q}}}(L)}{\mathrm{Sel}_{\mathcal{S}_{\mathsf{q}}}(L)} \hookrightarrow \frac{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)} \oplus \frac{\mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}$$

and

$$\mathrm{loc}^\dagger_{\mathsf{q}} : \frac{\mathrm{Sel}_{\mathcal{S}^{\dagger,\mathsf{q}}}(L^\dagger)}{\mathrm{Sel}_{\mathcal{S}^\dagger_{\mathsf{q}}}(L^\dagger)} \hookrightarrow \frac{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)} \oplus \frac{\mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}$$

have total rank two and annihilate each other under the induced Tate pairing by Proposition 2.7.2. Hence the image in the ordinary part is zero for both maps, and $d_{\mathsf{q}} < d_1$. However, by adding q, we have made $r_{\mathsf{q}} = 1$. Let $c \in \mathrm{Sel}_{\mathcal{F}(\mathsf{q})}(T_f)$ be a generator; since $\partial_{\mathsf{q}} c \neq 0$ by Proposition 3.3.3, c has nonzero component in the ϵ_{q} eigenspace for τ.

Now consider the set P of admissible ultraprimes s with sign $\epsilon_{\mathsf{s}} = \epsilon_{\mathsf{q}}$ such that $\mathrm{loc}_{\mathsf{s}} c \neq 0$. If, for any $\mathsf{s} \in \mathsf{P}$, $d_{\mathsf{qs}} \le d_{\mathsf{q}}$, then we may take $\mathsf{Q} = \mathsf{qs}$ and complete our induction step. For example, this will occur provided $d_{\mathsf{q}} > 0$, by the argument above; so without loss of generality, $d_{\mathsf{q}} = 0$ and $d_{\mathsf{qs}} = 1$ for all $\mathsf{s} \in \mathsf{P}$. By definition, we therefore have non-torsion elements $\phi(\mathsf{s}) \in \mathrm{Sel}_{\mathcal{S}(\mathsf{qs})}(L)$ and $\psi(\mathsf{s}) \in \mathrm{Sel}_{\mathcal{S}^\dagger(\mathsf{qs})}(L^\dagger)$ such that $\mathrm{loc}_{\mathsf{s}} \phi(s)$ and $\mathrm{loc}_{\mathsf{s}} \psi(s)$ do not lie in the unramified subspace of the ordinary cohomology.

Choose any $\mathsf{s}_1 \in \mathsf{P}$, and then choose $\mathsf{s}_2 \in \mathsf{P}$ such that $\mathrm{loc}_{\mathsf{s}_2} \phi(\mathsf{s}_1) \neq 0$ but $\mathrm{loc}_{\mathsf{s}_2} \psi(\mathsf{s}_1) = 0$. By another application of Proposition 3.3.3, $r_{\mathsf{qs}_1\mathsf{s}_2} = 1$, and a generator c' of $\mathrm{Sel}_{\mathcal{F}(\mathsf{qs}_1\mathsf{s}_2)}(T_f)$ again has nonzero component in the ϵ_{q} eigenspace. We now choose $\mathsf{s}_3 \in \mathsf{P}$ such that $\mathrm{loc}_{\mathsf{s}_3} c' \neq 0$, $\mathrm{loc}_{\mathsf{s}_3} \phi(\mathsf{s}_2) \neq 0$, and $\mathrm{loc}_{\mathsf{s}_3} \psi(\mathsf{s}_1) \neq 0$. Clearly $r_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = 0$. Note that $\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}^{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}(\mathsf{q})} = 3$; up to torsion, $\phi(\mathsf{s}_i)$ are generators. So to show that $d_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = d_{\mathsf{q}}$, it suffices to show that the images of $\phi(\mathsf{s}_i)$ form a rank-three subspace of

$$S := \bigoplus_{i=1}^{3} \frac{\mathsf{H}^1_{\mathrm{unr}\,+\,\mathrm{ord}}(\mathbb{Q}_{\mathsf{s}_i}, L)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{s}_i}, L)}$$

under the localization

$$\mathrm{loc} : \frac{\mathrm{Sel}_{\mathcal{S}^{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}}(\mathsf{q})(L)}{\mathrm{Sel}_{\mathcal{S}_{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}(\mathsf{q})}(L)} \hookrightarrow S.$$

By pairing $\phi(\mathsf{s}_i)$ and $\psi(\mathsf{s}_j)$ for $i \neq j$ and applying Proposition 2.7.2 once more, we see that $\mathrm{loc}_{\mathsf{s}_i} \phi(\mathsf{s}_j) \neq 0$ if and only if $\mathrm{loc}_{\mathsf{s}_j} \psi(\mathsf{s}_i) \neq 0$. Hence, the images of $\phi(\mathsf{s}_i)$ in S are of the form:

$$\begin{aligned} \mathrm{loc}(\phi(\mathsf{s}_1)) &= (0, *, \cdot) \\ \mathrm{loc}(\psi(\mathsf{s}_2)) &= (0, 0, *), \\ \mathrm{loc}(\psi(\mathsf{s}_3)) &= (*, 0, 0), \end{aligned}$$

where $*$ is nonzero and $\cdot$ may or may not be zero. This completes the inductive step since $d_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = d_{\mathsf{q}} < d_1$. $\square$

8. Proof of main results

For this section, let f be a modular form of weight two, level N, and trivial character, with ring of integers $\mathcal{O}_f$, and let $\wp \subset \mathcal{O}_f$ be an ordinary prime lying over $p \nmid 2N$. Denote by $\mathcal{O}$ the completion.

8.1. A result of Skinner-Urban.

The following result is a corollary to the proof of the main conjecture [47].

Theorem 8.1.1 (Skinner-Urban). *Let K be an imaginary quadratic field of discriminant prime to Np in which p splits. Assume that $\wp$ is ordinary for f and that:*

- *the mod $\wp$ representation $\overline{T}_f$ is absolutely irreducible;*
- *$N = N_1 N_2$, where every factor of N_1 is split in K and N_2 is the squarefree product of an odd number primes inert in K.*

If $\mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{W}_f)$ is Λ-cotorsion, then

$$\mathrm{char}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(1))^2$$

as ideals of Λ, where $\boldsymbol{\lambda}(1) \in \Lambda$ is the element constructed in (6.1).

Proof. We must explain some details and notations of [47], in which it is assumed that $\overline{T}_f$ is ramified at every $\ell | N_2$. As in [47], we let O_L be the ring of integers of a suitable finite extension of $\mathbb{Q}_p$ and consider f as a specialization of a suitable Hida family $\mathbf{f}$. This family is parametrized by $\mathbb{I}$, which is a normal domain and a finite integral extension of $O_L[\![W]\!]$. We write $\Gamma_K = \Gamma_K^+ \times \Gamma_K^-$ for the Galois group of the maximal $\mathbb{Z}_p$-extension of K and its decomposition into cyclotomic/anticyclotomic components. For a sufficiently large finite set of primes Σ, there is [47, Theorem 12.3.1] a three-variable p-adic L-function $\mathcal{L}_{\mathbf{f},K}^\Sigma \in \mathbb{I}[\![\Gamma_K]\!]$. (Here the superscript Σ refers to removing Euler factors at primes in Σ, or relaxing local conditions for a Selmer group.) Letting γ^- be a topological generator of Γ_K^-, we may expand:

$$\mathcal{L}_{\mathbf{f},K}^\Sigma = a_0 + a_1(\gamma^- - 1) + a_2(\gamma^- - 1)^2 + \dots \tag{48}$$

where $a_i \in \mathbb{I}[\![\Gamma_K^+]\!]$. Let $Ch_{K_\infty}^\Sigma(\mathbf{f}) \subset \mathbb{I}[\![\Gamma_K]\!]$ be the characteristic ideal of the three-variable Selmer group as considered in [47]. Skinner and Urban deduce

$$Ch_{K_\infty}^\Sigma(\mathbf{f}) \subset (\mathcal{L}_{\mathbf{f},K}^\Sigma) \tag{49}$$

by proving (see [47, Theorem 6.5.4, Proposition 12.3.6, Proposition 13.4.1]):

(1) If $P \subset \mathbb{I}[\![\Gamma_K]\!]$ is a height one prime which is not of the form $P_+\mathbb{I}[\![\Gamma_K]\!]$ for some $P_+ \subset \mathbb{I}[\![\Gamma_K^+]\!]$, then

$$\mathrm{ord}_P\, Ch_{K_\infty}^\Sigma(\mathbf{f}) \geq \mathrm{ord}_P(\mathcal{L}_{\mathbf{f},K}^\Sigma).$$

(2) If $\overline{T}_f$ is ramified at every $\ell | N_2$, then $\mathrm{ord}_P(\mathcal{L}_{\mathbf{f},K}^\Sigma) = 0$ for all height one primes P of the form $P_+\mathbb{I}[\![\Gamma_K]\!]$ for some $P_+ \subset \mathbb{I}[\![\Gamma_K^+]\!]$.

