Optimal Taxation in Overlapping Generations Economies with Aggregate Risk

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Abstract

How should governments leverage available policy instruments to raise revenue and share aggregate risk across generations? I address this question by developing a framework for analyzing optimal taxation in economies with overlapping generations (OLG) and stochastic government spending and productivity. I derive two new, opposing considerations in addition to the classic desire to smooth distortions. First, such economies lack Ricardian equivalence. This encourages governments to run balanced budgets, since deficits drive up interest rates and therefore future tax distortions. Second, the social planner has a redistributive motive across generations and thus faces an equity/efficiency tradeoff. I consider applications to three policy problems: financing of wars, intergenerational sharing of productivity risk, and intergenerational redistribution of trend productivity growth. I find that optimal policy in the first application features partial tax smoothing, with substantially higher labor taxes when government spending is high, but also substantial autocovariance of the labor tax rate. I demonstrate in the latter two applications the optimality of a Social Security program with procyclical benefits; this program is larger if trend productivity growth is more rapid and the planner is more inequality-averse.

Keywords: Optimal taxation, overlapping generations, intergenerational risk sharing, Ramsey taxation, tax smoothing, aggregate risk.

JEL Classification: H21, H23, H55, E62, H63

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1 Introduction

Most economic policies have differential effects on different generations. On the revenue side of the ledger, labor taxation's incidence falls primarily on the young, who make up a large portion of the labor force, while capital taxation's falls primarily on the old, who possess most of the economy's wealth. On the spending side, Medicare and Social Security explicitly redistribute from younger to older citizens, while education subsidies, child tax credits, and the Earned Income Tax Credit primarily benefit younger generations, even if such intergenerational redistribution is not those programs' intended purpose. Important national debates are currently being argued over possible reforms to many of these programs, most visibly Social Security, which faces a funding crisis in the near future. These debates point up a broad economic policy question: How should governments leverage available policy instruments—labor taxes, capital taxes, lump-sum transfers, and debt—to raise revenue and share aggregate risk across generations? This paper develops a framework for analyzing many variants of this problem.

Optimal distortionary taxation in a stochastic, general equilibrium model is not a novel question but instead a classic macroeconomic policy topic, having been extensively analyzed for economies featuring infinitely-lived households. In such models, intergenerational risk sharing is not a concern, leading to exclusive focus on efficiently raising revenue to fund exogenous, stochastic government spending, usually conceptualized as mandatory wars. This problem is often described as the "Ramsey taxation problem." The major contributions to this literature, detailed below, all derive different versions of the same policy guidance: Labor taxation should be nearly constant over time, or "smooth." Specifically, if complete insurance markets are available, then government budget shocks should be perfectly insured, leaving marginal distortionary costs of taxation constant over time; if not, then governments should borrow the full value of budget shocks, leaving marginal distortionary costs of labor taxation to follow a risk-adjusted random walk.

This result predicts well the empirical behavior of governments in response to large

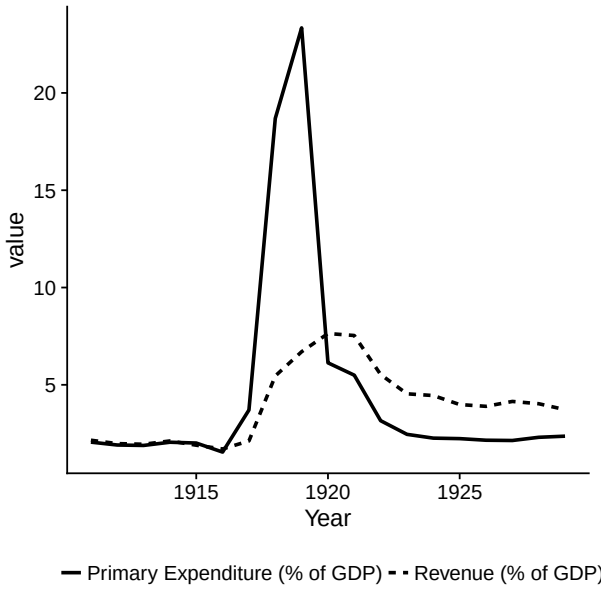
government spending shocks. Figure 1 shows the fiscal response to the largest spending shocks on record—the two World Wars—in the only two countries for which good data exists through the period. While the sample size is small and the data quite noisy, the graphs suggest that developed countries at least approximately follow the random walk advice of an incomplete markets Ramsey model. Indeed, one cannot reject a unit root for the path of revenue as a percentage of GDP.[1]

However, the real world is not, in fact, populated by one infinitely-lived cohort of households but rather by a series of overlapping generations, and these recommended policies are highly inequitable across those generations. Insurance against government spending shocks places that risk disproportionately on older generations, who are likely to have accumulated the most assets and therefore be the most likely counter-parties to the insurance contracts. Government borrowing instead places that risk disproportionately on younger and unborn generations, who will face steeper taxes in the future.

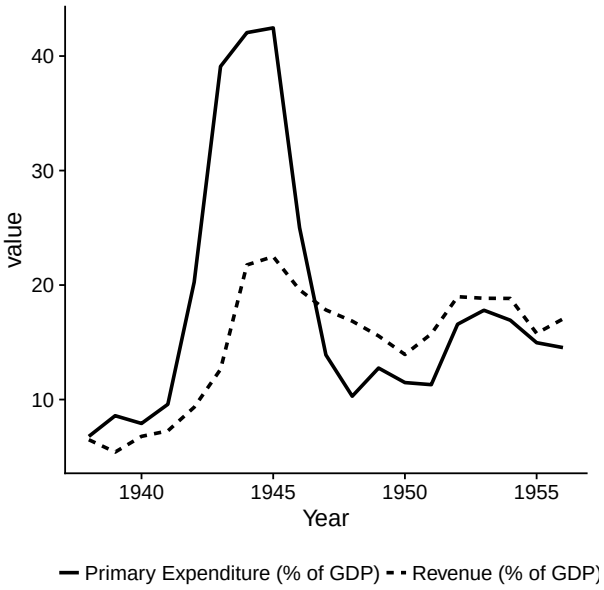
Furthermore, substantial portions of developed countries' government budgets are spent not on purchases of final goods and services, but on old-age transfers. The U.S., for example, spent 41% of its primary[2] federal budget on Social Security and Medicare in 2016 (Center on Budget and Policy Priorities, 2017). The optimality of such spending, and the extensive tax revenue and debt required to fund it, cannot be analyzed within the context of a representative-agent model. More broadly, the Ramsey taxation literature can safely ignore economies without government purchases but with other sorts of aggregate risk, so long as there is a representative agent. Since the welfare theorems apply and there is no heterogeneity, the *laissez-faire* competitive equilibrium is the only Pareto optimal outcome. However, when the economy consists of a series of overlapping generations, there is a continuum of Pareto optimal outcomes, some of which may be preferable to the *laissez-faire* competitive equilibrium from a distribution perspective.

[1] We will see in the numerical section that for certain values of the parameters one can obtain a similar path for optimal policy in an OLG model.

[2] excluding interest on the debt

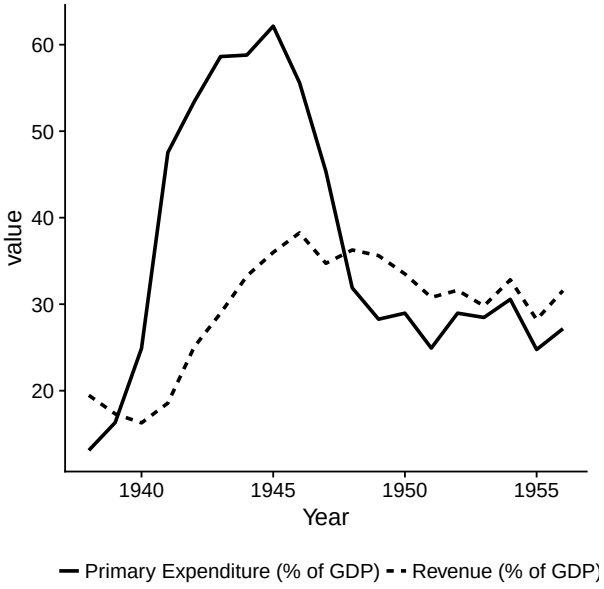


(a) World War I, U.S.



(b) World War II, U.S.

(c) World War I, U.K.



(d) World War II, U.K.

Figure 1: Primary expenditure and revenue as a percentage of GDP for four large government spending shocks.

The framework developed herein addresses these limitations of the existing optimal dynamic taxation literature by making a single, important change to the model: replacing the homogeneous, infinite-horizon households with a series of overlapping generations (OLG). These OLG households have a parameterized bequest motive and (weakly negative) lower limit on their net worth at end of life. Since such an economy no longer features a representative agent, one must posit a social planner with preferences across the utilities of different agents populating the economy. I assume that the planner has weakly inequality-averse preferences across these agents' expected utilities at birth. These assumptions nest those of the dynastic, or Ramsey, model as a special case, while also allowing investigation of the shortcomings of that model.

The implications of this change are far-reaching. Introducing OLG alters the planner's problem in two distinct, conceptually opposing ways. First, it removes "Ricardian equivalence." In a representative-agent model, all possible plans for raising a fixed present value of revenue have the same income effects on household behavior and macroeconomic aggregates. However, in an OLG model, different plans will extract more or less income from a given generation and thereby have different income effects on that generation's behavior. The most direct implication is increased incentive for the government to run a balanced budget, as deficits result in accumulation of government debt, which drives up the interest rate, distorts capital accumulation and intertemporal consumption choices, and requires higher taxes in the future to make the elevated interest payments. This effect is stronger when interest rates respond more strongly to government debt—when the intertemporal elasticity of substitution (IES) is low, when the elasticity of substitution between capital and labor is low, and when access to international capital markets is limited, though the model in this paper is of a closed economy. Additionally, different timing of taxation leads to different labor supplies for the various generations and, in turn, different labor tax revenues. Specifically, delaying taxation makes current generations feel richer and work less, thereby giving the government less tax revenue now, but makes future generations feel poorer and work

harder, thereby giving the government more tax revenue in the future. The net effect is of indeterminate sign, and is stronger when static income effects on labor supply are larger.

Second, as discussed above, OLG adds a redistribution motive to usual efficiency concerns, yielding an equity/efficiency tradeoff reminiscent of static, nonlinear optimal tax problems. This redistribution motive pushes toward smoother tax rates when government spending, and therefore taxes, are the primary reason for inequality across generations. When market forces lead to intergenerational inequality, this will push toward compensatory taxes that are higher on better-off generations. This effect is more pronounced when the planner is more inequality-averse.

The central analytical result of this paper shows how to incorporate these effects in forming optimal policy. In particular, the planner in an OLG economy considers all marginal effects of taxation: mechanical, substitution, and income. After summing these marginal effects at each point in time, the planner weights the result by the social marginal welfare weight of the generation paying the tax—the social value of an additional unit of consumption for that generation. He then smooths, or sets constant over time, that weighted sum—either state-by-state if markets are complete or in expectation if they are incomplete. By way of contrast, the social planner in a representative agent model need only smooth the marginal effect of linear taxation *above and beyond a hypothetical lump-sum tax*—the substitution effect, or "distortionary cost of taxation." This is not because the mechanical and income effects of the lump-sum component are irrelevant or not present, but rather because they are constant over time at the optimum—due to constant social marginal welfare weights and Ricardian equivalence—and thus are *necessarily* "smoothed" and cancel out.

I apply this framework to three distinct policy problems, chosen to highlight different aspects of optimal policy. First, I revisit the classic problem of optimal financing of wars, which highlights the partial tax smoothing properties of optimal policy in this framework. The loss of Ricardian equivalence pushes toward contemporaneous taxation to fund these wars, while the desire for intergenerational equity pushes back toward smooth taxes since

nonconstant taxes are the proximate cause of inequality. Second, I consider an economy without required government spending but with shocks to productivity. Here, taxes should be higher during higher-productivity periods and interest payments on government debt (interpreted as Social Security payments) should be procyclical to improve intergenerational equality.[3] Finally, I incorporate trend productivity growth, which creates a persistent desire to redistribute from later generations to earlier ones. This is performed by increasing the average levels of government debt, interest rates, and therefore payments to retirees, which is funded by increasing the average labor tax rate—a system highly reminiscent of Social Security. Such systematic intergenerational redistribution is larger for more inequality-averse planners, who perceive more benefit from such redistribution and are therefore willing to accept higher distortionary costs from funding it.

As a brief preview of these numerical results, Figure 2 considers the optimal policy response to a completely unanticipated, single-period shock to government spending—an unanticipated war of known duration and size. The figure shows that in the Ramsey model, the optimal response is to fund the war entirely with debt and then permanently increase taxation to cover the ensuing interest payments, but never to repay the principal. Optimal policy in overlapping generations models, on the other hand, features substantial contemporaneous taxation to fund the war, with a return to the previous level of taxation in the long run. The rate of that reversion is inversely related to the degree of concavity in the social welfare function of the planner (parameterized by ζ). In the limit of a "Rawlsian" planner who cares only about the worst-off generation (an infinitely concave social welfare function), taxes never revert to their previous level, but instead remain high to fund compensatory transfers, in the form of large interest payments on debt, to the old. This figure will be presented and analyzed again, with its assumptions more precisely specified, in Section 5; for now, merely consider how quantitatively and qualitatively different the optimal policies are for the representative-agent and OLG models.

[3]This pattern is reversed if income effects dominate.

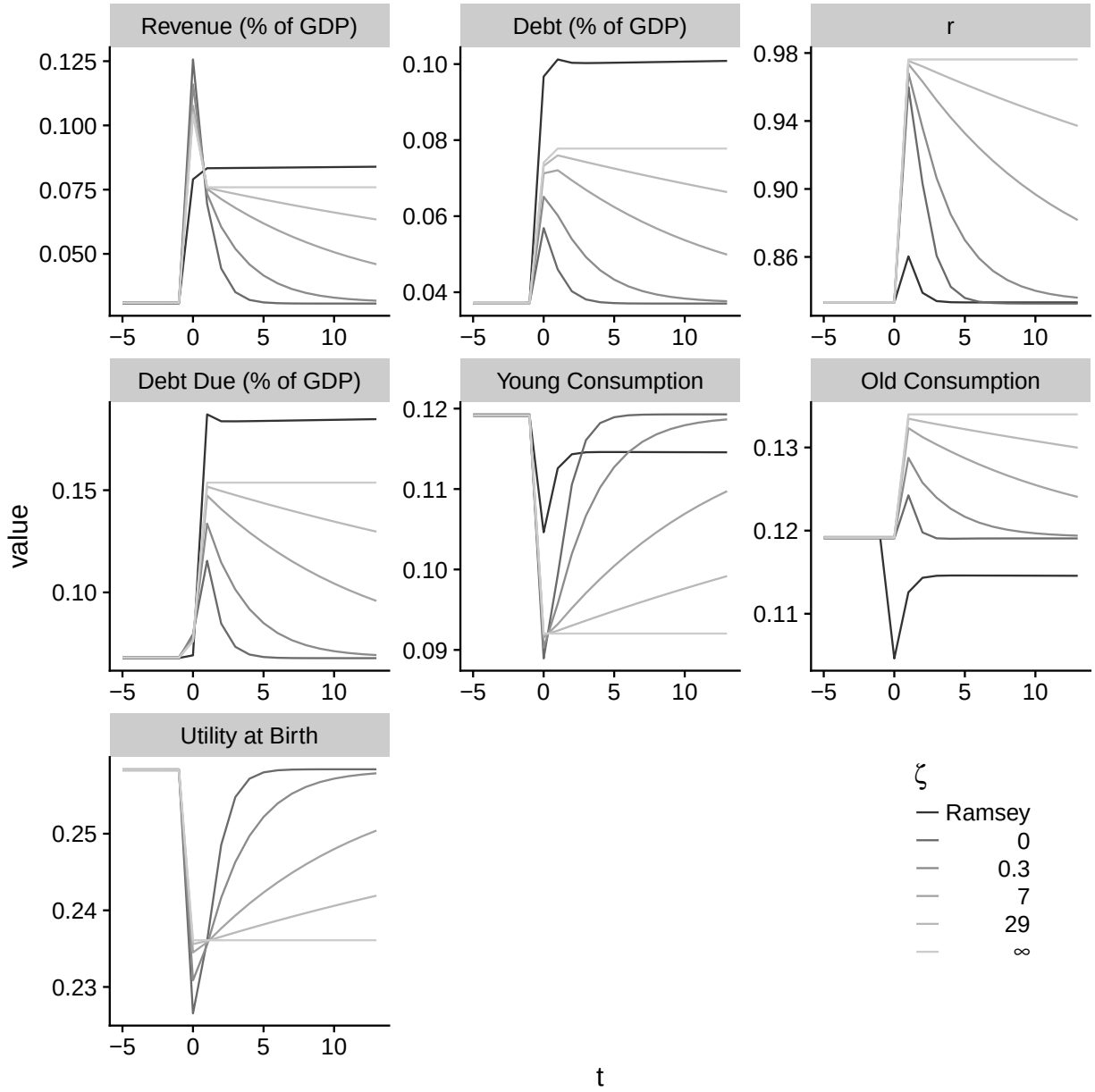


Figure 2: Optimal policy response to a completely unanticipated government spending shock equal in size to 10% of steady state GDP. These graphs show many economic variables each period, where period 0 is the period of the shock. Multiple models are considered: a Ramsey model, along with several OLG models featuring different planners with different levels of inequality-aversion, parameterized by ζ. All models assume isoelastic utility with an IES of 5.

1.1 Related Literature

This project relates to four major strands of economics literature.

Ramsey Taxation. The most directly related of these, referenced above, is commonly known as the Ramsey tax literature. This literature focuses on optimal linear taxation of a variety of goods in a model with a representative agent. Diamond and Mirrlees (1971) found this literature with consideration of the problem in a static context. Later, this problem was reformulated for a dynamic economy, with an eye toward aggregate uncertainty, by Barro (1979), who studies a reduced form model in which a planner seeks to minimize distortionary costs of taxation, which are assumed quadratic in the tax rate. This leads to a martingale property of tax rates. Additional papers in this strand consider richer models involving utility maximization and endogenous distortionary costs of taxation. Major findings include various versions of labor tax smoothing (Lucas and Stokey, 1983) and zero capital taxes (Judd, 1985; Chamley, 1986). A nice summary of this sort of problem can be found in Chari and Kehoe (1999).

Aiyagari et al. (2002) and, later, Farhi (2010) extend this literature by considering models with incomplete markets, in which the government does not have access to state-contingent debt to insure itself against adverse shocks. This leads to a smoothing of distortionary costs of taxation in expectation rather than state-by-state, and an employment of capital taxes[4] to hedge the government budget and manipulate interest rates. Aiyagari (1995) replaces aggregate uncertainty with idiosyncratic productivity shocks among infinitely-lived, *ex ante* homogeneous households and also finds that nonzero capital taxes are optimal.

Many papers analytically reconsider Ramsey taxation within the context of an OLG model (Atkinson and Sandmo, 1980; Escolano, 1992; Erosa and Gervais, 2002; Garriga, 2017). Other papers quantitatively investigate optimal policy in OLG models. For example, Conesa, Kitao and Krueger (2009) looks for optimal labor and capital taxes in a nonstochastic

[4] if capital is present, as in Farhi (2010) but not Aiyagari et al. (2002)

economy where policy must be constant and follow a given parametric form. While these papers might seem similar to the present one in concept, they differ in two critical ways. First, they do not consider aggregate uncertainty and so can only inform policy at a nonstochastic steady state or the transition thereto; they fail to provide guidance on the proper response to stochastic shocks to the government budget. Second, their focus is on capital taxes—especially whether they should be zero in steady state or transition; while I briefly treat capital taxes, my focus is primarily on the response of labor taxation to shocks. Finally, Weinzierl (2011) and Erosa and Gervais (2012) focus on how taxes should vary over the life cycle—a possibility I preclude in the present paper—in a Ramsey problem with idiosyncratic shocks but still no aggregate uncertainty.

