ECON3102-005
CHAPTER 6: ECONOMIC GROWTH:
THE SOLOW GROWTH MODEL (PART 1)

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Motivations

• Why do countries grow? Why are there poor countries? Why are there rich countries? Can poor countries be rich? If they cannot, why? If they can, why are they still poor?
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- As Robert Lucas put it, “Once you start thinking about growth, it’s hard to think about anything else.”

- We’ll use the framework we have learned and try to get some answers to the questions above. Now, there are some empirical facts that could help to motivate the discussion.
Observations

• Before the industrial revolution, standards of living differed little over time and across countries.
• Since the industrial revolution, per capita income growth has been sustained in the richest countries. In the US, average annual growth in per capita income has been about 2% since 1869.

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The Solow Growth Model

First, consider the consumers in the economy. We’ll add some dynamics here, as we analyze the economy in terms of the current and future periods. We also throw in some assumptions:

• Population $N$ grows at an exogenous rate $n$, following the equation $N' = (1 + n)N$, $\forall n > 1$.

• In each period, the consumer has one unit of time available. Consumers do not value leisure, so labor supply equals one. Then, the population equals the labor force: $N$ represents both the number of workers and the population, and $n$ is its growth rate.

• There is no government; consequently, no taxes.
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The Solow Growth Model (cont’d)

• Consumers receive $Y$, current real output, as income. They face the decision of how much of current income to save and how much to consume. We assume they consume a constant fraction of income:

$$C = (1 - s)Y, \quad s < 1,$$

where $C$ is current consumption, $s$ the savings rate, and current savings are $S = sY$. 
Consider the representative firm.

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- Since it is CRS, dividing (1) by \( N \) gives
  \[ \frac{Y}{N} = zF\left(\frac{K}{N}, \frac{N}{N}\right) = zF\left(\frac{K}{N}, 1\right) \]  
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- Here we let \( \frac{Y}{N} \) be output per worker, and \( \frac{K}{N} \) capital per worker. Then (2) tells that the output per worker depends on the capital per worker.
The Solow Growth Model (cont’d)

• Rewrite (2) as

\[ y = zf(k), \quad (3) \]

where \( y = Y/N, \ k = K/N, \ f(k) = F(k, 1). \)
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• Here, depreciation is denoted by $d$, where $0 < d < 1$.

• Now we can talk about dynamics. Given the depreciation rate, the capital stock changes over time according to

$$K' = (1 - d)K + I,$$

where $I$ denotes investment.
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In this economy there are two markets: labor and assets.

- In the labor market, current consumption goods are traded for current labor.
- In the assets market, current consumption goods are traded for capital.
- Capital is the asset in this economy, and consumers save by accumulating it.
The Solow Growth Model (cont’d)

- Note that the labor market clears at the inelastic supply of labor, \( N \). (It follows that \( w \) adjusts automatically.)

- Let \( S \) be the aggregate amount of savings in the current period. Then, the capital market is in equilibrium if \( S = I \); since \( S = Y - C \), this can be expressed as:
  \[
  Y = C + I.
  \]

- Substituting for \( I \) and \( C \) from equations, \( K' = (1 - d)K + I \) and \( C = (1 - s)Y \), gives
  \[
  Y = (1 - s)Y + K' - (1 - d)K.
  \]

  (5)

- Equation (5) says that future capital equals the amount of savings plus capital left over from the current period that has not depreciated.
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The Solow Growth Model (cont’d)

• Using equations, \( Y = zF(K, N) \) and \( K' = sY + (1 - d)K \), we have

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• We can divide by \( N \) to express in per worker terms

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\frac{K'}{N} = szF\left(\frac{K}{N}, 1\right) + (1 - d)\frac{K}{N},
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multiplying the first term by \( 1 = N'/N' \)

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\frac{K' \cdot N'}{N \cdot N'} = szF\left(\frac{K}{N}, 1\right) + (1 - d)\frac{K}{N},
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- which is

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k'(1 + n) = szf(k) + (1 - d)k.
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The Solow Growth Model (cont’d)

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- Dividing across by \( (1 + n) \) gives the key equation of the model:
\[ k' = \frac{\text{szf}(k)}{1 + n} + \frac{(1 - d)k}{1 + n} \] (*)
We want to find the steady state of the model. This is, the point at which $k' = k = k^*$. 
Steady States

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- Note that when we graph in $k'k$ space, any point that crosses the 45 degree line satisfies $k' = k$.  

![Diagram showing the 45 degree line and the point A where $k' = k = k^*$]
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Steady State in the Solow Growth Model

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• At the steady state, \( k = k^* \) and \( k' = k^* \); \( k^* \) is the equilibrium level of capital in the economy.
Suppose $k < k^*$. Then $k' > k$, and the capital stock increases from the current to the future period, until $k = k^*$. 

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• Suppose $k < k^*$. Then $k' > k$, and the capital stock increases from the current to the future period, until $k = k^*$.

• Here, current investment is relatively large with respect to depreciation and labor force growth.
Steady State Growth Rates

- What is the growth rate of $k^*$?

The answer: zero.

Why? Since it's a steady state, it won't move from there.

Another question: What is the growth rate of $y^*$?

The answer: zero.

Why? Since $k = k^*$ in the long run, output per worker is constant at $y^* = zf(k^*)$.

So, there's no growth here? Are we forgetting something?
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- The aggregate quantity of capital is \( K = k^*N \). Since \( k^* \) is constant and \( N \) grows at a rate \( n \), \( K \) should grow at a rate \( n \).
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• In this way, the Solow growth model is an exogenous growth model.