Although (2) does not apply, we claim that we may replace (49) by the weaker inclusion:

$$Ch_{K_\infty}^\Sigma(\mathbf{f}) \cdot (a_i) \supset (\mathcal{L}_{\mathbf{f},K}^\Sigma) \tag{50}$$

where a_i is any of the terms in (48). Indeed, because both sides of (50) are divisorial, it suffices to check that $\mathrm{ord}_P(\mathcal{L}_{\mathbf{f},K}^\Sigma) \leq \mathrm{ord}_P(a_i)$ for all P as in (2). But this is clear: if $(\mathcal{L}_{\mathbf{f},K}^\Sigma)$ is zero modulo P^k for such a prime P, then a_i is as well. By [50] a_i may be chosen so that its image under the specialization map $\mathbb{1} : \mathbb{I}[\![\Gamma_K^+]\!] \to O_L$ is nonzero. Fix such a choice $\widetilde{\alpha}$.

The divisibility (50) also (trivially) implies a divisibility for the Fitting ideal of the 3-variable Selmer group:

$$(\widetilde{\alpha}) Fitt_{K_\infty}^\Sigma(\mathbf{g}) \subset (\mathcal{L}_{\mathbf{g},K}^\Sigma). \tag{51}$$

Specializing (51) to the anticyclotomic variable, we obtain

$$\mathrm{char}_\Lambda \mathrm{Sel}^\Sigma(K_\infty, f) \subset L_p^\Sigma(K_\infty, f) \text{ in } \Lambda \otimes \mathbb{Q}_p,$$

where $L_p^\Sigma(K_\infty, f)$ is a certain Σ-primitive anticyclotomic L-function, and $\mathrm{Sel}^\Sigma(K_\infty, f)$ is the Σ-primitive Selmer group. Replacing [47, Proposition 3.3.19] by [40, Proposition A.2] (and using the hypothesis that the Selmer group is Λ-cotorsion), we may convert this to an imprimitive divisibility

$$\mathrm{char}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(1))^2 \text{ in } \Lambda \otimes \mathbb{Q}_p. \tag{52}$$

NB: The anticyclotomic p-adic L-function appearing in [47, §12.3.5], which emerges naturally from the specialization of the three-variable p-adic L-function, is normalized using Hida's canonical period, whereas $\boldsymbol{\lambda}(1)^2$ is the L-function constructed in [3], normalized using Gross's period. However, these L-functions differ only by a power of p. Similarly, the local cohomology groups $H^1(K_\ell, \mathbf{W}_f)$ for $\ell | N_2$ have characteristic ideal a power of $(\wp)$, so the choice of local condition at primes $\ell | N_2$ does not change the characteristic ideal in $\Lambda \otimes \mathbb{Q}_p$. See the appendix, and [40] for a detailed discussion.

To upgrade (52) to a divisibility in Λ, we simply note that the μ-invariant of $\boldsymbol{\lambda}(1)$ is 0 by [50]. $\square$

8.2. **The Heegner point main conjecture.** In this subsection, we prove the following main theorem.

Theorem 8.2.1. *Let f be a modular form of weight two, level N, and trivial character, with an ordinary prime $\wp$ of its ring of integers $\mathcal{O}_f$, and let K be an imaginary quadratic field. Assume:*

- *$N = N^+N^-$, where every factor of N^+ is split in K, and N^- is a squarefree product of primes inert in K.*
- *The residue characteristic p of $\wp$ does not divide $2D_KN$, and p splits in K.*
- *The modulo $\wp$ representation $\overline{T}_f$ associated to f is absolutely irreducible; if $p = 3$, assume that $\overline{T}_f$ is not induced from a character of $G_{\mathbb{Q}\sqrt{-3}}$.*

Then, for all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$ such that $(\boldsymbol{\kappa}(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \neq 0$, we have

$$\mathrm{rk}_\Lambda \operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}) = \mathrm{crk}_\Lambda \operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f) = 1$$

and

$$\mathrm{char}_\Lambda \left(\left(\operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \right)_{\mathrm{tors}} \right) = \mathrm{char}_\Lambda \left(\frac{\operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})}{(\boldsymbol{\kappa}(\mathsf{Q}))} \right)^2 \text{ in } \Lambda \otimes \mathbb{Q}_p.$$

For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$ such that $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$,

$$\mathrm{rk}_\Lambda \operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}) = \mathrm{crk}_\Lambda \operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f) = 0$$

and

$$\mathrm{char}_\Lambda \left(\operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \right) = (\boldsymbol{\lambda}(\mathsf{Q}))^2 \text{ in } \Lambda \otimes \mathbb{Q}_p.$$

If moreover the image of the $G_{\mathbb{Q}}$ action on $\overline{T}_f$ contains a nontrivial scalar, then the equalities hold in Λ.

Proof. Given f, apply Proposition 7.2.6 to the standard Selmer structure $(\mathcal{F}, \mathsf{S})$ on T_f (with local conditions the image of the Kummer map at all $\underline{v}$ such that $v | Np$). Let $\{Q_n, \epsilon_{Q_n}\}$ be a sequence of weakly admissible pairs representing the resulting pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$. Let g_n be the resulting sequence of newforms of level NQ_n obtained from Theorem 7.1.5 (and Proposition 7.2.2); g_n may only be defined for $\mathfrak{F}$-many n.

Step 1. $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$.

Proof. The prime (T), corresponding to the trivial character, does not lie in the exceptional set Σ for $\mathcal{F}_\Lambda$ (see the proofs of [33, Lemma 5.3.13] and [24, Lemma 2.2.7]). Hence $\operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}) = 0$, which by Theorem 3.4.6 and the nontriviality of $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ implies the claim. $\square$

Step 2. *For any fixed j,*

$$(\boldsymbol{\lambda}(\mathsf{Q})) \equiv (\boldsymbol{\lambda}_{g_n}(1)) \pmod{\wp^j, T^j}$$

for $\mathfrak{F}$-many n.