Mirrlees Taxation. The other half of optimal taxation literature focuses on nonlinear, redistributive taxation and is named after the seminal contribution of Mirrlees (1971). This paper considers the optimal nonlinear income tax in a static model featuring heterogeneous agents with a continuum of earning abilities. Many recent papers have added realism to this stylized model by considering a dynamic, life-cycle model with gradually unfolding uncertainty. Farhi and Werning (2013) considers households receiving idiosyncratic shocks to their exogenous wage over time and how these shocks might be optimally insured through a fully general, dynamic taxation system. Stantcheva (2016) adds endogenous human capital acquisition—and government policies thereto pertaining—to this model.

This literature relaxes the Ramsey assumptions of linear or affine taxation and representative households, but on the other hand assumes no aggregate uncertainty or general equilibrium effects; instead, it only requires that the government break even in expectation and assumes a constant interest rate and exogenous effective wages. Werning (2007) attempts a compromise by allowing aggregate uncertainty and (in parts of the paper) nonlinear taxation of heterogeneous households. However, this heterogeneity is perfectly persistent over time. That is, some households have higher earning ability than others, but after the

beginning of the model, households face no idiosyncratic uncertainty.

The model in the present paper features nonidentical households and, therefore, a redistributive motive for the planner, while allowing for aggregate uncertainty. However, heterogeneity and idisoyncratic uncertainty are present only in the form of birth cohort; each birth cohort is perfectly homogeneous, and no idiosyncratic uncertainty exists after birth.

Intergenerational Risk Sharing. Third is a literature explicitly focusing on issues of overlapping generations and intergenerational risk sharing. Samuelson (1958) and Diamond (1965) lay the groundwork for the OLG model I use throughout the paper and derive certain results about optimal taxes, but in a context that allows achievement of a first-best outcome through lump-sum taxation. Piketty and Saez (2013) and Farhi and Werning (2010) consider optimal bequest or inheritance taxation in the context of heterogeneous earning ability; I abstract from bequest taxation in the present project, and my numerical simulations focus on calibrations without a bequest motive. Farhi et al. (2012) considers a government lacking access to a commitment technology in an overlapping generations framework, which leads to an incentive to redistribute capital *ex post* in a model featuring intra-cohort heterogeneity but no aggregate uncertainty.

A related literature considers intergenerational risk sharing from outside the context of optimal taxation. Green (1977) asks whether social insurance against uncertain population growth in an overlapping generations model could be designed in a Pareto-improving manner; he concludes that such is analytically possible, but not for reasonable values of the parameters. Ball and Mankiw (2007) consider the problem of intergenerational risk sharing not from the point of view of a utilitarian social planner, but rather by asking what the complete markets outcome would be if individuals could trade Arrow-Debreu securities behind a veil of ignorance. They then consider implementing that allocation using more conventional taxation tools. Other, even more abstract papers (Gordon and Varian, 1988) exist as well.

OLG Tax Incidence. Finally is the literature assessing the incidence and welfare effects of policy perturbation in large-scale, quantitative, OLG simulation models. Auerbach and Kotlikoff (1987), Kotlikoff, Smetters and Walliser (1999), and Altig et al. (2001) consider numerous policy reforms, either small or large, in a 55-generation OLG model with perfect forsight—that is, without uncertainty—and analyze the transition path. They then assess which demographic groups are better- and worse-off under the reform. Though these papers involve models that are much richer than the present paper's, they are fundamentally answering an incidence question rather than an optimal policy question, while also abstracting from uncertainty. Another set of similar papers directly concerns Social Security—either its optimality, or consideration of specific reforms—rather than broader policy questions. As an example of the former, Harenberg and Ludwig (2014) find that introduction of a pay-as-you-go ("PAYGO") Social Security system is optimal when there are interacting idiosyncratic and aggregate risks, in contrast to, among others, Krueger and Kubler (2006), which considers only aggregate risk. As an example of the latter, Feldstein (1998) covers the subject of privatizing Social Security—one oft-discussed reform—in great detail.

The remainder of the paper is ordered as follows: In Section 2, I describe the economy and formally state the problem faced by the social planner. In Section 3, I build intuition by characterizing optimal policy in two cases leading to extreme policies: quasilinear utility, and log-separable utility. Section 4 characterizes optimal policy more generally, in models with either recursively complete or incomplete markets. Section 5 applies the model numerically to three different policy problems to give concreteness to the discussion. Finally, robustness to addition of further generations is tested in Section 6, and Section 7 concludes. All proofs can be found in the Appendix.

2 Model

In this section, I describe the economy and the policies available to the government. Afterwards, I define a few pieces of notation that will be useful for subsequent discussion of optimal policy. The economy is a closed, neoclassical overlapping generations (OLG) economy with two generations, aggregate risk, discrete time, elastic labor supply, and capital. The nature of the overlapping generations component is designed to nest a traditional infinite-horizon model. There are two variants of the model: one with recursively complete markets, and one with incomplete markets; these variants differ only in the available assets and policy instruments.

2.1 Uncertainty

Aggregate risk is described by a discrete set of states $s_t \in \mathcal{S}$ and histories of those states $s^t = (s_0, s_1, ...s_{t-1}, s_t)$. The state of the world s_t evolves according to a Markov process with transition matrix $\boldsymbol{P}$. Exogenous, required government spending g and labor-augmenting productivity A are each functions of the state of the world s_t (not its history), which captures any uncertainty, and t, which captures any trend growth: $g(s^t) = g(s_t, t)$ and $A(s^t) = A(s_t, t)$. There is no idiosyncratic uncertainty.

2.2 Available Assets

The only asset in positive net supply is productive and risky capital. In addition, other assets in zero net supply exist as below, depending on the variant of the model.

Complete Markets At each history s^t, a market opens for a set of state-contingent assets delivering consumption at date $t+1$. Specifically, for each $s_{t+1} \in \mathcal{S}$, there exists an asset which delivers one unit of consumption at history (s^t, s_{t+1}) and costs $q(s^t, s_{t+1})$ units of consumption at history s^t.

Despite the name, this model does not feature truly complete markets. Individuals may not purchase insurance against the generation or state of the world into which they are born. This limits the ability of a social planner to efficiently distribute risk between generations and generates the fundamental economic problem this model is designed to analyze.

Incomplete Markets The only other asset is a one-period risk free bond. For each s^t, this bond costs one unit of consumption at history s^t and delivers $R^f(s^t)$ units of consumption at all $s^{t+1} \succeq s^t$.

2.3 Agents

The economy consists of three types of agents: households, firms, and the government.

Government. I abstract from commitment issues and focus on a government with access to a perfect commitment technology.

Complete Markets. The government has access to linear taxes on labor income $\tau^L(s^t)$ and capital income $\tau^K(s^t)$, as well as non-negative, lump-sum transfers to the young $T(s^t)$.[5] Capital income taxes may be state-contingent, and as a result it can be assumed without loss of generality that they are levied on gross capital income. The government also has access to the state-contingent asset market, allowing it to structure its portfolio of assets and debt in such a way as to provide insurance. Thus, the government's budget constraint at history s^t is

$$b(s^t) + g(s_t, t) + T(s^t) \leq \sum_{s^{t+1} \succeq s^t} q(s^{t+1})b(s^{t+1}) + \tau^L(s^t)w(s^t)\ell(s^t) + \tau^K(s^t)R^K(s^t)k(s^{t-1}) \quad (1)$$

[5]This is equivalent to assuming age-dependent lump-sum transfers $T^y(s^t)$ and $T^o(s^t)$; since markets are recursively complete, anticipated transfers to the old can simply be converted into their present value by the young.

where b is the government's debt due, w is the pre-tax wage, ℓ is total labor supply, R^K is the gross return to capital, and k is the level of capital. Without loss of generality, I assume that state-contingent assets are untaxed.

Incomplete Markets. The government behaves similarly if markets are incomplete, with a few important changes. Most importantly, the government no longer has access to the state-contingent asset market; it can only trade the risk free bond. Second, capital taxes may no longer be state contingent but must be set one period in advance. Finally, lump-sum transfers may no longer be assumed to accrue only to the young, since households cannot convert anticipated, but uncertain, old age transfers into their present value. Thus, the government's budget constraint becomes

$$R^f(s^{t-1})b(s^{t-1})+g(s_t,t)+T^y(s^t)+T^o(s^t) \leq b(s^t)+\tau^L(s^t)w(s^t)\ell(s^t)+\tau^K(s^{t-1})R^K(s^t)k(s^{t-1}), \tag{2}$$

where b is now the government's debt issued, and other variables are unaltered. To avoid an unrealistic outcome in which the government accumulates sufficient assets to pay for all expenditures with interest on those assets, I impose a lower bound $\underline{b}(t)$ (which should be thought of as nonpositive) on government debt. To seriously enforce the idea that government debt is risk free, I impose an upper bound $\overline{b}(t)$ as well. Similar constraints are imposed by Aiyagari et al. (2002) and Farhi (2010).

Households. A series of overlapping generations of households live for two periods each, and there is no population growth. I abstract from the desire for intra-cohort redistribution by assuming that all members of a cohort are identical. They inherit $z(s^t)$, provide labor $\ell(s^t)$, and consume $c^y(s^t)$ during youth. During old age they consume $c^o(s^{t+1})$ and leave a bequest

$$z(s^{t+1}) \geq \underline{z}(s_{t+1}, t+1). \tag{3}$$

They face a market wage $w(s^t)$, a lump-sum transfer $T(s^t)$, and a vector of asset prices $\boldsymbol{q}(s^{t+1})$. A household born at history s^t ranks allocations recursively according to[6]

$$U(s^t) = u(c^y(s^t), \ell(s^t)) + \beta \mathbb{E}_t[u(c^o(s^{t+1}), 0)] + \delta \mathbb{E}_t[U(s^{t+1})]. \tag{4}$$

δ parameterizes the degree of altruism toward offspring, or bequest motive, while $\underline{z}(s_t, t)$ limits the extent to which households may pass on debt to their heirs.[7] $\delta = \beta$ and $\underline{z} = -\infty$ corresponds to the traditional infinite-horizon model, with each generation allowed to die in an unlimited amount of debt but caring equally about its offspring as its old-age self. $\delta = 0$ and $\underline{z} = 0$ corresponds to the stark OLG economy, with households that do not care about their offspring and are not allowed to die in debt.

If markets are complete, households face a single budget constraint in present value terms:

$$c^y(s^t) + \sum_{s^{t+1} \succeq s^t} q(s^{t+1})(c^o(s^{t+1}) + z(s^{t+1})) \leq T(s^t) + z(s^t) + (1 - \tau^L(s^t))w(s^t)\ell(s^t) \tag{5}$$

If markets are incomplete, households face period-by-period budget constraints:

$$c^y(s^t) + k(s^t) + b(s^t) \leq T^y(s^t) + z(s^t) + (1 - \tau^L(s^t))w(s^t)\ell(s^t) \tag{6}$$

$$c^o(s^{t+1}) + z(s^{t+1}) \leq T^o(s^{t+1}) + R^f(s^t)b(s^t) + (1 - \tau^K(s^t)R^K(s^{t+1})k(s^t) \quad \forall s^{t+1} \succeq s^t \tag{7}$$

Households thus maximize (4) subject to (5) and (3) if markets are recursively complete, or subject to (6), (7), and (3) if markets are incomplete.

Firms. Firms have a constant returns to scale production technology $F(k, \ell; A)$, including returned capital net of depreciation. Facing given prices $w(s^t)$ for wages and $R^K(s^t)$ for

[6] I have imposed the assumption that utility is time-separable for ease of exposition. The results could easily be extended to cover non-time-separable utility.

[7] It should be thought of as nonpositive, though I do not formally impose that assumption.

capital, firms simply statically maximize over k and ℓ

$$F(k, \ell; A) - w(s^t)\ell - R^K(s^t)k.$$

2.4 Equilibrium and Implementability

Equilibrium is standard, consisting of prices, policies, and quantities such that agents optimize subject to budget constraints and markets clear. The formal definition is given in Appendix A.

To this point, I have been deliberately vague about the issue of the first period policy problem. As mentioned earlier, I assume a government with access to perfect commitment. This is not an innocuous assumption. As is typical of similar optimal taxation problems, the optimal unrestricted policy is time-inconsistent; the government wishes to confiscate wealth initially, but promise to never do so in the future. Fortunately, government behavior in the initial period is not important to the analytical results to follow, nor to many of the numerical simulation results, assuming a sufficient "start up period" that is ignored. However, in certain of the numerical simulations, initial conditions remain important in perpetuity. To ensure more realistic numerical solutions, I will forbid capital confiscation in the first period by assuming a zero capital tax rate, which is either exactly or approximately the nonstochastic steady-state optimal capital tax rate.

The optimal policy problem will be formulated using the well-known "primal approach," in which I search directly for the optimal allocation and afterward derive the supporting policies. To do so, I must characterize which allocations are achievable for some set of policies.

Definition 2.1 (Implementable Allocation) *An allocation is implementable if there exists a policy equilibrium of which it is a part.*

The conditions that characterize implementable allocations are similar between the complete and incomplete markets models, but I will specify both sets of conditions fully, as these

form the basis of the social planner's problems to follow.

Proposition 2.1 (Implementability Conditions—Complete Markets) *An allocation* $\{c^o(s^t), c^y(s^t), \ell(s^t), k(s^t), z(s^t)\}_{t\geq 0}$ *is implementable in the complete markets model iff it*

- *Satisfies the resource constraint at each* $s^t, t \geq 0$*:*

$$c^y(s^t) + c^o(s^t) + g(s_t, t) + k(s^t) \leq F(k(s^{t-1}), \ell(s^t); A(s_t, t)) \tag{8}$$

- *Satisfies the implementability condition at each* $s^t, t \geq 0$*:*

$$u_c^y(s^t)c^y(s^t) + u_\ell^y(s^t)\ell(s^t) + \beta\mathbb{E}_t[u_c^o(s^{t+1})(c^o(s^{t+1}) + z(s^{t+1}))] \geq u_c^y(s^t)z(s^t) \tag{9}$$

- *Satisfies the following constraint on the initial old:*

$$c^o(s_0) + z(s_0) \geq b(s_0) + F_K(k_-, \ell(s_0); A(s_0))k_- \tag{10}$$

- *Satisfies the optimal bequest conditions at each* $s^t, t \geq 0$*:*

$$\beta u_c^o(s^t) \geq \delta u_c^y(s^t) \tag{11a}$$

$$z(s^t) \geq \underline{z} \tag{11b}$$

$$(\beta u_c^o(s^t) - \delta u_c^y(s^t))(z(s^t) - \underline{z}) = 0 \tag{11c}$$

Proposition 2.2 (Implementability Conditions—Incomplete Markets) *An allocation* $\{c^o(s^t), c^y(s^t), \ell(s^t), k(s^t), z(s^t)\}_{t\geq 0}$ *is implementable in the incomplete markets model iff it, together with some sequence* $\{b(s^t)\}_{t\geq 0}$

- *Satisfies the resource constraint (8) at each* $s^t, t \geq 0$

- *Satisfies the implementability condition for the young at each $s^t, t \geq 0$:*

$$u_c^y(s^t)(c^y(s^t) + k(s^t) + b(s^t)) + u_\ell^y(s^t)\ell(s^t) \geq u_c^y(s^t)z(s^t) \tag{12}$$

and the implementability condition for the old at each $s^t, t \geq 1$:

$$c^o(s^t) + z(s^t) \geq \frac{u_c^y(s^{t-1})}{\beta \mathbb{E}_{t-1}[u_c^o(s^t)]} b(s^{t-1}) + \frac{F_K(s^t)u_c^y(s^{t-1})}{\beta \mathbb{E}_{t-1}[F_K(s^t)u_c^o(s^t)]} k(s^{t-1}) \tag{13}$$

- *Satisfies the following constraint on the initial old:*

$$c^o(s_0) + z(s_0) \geq b_{-1} + F_K(k_-, \ell(s_0); A(s_0))k_- \tag{14}$$

- *Satisfies the optimal bequest conditions (11) at each $s^t, t \geq 0$*

- *Satisfies the debt limits at each $s^t, t \geq 0$:*

$$\underline{b}(t) \leq b(s^t) \leq \overline{b}(t) \tag{15}$$

2.5 Social Planner

There exists a social planner, who must choose among the implementable allocations. He ranks allocations according to

$$\Delta^{-1} W[u(c_-^y, \ell_-) + \beta u(c^o(s_0), 0)] + \mathbb{E}_0 \sum_{t=0}^{\infty} \Delta^t W[u(c^y(s^t), \ell(s^t)) + \beta \mathbb{E}_t u(c^o(s^{t+1}), 0)] \tag{16}$$

where $W(\bullet)$ is a social welfare function, usually assumed to be weakly concave, and c_-^y and ℓ_- are young consumption and labor from last period. A few comments about the social planner's objective function are in order. First, he discounts later generations relative to earlier ones according to a social discount factor Δ. This should not be seen as a strong

political economy assumption, but instead simply as a requirement to ensure a solution. Second, he values only that utility derived directly by the households—not the altruism they feel toward their descendants. This makes sense given that the planner directly values those descendants. Third, the argument of the social welfare function is *ex ante* expected lifetime utility. This allows a planner to have redistributive preferences across generations and across cohorts born at different histories s^t at the same time t, but not across households born at the same history s^t but experiencing different shocks in old age s_{t+1}. While inserting realized *ex post* utility as the argument of $W(\bullet)$ would allow such, it also fails to respect individual preferences over risk; if $W(\bullet)$ were strictly concave and the argument were realized *ex post* utility, then the social planner would be more risk averse than households and likely choose a constrained Pareto inefficient allocation. This assumption—that the planner respects individual preferences over risk—plays an important role by reducing the planner's desire to insure individuals against shocks they may face in old age relative to a more naive view one might take. Finally, if $W(\bullet)$ is strictly concave, notice that the planner's objective is not time-separable. For example, if an existing cohort of households experienced very poor youths, the government may wish to compensate them in their old age—a motive that cannot be captured in a time-separable objective.

While this set of assumptions is not completely general in that it does not trace the entire Pareto frontier—one could instead consider a set of Pareto weights across all possible cohorts such that the weights sum to one—it allows a reasonable degree of generality while preserving a structure that lends itself to a recursive formulation and numerical solution.

The planner thus faces the following problems:

Problem 2.1 (Complete Markets Planning Problem) *The social planner maximizes (16) subject to (8), (9), (10), and (11).*

Problem 2.2 (Incomplete Markets Planning Problem) *The social planner maximizes (16) subject to (8), (12), (13), (14), (11), and (15).*

These optimization problems are not generally convex. As a result, first order conditions are necessary, but not sufficient, for an optimum.

2.6 Notation

Here, I introduce some additional notation I will use throughout my analysis of optimal policy in this model. First, I attach the following Lagrange multipliers to the constraints associated with the two planning problems:

- $\Delta^t \psi(s^t)$ to the resource constraint (8)
- $\Delta^t \mu(s^t)$ to the implementability condition for the generation born at s^t in the complete markets model (9)
- $\Delta^t \mu^y(s^t)$ to the implementability condition for the young at s^t in the incomplete markets model (12)
- $\Delta^{t-1} \beta \mu^o(s^t)$ to the implementability condition for the old at s^t in the incomplete markets model (13)

Second, since I will no longer discuss pre-tax wages and instead write them as the marginal product of labor, I reuse $w(s^t) \equiv W'(s^t)$ as the derivative of the social welfare function for the generation born at s^t with respect to expected lifetime utility, while $\tilde{w}(s^t) \equiv u_c^y(s^t) w(s^t)$ represents the social marginal welfare weight for the cohort born at s^t—the value to the planner of an extra dollar of consumption for that cohort.