Proof. Recall notations of (5.1) and (6.1). By definition, the image of $\boldsymbol{\lambda}(\mathsf{Q})$ modulo $(\wp^j, T^j)$ is a map $\mathrm{Gal}(K_j/K) \to \mathcal{O}$ obtained, for $\mathfrak{F}$-many n, by evaluating a surjective map $F_n : M_{Q_n} \otimes_{\mathbb{T}^{Q_n}} \mathcal{O}(f) \to \mathcal{O}(f)/\wp^j$ of $\mathbb{T}^{Q_n}$-modules at certain CM points, where $\mathcal{O}(f)$ is $\mathcal{O}$ with $\mathbb{T}^{Q_n}$-action by f. Recall that the map is chosen to factor through multiplicatation by $\mathcal{O}(f)/\pi^{j+C}$ for the constant C of Lemma 5.1.4, and that $\pi^C(M_{Q_n} \otimes_{\mathbb{T}^{Q_n}} \mathcal{O}(f))$ is principal. When g_n has a sufficiently deep congruence to f, $\mathcal{O}(g_n)/\pi^{j+C} = \mathcal{O}(f)/\pi^{j+C}$ as $\mathbb{T}^{Q_n}$-modules, and the composite $G_n : M_{Q_n} \to \mathcal{O}(g_n) \to \mathcal{O}(g_n)/\pi^j$ induces a unit multiple of F_n, where $M_{Q_n} \to \mathcal{O}(g_n)$ is the quaternionic modular form associated to g_n. But G_n is the very map whose evaluation at CM points is used to define $\boldsymbol{\lambda}_{g_n}(1)$, and the claim follows. $\square$

Step 3. *For any fixed j,*

$$\mathrm{Fitt}_\Lambda \operatorname{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \equiv \mathrm{Fitt}_\Lambda \operatorname{Sel}_{\mathcal{F}_{g_n,\Lambda}}(\mathbf{W}_{g_n})^\vee \pmod{\wp^j, T^j}$$

for $\mathfrak{F}$-many n.

Proof. Since fitting ideals are stable under base change and $\overline{T}_f$ has no G_K-fixed points, it suffices to show that

$$\mathrm{Fitt}_\Lambda\left(\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f[\pi^j, T^j])\right) = \mathrm{Fitt}_\Lambda\left(\mathrm{Sel}_{\mathcal{F}_{g_n,\Lambda}}(\mathbf{W}_{g_n}[\pi^j, T^j])\right) \tag{53}$$

for $\mathfrak{F}$-many n. Note that, for $\mathfrak{F}$-many n, $\mathbf{W}_f[\pi^j, T^j] = \mathbf{W}_{g_n}[\pi^j, T^j]$ as finite Galois modules, and

$$\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f[\pi^j, T^j])$$

is isomorphic to a submodule of $H^1(K^{\Sigma\cup Q_n}/K, \mathbf{W}_f[\pi^j, T^j])$ defined by certain local conditions. We will show that these local conditions coincide with the ones defining $\mathrm{Sel}_{\mathcal{F}_{g_n,\Lambda}}(\mathbf{W}_{g_n}[\pi^j, T^j])$. At a j-admissible prime $q_n|Q_n$, this is clear (cf. the proof of Construction 5.2.2(1)). At $v|N$, the local conditions are simply the kernels

$$\ker\left(H^1(K_v, \mathbf{W}_f[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_f)\right), \ \ker\left(H^1(K_v, \mathbf{W}_{g_n}[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_{g_n})\right).$$

If $v|N$ is a prime of multiplicative reduction for f, then $\mathbf{W}_f = \mathbf{W}_{g_n}$ as G_{K_v} modules for $\mathfrak{F}$-many n, so the local conditions clearly coincide. For other places of bad reduction, the inertia co-invariants T_{f,I_v} and T_{g_n,I_v} are finite, and we may assume they are isomorphic, say with exponent bounded by π^{M-j} for some $M \geq 0$. It follows that the local conditions are also given by the kernels of

$$\ker\left(H^1(K_v, \mathbf{W}_f[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_f[\pi^M])\right), \ \ker\left(H^1(K_v, \mathbf{W}_{g_n}[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_{g_n}[\pi^M])\right),$$

which also agree for $\mathfrak{F}$-many n.

For primes $v|p$, it suffices to compare the kernels

$$\left(H^1(K_v, \mathrm{gr}\,\mathbf{W}_f[\pi^j, T^j]) \to H^1(K_v, \mathrm{gr}\,\mathbf{W}_f)\right), \ \ker\left(H^1(K_v, \mathrm{gr}\,\mathbf{W}_{g_n}[\pi^j, T^j]) \to H^1(K_v, \mathrm{gr}\,\mathbf{W}_{g_n})\right).$$

A similar argument as above applies provided that $H^0(K_v, \mathrm{gr}\,\mathbf{W}_f) = \prod_{w|v} H^0(K_{\infty,w}, \mathrm{gr}\,W_f)$ is finite, which it is because a_p cannot be a root of unity. □

Step 4. *Conclusion of the proof.*

Step 1 shows that N^-Q_n is the squarefree product of an odd number primes inert in K for $\mathfrak{F}$-many n, and by Theorem 3.4.6 applied to the Euler system $(\boldsymbol{\kappa}_{g_n}, \boldsymbol{\lambda}_{g_n})$ for T_{g_n}, $\mathrm{Sel}_{\mathcal{F}_{g_n,\Lambda}}(\mathbf{W}_{g_n})$ is then Λ-cotorsion. By Theorem 8.1.1, for such n we have:

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}_{g_n,\Lambda}}(\mathbf{W}_{g_n})^\vee \subset (\boldsymbol{\lambda}_{g_n}(1))^2 \subset \Lambda. \tag{54}$$

On the other hand, by Theorem 3.4.6, the theorem would follow from:

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(\mathsf{Q}))^2 \subset \Lambda. \tag{55}$$

For the passage between characteristic ideal and fitting ideals, recall that the characteristic ideal of any Λ-module is the smallest divisorial ideal containing the Fitting ideal; cf. [47, Corollary 3.2.9]. Steps 2 and 3 allow us to pass from (54) to (55). □

Corollary 8.2.2. *Under the hypotheses of Theorem 8.2.1, if additionally $\nu(N^-)$ is even, then the Heegner point main conjecture holds for f in $\Lambda \otimes \mathbb{Q}_p$; that is, there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}(K_\infty, A_f[\wp^\infty])^\vee \approx \Lambda \oplus M \oplus M$$

for some torsion Λ-module M, and

$$\mathrm{char}_\Lambda\left(\frac{\mathrm{Sel}(K, T_f \otimes \Lambda)}{\Lambda\boldsymbol{\kappa}(1)}\right) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. If additionally the image of the Galois action on $\overline{T}_f$ contains a nontrivial scalar, then the equality is true in Λ.

Corollary 8.2.3. *Under the hypotheses of Theorem 8.2.1, the bipartite Euler system $(\kappa(1,\cdot), \lambda(1,\cdot))$ of (6.2.10) is nontrivial.*

Proof. Let $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ be such that $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f) = 0$, where again $\mathcal{F}$ is the standard Selmer structure for T_f. As noted in Step 1 of the proof of Theorem 8.2.1, we have $\mathbb{1}(\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{W}_f)) \neq 0$, so $\mathbb{1}(\boldsymbol{\lambda}(\mathsf{Q})) \neq 0$; this implies $\lambda(1, \mathsf{Q}) \neq 0$ by Remark 6.2.12. □

Corollary 8.2.3 is generalized by Theorem A.1.1 of the appendix.

8.3. **Kolyvagin's conjecture.** Let f, $\wp$, K, and N^+N^- be as in (3.1.1) and (5.1.1).

8.3.1. For any $\mathsf{m} \in \mathsf{K}$, define the m-transverse Selmer ranks

$$r^{\pm}_{\mathsf{m}} = \mathrm{rk}_{\mathcal{O}} \, \mathrm{Sel}_{\mathcal{F}(\mathsf{m})}(T_f)^{\pm}, \tag{56}$$

where $\pm$ refers to the τ eigenvalue; note that this is well-defined because the local conditions defining $\mathcal{F}(\mathsf{m})$ are all τ-stable. When $\mathsf{m} = 1$, the $r^{\pm}_1$ are the classical Selmer ranks of f.

Proposition 8.3.2. *For all* $\mathsf{ml} \in \mathsf{K}$*, and for each* $\delta \in \{\pm\}$*, either:*

- $r^{\delta}_{\mathsf{ml}} = r^{\delta}_{\mathsf{m}} - 1$, $\mathrm{loc}^{\delta}_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{m})}(T_f))^{\delta} \neq 0$, *and* $\partial^{\delta}_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{ml})}(T_f))^{\delta} = 0$.
- $r^{\delta}_{\mathsf{ml}} = r^{\delta}_{\mathsf{m}} + 1$, $\mathrm{loc}^{\delta}_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{m})}(T_f))^{\delta} = 0$, *and* $\partial^{\delta}_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{ml})}(T_f))^{\delta} \neq 0$.

Proof. If $\mathcal{F}^{\mathsf{l}}(\mathsf{m}, \mathsf{Q}) = \mathcal{F}(\mathsf{ml}, \mathsf{Q}) + \mathcal{F}(\mathsf{m}, \mathsf{Q})$ and $\mathcal{F}_{\mathsf{l}}(\mathsf{m}, \mathsf{Q}) = \mathcal{F}(\mathsf{ml}, \mathsf{Q}) \cap \mathcal{F}(\mathsf{m}, \mathsf{Q})$, then we have a τ-equivariant exact sequence

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{l}}(\mathsf{m},\mathsf{Q})}(T_f) \to \mathrm{Sel}_{\mathcal{F}^{\mathsf{l}}(\mathsf{m},\mathsf{Q})}(T_f) \to \mathsf{H}^1(K_{\mathsf{l}}, T_f),$$

where the image of the final arrow has rank two and is self-annihilating under the local Tate pairing by Propositions 2.7.2 and 2.7.4. Since the Tate pairing of two classes with opposite τ eigenvalues is necessarily zero, the proposition follows. □

Lemma 8.3.3. *Suppose given elements* $c^{\pm} \in \mathsf{H}^1(K, T_f)^{\pm}$*, where* $\pm$ *is the* τ *eigenvalue. Then there exists a Kolyvagin-admissible ultraprime* l *such that*

$$c^{\pm} \neq 0 \implies \mathrm{loc}^{\pm}_{\mathsf{l}} \, c^{\pm}.$$

If (sclr) holds for T_f*, then the same is true for elements* $c^{\pm} \in \mathsf{H}^1(K, T_f/\pi^j)$*.*

Proof. The proof of Theorem 3.3.4 applies almost verbatim, except that in the proof of Lemma 3.3.6 we will have two homomorphisms $\phi^{\pm} \in \mathrm{Hom}_{G_K}(G_L, \overline{T}_f)^{\pm}$, and we must choose $g \in G_L$ so that $\phi^{\epsilon}(g)$ has nonzero component in the ϵ eigenspace of τ for both signs ϵ (unless ϕ^{ϵ} is itself 0); for each ϵ, this condition is satisfied outside a proper subgroup of G_L, so indeed there exists $g \in G_L$ such that both conditions are satisfied. With this modification, the rest of the proof applies unchanged. □

Lemma 8.3.4. *Suppose that the bipartite Euler system* $(\kappa(1, \cdot), \lambda(1, \cdot))$ *of (6.2.10) is nontrivial. Then, for all* $\mathsf{m} \in \mathsf{K}$*,* $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ *is nontrivial.*

In particular, for all $\mathsf{m} \in \mathsf{K}$ *and* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}$*:*

$$\mathrm{rk}_{\mathcal{O}} \, \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) \leq 1 \iff \begin{cases} \kappa(\mathsf{m}, \mathsf{Q}) \neq 0, & \nu(N^-) + |\mathsf{Q}| \text{ even} \\ \lambda(\mathsf{m}, \mathsf{Q}) \neq 0, & \nu(N^-) + |\mathsf{Q}| \text{ odd.} \end{cases}$$

Proof. Proposition 6.2.11 implies that, for fixed m, the pair $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ forms a bipartite Euler system with sign $\nu(N^-)$ for the self-dual Selmer structure $(\mathcal{F}(\mathsf{m}), \mathsf{S} \cup \mathsf{m})$ on T_f. We will prove that, for any $\mathsf{ml} \in \mathsf{K}$, if $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ is nontrivial then so is $(\kappa(\mathsf{ml}, \cdot), \lambda(\mathsf{ml}, \cdot))$.

Choose $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}_{\mathsf{m}}$ such that $\mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) = 0$ and $\mathsf{l} \notin \mathsf{Q}$; this is possible by Corollary 3.3.7. By Proposition 8.3.2, we may choose a nonzero

$$d \in \mathrm{Sel}_{\mathcal{F}(\mathsf{ml},\mathsf{Q})}(T_f).$$

Applying Theorem 3.3.4 to d, let q be admissible with sign ϵ_{q} such that $\mathsf{q} \notin \mathsf{Qml}$ and $\mathrm{loc}_{\mathsf{q}} \, d \neq 0$. By Proposition 3.3.3 for the Selmer structures $\mathcal{F}(\mathsf{m}, \mathsf{Qq})$ and $\mathcal{F}(\mathsf{m}, \mathsf{Q})$,

$$\mathrm{rk}_{\mathcal{O}} \, \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Qq})}(T_f) = 1. \tag{57}$$

Hence, by hypothesis, $\kappa(\mathsf{m}, \mathsf{Qq})$ generates $\mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Qq})}(T_f)$ up to finite index, and in particular $\partial_{\mathsf{q}} \kappa(\mathsf{m}, \mathsf{Qq}) \neq 0$. Now, taking the sum of local pairings and using Proposition 2.7.2,

$$0 = \sum_{\mathsf{v}} \langle d, \kappa(\mathsf{m}, \mathsf{Qq}) \rangle_{\mathsf{v}} = \langle d, \kappa(\mathsf{m}, \mathsf{Qq})_{\mathsf{l}} + \langle d, \kappa(\mathsf{m}, \mathsf{Qq}) \rangle_{\mathsf{q}}. \tag{58}$$

Since the latter pairing is nonzero by construction, the former is as well, and so, by Proposition 6.2.11(1),

$$\mathrm{Res}_{\mathsf{l}} \, \kappa(\mathsf{m}, \mathsf{Qq})) \neq 0 \implies \kappa(\mathsf{ml}, \mathsf{Qq}) \neq 0.$$

□

8.3.5. For any $\mathsf{m} \in \mathsf{K}$, define the vanishing order of the Kolyvagin system at m:

$$\nu_{\mathsf{m}} = \begin{cases} \min\{|\mathsf{n}| \,:\, \mathsf{n} \in \mathsf{K},\ \lambda(\mathsf{n} \cup \mathsf{m}, 1) \neq 0\}, & \nu(N^-) \text{ odd}, \\ \min\{|\mathsf{n}| \,:\, \mathsf{n} \in \mathsf{K},\ \kappa(\mathsf{n} \cup \mathsf{m}, 1) \neq 0\}, & \nu(N^-) \text{ even}. \end{cases} \tag{59}$$

Corollary 8.3.6. *If $(\kappa(1,\cdot), \lambda(1,\cdot))$ is nontrivial, and in particular under the hypotheses of Theorem A.1.1, we have for all $\mathsf{m} \in \mathsf{K}$:*

- *If $\nu(N^-)$ is odd, then $\nu_{\mathsf{m}} = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\}$ and $r_{\mathsf{m}}^\pm \equiv \frac{\epsilon_f - 1}{2} \pmod 2$.*
- *If $\nu(N^-)$ is even, then $\nu_{\mathsf{m}} = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - 1$ and $\epsilon_f \cdot (-1)^{|\mathsf{m}| + \nu_{\mathsf{m}} + 1}$ is the larger τ eigenspace.*

In particular, if $\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}(K, T_f) = 1$, then $L'(f/K, 1) \neq 0$.