The formulas discussed in the next section make frequent reference to the (marginal) distortionary cost of taxation;[8] this means the marginal dead weight loss associated with

[8] Many economists—especially those focused on static economies—are more familiar with optimal tax expressions in terms of the tax itself, or wedges, often in the form $\frac{\tau}{1-\tau}$. If there are no wealth effects, there does exist a nice relationship between the distortionary cost of labor taxation and the labor wedge:

$$\varepsilon \frac{\tau^L}{1-\tau^L} = \frac{\tilde{\mu}}{1+\tilde{\mu}},$$

where ε is the elasticity of labor supply with respect to the net-of-tax wage. However, for more complex cases,

raising an additional dollar of revenue through a particular tax; put another way, it is the cost above and beyond a lump-sum tax that mechanically[9] raises the same amount of revenue. At the optimum, this must be equal to the improvement in the planner's objective function that would occur by allowing a dollar of non-distortionary, lump-sum taxation, normalized by the improvement in the planner's objective function that would occur through an extra dollar of consumption for the generation being taxed. Therefore, distortionary costs can be written in terms of other variables as follows:

Definition 2.2 (Distortionary Cost of Labor Taxation) *The (marginal) distortionary cost of labor taxation on the generation born at s^t, denoted $\tilde{\mu}(s^t)$, is defined as*

$$\tilde{\mu}(s^t) \equiv \frac{\mu(s^t)}{w(s^t)} \tag{17}$$

$$\tilde{\mu}(s^t) \equiv \frac{\mu^y(s^t)}{w(s^t)} \tag{18}$$

for the complete and incomplete markets cases, respectively.

Definition 2.3 (Distortionary Cost of *Ex Post* Capital Taxation) *The distortionary cost of* ex post *capital taxation on the generation dying at s^{t+1}, denoted $\hat{\mu}(s^{t+1})$, is defined as*

$$\hat{\mu}(s^{t+1}) \equiv \frac{\mu(s^t)}{w(s^t)} \tag{19}$$

$$\hat{\mu}(s^{t+1}) \equiv \frac{\mu^o(s^{t+1})}{w(s^t)u_c^o(s^{t+1})} \tag{20}$$

for the complete and incomplete markets cases, respectively.

The latter definition is less intuitive than the former, and warrants further discussion. First, in the case of complete markets, notice that the cost of capital taxation at any $s^{t+1} \succeq s^t$

such a simple relationship doesn't exist, and so the below expressions involving $\tilde{\mu}$ cannot be replaced with simple expressions involving τ^L. Additionally, $\tilde{\mu}$ has a much stronger intuitive meaning within the context of tax smoothing. Thus, I will work with $\tilde{\mu}$ in analytical expressions, though in numerical simulations I will give τ^L for concreteness.

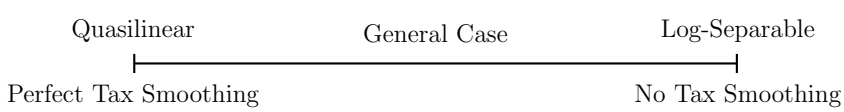
[9] without accounting for behavioral response

is always the same as the cost of labor taxation at s^t. This is because at the optimum, revenue must be extracted from the generation born at s^t in an efficient way, or the resulting allocation will be constrained inefficient. Thus, the distortionary cost of all taxes applied to the same generation must be equal. Meanwhile, in the incomplete markets model, *ex post* capital taxes are not allowed; the capital tax rate must be set one period in advance. Nonetheless, one can consider the cost of raising such a tax if it were allowed; that is what $\hat{\mu}(s^{t+1})$ captures, and it matches the marginal value of allowing a small lump-sum tax on the old in only that particular state of the world. Since each such instrument is not actually available, the distortionary costs of all such instruments need not equal each other, or of labor taxation in the previous period.

Finally, from this point forward, I reduce dependence on s^t to a subscript t for all variables when it does not lead to a substantial loss in clarity.

3 Two Polar Cases

Before characterizing optimal policy for the general case, I build intuition for the main results by discussing optimal policy in two extreme cases: quasilinear utility and a utilitarian planner; and log-separable utility, a utilitarian planner, and no capital.



We will see that in the former case, the usual, perfect tax smoothing results obtain, while in the latter, policy is formed entirely period-by-period, with no tax smoothing or intertemporal concerns whatsoever. In both cases, I will focus on a pure OLG model—no bequest motive ($\delta = 0$), and a minimum bequest of 0 ($\underline{z}(s_t, t) = 0$).

3.1 Quasilinear Utility, Utilitarian Planner, $\beta = \Delta$

Proposition 3.1 *Assume that period utility is quasi-linear:*

$$u(c, \ell) = c - v(\ell),$$

where $v(\bullet)$ is increasing and convex, with negative consumption permitted. Further assume the planner is utilitarian with $\beta = \Delta$. If markets are recursviely complete, then for $t \geq 0$, the distortionary cost of labor taxation is constant

$$\tilde{\mu}_t = \tilde{\mu}_{t+1}$$

and ex ante capital taxes are zero

$$\mathbb{E}_t[\tau^K_{t+1} F_{K,t+1}] = 0.$$

If markets are incomplete, then the distortionary cost of labor taxation is a martingale away from debt limits

$$\tilde{\mu}_t = \mathbb{E}_t[\tilde{\mu}_{t+1}] + \nu^u_t - \nu^l_t$$

and capital taxes satisfy

$$\frac{\tau^K_t}{1 - \tau^K_t} = \frac{\mathrm{Cov}_t[\tilde{\mu}_{t+1}, k_t F_{KK,t+1}] - \frac{\mathbb{E}_t F_{KK,t+1}}{\mathbb{E}_t F_{K,t+1}} \mathrm{Cov}_t[\tilde{\mu}_{t+1}, k_t F_{K,t+1}]}{\mathbb{E}_t[(1 + \tilde{\mu}_{t+1}) F_{K,t+1}]}.$$

All of these results except the last are highly familiar elements of optimal tax literature; the last is equivalent to that in Farhi (2010), which serves as the benchmark for optimal taxation with incomplete markets. The standard results are preserved because the two effects that overlapping generations introduce are both eliminated in this specification. First, the

planner has no redistributive preferences—all generations have the same social marginal welfare weight, since the planner is utilitarian and there is no deminishing marginal utility—so policy is constructed to maximize efficiency, as in a classic dynastic model. Second, quasilinear utility carries no income effects, thereby eliminating any difference in household behavior between OLG and dynastic versions of the model. With these two new effects nullified, the model resembles the classic Ramsey model.

3.2 Log-Separable Utility, No Capital, Utilitarian Planner, $\beta = \Delta$

Proposition 3.2 *Assume that period utility is log-separable:*

$$u(c, l) = \log c - v(\ell),$$

where $v(\bullet)$ is increasing and convex. Further assume the planner is utilitarian with $\beta = \Delta$, and there is no capital:

$$F(k, \ell; A) = A\ell.$$

If markets are recursively complete, then for $t \geq 1$, the distortionary cost of taxation (as well as the allocation and tax rate) depends only on the current state, and not on the stochastic process for government spending or initial conditions, so long as lump-sum transfers are not used.

The reasons for this stark result, which states that standard tax smoothing has no role in this economy,[10] are twofold. First, with log-separable utility and a single working period per cohort, income and substitution effects of labor taxation cancel out, leaving labor effort unaffected. This means the government can set whatever labor tax rate it sees fit without any consequences for GDP. It is worth emphasizing that this does *not* mean that labor taxation is without efficiency costs; the first best would involve an *increase* in labor effort

[10]For emphasis, there is nothing special about this setup for an infinite-horizon economy. Perfect tax smoothing would apply as usual.

during a war. This serves to highlight a key point made in the more general analysis: With overlapping generations, income and mechanical effects matter—not just the standard substitution effects.

Second, log-separable utility has an intertemporal elasticity of substitution (IES) of 1. Combined with the availability of complete markets, this means that old consumpiton at s^{t+1} can be "chosen" by the planner, completely independently of the allocation at s^t as well as the allocations at other successors of s^t. It is this latter property—the fact that old consumption at s^{t+1} can be set independently of allocations at other possible sucessors of s^t—that the incomplete markets model lacks and, therefore, prevents it from having this extreme property.

These two properties, combined with the lack of capital, give the planner a completely time-separable problem.[11]

4 Properties of Optimal Policy in the General Case

Having discussed two special, polar cases—one involving perfect, traditional tax smoothing, and another in which current policy depends only on the present shock—I now address the properties of optimal policy in the general case. It features elements of both extreme cases presented last section: tax smoothing for usual efficiency reasons, and taxes that depend on the current shock due to income effects and the loss of Ricardian equivalence; meanwhile, the redistributive motive can push toward smoother or less smooth taxes depending on the

[11] It is worth noting that the sense in which optimal policy depends only on the present period here is quite different from a similarly-worded finding in Lucas and Stokey (1983). In that paper, optimal policy depends on the initial government budget position and the Markov process for government spending (through the sufficient statistic of the multiplier on the implementability condition) as well as the current government spending shock. This means that within any simulation of the model, optimal policy will feature the same tax rates in all periods that share the same government spending shock, but not across simulations that feature different initial conditions or different Markov processes. In constrast, optimal policy in the present model depends only on the current government spending shock; neither initial conditions nor the properties of the Markov process are relevant. Moreover, in Lucas and Stokey (1983), the distortionary cost of taxation is constant over time—the usual tax smoothing result—and depends on the initial conditions and the Markov process for government spending. In contrast, the distortionary cost of taxation here is *not* constant, and also depends only on the current government spending shock, and neither the initial conditions nor the properties of the Markov process.

nature of shocks. At the optimum, the planner chooses the best possible tradeoff among these goals in a way this section will make precise. I first discuss the mathematically simpler case of recursively complete markets before considering the slightly messier case of incomplete markets, though the cases are conceptually similar.

4.1 Recursively Complete Markets

4.1.1 Labor Taxes

I begin by characterizing optimal labor taxes. When markets are complete, the government can reform its policies in any way that is budget neutral in present value, raising revenue at any combination of histories it sees fit. A candidate policy is optimal only if no such reform improves welfare. These results focus on the neutrality of a small reform in which the government increases labor taxes at history s^t and uses the proceeds to reduce labor taxes at a particular $s^{t+1} \succeq s^t$. Since these two histories are temporally adjacent, a "government Euler equation" with respect to a particular state-contingent asset emerges. The nature of such an Euler equation depends strongly on whether the bequest requirement binds.

Proposition 4.1 *If markets are complete and the bequest requirement does not bind at s^{t+1}, then the distortionary cost of labor taxation satisfies*

$$q(s^{t+1})\tilde{w}_t\tilde{\mu}_t = Pr_t(s^{t+1})\Delta\tilde{w}(s^{t+1})\tilde{\mu}(s^{t+1}). \tag{21}$$

This equation resembles the standard tax smoothing result. It differs only to the extent that

$$q(s^{t+1})\tilde{w}_t \neq Pr_t(s^{t+1})\Delta\tilde{w}(s^{t+1}).$$

Expanding q and noting that bequests are set optimally when the bequest requirement does not bind shows this is equivalent to the relation

$$\delta w_t \neq \Delta w(s^{t+1}).$$

That is, at histories at which the bequest requirement does not bind, the planner deviates from tax smoothing only to the extent to which the relative social weighting of the two adjacent generations differs from the relative private weighting. If the planner is utilitarian and $\delta = \Delta$, we recover the Ramsey tax smoothing result. These possibly differing weights embody the essence of the redistributive motive—one of the two effects introduced by OLG. The other—the loss of Ricardian equivalence—does not factor in this relation. Since the bequest requirement does not bind, households are, in fact, proportionately Ricardian; anticipated taxes at s^{t+1} are felt proportionately by the generation born at s^t and lead to an adjustment of their consumption path.

Proposition 4.2 *If markets are complete and the bequest requirement binds at s^{t+1}, then the distortionary cost of labor taxation satisfies*

$$q_{t+1}\,\tilde{w}_t \overbrace{\left\{ \underbrace{1}_{\substack{\text{Mech.}\\ \text{Effect}}} + \tilde{\mu}_t \left[\underbrace{1}_{\substack{\text{Subst.}\\ \text{Effect}}} - \underbrace{\sigma^o_{t+1}\left(1 + \frac{z_{t+1}}{c^o_{t+1}}\right)}_{\substack{\text{Dynamic Income}\\ \text{Effect}}} \right] \right\}}^{\text{M.C. of labor taxation at } s^t} = Pr(s^{t+1})\Delta\, \overbrace{\tilde{w}_{t+1} \left\{ \underbrace{1}_{\substack{\text{Mech.}\\ \text{Effect}}} + \tilde{\mu}_{t+1}\eta_{t+1} \right\}}^{\text{M.C. of labor taxation at } s^{t+1}} \quad (22)$$

where

$$\sigma^j_t = -\frac{u^j_{cc,t} c^j_t}{u^j_{c,t}} = \frac{1}{IES^j_t}$$

and

$$\begin{aligned}\eta_{t+1} =& 1 - \sigma^y_{t+1}\left(1 - \frac{\underline{z}_{t+1}}{c^y_{t+1}}\right) + \frac{u^y_{c\ell,t+1}}{u^y_{c,t+1}}\ell_{t+1} \\ =& \underbrace{1}_{\substack{\text{Subst.}\\ \text{Effect}}} + \underbrace{\ell_{t+1}\left(\frac{u^y_{c\ell,t+1}}{u^y_{c,t+1}} + F_{L,t+1}(1-\tau^L_{t+1})\frac{u^y_{cc,t+1}}{u^y_{c,t+1}}\right)}_{\text{Static Income Effect}} + \underbrace{\sigma^y_{t+1}\frac{F_{L,t+1}(1-\tau^L_{t+1})\ell_{t+1} + \underline{z}_{t+1} - c^y_{t+1}}{c^y_{t+1}}}_{\text{Dynamic Income Effect}}.\end{aligned}$$

This expression considers the same policy reform but focuses on periods when the bequest requirement does bind—all periods in a strict OLG model. The left hand side represents the marginal cost of labor taxation per dollar raised at history s^t, and the right hand side represents the marginal cost at history s^{t+1}.

The former has three components: the mechanical effect of the tax, the distortionary cost or substitution effect of the tax, and the dynamic income effect of the tax, which acts through changing asset prices and is less familiar in optimal tax analysis. Specifically, the reform under consideration reduces old consumption at s^{t+1} but does not alter young consumption at s^t.[12] Thus, $q(s^{t+1})$ rises, making households born at s^t feel poorer, work harder, and offset the distortionary cost of taxation. This dynamic income effect is proportional to the inverse of the IES, which governs how much asset prices respond to changes in consumption trajectories.

The right hand side has four distinct components. First is the mechanical effect, and η_{t+1} encompasses the other three: the standard distortionary cost or substitution effect, the standard (static) income effect which is countervailing and proportional to ℓ_{t+1}, and a dynamic income effect similar to the one above. Under this perturbation, consumption at s^{t+1} falls but consumption at s^{t+2} stays the same for all $s^{t+2} \succeq s^{t+1}$. Thus, the price of consumption at all histories $s^{t+2} \succeq s^{t+1}$ falls. This makes the household born at s^{t+1} feel

[12] To see this, notice that since the government has directed all of the new revenue, ϵ, in this reform to reduction in taxes at s^{t+1}, then $b(s^{t+1})$ drops by $\frac{\epsilon}{q(s^{t+1})}$ with the government budget position unaltered at all other histories. Market clearing requires that $c^o(s^{t+1}) + z(s^{t+1})$ must also reduced by $\frac{\epsilon}{q(s^{t+1})}$, which precisely accounts for the lost income from the tax, leaving young consumption at s^t unchanged.

richer in present value terms, decrease labor supply, and partially offset the standard income effect/exacerbate the substitution effect. This effect is proportional to saving, and again proportional to the inverse of the IES.

The final ingredient is the social marginal welfare weight, which captures the idea that the planner may care more about costs imposed on one generation than another. The sum of all effects on each cohort, weighted by this social marginal welfare weight, must be equal at the optimum in present value terms. Otherwise, the planner could benefit from a small reform.

This central result makes two conceptual alterations to the standard tax smoothing result of the Ramsey model—or the quasilinear case from last section—directly corresponding to the two new considerations introduced by the OLG framework. First, it takes into account income effects, which reduce the distortionary costs of taxation. These could be ignored in the Ramsey model, or if bequest limits do not bind, since they are constant in present value terms due to Ricardian equivalence, but cannot be ignored in an OLG model if bequest requirements bind. Second, it weights by the social marginal welfare weight, which captures the desire to redistribute. Social marginal welfare weights will generically be nonconstant even if the planner is utilitarian, since if the bequest constraint binds, marginal utilities of consumption will be unequal across generations. This weighting also necessitates adding back in mechanical effects, which are implicitly subtracted from both sides in the Ramsey case where social marginal welfare weights are constant.

These two alterations have directly opposing effects. A larger departure from Ricardian equivalence, or stronger income effects, leads to a smaller (or negative) weight on the distortionary cost of taxation, thereby encouraging higher taxes on less well off generations. Viewed from the government budget perspective, the "less Ricardian" are households, the larger the distortion in interest rates associated with government saving and borrowing, which makes unbalanced government budgets and tax smoothing less attractive. Furthermore, static income effects add other consequences to deferring tax collections, as current

generations feel wealthier and work less, further harming the government budget position in the short run, while future generations feel poorer and increase their labor supply, helping the government budget position in the future; the net effect is indeterminate. On the other hand, a more concave social welfare function means a stronger redistributive motive, lower taxes on less well-off generations, and therefore counteracts the loss of Ricardian equivalence. This effect can push for smoother taxes if the main source of inequality across generations is different tax rates; it can push for less smooth taxes if generations' pre-tax wages are highly unequal.

4.1.2 Capital Taxes

One could instead consider a reform that increases labor taxes at history s^t and uses the proceeds to reduce capital taxes by an equal amount at all $s^{t+1} \succeq s^t$. This reform should be welfare-neutral at the optimum, and analyzing it reveals properties of optimal capital taxes. As the result of Chamley (1986) and Judd (1985) is quite familiar, I focus here on the stark OLG case.

Proposition 4.3 *If markets are complete and bequest limits bind at zero ($\underline{z}(s_t, t) = 0$, and $\delta = 0$), then optimal capital taxes satisfy*

$$\sum_{s_{t+1}} q_{t+1} F_{K,t+1} \tau^K_{t+1} = \frac{\tilde{\mu}_t}{1 + \tilde{\mu}_t} \left\{ \eta_t - 1 + \sum_{s_{t+1}} q_{t+1} F_{K,t+1} \sigma^o_{t+1} \right\}$$

Recall that $\eta_t - 1$ includes static income effects (which are negative) and dynamic income effects (which are positive). Thus, this proposition says that ex ante capital taxes are more positive when any of the following is true:

- Static income effects are smaller. Raising capital taxes causes households to save less and consume more in youth. This causes households to supply less labor and therefore

pay less labor income tax, which hurts the government budget. This effect is muted when static income effects are smaller.

- Dynamic income effects are larger. Raising capital taxes means that consumption in old age is more expensive. This makes households feel poorer, which causes them to supply less labor and therefore pay more labor income tax, which helps the government budget.

- The IES is smaller. The larger the IES, the greater the distortion of capital accumulation that comes from an increase in capital taxes.