Proof. Note that the parity statement in Theorem 3.3.8 implies $r_{\mathsf{m}}^+ + r_{\mathsf{m}}^- \equiv \nu(N^-) + 1 \pmod 2$ for all $\mathsf{m} \in \mathsf{K}$ by Proposition 8.3.2. So if we have $\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{mn})}(T_f) \leq 1$, for some $\mathsf{n} \in \mathsf{K}$ such that $\mathsf{m} \cap \mathsf{n} = \emptyset$, then the τ-equivariant localization map

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{m})}(T_f) \to \oplus_{\mathsf{l} \in \mathsf{n}} H^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f)$$

has kernel of rank either zero (if $\nu(N^-)$ is odd), or at most one (if $\nu(N^-)$ is even). It follows that $\nu_{\mathsf{m}} \geq \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\}$ in the former case and $\nu_{\mathsf{m}} \geq \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - 1$ in the latter. The opposite inequality follows readily from repeated applications of Proposition 8.3.2 and Lemma 8.3.3. The additional claims about parity and τ eigenvalues follow from Proposition 6.2.7.

For the final statement, take $\mathsf{m} = 1$. The corollary implies $\nu_1 = 0$, so $\kappa(1,1) \neq 0$. Since $\kappa(1,1)$ is the Kummer image of the classical Heegner point $y_K \in E(K)$, the result follows from the Gross-Zagier theorem of [52]. ☐

It remains to relate the vanishing of the patched Kolyvagin classes to the classical vanishing order

$$\nu_{\mathrm{classical}} := \begin{cases} \min\left\{\nu(m) \,:\, \overline{\lambda}(m, 1) \neq 0\right\}, & \nu(N^-) \text{ odd}, \\ \min\left\{\nu(m) \,:\, \overline{c}(m, 1) \neq 0\right\}, & \nu(N^-) \text{ even}. \end{cases} \tag{60}$$

Corollary 8.3.7. *If $(\kappa(1,\cdot), \lambda(1,\cdot))$ is nontrivial, and in particular under the hypotheses of Theorem A.1.1, $\nu_{classical}$ is finite. If (sclr) holds for f, then $\nu_{classical} = \nu_1$, and in particular:*

- *If $\nu(N^-)$ is odd, then $\nu_{classical} = \max\left\{r_1^+, r_1^-\right\}$ and $r_1^\pm \equiv \frac{\epsilon_f - 1}{2} \pmod 2$.*
- *If $\nu(N^-)$ is even, then $\nu_{classical} = \max\left\{r_1^+, r_1^-\right\} - 1$ and $\epsilon_f \cdot (-1)^{1 + \nu_{classical}}$ is the larger τ eigenspace.*

Proof. The finiteness of the classical vanishing order is clear by construction: if a patched Kolyvagin class or element is nontrivial, then infinitely many of the classical Kolyvagin classes or elements defining it are nontrivial. This also shows $\nu_{\mathrm{classical}} \leq \nu_1$. We will check that equality holds under the condition (sclr). Suppose first $\nu(N^-)$ is even. We abbreviate by $c_j(m,1) \in H^1(K, T_j)$ the image of $\overline{c}(m,1)$ when $v_\wp(I_m) \geq j$. Given some nonzero $c_j(m,1)$, one may show as in [34, p. 309] that there exist classes $c_j(m_n, 1) \neq 0$ with $v_\wp(I_{m_n}) \to \infty$ and $\nu(m_n) = \nu(m)$. (In [34], additional hypotheses are put on the image of the Galois action, but the argument goes through by invoking Lemma 8.3.3.) In particular, the sequence m_n defines a nonzero $\kappa(\mathsf{m}, 1)$ witnessing $\nu_1 \leq \nu_{\mathrm{classical}}$.

Now suppose that $\nu(N^-)$ is odd, and that $\lambda_j(m,1) \neq 0$ where $\nu(m) = \nu_{\mathrm{classical}}$. We choose an auxiliary $\{\mathsf{q}, \epsilon_{\mathsf{q}}\} \in \mathsf{N}$ with the following properties:

- ϵ_{q} is the sign of the larger τ eigenspace in $\mathrm{Sel}_{\mathcal{F}}(T_f)$.
- The localization map $\mathrm{loc}_{\mathsf{q}}$ is trivial on $\mathrm{Sel}_{\mathcal{F}}(T_f)$.

To ensure the second condition, we may choose $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ to be a complex conjugation. Let $\{q_n, \epsilon_{q_n}\}$ represent $\{\mathsf{q}, \epsilon_{\mathsf{q}}\}$ as in Remark 5.1.6. Once again, the argument of [34, p. 309] implies that, for each n, there exists m_n with $c_j(m_n, q_n) \neq 0$ and $v_\wp(I_{m_n}) \to \infty$. We therefore obtain a nonzero patched class $\kappa(\mathsf{m}, \mathsf{q})$ with $|\mathsf{m}| = \nu_{\mathrm{classical}}$. Repeating the argument of Lemma 8.3.4, it follows that $\nu_1 \leq \nu_{\mathrm{classical}} + 1$. For contradiction, we assume that

$$\nu_{\mathrm{classical}} = \nu_1 - 1 = r^{\epsilon_{\mathsf{q}}} - 1.$$

This implies $\partial_{\mathsf{q}} \kappa(\mathsf{m}, \mathsf{q}) = 0$, so by Lemma 8.3.4 and Proposition 3.3.3, we conclude

$$\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f) = 2.$$

However, by Proposition 8.3.2 and the assumption $|\mathsf{m}| = \nu_1 - 1 = r^{\epsilon_{\mathsf{q}}} - 1$, $\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f)^{\epsilon_{\mathsf{q}}}$ is odd, hence equal to one. Proposition 8.3.2 then implies

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f)^{\epsilon_{\mathsf{q}}} \subset \mathrm{Sel}_{\mathcal{F}}(T_f)^{\epsilon_{\mathsf{q}}},$$

so

$$\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f)) = 0$$

by the choice of q. However, this contradicts Proposition 3.3.3, so we must have $\nu_1 = \nu_{classical}$.

□

Appendix A. Kolyvagin's conjecture for inert or non-ordinary p

A.1. **The main result.** In this appendix, we shall prove the following:

Theorem A.1.1. *Let f be a non-CM modular form of weight two, level N, and trivial character, with a prime $\wp$ of its ring of integers $\mathcal{O}_f$, and let K be an imaginary quadratic field. Assume:*

- *$N = N^+N^-$, where every factor of N^+ is split in K, and N^- is a squarefree product of an even number of primes inert in K.*
- *The residue characteristic p of $\wp$ does not divide $2D_K N$.*
- *The modulo $\wp$ representation $\overline{T}_f$ associated to f is absolutely irreducible; if $p = 3$, assume that $\overline{T}_f$ is not induced from a character of $G_{\mathbb{Q}\sqrt{-3}}$.*
- *If p is inert in K or or a_p is not a $\wp$-adic unit, then there exists some prime $\ell_0 || N$.*
- *If a_p is not a $\wp$-adic unit, then either ℓ_0 may be chosen above so that A_f has non-split toric reduction at ℓ_0, or the image of the Galois action on T_f contains a conjugate of $SL_2(\mathbb{Z}_p)$.*

Then $(\kappa(1,\cdot), \lambda(1,\cdot))$ is nontrivial.

A.1.2. If p is split in K, then this is simply Corollary 8.2.3. If p is non-ordinary or inert in K, then the anticyclotomic main conjecture is currently not known in full generality; however, since all we are interested in specialization the trivial character, we will show that the result may instead be obtained, more circuitously, by combining main conjectures for quadratic twists of f. The proof applies equally well to the split ordinary case.

A.2. **Comparing periods.**

A.2.1. Let f be a modular form of weight two, level N, and trivial character, with ring of integers $\mathcal{O}_f$ of its coefficient field, and let $\wp \subset \mathcal{O}_f$ be an ordinary prime lying over $p \nmid 2N$, with associated completion $\mathcal{O}$; we assume that $\overline{T}_f$ is absolutely irreducible. There are two ways to normalize the anticyclotomic p-adic L-function, as explained in [50, 40]. For any factorization $N = N_1 N_2$, where N_1 and N_2 are coprime, the congruence ideal $\eta_f(N_1, N_2) \subset \mathcal{O}$ is defined as

$$\pi_f(\mathrm{Ann}_{\mathbb{T}_{N_1,N_2}}(\ker \pi_f))\cdot, \tag{61}$$

where $\pi_f : \mathbb{T}_{N_1,N_2} \to \mathcal{O}$ is the projection giving the Hecke eigenvalues of f. Hida's canonical period [23] is defined (up to p-adic units) by:

$$\Omega_f^{can} = \frac{(f,f)}{\eta_f(N,1)}, \tag{62}$$

where (f,f) is the Peterson inner product.

On the other hand, if $N = N_1 N_2$ where N_2 is squarefree with an odd number of prime factors, then the f-isotypic part of the Hecke module $\mathbb{Z}_p[X_{N_1,N_2}]$ is free of rank one, generated by an element φ_{f,N_2}. Gross's period is defined as:

$$\Omega_{f,N_2} = \frac{(f,f)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2} \rangle}, \tag{63}$$

where $\langle \cdot, \cdot \rangle$ is the canonical intersection pairing on $\mathbb{Z}_p[X_{N_1,N_2}]$. This period occurs naturally in anticyclotomic Iwasawa theory due to the well-known special value formula of Gross.

Proposition A.2.2. *Let K be an imaginary quadratic field of discriminant prime to Np, and suppose that $N = N^+N^-$ where all factors of N^+ are split in K, and N^- is a squarefree product of an odd number primes inert in K. Then $L(f/K, 1) \in \overline{\mathbb{Q}} \cdot \Omega_{f,N^-}$ and the element $\lambda(1) \in \Lambda$ constructed in (6.2.10) satisfies:*

$$\frac{L(f/K, 1)}{\Omega_{f,N^-}} = \lambda(1)^2 \tag{64}$$

up to p-adic units.

A.2.3. Consider the Néron model $\mathcal{A}_f$ of A_f over $\mathbb{Z}_\ell$. The Tamagawa numbers over K are:

$$t_f(\ell) = \lg_{\mathcal{O}} \Phi(\mathcal{A}_f)(\mathcal{O}_K/\ell)_\wp \tag{65}$$

We also require a variant:

$$c_f(\ell) = \lg_{\mathcal{O}} \Phi(\mathcal{A}_f)(\overline{\mathbb{F}}_\ell)_\wp \tag{66}$$

The number $c_f(\ell)$ is the maximal exponent e such that $A_f[\wp_f^e]$ is unramified at ℓ. If ℓ is inert in K, then $t_f(\ell) = c_f(\ell)$. The following theorem generalizes [44, 28] and [40, Theorem 6.8].

Proposition A.2.4. *Suppose $N = N_1N_2$ where N_2 is squarefree with an odd number of prime factors and coprime to N_1. For any $\ell_0 || N$, we have:*

$$v_\wp \eta_f(N, 1) - v_\wp\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle \geq \sum_{\ell | N_2} c_f(\ell) - \sigma(N_2)c_f(\ell_0).$$

Proof. For a decomposition $N = N_1'N_2'$ with N_2' the squarefree product of an even number of primes that do not divide N_1', one defines $\delta(N_1', N_2') \subset \mathcal{O}$ to be the "degree" of an optimal modular parametrization $J_{N_1',N_2'} \to A_f$ as explained in [40, 28]. By [40, Proposition 6.6], we have:

$$v_\wp\delta(N, 1) = c_f(\ell_0) + v_\wp\langle\varphi_{f,\ell_0}, \varphi_{f,\ell_0}\rangle. \tag{67}$$

On the other hand, [15, Lemma 4.17] implies that:

$$v_\wp\eta_f(N/\ell_0, \ell_0) \geq v_\wp\langle\varphi_{f,\ell_0}, \varphi_{f,\ell_0}\rangle. \tag{68}$$

Because $\mathbb{T}_{N/\ell_0,\ell_0}$ is a quotient of $\mathbb{T}_{N,1}$, we conclude that:

$$v_\wp\eta_f(N, 1) \geq v_\wp\delta(N, 1) - c_f(\ell_0). \tag{69}$$

We apply [40, Proposition 6.6] again, this time to the decomposition $N = N_1N_2$ and any $r|N_2$. This yields:

$$v_\wp\delta(N_1r, N_2/r) = c_f(r) + v_\wp\langle\varphi_{f,N_2}, \varphi_{f,N_2}\rangle. \tag{70}$$

If N_2 is prime, this is sufficient to conclude. If not, we may choose $r \neq \ell_0$.

The results of Ribet-Takahashi and Khare [44, 28] imply that:

$$v_\wp\delta(N, 1) \geq v_\wp\delta(N_1r, N_2/r) + \sum_{\ell|N_2/r} c_f(\ell) - \sigma(N_2/r) \cdot c_f(\ell_0). \tag{71}$$

(If $\ell_0|N_1$, we are using the fact that both r and ℓ_0 exactly divide N_1r.) Combining (69), (70), and (71) completes the proof.

□

Remark A.2.5. If ℓ_0 is residually ramified, the inequality is an equality. In [53, Theorem 6.4], more restrictive conditions are given under which this result holds.

A.2.6. We will also require the following related result:

Proposition A.2.7. *Let f and $\wp$ be as above and suppose $\ell_0||N$. Then, in the notation of the proof of Theorem A.2.4,*

$$v_\wp\eta_f(N, 1) - v_\wp\langle\varphi_{f,\ell_0}, \varphi_{f,\ell_0}\rangle \leq c_f(\ell_0).$$

Proof. Let $J = J_0(N)$ be the modular Jacobian and let $\mathbb{T}$ be the full Hecke algebra of level N. Write $\pi : \mathbb{T}_{\mathfrak{m}} \to \mathcal{O}_{f,\wp}$ for the projection associated to g and let I be its kernel. The claim will follow from (67) once we establish

$$v_\wp \eta_f(N,1) \leq v_\wp \delta(N,1). \tag{72}$$

Indeed, if $J \to A$ is an optimal parametrization, then the dual map $A^\vee \to J^\vee$ is an inclusion. The composition

$$\phi : J \to A \to A^\vee \to J^\vee \xrightarrow{w_N} J$$

is a Hecke-equivariant endomorphism; by (22), its image in $\mathrm{End}(J)_{\mathfrak{m}}$ may be identified with some $y \in \mathbb{T}_{\mathfrak{m}}$. Because $\mathrm{im}\,\phi \subset J[I]$, we have $y \in \mathrm{Ann}(I)$. By the definition of $\delta(N,1)$,

$$(\pi(y)) = \delta(N,1) \subset \mathcal{O}.$$

This implies (72). □

A.3. Cyclotomic Iwasawa theory: ordinary case.

A.3.1. Let $\Lambda_{\mathbb{Q}_\infty} = \mathcal{O}[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$ be the cyclotomic Iwasawa algebra. We denote by $\mathbb{1} : \Lambda_{\mathbb{Q}_\infty} \to \mathcal{O}$ and $\mathbb{1} : \Lambda \to \mathcal{O}$ the specializations at the trivial character. If $\wp$ is ordinary and Σ is a finite set of rational primes, we consider the Σ-ordinary cyclotomic Selmer group

$$\mathrm{Sel}(\mathbb{Q}_\infty, W_f) = \ker\left(H^1(\mathbb{Q}, \mathbf{W}_f) \to \prod_{v \notin \Sigma \cup \{p\}} H^1(I_v, \mathbf{W}_f) \times \frac{H^1(\mathbb{Q}_p, \mathbf{W}_f)}{H^1_{\mathrm{ord}}(\mathbb{Q}_p, \mathbf{W}_f)}\right),$$

and denote by $Ch^\Sigma_{\mathbb{Q}_\infty,f} \subset \Lambda_{\mathbb{Q}_\infty}$ the characteristic ideal of its Pontryagin dual.