The magnitude of the capital taxes—positive or negative—increases when the labor tax rate, and therefore the marginal distortionary cost of labor taxation, is higher, because the effects on the government budget are amplified. There is a classic case in which these three effects offset and ex ante capital taxes are precisely zero:

Corollary 4.4 *If preferences exhibit constant relative risk aversion σ over consumption, and are separable between labor and consumption, then ex ante capital taxes between t and $t+1$ are zero if bequest limits bind at zero.*

This draws upon the static Ramsey tax intuition: Goods should be taxed at rates inversely related to their elasticities of demand. If utility is isoelastic over consumption and separable from labor, then all states' consumption should be taxed equally, which means that capital should not be taxed.

4.2 Incomplete Markets

Moving to incomplete markets does not change any fundamental intuition. It merely adds complexity in that no feasible perturbation affects just two contiguous histories; generically, any perturbation affecting history $s^{t+1} \succeq s^t$ will affect $(s^t, s_{t+1}) \forall s_{t+1} \in \mathcal{S}$. Thus, expected returns to *both* assets (risk free debt and capital) will generally be affected. These effects must be considered when weighing a policy perturbation.

4.2.1 Labor Taxes

Similar to, but slightly different from, the complete markets case, I will present two expressions that enforce the neutrality of a small reform in which the government raises labor taxes at history s^t, which it uses to reduce its risk free debt and, in turn, reduce labor taxes collected at all $s^{t+1} \succeq s^t$ by an equal amount. At the optimum, this reform must be welfare-neutral. The expressions differ, as those in the last section did, regarding whether the bequest constraint binds at s^{t+1}. I begin with an expression analogous to (21):

Proposition 4.5 *If markets are incomplete and the bequest requirement does not bind for any $s^{t+1} \succeq s^t$, then the distortionary cost of labor taxation satisfies*

$$\tilde{w}_t \tilde{\mu}_t = \Delta R_t^f \mathbb{E}_t[\tilde{w}_{t+1} \tilde{\mu}_{t+1}] + \nu_t^u - \nu_t^l \tag{23}$$

where R_t^f is the gross risk free rate between t and $t+1$.

This expression sums (21) across all $s^{t+1} \succeq s^t$, with an adjustment $\nu_t^u - \nu_t^l$ for the possibility of binding government debt limits. The sum must be taken since market incompleteness prevents the government from performing any of the reforms captured by (21) separately. Nonetheless, the intuition is identical, so long as the government has access to credit: The planner trades off the gap between private intergenerational weighting and social intergenerational weighting against the ratio of distortionary costs, taking into account any covariance between the two.

As with complete markets, this expression differs from its Ramsey equivalent—that distortionary costs of taxation are a risk-adjusted martingale—only to the extent that $\delta w_t \neq \Delta w_{t+1}$. Thus, the only alteration induced by OLG is the motive for redistribution; when the request requirement does not bind, households are indeed proportionally Ricardian.

Next, consider an expression analogous to (22):

Proposition 4.6 *If markets are incomplete, bequest limits bind at zero, and government debt limits do not bind, then the distortionary cost of labor taxation satisfies*

$$\tilde{w}_t \left\{ 1 + \tilde{\mu}_t \left(1 - \underbrace{\epsilon_{b_t}^{R_t^f}}_{\substack{\text{Updated Dynamic} \\ \text{Income Effect}}} \right) + \underbrace{\frac{\partial \tau_t^K}{\partial b_t}}_{\substack{\text{Size of Capital Tax} \\ \text{Windfall}}} \sum_{s_{t+1}} \left[\underbrace{q_{t+1} F_{K,t+1} k_t \hat{\mu}_{t+1}}_{\substack{\text{Benefit of Capital Tax} \\ \text{Windfall}}} \right] \right\}$$

$$= \Delta R_t^f \mathbb{E}_t \left\{ \tilde{w}_{t+1} \left[1 + \tilde{\mu}_{t+1} \eta_{t+1} + \underbrace{\frac{\sigma_{t+1}^y (1 - \tau_{t+1}^K)}{c_{t+1}^y}}_{\substack{\text{Size of Capital Tax} \\ \text{Reduction}}} \left(\begin{array}{c} \sum_{s_{t+2}} \left[\underbrace{q_{t+2} F_{K,t+2} k_{t+1} \hat{\mu}_{t+2}}_{\substack{\text{Cost of Capital Tax} \\ \text{Reduction}}} \right] \\ - \underbrace{\frac{\tilde{\mu}_{t+1} k_{t+1}}{1 - \tau_{t+1}^K}}_{\substack{\text{Relative to Previous} \\ \text{Dynamic Income Effect}}} \end{array} \right) \right] \right\} \tag{24}$$

where q_{t+1} is the price of a state-contingent asset delivering consumption at s^{t+1}, were such an asset available, and

$$\epsilon_{b_t}^{R_t^f} = \frac{\partial R_t^f}{\partial b_t} \frac{b_t}{R_t^f} = -R_t^f b_t \frac{\mathbb{E}_t u_{cc,t+1}^o}{\mathbb{E}_t u_{c,t+1}^o}.$$

Relative to summing (22) over all $s^{t+1} \succeq s^t$, this expression involves several alterations, highlighted with underbraces, though the overall intuition is unaltered. First, the left hand side replaces σ_{t+1}^o with $\epsilon_{b_t}^{R_t^f}$, which plays the same role: It captures the income effect of a tax increase at date t, with all of the associated drop in consumption occurring at date $t+1$.[13] When the government raises taxes at date t, fewer bonds are issued, necessitating a drop in the interest rate, which makes households born at t feel poorer. This in turn makes them work harder, offsetting some of the distortionary effect—a dynamic income effect. The right hand side needs no similar adjustment from the summed (22), because the reduction in c_{t+1}^y

[13] To see that they are equivalent, notice that since σ_{t+1}^o is the inverse of the intertemporal elasticity of substitution for the old at s^{t+1}, it is also the percentage that the gross return on the asset paying off at s^{t+1} changes when $c^o(s^{t+1})$—i.e., the amount of that asset issued—increases by one percent, holding fixed $c^y(s^t)$. Similarly, $\epsilon_{b_t}^{R_t^f}$ is the percentage that the risk free gross return changes when b_t increases by one percent (and therefore all $c^o(s^{t+1})$ increases by $1\% \cdot b_t R_t^f$).

has the same effect on all of the state-contingent asset returns as it does on the risk free rate.

Second, $\frac{\partial \tau_t^K}{\partial b_t} \sum_{s_{t+1}} [q_{t+1} F_{K,t+1} k_t \hat{\mu}_{t+1}]$ was added to the left hand side. This reflects the fact, just discussed, that a perturbation in which tax revenue increases at t and falls by R_t^f at $t+1$ necessitates a drop in the interest rate. But in equilibrium, the after-tax return on capital must drop as well. This could occur through increase in capital accumulation, but this throws off the perturbation under consideration. Instead, equilibrium can be achieved via an increase in capital taxes equal to $-\frac{\partial \tau_t^K}{\partial b_t}$,[14] which in turn leads to an increase in capital tax collections at date $t+1$ by $-\frac{\partial \tau_t^K}{\partial b_t} F_{K,t+1} k_t$. These increased collections act as free lump-sum taxes on the old at date $t+1$, since they were enacted precisely to *prevent* behavior from changing; therefore, they represent a gain for the government budget worth $q_{t+1}\hat{\mu}_{t+1}$ more than their cost to households, by definition of $\hat{\mu}$.

Third, $\frac{\sigma_t^y (1-\tau_t^K)}{c_t^y} \sum_{s_{t+1}} [q_{t+1} F_{K,t+1} k_t \hat{\mu}_{t+1}]$ was added the right hand side. This reflects a similar idea—increased labor taxes, with the entire drop in consumption occurring during youth, cause an increase in the risk free rate, which in turn must cause a decrease in capital taxes to leave capital accumulation unaffected—though the magnitude and sign are different. $-\sigma_{t+1}^y$ is the elasticity of both the interest rate *and* the after tax expected capital return (holding fixed capital accumulation and old consumption at $t+2$) with respect to c_{t+1}^y—they are both affected to the same extent. To convert this elasticity to a change in the capital tax rate, it should be divided by c_{t+1}^y and multiplied by $-(1-\tau_{t+1}^K)$, meaning

$$\left.\frac{d\tau_{t+1}^K}{dc_{t+1}^y}\right|_{k_{t+1}, c_{t+2}^o} = \frac{\sigma_{t+1}^y (1-\tau_{t+1}^K)}{c_{t+1}^y}.$$

As expected, this term is positive, whereas the term on the left hand side is negative, reflecting that capital taxes *drop* between $t+1$ and $t+2$ which hurts the government budget position, whereas they *rose* between t and $t+1$ which helped the government budget position.

Finally, $\sigma_{t+1}^y \tilde{\mu}_{t+1} \frac{k_{t+1}}{c_{t+1}^y}$ is subtracted from the right hand side. This is to avoid double-counting the loss that occurs from the drop in capital taxes. The third alteration, discussed

[14] recall that b_t drops if taxes are increased at date t

in the previous paragraph, captures the entire effect of a change in capital taxes, holding fixed capital accumulation and, therefore, pre-tax capital returns. However, recall η_{t+1} includes a "dynamic income effect" that captures the fact that when young consumption at date $t+1$ drops, interest rates rise, making households feel richer and work less, hurting the government budget position. η_{t+1} applies this effect to all of saving—both risk free bonds and capital—while the second term discussed above already accounts for this effect on capital. Thus, this final term is subtracted to counteract that.[15]

Though this expression appears far more complicated than (22), it captures the same intuition: If bequest limits bind, then optimal labor taxes must be set to balance mechanical, substitution, and income effects, with the latter further broken down into static and dynamic income effects. The change to incomplete markets merely makes the expression for dynamic income effects more complicated, since even the simplest allowable perturbation affects both assets. The equation still preserves the distinct contribution of each of OLG's two new effects: the lack of Ricardian equivalence, and the desire for redistribution.

4.2.2 Capital Taxes

As with complete markets, one can consider a policy perturbation wherein labor taxes at date t are increased and capital taxes between dates t and $t+1$ are cut. At the optimum, this must be welfare-neutral, and gives rise to the following result.

Proposition 4.7 *If markets are incomplete, bequest limits bind at zero, and government*

[15] This term wasn't necessary on the left hand side because $\epsilon_{b_t}^{R_t^f}$, as an elasticity, only is proportional to b_t and not all of saving $b_t + k_t$.

debt limits do not bind, then capital taxes satisfy

$$\frac{\tau_t^K}{1-\tau_t^K} = \frac{\left\{\begin{array}{c} \mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]\tilde{\mu}_t\left[\left(\eta_t - 1 - \sigma_t^y \frac{k_t}{c_t^y}\right) + \frac{1}{1-\tau_t^K}\epsilon_{k_t}^{R_t^f}\frac{b_t}{k_t}\right] \\ +\frac{1}{1-\tau_t^K}\mathbb{E}_t\left[F_{K,t+1}u^o_{c,t+1}\epsilon_{k_t}^{1-\tau_t^K}\hat{\mu}_{t+1}\right] \\ +\mathrm{Cov}_t[\hat{\mu}_{t+1}, k_t F_{KK,t+1}u^o_{c,t+1}] - \frac{\mathbb{E}_t[F_{KK,t+1}u^o_{c,t+1}]}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]}\mathrm{Cov}_t[\hat{\mu}_{t+1}, k_t F_{K,t+1}u^o_{c,t+1}] \end{array}\right\}}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}(1+\hat{\mu}_{t+1})]} \tag{25}$$

This equation is related to the benchmark result in Farhi (2010), but noticeably different. The last row of the numerator is perfectly analogous to that paper's "hedging term," and encourages higher capital taxation when the return to capital is correlated with the government's desire for funds, tempered by any adjustement in capital accumulation. That paper's other term was labeled the "intertemporal term," and "reflects the possibility of manipulating interest rates." However, the essence of that term was that capital taxes should be raised if there is an adverse shock at date t, and not if there isn't; this leads to higher interest rates for the government if there is an adverse shock, but lower interest rates if not. Two complexities make this term not applicable to the present model. First, those two different levels of capital taxes fall on different cohorts, which necessitates consideration of welfare weights or a different policy perturbation. Second, while such policy plans may be "anticipated" by households the period before, they are not *internalized*, since they will only affect the newly born generation.[16] Instead of this intertemporal term, the first row contains the income effects of such a policy perturbation, which does not need to be considered in a representative-agent model.

[16]This is, of course, an implication of the extreme assumption that households live for only two periods, and make meaningful economic choices in only one. Nonetheless, the intuition would remain that the effects of anticipated capital tax policy in the future would not be fully internalized by generations alive today, which would mitigate the value of the intertemporal term.

4.3 Summary

In this section, we saw that moving from the Ramsey model to an OLG model introduces additional effects the planner must consider when setting optimal taxes. First, taxes have (nonconstant) static and dynamic income effects due to loss of Ricardian equivalence. The former embodies the classic offsetting income effect in labor tax analysis, while the latter captures the fact that altering the policy path alters asset prices, which households might find either desirable or undesirable, and adjust their labor supply accordingly. Second, redistribution motives generally exist—even when the planner if utilitarian if there is diminishing marginal utility of consumption. Though these effects are present in both complete and incomplete markets models, the expressions satisfied by optimal policies are more complicated in the latter case, since attention must be given to the which perturbations are possible (and therefore which perturbations must be welfare-neutral at the optimum). Additionally, policy perturbations will affect all asset prices if markets are incomplete, which must be taken into account.

5 Three Applications to Policy Questions

Having discussed in detail the analytic properties of optimal policy in models with overlapping generations, I now give more concrete understanding of how governments ought to conduct policy in such models through numerical application to three important, conceptually distinct, policy questions:

1. How should the government fund a stochastic sequence of required government expenditure, such as wars?

2. How might the tax system be used to optimally share the risk of productivity shocks across generations?

3. To what extent should a "Social Security" system be constructed to redistribute from

later, presumably wealthier, generations to earlier, presumably poorer generations, and how should that system respond to productivity shocks?

As discussed in the analytical section, we will see broadly that the government institutes more strongly redistributive policies when the IES—a proxy for the government's ability to borrow without altering interest rates too much—is low and the degree of inequality-aversion in the social welfare function is high; under these conditions, deviating from a balanced budget is the least expensive in efficiency terms and the most rewarding. We will also see that each of these policy questions highlights different aspects of how the planner uses the instruments available to achieve his optimum.

For all applications, the uncertainty should be envisioned as worldwide. I have assumed a closed economy, which means that the smaller the country and the more local the shock, the better the insurance terms on worldwide capital markets. Shocks that are perfectly insurable abroad are not the focus of this analysis.

Calibration. Before engaging with these specific applications, I briefly describe the functional forms, general calibrations, and numerical methods I employ. First, a caveat: The model developed in the previous sections and applied here is clearly highly stylized. Thus, these calibrations were not chosen to be especially "realistic," as such realism is beyond the scope of this paper and deserves its own attention in future work. Results presented in this section, therefore, should not be taken too literally as quantitative recommendations for actual policy, but rather as illustrations of the properties of optimal policy that are induced by the introduction of OLG to the Ramsey model.

In my benchmark simulations, I use household utility functions from Chari, Christiano and Kehoe (1994) and later Farhi (2010):

$$u(c, \ell) = (1 - \gamma)\log(c) + \gamma \log(1 - \ell),$$

where $\gamma = 0.75$. The pure rate of time preference for both households and the planner is

2% per year, or, since a period corresponds to roughly thirty years, $\Delta = \beta = 0.98^{30}$. In sensitivity analysis I will consider alternative values for the IES, which is not possible within the context of a log-log utility function. In such analyses, I use the following utility function:

$$u(c, \ell) = \frac{c^{1-\sigma}}{1-\sigma} - \eta \frac{\ell^{1+\varphi}}{1+\varphi},$$

where $1/\sigma$ varies and captures the IES, $1/\varphi$ represents the Frisch elasticity of labor supply which I set to 1/2, and $\eta = 1$.[17] I refer to the prior calibration as "log-log" utility, and the latter as "isoelastic" utility.

I use different functional forms for the social welfare function depending on whether the IES is 1. In either case, ζ parameterizes the degree of inequality-aversion (concavity) of the social welfare function, with $\zeta = 0$ representing a utilitarian planner and $|\zeta| \to \infty$ representing a "Rawlsian" planner who cares only about the worst-off generation. If IES = 1, I use the CARA function

$$W(u) = \begin{cases} u & \text{if} \quad \zeta = 0 \\ -\frac{1}{\zeta}\exp(-\zeta u) & \text{if} \quad \zeta > 0 \end{cases}$$

If IES $\neq 1$, I use the CRRA function, altered according to Kaplow (2003)

$$W(u) = \begin{cases} \frac{[(1-\sigma)u]^{-\zeta}u}{1-\zeta} & \text{if} \quad \zeta \neq 1 \\ \ln u & \text{if} \quad \zeta = 1 \end{cases}$$

To ensure the welfare function has its usual properties, ζ should be weakly positive if $\sigma < 1$, and ζ should be weakly negative if $\sigma > 1$. These choices make the welfare function scale invariant, which will play a crucial role in developing a recursive representation of the planner's problem when trend productivity growth is introduced.

[17] I alter the function to make it consistent with balanced growth by multiplying the disutility of labor by $\exp(at)^{1-\sigma}$, where a is the trend growth rate of labor-augmenting productivity.

Production is Cobb-Douglas with $\alpha = 1/3$ and depreciation of 8% per year, meaning

$$F(K, L; A) = K^{1/3}(AL)^{2/3} + 0.92^{30}K.$$

For simplicity and intuition, I consider only models that are purely dynastic or purely OLG. A "Ramsey" model sets $\delta = \beta, \underline{z} = -\infty$, and an "OLG" model sets $\delta = 0, \underline{z} = 0$. If markets are incomplete, I must specify the upper and lower debt limits; I follow Farhi (2010) and set them at 50% and -10%[18] of the average across states of the first best GDP for that state if it were absorbing. To avoid issues with the initial period, I always allow the system to run for 100 periods in the "good" state prior to any results that are shown. I will discuss the calibration of the stochastic process within the context of each application.[19]

Solution Method. To compute a numerical solution to the plannning problem, I first reformulate Problems 2.1 and 2.2 recursively, the details of which can be seen in Appendix D.1. I solve the resulting Bellman equations on rectangular, bounded state spaces, verifying that the bounds do not bind, and that altering them does not substantially alter the results.

My solution method is collocation using Chebyshev polynomials on a sparse grid.[20] Broadly, collocation techniques parameterize the value function as a weighted sum of N smooth functions (in this case, Chebyshev polynomials). Then, N points—the collocation points, or nodes—are chosen inside the state space. Finally, I solve for weights, using a value function iteration approach, such that the Bellman equation holds exactly at those N points if the value function is represented by the weighted sum. I choose the N points and the set of N Chebyshev polynomials according to the sparse grid method of Smolyak (1963).[21] Finally

[18]Farhi (2010) chose 100% and -20%, but those values are for the amount of debt *issued*, and the interest rates are nearly double in this model due to the long period length.

[19]Though I consider many calibrations for utility and social welfare functions, for the sake of brevity, I present only a selected few of these in the main text; the curious reader may find many others and brief discussion thereof in the online Simulation Appendix.

[20]Excellent references include Miranda and Fackler (1997), Mark (2004). The classic book on numerical methods, including this one, is Judd (1998).

[21]Excellent references include Krueger and Kubler (2004) and Judd et al. (2014).

I ensure that the grid is sufficiently dense such that the Bellman equation holds quite closely at points other than the collocation nodes. Further details can be found in Appendix D.2.

5.1 Stochastic Government Expenditure

Determining optimal financing of government expenditure shocks, such as wars,[22] has been a classic application of Ramsey optimal taxation models. In such standard models, shocks to government expenditure should be financed entirely through insurance if markets are complete (Chari, Christiano and Kehoe, 1994), and entirely through debt if markets are incomplete (Farhi, 2010). These stark benchmark policy prescriptions make this question an excellent laboratory for exploring the tax smoothing properties of the present OLG model.