From the work of Skinner and Urban, we deduce the following result.

Theorem A.3.2 (Skinner-Urban)**.** *Let K be an imaginary quadratic field of discriminant prime to Np in which p splits. Assume that $\wp$ is good ordinary for f and that:*

- *the mod $\wp$ representation $\overline{T}_f$ is absolutely irreducible;*
- *$N = N_1N_2$, where every factor of N_1 is split in K and N_2 is the squarefree product of an odd number primes inert in K.*

Then there exists an element $\alpha \in \Lambda_{\mathbb{Q}_\infty}$ such that $\mathbb{1}(\alpha)$ divides

$$\frac{\Omega_{f,N_2}}{\Omega_f^{can}} \sim \frac{\eta_f(N,1)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle}$$

in $\mathcal{O}$ and

$$(\alpha)Ch_{\mathbb{Q}_\infty,f}Ch_{\mathbb{Q}_\infty,f\otimes\chi_K} \subset (L_p(\mathbb{Q}_\infty,f))(L_p(\mathbb{Q}_\infty,f\otimes\chi_K)).$$

Proof. Recall the divisibility established in the course of the proof of Theorem 8.1.1 for the Fitting ideal of the 3-variable Selmer group:

$$(\widetilde{\alpha})Fitt^\Sigma_{K_\infty}(\mathbf{g}) \subset (\mathcal{L}^\Sigma_{\mathbf{f},K}), \tag{73}$$

where $\widetilde{\alpha} \in \mathbb{I}[\Gamma_K^+]$ may be chosen such that $\widetilde{\alpha}$ specializes to a unit multiple of $\Omega_{f,N_2}/\Omega_f^{can}$ at the trivial character (by [50]). By Lemma 3.2.5, Corollary 3.2.9(i), and Corollary 3.2.20(iii) of [47], specializing to the cyclotomic variable yields a divisibility

$$(\alpha)Ch^\Sigma_{\mathbb{Q}_\infty,f}Ch^\Sigma_{\mathbb{Q}_\infty,f\otimes\chi_K} \subset (L_p^\Sigma(\mathbb{Q}_\infty,f))(L_p^\Sigma(\mathbb{Q}_\infty,f\otimes\chi_K)), \tag{74}$$

where α is the image of $\widetilde{\alpha}$. The desired divisibility for the imprimitive L-functions and Selmer groups follows by [47, Proposition 3.2.18]. □

A.4. Cyclotomic Iwasawa theory: general case.

A.4.1. Kato gave a formulation [27] of the cyclotomic main conjectures which also applies to non-ordinary primes. For any finite set of rational primes Σ, consider the strict Selmer group:

$$\mathrm{Sel}_{\mathrm{str}}(\mathbb{Q}_\infty, W_f) = \ker\left(H^1(\mathbb{Q}, \mathbf{W}_f) \to \prod_{v \notin \Sigma \cup \{p\}} H^1(I_v, \mathbf{W}_f) \times H^1(\mathbb{Q}_p, \mathbf{W}_f)\right),$$

and denote by $Ch^{\Sigma}_{\mathrm{Kato},f} \subset \Lambda_{\mathbb{Q}_\infty}$ the characteristic ideal of its Pontryagin dual.

The Iwasawa cohomology $H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbf{T})$ (for Σ the set of primes dividing Np) is free of rank one when $\overline{T}_f$ is absolutely irreducible, and under that hypothesis Kato defined an element $z_{\mathrm{Kato}} \in H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbf{T})$ which is closely related to L-values of f.

Remark A.4.2. In [27, Theorem 12.4], it is only asserted that z_{Kato} is integral when the image of the Galois action on T_f contains a conjugate of $SL_2(\mathbb{Z}_p)$. However, the proof (13.14 of *loc. cit.*) only requires the fact that any two $\mathcal{O}$-lattices in $T_f \otimes \mathbb{Q}_p$ are homothetic, which holds whenever $\overline{T}_f$ is absolutely irreducible. Note that $z_{\mathrm{Kato}} = \mathbf{z}_\gamma^{(p)}$ in the notation of [27], where $\gamma \in T_f$ is the sum of any generators of the two eigenspaces for complex conjugation.

A.4.3. Kato's main conjecture [27, Conjecture 12.10] then asserts that:

$$\mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}}\left(\frac{H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbf{T})}{\Lambda_{\mathbb{Q}_\infty} \cdot z_{\mathrm{Kato}}}\right) = Ch_{\mathrm{Kato},f}. \tag{75}$$

If $\wp$ is ordinary, then this conjecture is equivalent to the usual cyclotomic main conjecture for f by [27, 17.13]. We abbreviate by $Z_{\mathrm{Kato},f} \subset \Lambda_{\mathbb{Q}_\infty}$ the left side of (75).

The following is the analogue of Theorem A.3.2 in the non-ordinary case, due to Wan.

Theorem A.4.4 (Wan). *Let K be an imaginary quadratic field in which p splits. Assume that $\wp$ is a good, non-ordinary prime for f and that:*

- *For all primes $\ell | N$ ramified in K, that $T_f|_{G_{\mathbb{Q}_\ell}}$ is Steinberg with sign -1. At least one such prime exists.*
- *For all $\ell | N$, ℓ is either split or ramified in K.*

Then

$$Ch_{\mathrm{Kato},f} \cdot Ch_{\mathrm{Kato},f\otimes\chi_K} \subset Z_{\mathrm{Kato},f} \cdot Z_{\mathrm{Kato},f\otimes\chi_K} \text{ in } \Lambda_{\mathbb{Q}_\infty}.$$

Proof. This is proven in [51, p. 29]; compare to the proof of Corollary 3.32, where it is assumed that $Z_{\mathrm{Kato},f\otimes\chi_K} \subset \mathrm{Char}_{\mathrm{Kato},f\otimes\chi_K}$. □

Additionally, Kato [27] has proven one direction of his conjecture in our setting:

Theorem A.4.5 (Kato). *Let f be a modular forms of weight two, level N, and trivial character, and $\wp \subset \mathcal{O}_f$ a prime of good reduction with odd residue characteristic. Then $Ch_{\mathrm{Kato},f} \neq 0$ and*

$$Z_{\mathrm{Kato},f} \subset Ch_{\mathrm{Kato},f}$$

in $\Lambda_{\mathbb{Q}_\infty} \otimes \mathbb{Q}_p$. In particular, if a_p is a $\wp$-adic unit, then $\mathrm{Sel}(\mathbb{Q}_\infty, W_f)$ is $\Lambda_{\mathbb{Q}_\infty}$-cotorsion and

$$L_p(\mathbb{Q}_\infty, f) \subset Ch_{\mathbb{Q}_\infty,f}$$

in $\Lambda_{\mathbb{Q}_\infty} \otimes \mathbb{Q}_p$. If the image of the Galois action on T_f contains $SL_2(\mathbb{Z}_p)$, then all of the inclusions hold in $\Lambda_{\mathbb{Q}_\infty}$.

A.4.6. Denote by $\mu(f)$ the μ-invariant of $Ch_{\mathbb{Q}_\infty,f}$ or $Ch_{\mathrm{Kato},f}$ in the ordinary or non-ordinary case, respectively. To control the powers of $\wp$ in Theorem A.4.5, we will use the following.

Lemma A.4.7. *Let f and g be modular forms of weight two and trivial character such that $\overline{T}_f$ is absolutely irreducible. Suppose that f and g have a congruence modulo $\wp^j$, i.e. there is a common completion $\mathcal{O}$ of $\mathcal{O}_f$ and $\mathcal{O}_g$ and, in some basis, a congruence of $\mathcal{O}$-valued associated Galois representations*

$$T_f \equiv T_g \pmod{\wp^j}.$$

If $\mu(f) < j$, then $\mu(g) = \mu(f)$.