5.1.1 Unanticipated Spending Shock

To build intuition, I first consider the simplest version of government spending uncertainty—a completely unanticipated spending shock (a "war"), known to last a single period, with zero government spending before or after. I assume that the economy is in steady state prior to the war[23] The war costs 10% of steady state GDP.[24]

Figure 3 shows the optimal policy response to this unanticipated shock for a constant IES of 5 for a variety of social planners.[25], while Figure 4 shows the optimal response for

[22] There are multiple possible ways to model a war or other government expenditure in the context of the present model. For simplicity, I consider government expenditure to be a pure, required resource cost for the economy, having no effect on utility. On the other hand, one might consider government expenditure to be purchases of public goods, which surely enter household utility, but also are presumably optimally chosen rather than required and thus require a far richer model.

[23] For Ramsey models, which have a continuum of steady states, I choose the one with the same government debt as the OLG steady state to which it is compared.

[24] Given the completely unanticipated nature of the shock, and completely deterministic economy thereafter, there is no distinction between complete and incomplete markets, or between household holdings of capital and risk free bonds. Nonetheless, a choice must be made about how household balance sheets respond to the unanticipated shock in the period of the shock. I make the simplest possible assumption, which is that households officially hold risk free debt (paying $1/\beta$ gross interest) in the pre-shock steady state; thus, old consumption in the period of the shock is bounded below by its steady state level in the OLG model. This eliminates the ability of the planner to decrease old consumption by decreasing young labor supply, which makes the solution more realistic.

[25] I have chosen this (perhaps unrealistically high) IES to allow the government to raise an unlimited amount of revenue from debt issuance and thereby give the government a "real choice" over how to finance

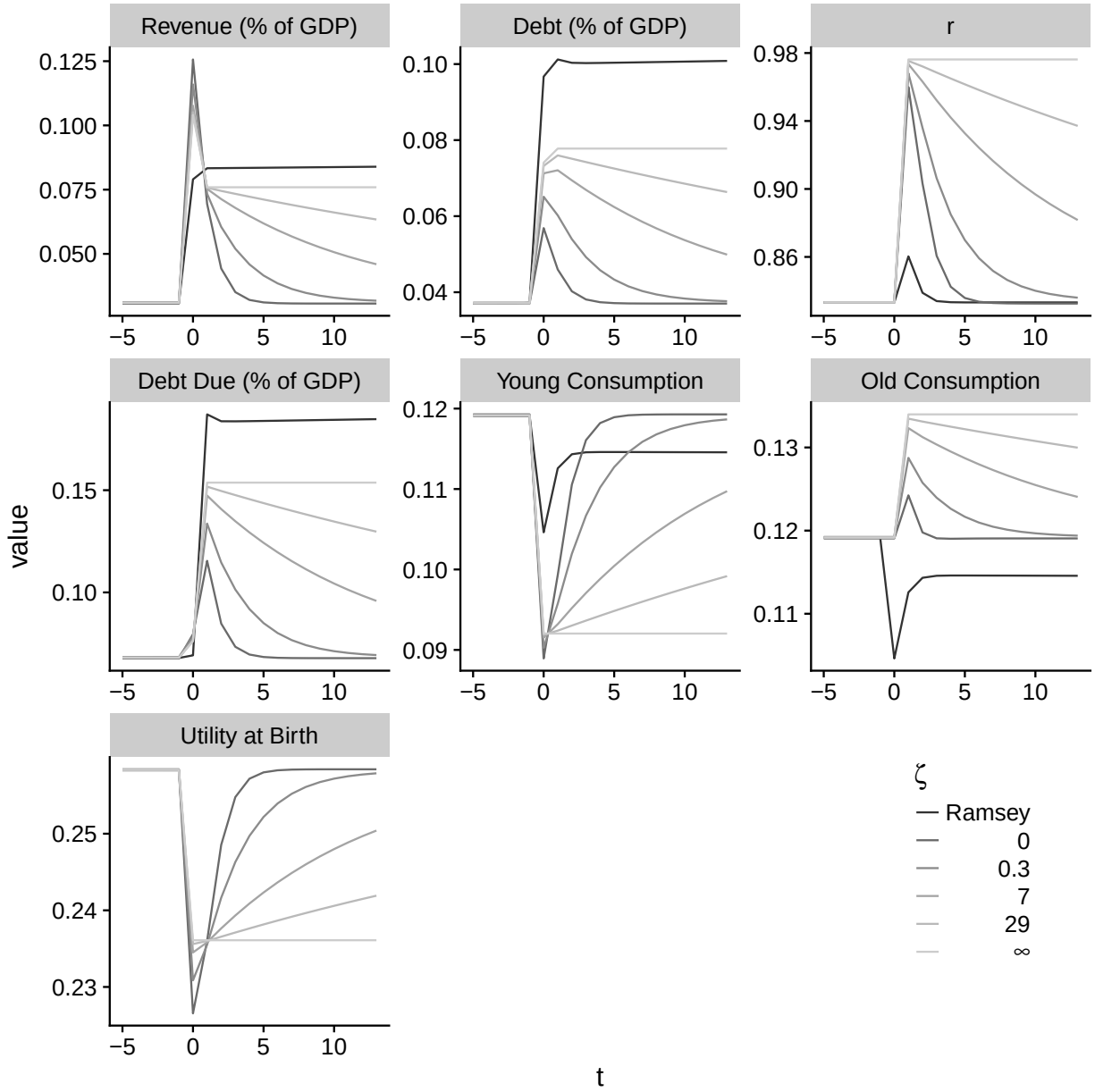


Figure 3: Optimal policy response to a completely unanticipated government spending shock equal in size to 10% of steady state GDP. These graphs show many economic variables each period, where period 0 is the period of the shock. Multiple models are considered: a Ramsey model, along with several OLG models featuring different planners with different levels of inequality-aversion, parameterized by ζ. All models assume isoelastic utility with an IES of 5.

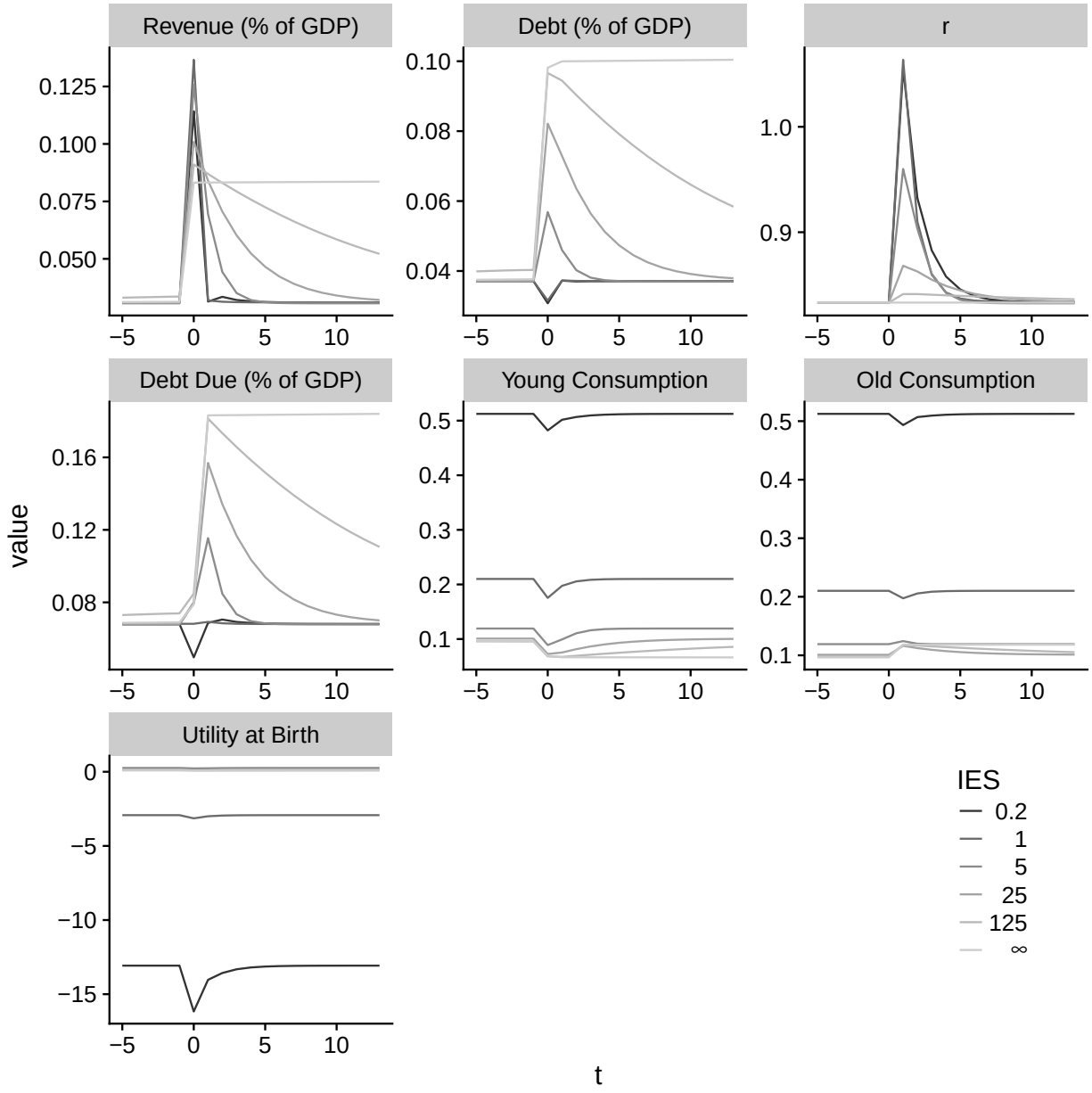


Figure 4: Optimal policy response to a completely unanticipated government spending shock equal in size to 10% of steady state GDP. These graphs show many economic variables each period, where period 0 is the period of the shock. The figure compares OLG models featuring a utilitarian planner but a variety of IESs.

a utilitarian planner for a variety of IESs. Together, these figures illustrate the difference in optimal policy response between a Ramsey model and an OLG model, while highlighting through comparative statics the two economic forces introduced by OLG.

The clearest distinction is between the OLG models and the Ramsey model. In the Ramsey model, the war is paid for over the rest of history, with taxes and debt constant from period 2 onward at a much higher level than previous to the war. On the other hand, in all OLG models except the ones featuring a Rawlsian planner or an infinite IES, taxes are higher during the war than afterward, with taxes eventually returning to the pre-war level.

The reason for the temporary increase in taxation is the loss of Ricardian equivalence. Financing the war with debt drives up the interest rate, which increases the amount of revenue that must be raised later and the associated efficiency cost. This effect is strongest when the IES is low, since interest rates must move substantially to induce households to hold more government debt.[26] The price of this contemporaneous taxation is inequality across generations, which is visibly falling with the IES.

On the other hand, the planner's desire for equality across generations pushes back toward more constant labor tax rates. In this model, in which productivity is constant, labor taxes play a large role in determining the welfare of a given generation. Thus, a more inequality-averse planner cares more about maintaining a constant tax, which we see in Figure 3.[27] Additionally, a more inequality-averse planner reverts more slowly to steady-state taxation because he attempts to compensate generations that faced high taxes in youth with more consumption in retirement; to do this, he offers a better interest rate on debt, which in turn requires raising taxes on the subsequent generation. Inequality across generations is,

the war in the short run. Similar properties could be generated by allowing the government to borrow from abroad, or choosing a more elastic production function.

[26] This should not be interpreted strictly as a statement about the IES; rather, it shows that the government will finance the war with more debt if it has freer access to debt markets. This could occur through easier crowding out of consumption (higher IES), investment (greater elasticity of substitution between labor and capital), or net exports (greater ability to borrow on worldwide capital markets in an open economy generalization).

[27] The reason for the spike in tax rates at date 0 even for a Rawlsian planner is that capital also affects welfare by raising the pre-tax wage. An inequality-averse planner thus places higher taxes on generations with access to more capital—in this case, the generation born at date 0.

intuitively, falling with the planner's degree of inequality aversion.

5.1.2 Stochastic Government Spending, Complete Markets

Next, I simulate a complete markets model with more conventionally stochastic government spending, which allows some preparation for shocks. I assume i.i.d., equal probability draws of government spending that is 20% ("peace") or 25% ("war") of steady-state GDP and log-log utility. Simulations with other IESs can be found in the Appendix; they are qualitatively similar and demonstrate the same comparative statics as with unanticipated spending shocks above. I simulate three cases: an OLG model with a utilitarian planner, an OLG model with a concave planner ($\zeta = 24$), and a Ramsey model as a benchmark.

I begin by presenting graphs (Figure 5) showing the evolution of several variables during state transitions. This is done by alternating between four periods in the low spending state and four periods in the high spending state.[28]

First, I compare the OLG models, taken together, to the classic Ramsey model. As in the unanticipated, deterministic example, we see that the labor tax rate is far more variable in the OLG models, reflecting some use of contemporaneous labor taxation to fund government expenditure shocks, rather than pure insurance. Whereas these shocks can be fully smoothed in the Ramsey model by insurance, such a scheme would place the burden of such shocks entirely on the old at the time of the shock. Thus, debt payments are far less variable in the OLG models, which implements this lesser use of insurance. As a result of the greater variance of labor tax rates, however, the OLG economies perform worse on average, since the taxation system is less efficient—the classic equity/efficiency trade-off. The other major difference is in the trajectory of debt. In the Ramsey model, growth of issued debt is procyclical (anticorrelated with government spending), reflecting the benefit of insurance against government expenditure; in the OLG models, however, this effect is substantially reduced, begetting a more intuitive countercyclical trajectory.

[28] Of course, this pattern is not anticipated by the planner.

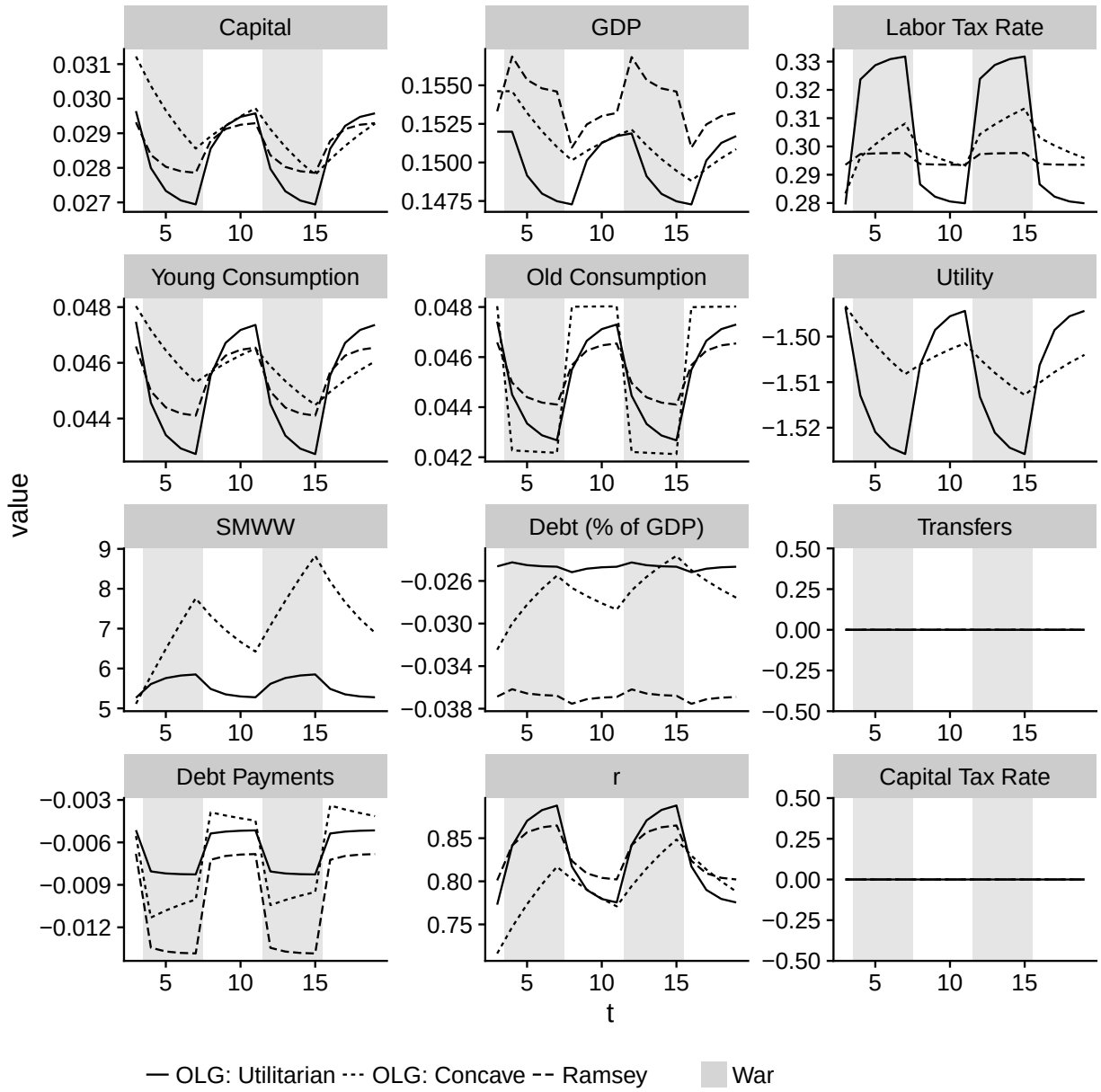


Figure 5: This figure captures the salient features of state transitions in the complete markets model of stochastic government expenditure when utility is log-log. Shading marks periods of high government spending ("wars"). The concave planner has $\zeta = 24$. The model was first allowed to run for 100 periods in state 1 to arrive at a steady state. The Ramsey model was started with debt equal to -2.4% of nonstochastic steady state GDP, which corresponds to the level of government debt in the OLG model after 100 periods of low government spending; initial conditions are irrelevant for the OLG models.

Next, I compare the OLG models to each other. The primary difference here is that the labor tax rate is more volatile, as expected, if the planner is utilitarian. As in the deterministic case above, a concave planner has a desire to compensate generations that were young—and therefore heavily taxed—during a period of high government spending. He does so by raising more revenue during the ensuing low-spending period and using it to both pay a better interest rate to retirees and lower taxes in the previous period via the insurance channel. The result is that the concave planner achieves more equity across generations and smoother labor taxation.

Figure 6 shows the results of randomly simulating the model across ten thousand periods, and offers a different view of the tax smoothing properties of these models. The Ramsey model features tax rates that are quite close to constant. On the other hand, the OLG models feature labor tax rates that are positively autocorrelated and decidedly non-constant, even conditional on s_t and s_{t-1}. The fact that they are non-constant is attributable to the loss of Ricardian equivalence, while the autocorrelation is attributable to the redistributive motive and the usual efficiency concern.

5.1.3 Stochastic Government Spending, Incomplete Markets

The incomplete markets restriction might seem to be irrelevant in this case, as there are two assets (capital and risk free debt) and two states, while debt as a percentage of GDP was sufficiently small so as to not violate any imposed debt limits. However, recall that the log-log utility function features constant labor supply if there are no lump-sum transfers and, absent productivity shocks, this implies that capital is actually riskless; thus the available assets do *not* span the state space in the log-log case. Additionally, while households may have access to complete markets over shocks received in old age, the government does not; the government may only directly interact with the risk free debt market, and faces position limits in even that market.[29]

[29]The government may tax capital. However, taxing even the gross return to capital is not equivalent to pure ownership of the capital, as it distorts the return to capital felt by households and thus their saving

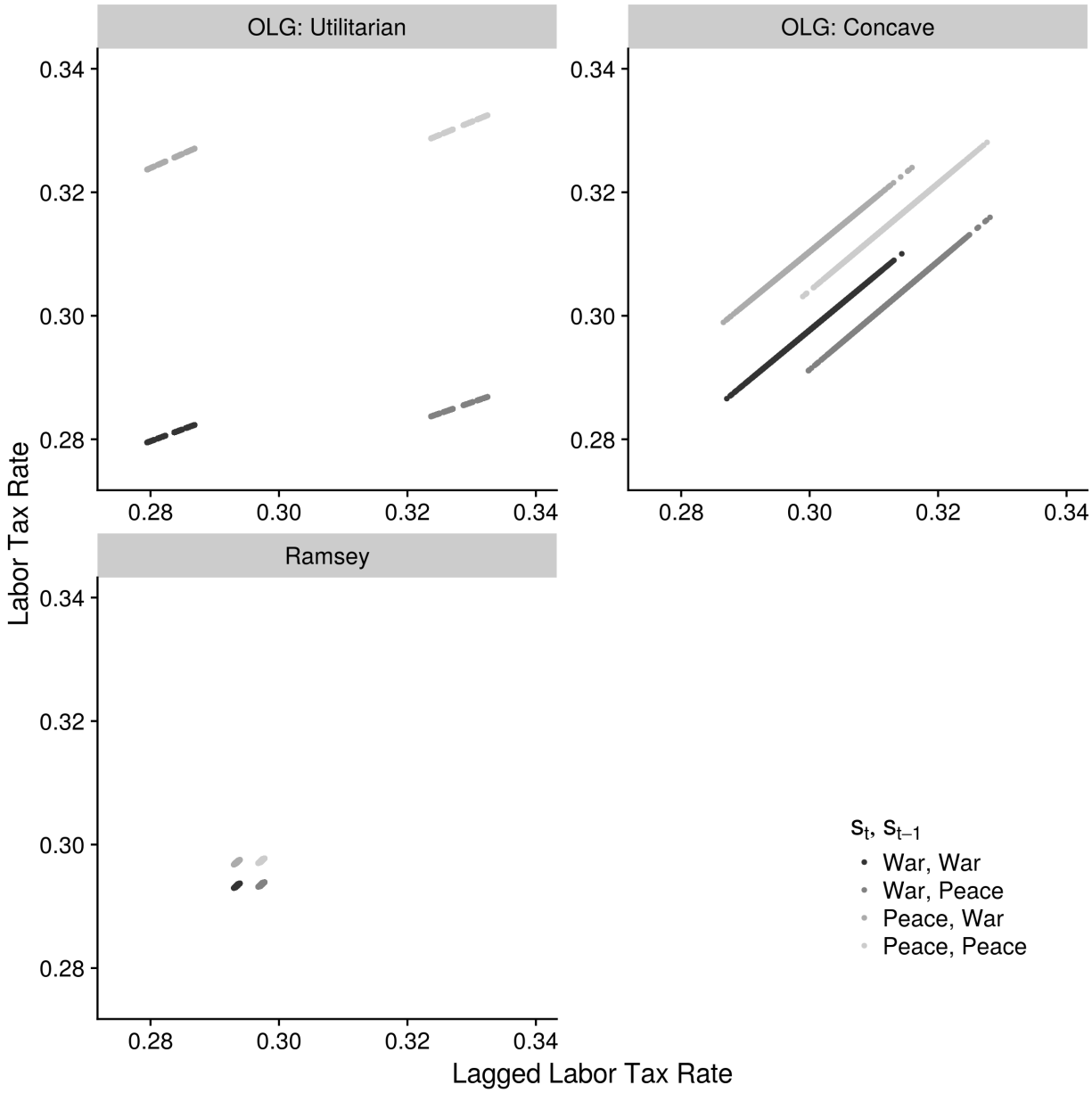


Figure 6: This figure presents the results of a random simulation of the complete markets model of stochastic government expenditure when utility is log-log. The model is simulated over 10000 periods (after a 100-period startup period that is excluded). The concave planner has $\zeta = 24$. The lagged labor tax rate is on the horizontal axis, and the current labor tax rate is on the vertical axis. The different shades of gray represent different combinations of the state s and lagged state s_{-1}. The Ramsey model was started with debt equal to -2.4% of nonstochastic steady state GDP; initial conditions are irrelevant for the OLG models.

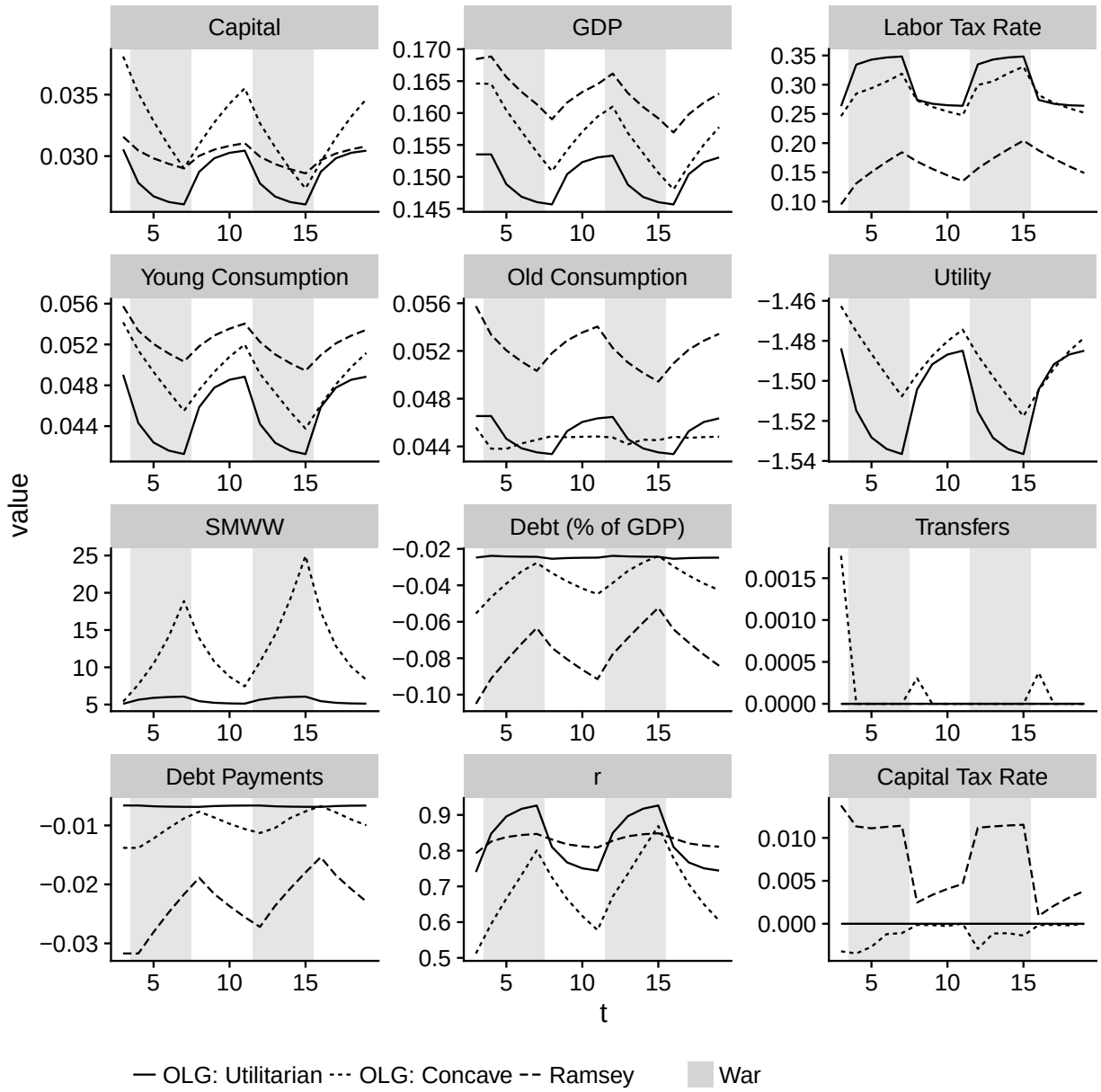


Figure 7: This figure captures the salient features of state transitions in the incomplete markets model of stochastic government expenditure when utility is log-log. Shading marks periods of high government spending ("wars"). The concave planner has $\zeta = 24$. The model was first allowed to run for 100 periods in state 1 to arrive at a steady state. The Ramsey model was started with debt equal to -2.4% of nonstochastic steady state GDP, which corresponds to the level of government debt in the OLG model after 100 periods of low government spending; initial conditions are irrelevant for the OLG models.

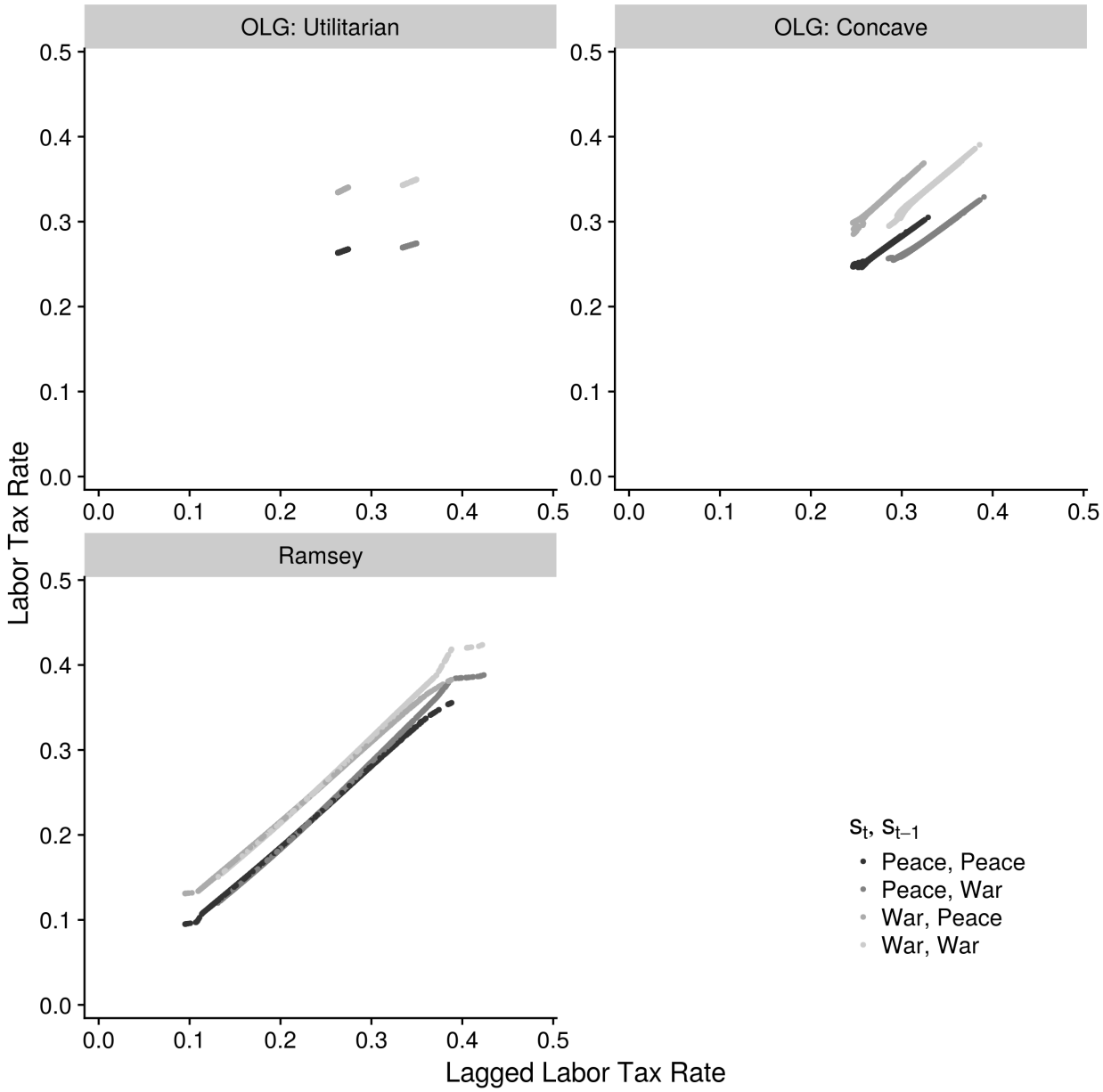


Figure 8: This figure presents the results of a random simulation of the incomplete markets model of stochastic government expenditure when utility is log-log. The model is simulated over 10000 periods (after a 100-period startup period that is excluded). The concave planner has $\zeta = 24$. The lagged labor tax rate is on the horizontal axis, and the current labor tax rate is on the vertical axis. The different shades of gray represent different combinations of the state s and lagged state s_{-1}.

Figures 7 and 8 respectively show the behavior of several variables across state transitions, and the autocorrelation structure of labor taxes. Perhaps most striking is that the graphs are qualitatively quite similar to the equivalent complete markets graphs: The transitions exhibit most of the same properties, and the scatterplot exhibits the same autocorrelation structure within combinations of spending and lagged spending, while having these clusters of points spread out across the graph. This similarity might initially seem surprising, but introducing the OLG structure to the model in a sense introduces a substantial amount of market incompleteness, even if the model features recursively complete markets. This is because, while the government *can* insure itself completely against government spending shocks, it *chooses* not to due to equity concerns, rendering a similar position to that of a Ramsey, incomplete markets problem. Put another way, individuals cannot insure themselves against the state of the world into which they are born, and the government cannot insure itself against shocks using the entire collection of agents as counterparties, which makes markets substantively incomplete.

The one qualitative difference from complete markets is that the government makes use of some lump-sum transfers in old age. Specifically, upon exiting a war, the planner compensates the old with a very small lump-sum transfer. The reason that transfer is not larger is that the efficiency cost of these transfers is quite large; they reduce labor supply, through the income effect, in the previous period. A much more nuanced discussion of old age transfers and state contingent debt can be found in the sections below on productivity shocks and trend productivity growth—applications that better lend themselves to study of this phenomenon.

5.2 Productivity Shocks

Next, consider an economy in which productivity is uncertain but has no trend growth, and in which there are no required government expenditures. If households are dynastic, the welfare

incentives.

theorems apply, and there is no need for any intervention by the government. However, in an OLG model, the planner faces a problem of optimally sharing the productivity risk across generations. Since this application involves only the single use for government policy, it is ideal for examining how the planner makes use of the instruments available to share risk intergenerationally. The producitivity shocks are calibrated as equal probability i.i.d. draws of $A = 1.08$ and $A = 0.92$.[30]

5.2.1 Complete Markets

I begin with the complete markets model, in which the government may issue state-contingent debt. Figure 9 shows the evolution of various variables during state transitions, and captures the essence of the government's policies. If the planner is utilitarian, then the desire for intergenerational risk sharing is fairly small, stemming only from diminishing marginal utility of consumption. Furthermore, retirees already share in productivity shocks to the extent that capital returns are affected. As a result, tax rates are nearly constant.[31]

In this model and the following one with trend productivity growth, government purchases are zero. Thus, payments on government debt should be interpreted as Social Security (SS) payments. They are a saving instrument for households with the government as the counterparty; they do not directly disincentivize labor through the income effect because disbursements are proportional to saving and, therefore, labor supply. If the planner is utilitarian, these SS payments are procyclical, but very mildly so. This causes retirees to share in the ups and downs of the economy to a greater extent than they would privately choose through risky capital alone.

Considering an explicitly inequality averse planner accentuates the policy response. Labor tax rates and SS payments are strongly procyclical, with fluctuations in the latter representing well over 10% of steady state consumption. Additionally, very small lump-sum transfers

[30] 16% is approximately the worst underperformance of trend over a 30 year period during the previous century or so.

[31] It is worth keeping in mind, throughout this section and the next, that labor taxes in an OLG economy might be constant and positive even without uncertainty, to prevent dynamic inefficiency.

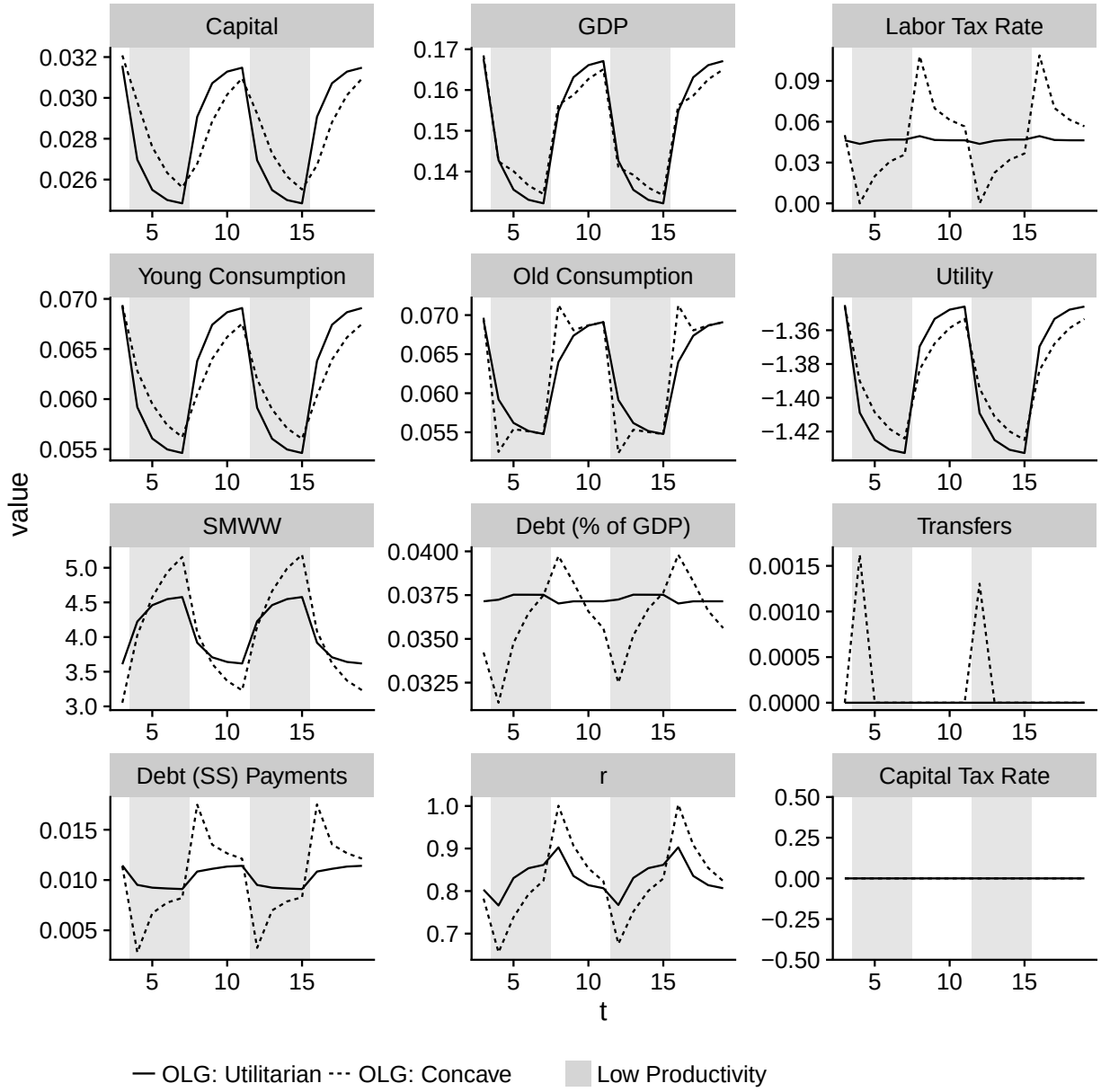


Figure 9: This figure captures the salient features of state transitions in the complete markets model of stochastic productivity when utility is log-log. Shading marks periods of low productivity. The concave planner has $\zeta = 4$. The model was first allowed to run for 100 periods in the good state to arrive at a steady state.

are granted to young households during the first period of a low productivity spell. This is funded through a sharp reduction in SS payments to the old and partially insures households against the state into which they are born.