Proof. For the sake of notation, assume $\wp$ is ordinary; this makes no difference to the proof. By [18], $\mu(f)$ is also the μ-invariant of $Ch^{\Sigma}_{\mathbb{Q}_\infty,f}$ for any finite set of primes Σ, and likewise for g. If Σ contains all primes dividing the level of either f or g, then we have:

$$\mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty, W_f)[\wp^j] \simeq \mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty, W_g)[\wp^j] \tag{76}$$

as $\Lambda_{\mathbb{Q}_\infty}$-modules. Let M_f and M_g be the Pontryagin duals of $\mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty, W_f)$ and $\mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty, W_g)[\wp^j]$, respectively, and let $\mathfrak{P} = (\wp) \subset \Lambda_{\mathbb{Q}_\infty}$. Then we have a congruence

$$M_f \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j \simeq M_g \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j.$$

Since $\mu(f) = \lg M_{f,(\mathfrak{P})} < j$, where $(\mathfrak{P})$ denotes the localization,

$$M_{f,(\mathfrak{P})} \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j = M_{f,(\mathfrak{P})} \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^{j-1}, \tag{77}$$

which implies the same for g. Therefore $M_{g,(\mathfrak{P})} \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j = M_{g,(\mathfrak{P})}$ and the result follows. □

Proof of Theorem A.1.1. Let us suppose first that A_f has non-split toric reduction at ℓ_0 if $\wp$ is non-ordinary. Fix once and for all an auxiliary quadratic imaginary field $\mathcal{K}$, not contained in the fixed field $K(T_f)$, such that:

- If $\wp$ is ordinary, then ℓ_0 is inert in $\mathcal{K}$ and every other factor of Np is split in $\mathcal{K}$.
- If $\wp$ is non-ordinary, then ℓ_0 is ramified in $\mathcal{K}$ and every other factor of Np is split in $\mathcal{K}$.

As in the proof of Theorem 8.2.1, begin by applying Proposition 7.2.6 and Theorem 7.1.5 to obtain some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, represented by Q_n, and a resulting sequence of newforms g_n of NQ_n; we make sure to choose each $\mathsf{q} \in \mathsf{Q}$ such that $\mathrm{Frob}_{\mathsf{q}}$ has trivial image in $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$, which is clearly possible.

Claim. There exists a constant C, depending only on f, such that

$$v_\wp\left(\frac{L(g_n/K,1)}{\Omega^{can}_{g_n}}\right) \leq \lg_{\mathcal{O}} \mathrm{Sel}(K, W_{g_n}) + \sum_{\ell | NQ_n} t_{g_n}(\ell) + C \tag{78}$$

for $\mathfrak{F}$-many n.

Proof of claim. Consider first the ordinary case. By Lemma A.4.7 and Theorem A.4.5,

$$\wp^{\mu(f\otimes\chi_{\mathcal{K}})} \cdot (L_p(\mathbb{Q}_\infty, g_n \otimes \chi_{\mathcal{K}})) \subset Ch_{\mathbb{Q}_\infty, g_n\otimes\chi_{\mathcal{K}}} \text{ in } \Lambda_{\mathbb{Q}_\infty} \tag{79}$$

for $\mathfrak{F}$-many n. By Theorem A.3.2 for g_n, for $\mathfrak{F}$-many n we have

$$(\alpha) \cdot \wp^{\mu(f\otimes\chi_K)} \cdot Ch_{\mathbb{Q}_\infty,g_n} \cdot Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_{\mathcal{K}}} \subset (L_p(\mathbb{Q}_\infty, g_n)) \cdot Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_{\mathcal{K}}}, \tag{80}$$

where, by Proposition A.2.7, $\mathbb{1}(\alpha)$ divides $\wp^{c_f(\ell_0)}$ in $\mathcal{O}$. Since $Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_{\mathcal{K}}} \neq 0$, and since characteristic ideals are divisorial, we conclude that

$$(\alpha) \cdot \wp^{\mu(f\otimes\chi_{\mathcal{K}})} \cdot Ch_{\mathbb{Q}_\infty,g_n} \subset (L_p(\mathbb{Q}_\infty, g_n)). \tag{81}$$

Applying the same argument to $g_n \otimes \chi_K$, we have:

$$(\alpha)^2 \cdot \wp^{\mu(f\otimes\chi_K\otimes\chi_{\mathcal{K}})+\mu(f\otimes\chi_{\mathcal{K}})} \cdot Ch_{\mathbb{Q}_\infty,g_n} \cdot Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_K} \subset (L_p(\mathbb{Q}_\infty, g_n) \cdot (L_p(\mathbb{Q}_\infty, g_n \otimes \chi_K)). \tag{82}$$

The result now follows from standard interpolation properties of both sides of (82), cf. e.g. [47, Theorem 3.6.11].

The non-ordinary case is similar: combining Theorem A.4.4 and Theorem A.4.5, we have for $\mathfrak{F}$-many g_n

$$\wp^{\mu(f\otimes\chi_{\mathcal{K}})} \cdot Ch_{\mathrm{Kato},g_n} \subset Z_{\mathrm{Kato},g_n},$$

and likewise for the twist $g_n \otimes \chi_K$. Since Z_{Kato,g_n} is principal, the result follows as in [51, Corollary 3.35]. Note that the p-adic Tamagawa factor appearing there is trivial by [2, Proposition II.2]. □

As in Step 3 of the proof of Theorem 8.2.1, $\#\,\mathrm{Sel}(K, W_{g_n}) = \#\,\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f) < \infty$ for $\mathfrak{F}$-many n (the local conditions at $v|N$ may be compared in the same way, and at $v|p$ we use [20, Lemma 7]).

Now, by combining the claim above with Proposition A.2.4, we have for $\mathfrak{F}$-many n:

$$v_\wp\left(\frac{L(g_n/K,1)}{\Omega_{g_n,N^-Q_n}}\right) \leq \lg_{\mathcal{O}} \mathrm{Sel}(K, W_{g_n}) + \sum_{\ell | N^+} t_{g_n}(\ell) + C' \tag{83}$$

for a constant C' that does not depend on n. In particular, for $\mathfrak{F}$-many n, $L(g_n/K,1) \neq 0$, which by parity considerations implies that $\nu(N^-) + |\mathsf{Q}|$ is odd. Exactly as in Step 2 of the proof of Theorem 8.2.1, we then conclude from (83) that $\lambda(1,\mathsf{Q}) \neq 0$.

If we assume instead that $\wp$ is non-ordinary but the image of the Galois action on T_f contains a conjugate of $SL_2(\mathbb{Z}_p)$, then, rather than choosing $\mathcal{K}$ at the beginning, we choose $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ as in the proof of Theorem 8.2.1, but we also ensure that $\epsilon_{\mathsf{Q}}(\mathsf{q}) = -1$ for at least one q. If $\{Q_n, \epsilon_{Q_n}\}$ represents $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}$ and $q_n|Q_n$ represents q, then for each n, we choose an auxiliary imaginary quadratic field $\mathcal{K}_n$ such that q_n is ramified in $\mathcal{K}_n$ and all other factors of NQ_np are split. Note that, by the proof of [17, Lemma 6.15], the image of the Galois action on T_{g_n} contains a conjugate of $SL_2(\mathbb{Z}_p)$ for $\mathfrak{F}$-many n. The argument is then the same as above; $\mu(f \otimes \chi_{\mathcal{K}_n})$ and $\mu(f \otimes \chi_{\mathcal{K}_n} \otimes \chi_K)$ may not be uniformly bounded in n, but by Kato's result these error terms are not needed under the large-image hypothesis. □

Remark A.4.8. In all cases, we crucially use the prime ℓ_0 to make the error terms uniform in n.

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