Perhaps counterintuitively, SS payments do not insure the old against shocks to their retirement income—a common justification for SS in policy debates—but instead do precisely the opposite. However, recall that the social planner's welfare function was constructed to respect private preferences over risk. Thus, the planner has no motivation to decrease the amount of risk assumed by retirees. Rather, it is socially optimal for retirees to assume *more* risk than is privately optimal, so as to reduce the risk borne by future generations. This is especially true in the first period of a low productivity spell[32] when the planner is concave, since retirees in such a period will have enjoyed high utility during youth, thereby reducing their social marginal welfare weight.

Figure 10 shows the autocorrelation structure of tax rates in this model and is qualitatively similar to the stochastic government spending application. The main difference is that it is never optimal to set a negative labor tax rate; instead, a lump-sum transfer shsould be granted. Figure 11 shows the autocorrelation structure of the SS payments. All of these policies are highly autocorrelated contingent on the sequence of shocks received, which highlights the continued role for smoothing of distortions. However, in these simulations with a concave planner we see a negative *overall* correlation of the tax rate and SS payment, which shows that in certain economies we may observe oscillatory, rather than smooth, tax systems due to the redistributive motive.

Comparative Static: IES How do these risk-sharing policies change with the IES, a measure of the government's ability to access debt markets at little cost? Figure 12 compares the evolution of models with different IESs and points to several important differences. Moving from log utility to a higher IES accentuates the above policy behavior: SS payments fall in bad periods and rise in good periods to distribute risk between generations; these

[32] and vice-versa

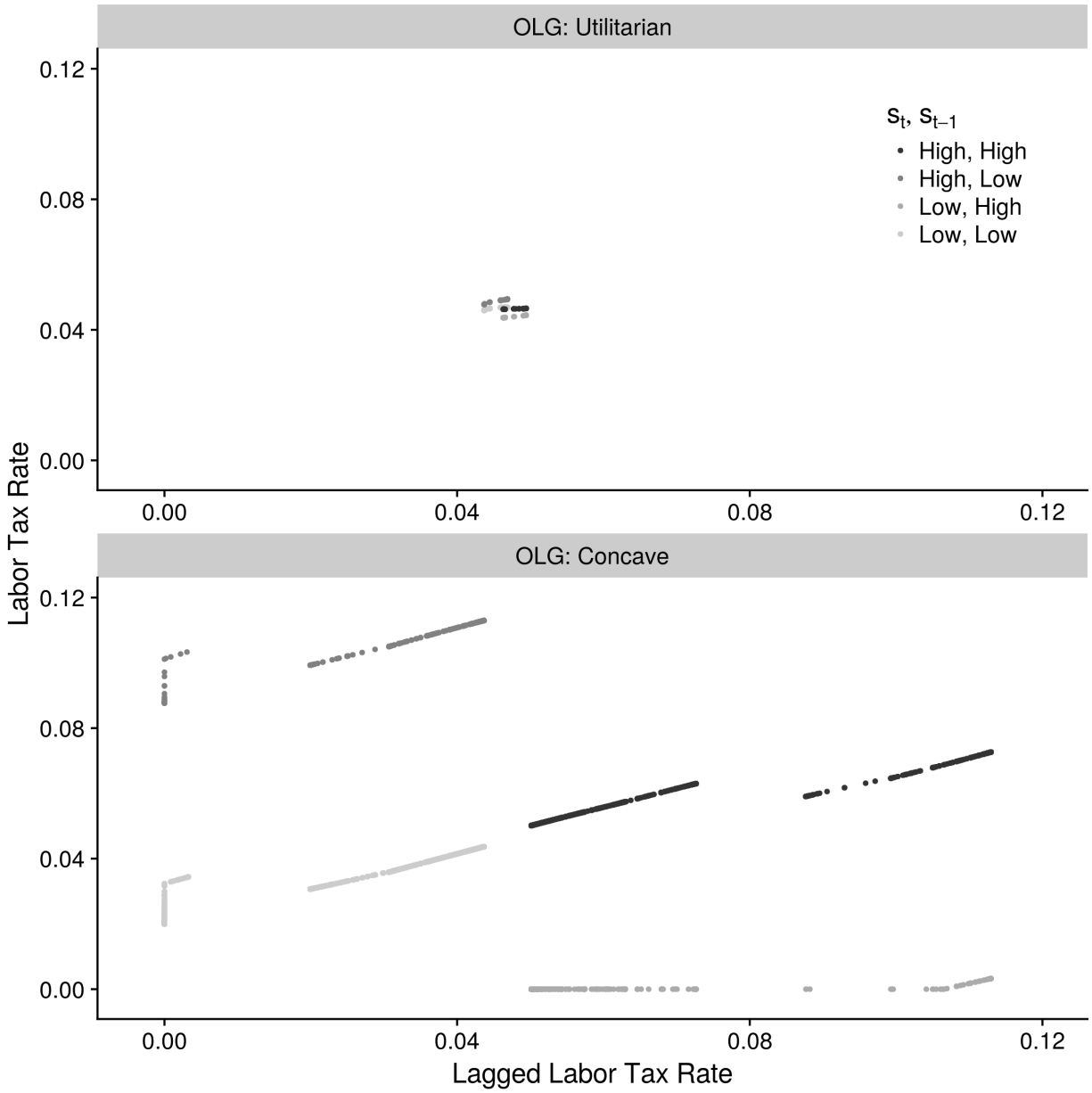


Figure 10: This figure presents the results of a random simulation of the complete markets model of stochastic productivity when utility is log-log. The model is simulated over 10000 periods (after a 100-period startup period that is excluded). The concave planner has $\zeta = 4$. The lagged labor tax rate is on the horizontal axis, and the current labor tax rate is on the vertical axis. The different shades of gray represent different combinations of the state s and lagged state s_{-1}.

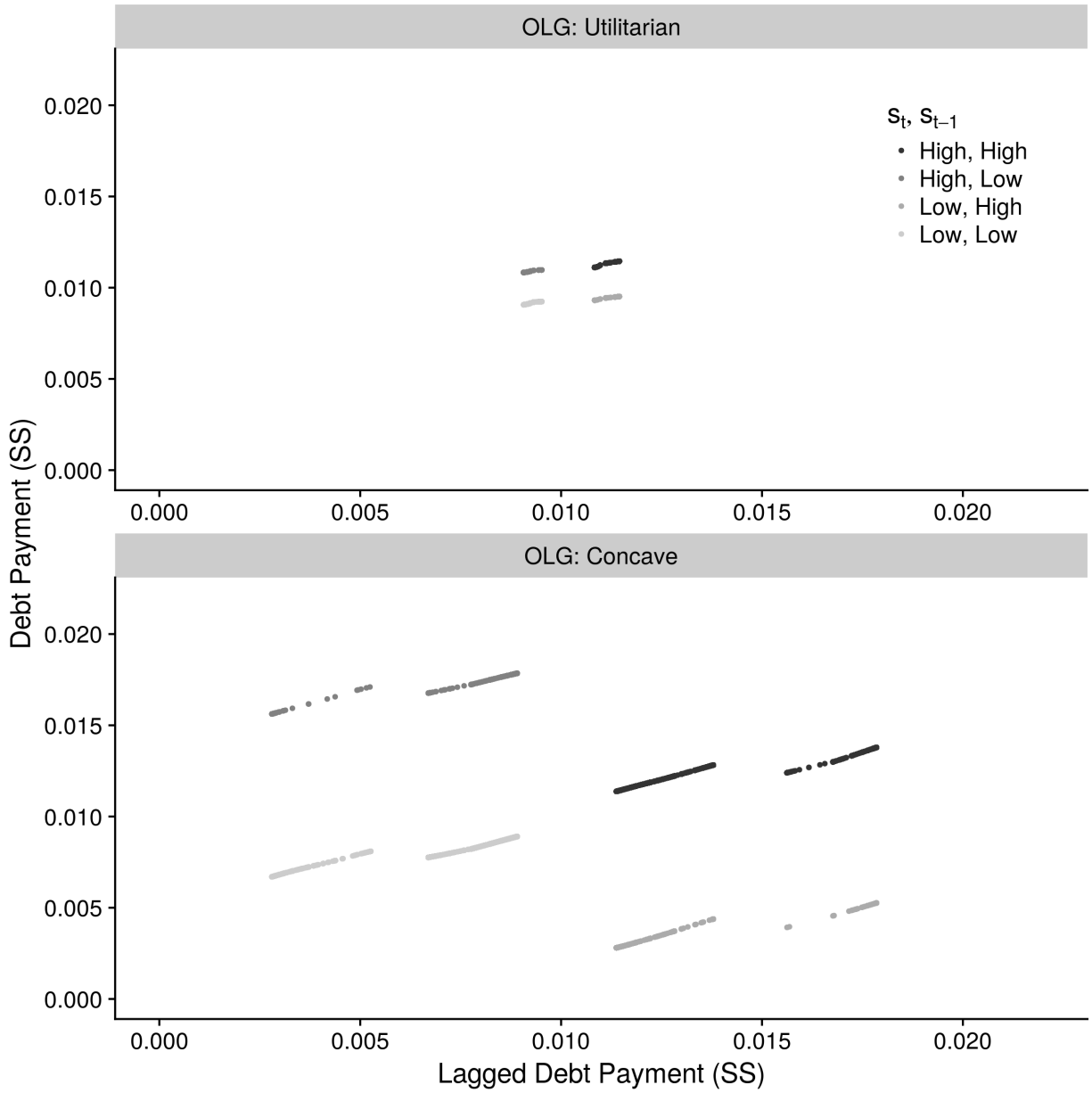


Figure 11: This figure presents the results of a random simulation of the complete markets model of stochastic productivity when utility is log-log. The model is simulated over 10000 periods (after a 100-period startup period that is excluded). The concave planner has $\zeta = 4$. The lagged debt (SS) payment is on the horizontal axis, and the current debt (SS) payment is on the vertical axis. The different shades of gray represent different combinations of the state s and lagged state s_{-1}.

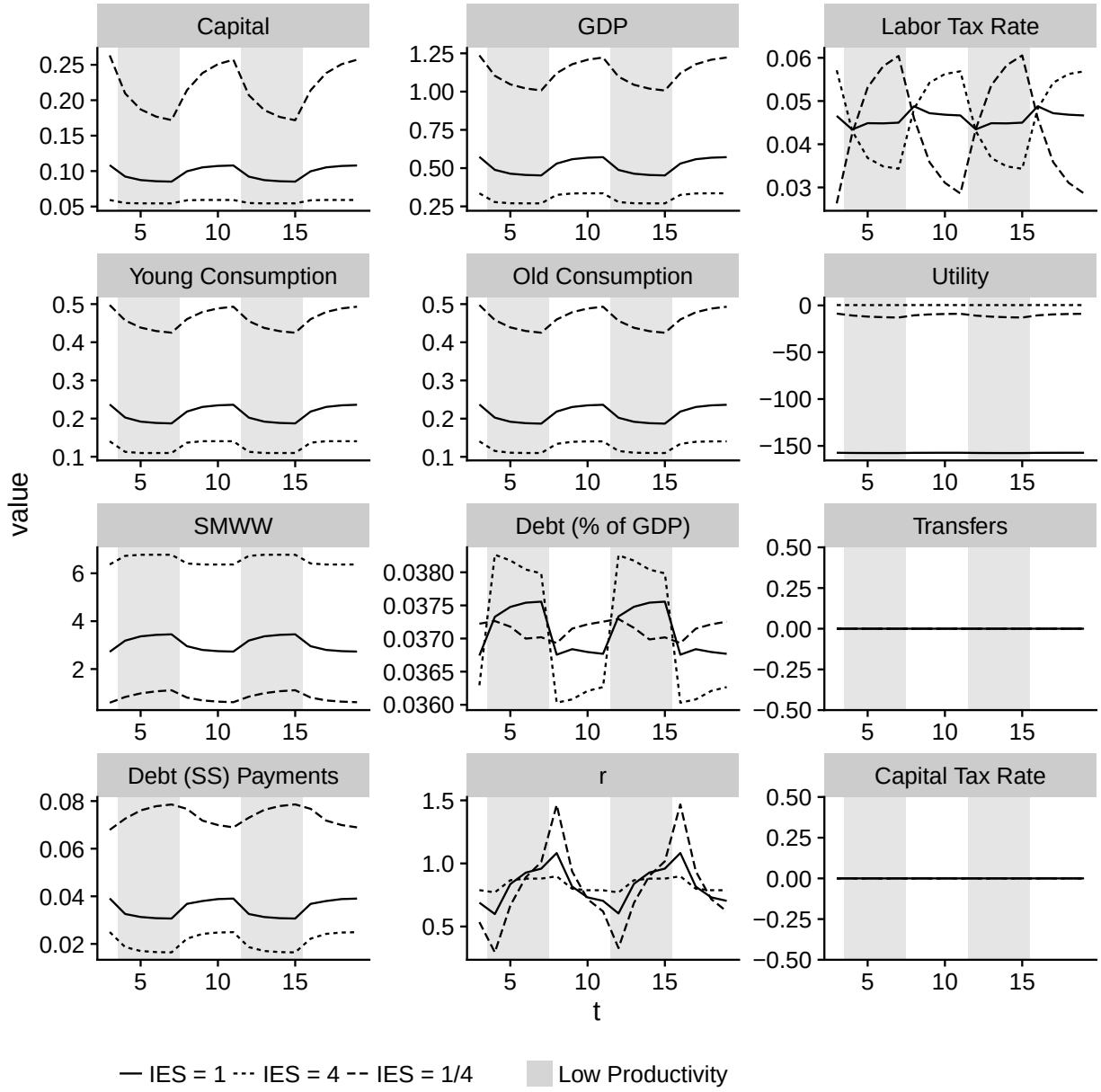


Figure 12: This figure captures the salient features of state transitions in the complete markets model of stochastic productivity for isoelastic utility with a variety of IESs and a utilitarian planner. Shading marks periods of low productivity. The model was first allowed to run for 100 periods in the good state to arrive at a steady state.

fluctuations are funded primarily by changes in the labor tax rate. However, moving to a lower IES appears to reverse the nature of government policy: SS payments and the labor tax rate *rise* during periods of low productivity. This is because income effects are now stronger than substitution effects; as a result, (22) says that taxes should be *higher* on less well-off generations.

5.2.2 Incomplete Markets

If markets are incomplete, and the government only has access to risk-free debt, then optimal policy looks very different, as evidenced in Figure 13.[33] The government can now no longer make state-contingent SS payments for the purpose of intergenerational risk sharing without resorting to lump-sum transfers. However, such lump-sum transfers are highly inefficient; they are granted regardless of labor supply in the previous period and thus disincentivize labor via the income effect. As a result, we see that the government makes limited use of this instrument, though slightly greater when the planner is more inequality-averse. Labor taxes are weakly countercyclical when the planner is utilitarian, as the government must raise taxes to achieve the same revenue during downturns.[34] They are strongly procyclical when the planner is concave, as this provides insurance to the unborn; the lost revenue is replaced with steeply increasing borrowing during downturns, the inefficiency of which is justified for the more inequality-averse planner.

Why are incomplete markets policies so different from complete markets policies for this application, when they were so similar for the application to government spending shocks? In the previous application, state-contingent debt was used for efficiency reasons—to insure the government budget constraint—and had unfortunate distributional consequences—it disproportionately placed risk on the old. As a result, the government made less use of state-contingent debt than in a dynastic model, and the incomplete markets restriction was

[33]For this example, I have made the shocks larger, $A \in \{1.12, 0.88\}$ to make the lump-sum transfer instrument active.

[34]Recall that the government issues debt to prevent dynamic inefficiency, and thus must make the interest payments on that debt.

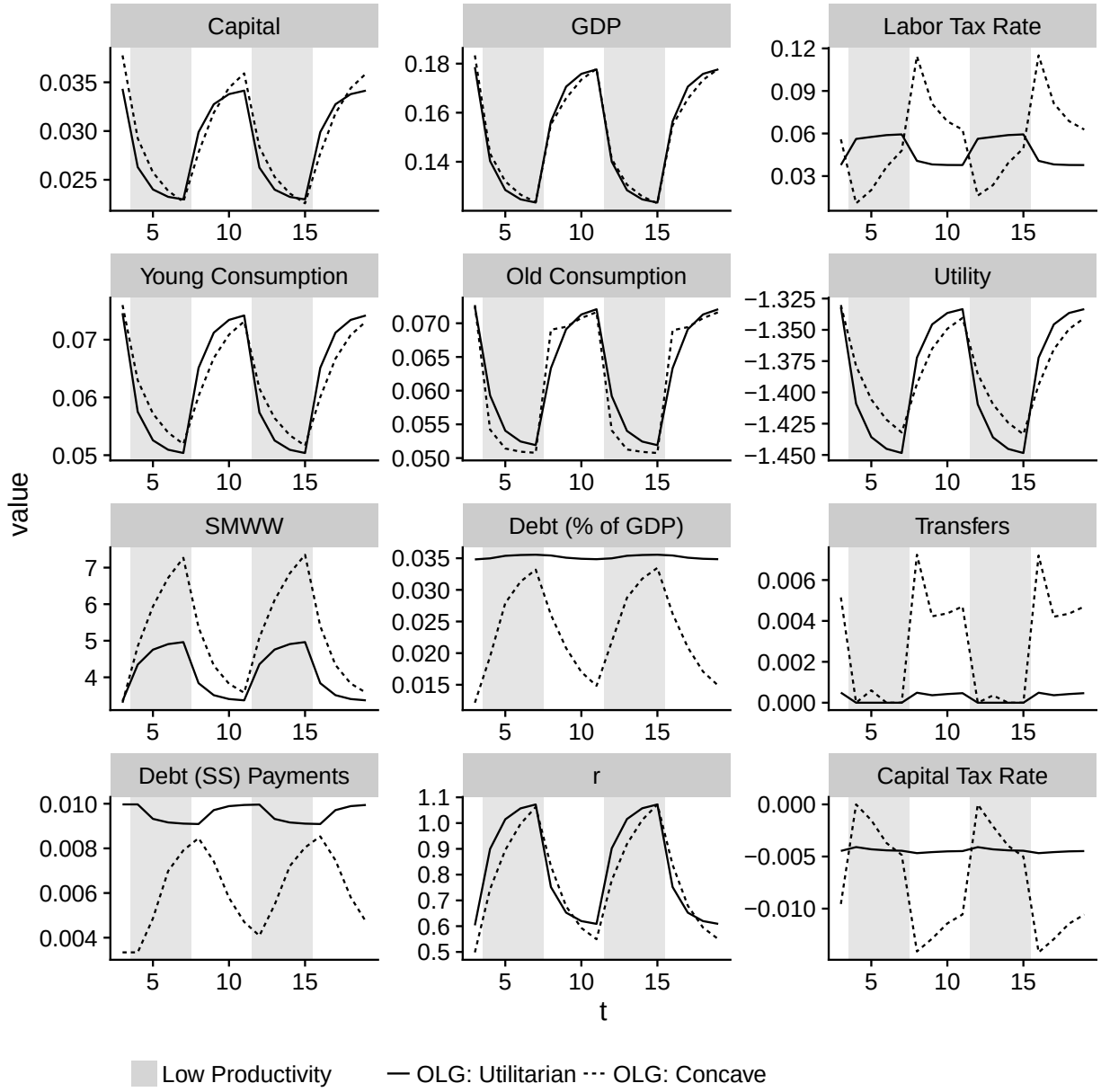


Figure 13: This figure captures the salient features of state transitions in the incomplete markets model of stochastic productivity when utility is log-log. Shading marks periods of low productivity. The concave planner has $\zeta = 4$. The model was first allowed to run for 100 periods in the good state to arrive at a steady state.

less important. In the present application, state-contingent debt plays the opposite role—it is used precisely for its distributional properties. As a result, the inequality-averse planner makes heavy use of that instrument when it is available, and policy changes greatly when it is not.

5.3 Productivity Growth

A common argument for an intergenerational transfer system like Social Security in the U.S. is that, if productivity is growing on average, then future generations will be wealthier than current ones. Thus, a system that redistributes from future generations to earlier ones is socially desirable. By adding a deterministic trend to productivity, the present model is well-suited to analyzing how such a system ought to look and what it depends on. Preserving shocks to productivity around such a trend from the previous application allows analysis of how such a SS system should respond to productivity shocks—a major policy question at hand today, after a period of slower than average productivity growth. Thus, I add deterministic growth in A of 2.1% per year (the long run U.S. average) to the previous model.

Figure 14 depicts optimal policy in this model if the government has access to state-contingent debt. While preserving all properties of policy in the previous model, trend growth introduces several new effects. First and most importantly, labor taxes and SS payments are much higher, especially when the planner is inequality-averse. This is evidence of precisely the intuition stated above—redistribution from richer, future (younger) generations to poorer, current (older) generations, the motive for which grows with the planner's inequality aversion. This comes at a price however—a higher interest rate, which in turn crowds out capital formation and lowers utility, all the more when the planner is more inequality averse. It is worth noting that the intergenerational risk-sharing carried out by state-contingent SS payments is amplified by the trend growth. This is because, with this calibration, the "stakes" are higher—fluctuations are 8% above or below trend, which are

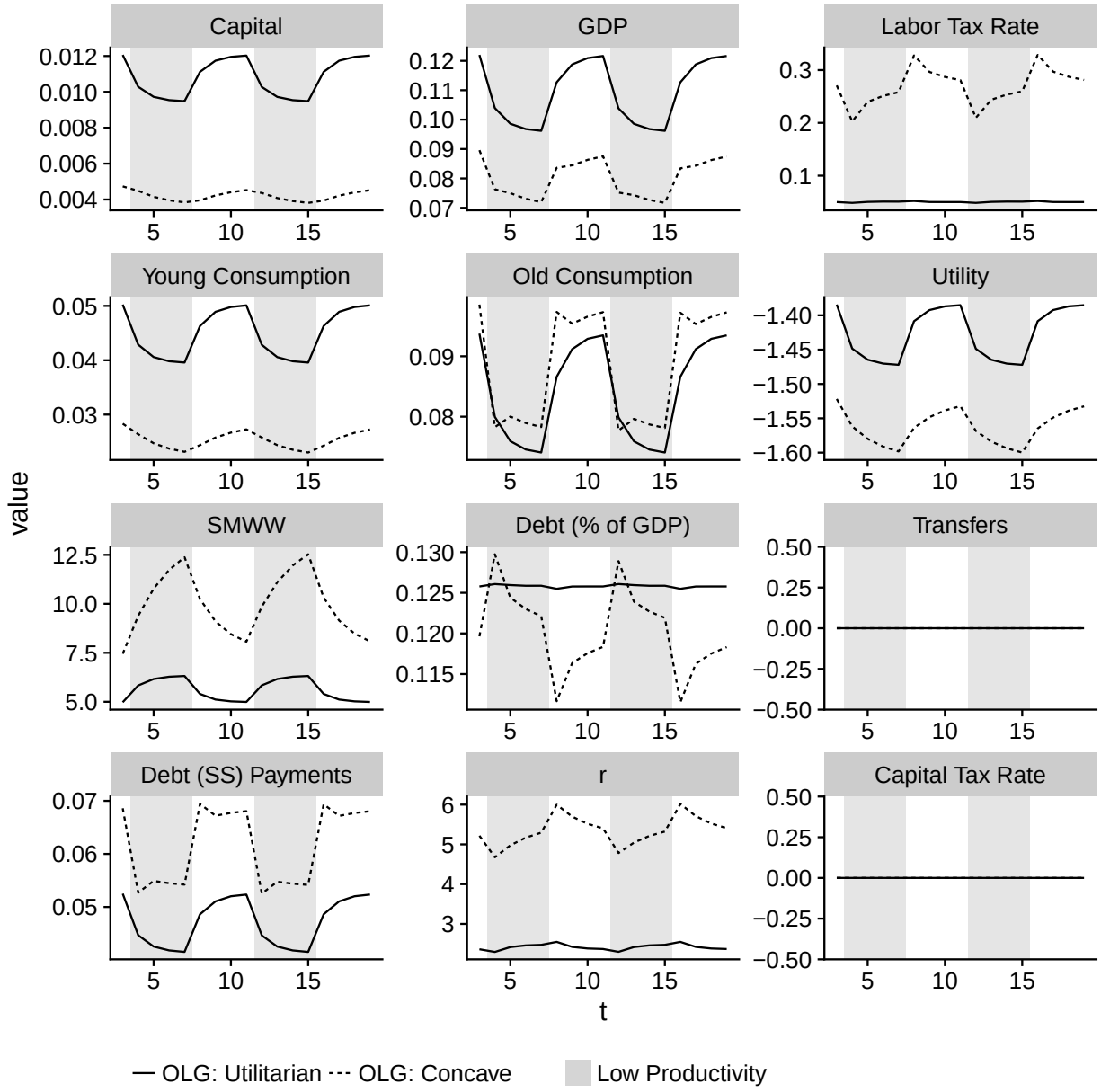


Figure 14: This figure captures the salient features of state transitions in the complete markets model of stochastic productivity with trend growth of 2.1% per year when utility is log-log. Shading marks periods of low productivity. The concave planner has $\zeta = 4$. The model was first allowed to run for 100 periods in the good state to arrive at a steady state. All variables detrended, with old consumption and debt (SS) payments detrended by the previous period's expected productivity.

very large relative to the earnings of the previous generation.

Figure 15 depicts optimal policy in this model if the government does not have access to state-contingent debt. Importantly, trend growth does *not* preserve all properties of policy in the previous model in this incomplete markets case. Specifically, the intergenerational risk sharing that was implemented via lump-sum transfers has disappeared. When labor tax rates are higher, which they are here to redistribute the trend growth in productivity, the income effect of lump-sum transfers has a larger effect on the government budget and, therefore, a larger efficiency cost. Thus, the planner decides to forgo intergenerational risk sharing. This highlights an important tradeoff between intergenerational risk sharing and deterministic intergenerational redistribution faced by a planner without access to state-contingent debt, but not by a planner with access to state-contingent debt.

5.4 Summary

Though each of these applications gives rise to substantially different optimal policies, the ideas are similar: An OLG planner, like a Ramsey planner, cares about smoothing distortionary costs of taxation, but also cares about intergenerational equity to a degree parameterized by his inequality-aversion. Income effects, however, often impair his ability to achieve such equity, and are higher for lower IESs with constant-IES utility functions.

One final note: Access to state-contingent debt changes policy substantially—especially for the applications to productivity shocks and growth—which prompts a discussion of whether such a policy instrument is realistic. Traditionally, in Ramsey models focused on government spending shocks, such state-contingent debt is widely seen as an unrealistic ability for the government to default on its debt in bad states of the world in a manner that is anticipated and not penalized via future difficulties obtaining credit; it is assumed for mathematical convenience. However, I argue that in the context of productivity shock and growth applications, such state-contingent debt is quite realistic; it represents the government's ability to issue different-sized SS checks depending on the state of the world. Though U.S. law

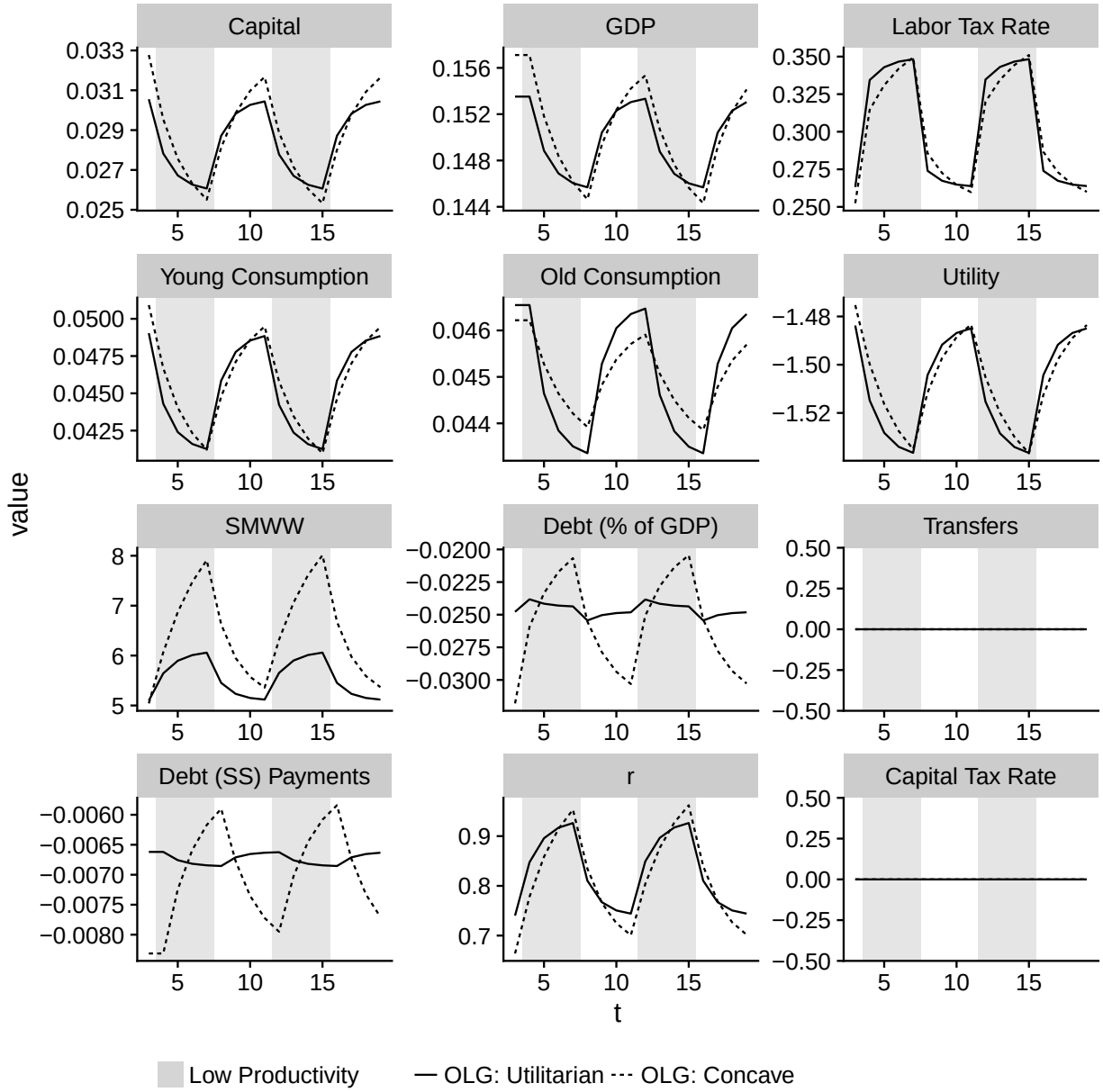


Figure 15: This figure captures the salient features of state transitions in the incomplete markets model of stochastic productivity with trend growth of 2.1% per year when utility is log-log. Shading marks periods of low productivity. The concave planner has $\zeta = 4$. The model was first allowed to run for 100 periods in the good state to arrive at a steady state. All variables detrended, with old consumption and debt (SS) payments detrended by the previous period's expected productivity.

does not explicitly specify SS as a state-contingent benefit that varies with the economoy's performance in late middle age or after retirement, it is quite reasonable to think that those benefits will be revised upward via new legislation if the economy performs unexpectedly well. The reverse seems unlikely due to loss aversion and political economy concerns, which are beyond the scope of this paper. But overall, the ability of the government to change SS payments in a proportional manner seems far more realistic than issuing true lump-sum transfers to retirees if the economy performs unexpectedly well.

6 Robustness: More than Two Generations

The two-generation model considered to this point has limited concrete applicability, as each period represents roughly thirty years. To check whether the foregoing results are qualitatively robust to a greater number of generations, and therefore shorter period length, I extend the complete markets model to feature a generic number of generations, but no bequest motive for simplicity. As this is merely a robustness test, I focus only on the first application: a model with stochastic government expenditure, and constant productivity. I consider only the simple, complete markets case with a utilitarian planner.

Most aspects of the model are unchanged. Instead of two generations, one working and one retired, there are now J_2 generations, of which J_1 are working and $J_2 - J_1$ are retired, indexed by $j = 1, 2, ..., J_2$. Households' preferences and budget sets are precisely as before, properly extended to account for the further generations. The planner still has access to linear taxes on labor, state-contingent linear taxes on capital, age-contingent lump-sum transfers (which can without loss of generality be assumed to only be given to the youngest generation), and state-contingent debt.

Though the models are similar, an important complexity is added: Multiple agents trade in the same markets (for labor and state-contingent securities) at the same time, and an allocation is only implementable if it is compatible with all trading agents facing the same

prices. In light of this complexity, I follow Werning (2007) and use the "market weights" approach to characterize the implementable allocations and formulate the planner's problem. Policy equilibria, implementable allocations, and the planner's problem are formally defined and proven in Appendix E.

6.1 Calibration

For results to be comparable, care must be taken when altering the number of generations, and implicitly the length of a period, to leave fundamental economic parameters unaltered. With this in mind, I choose the same log-log calibration of utility and Cobb-Douglas calibration of production, while changing the stochastic process so that the following are kept constant:

- The probability of a war per unit time
- Expected government expenditure as a percentage of steady-state GDP per unit time
- The variance of government expenditure as a percentage of steady-state GDP per unit time

Intuitively, this means that wars become less likely but more costly as the length of a period is shortened. β, Δ, and depreciation are also revised accordingly. I consider models with 3 generations (2 working), 4 generations (2 working), and 5 generations (3 working).

6.2 Results

Figure 16 shows how various variables evolve over state transitions in models with two through five generations. The results are very qualitatively similar to each other, suggesting that the nature of the system does not qualitatively depend on the number of generations. This provides reassurance that the assumption of two generations is not qualitatively important.

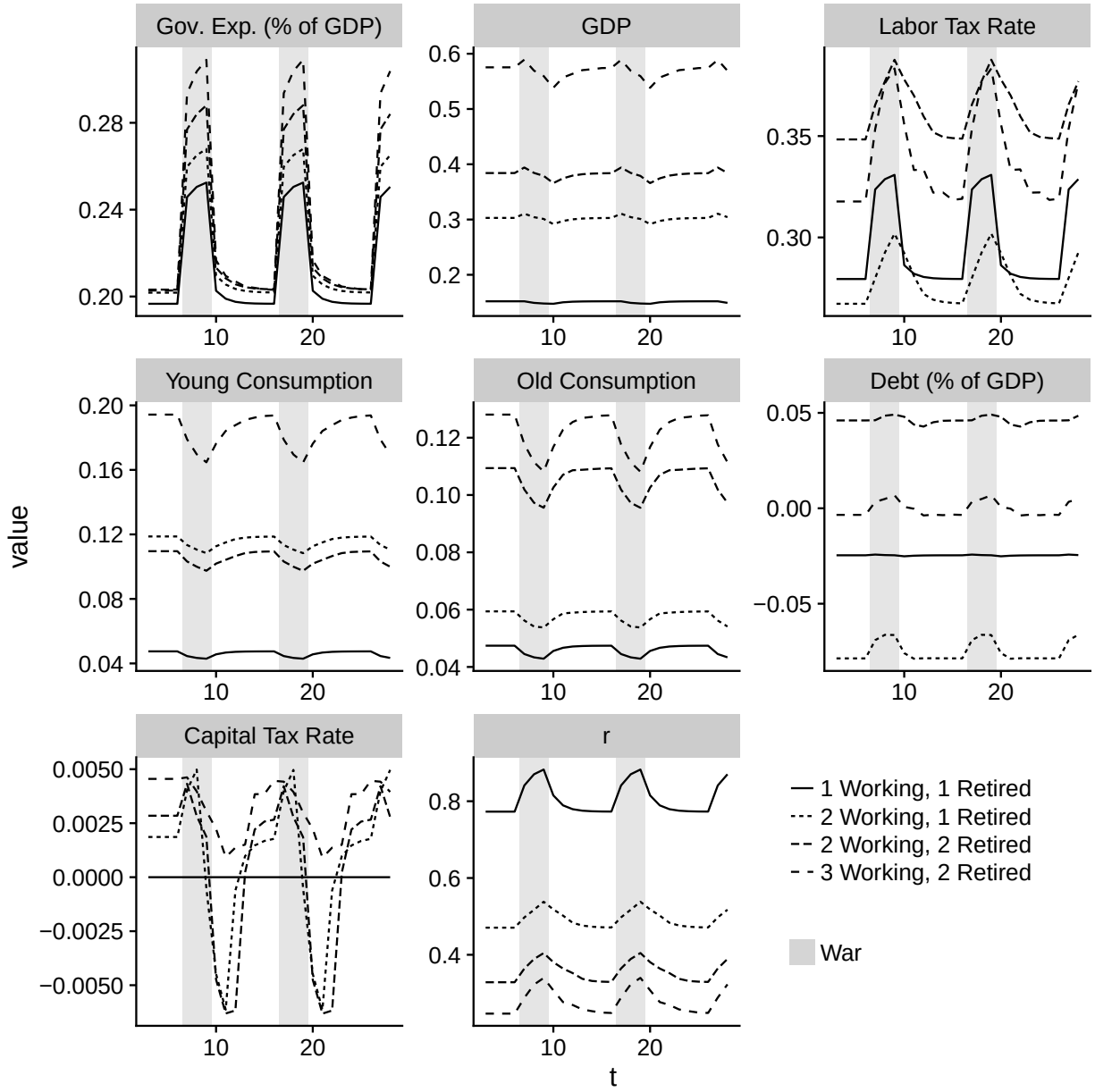


Figure 16: This figure captures the salient features of state transitions in the complete markets model of stochastic government expenditure when utility is log-log and the planner is utilitarian, with the number of generations ranging from 2 to 5. Shading marks periods of high government spending ("wars"). The model was first allowed to run for 100 periods in state 1 to arrive at a steady state.

There are two qualitative changes. First, government debt is not always negative in steady state, which calls into question the "buffer stock" intuition of a risk-averse planner. However, the planner is also interested in avoiding dynamic inefficiency, which may require positive steady state debt depending on the discount rate and return to capital, which vary across these models. Second, capital taxes are no longer always zero, consistent with the intuition of Erosa and Gervais (2002). In an OLG model, the planner will generically wish to tax different generations at different rates; if that is forbidden, as here, capital taxes may fill a similar desire to redistribute across generations.[35]

Figure 17 shows the autocorrelation structure of labor taxes in models with more generations. Models involving 2, 3, or 4 generations have similar results, though shorter periods lead to slightly lower variation in optimal tax rates. Additionally, a greater number of generations and a smaller discount rate means that a longer history is relevant. The model with 5 generations is, however, noticeably different, having more of a cloud-like structure rather than a tree-like structure. This noise could come from two different areas. First, this model, having six continuous state variables in its recursive representation, is subject to larger numerical error than the others. But more economically significantly, in this model, a single lag of the labor tax rate fails to capture all of the tax rates previously faced by currently working generations. This could explain the higher variation in the tax system.

Overall, the results of this section suggest that the assumption of two generations throughout most of the paper is not qualitatively important, and that the results may be applied to models with more generations and shorter periods.

[35]Erosa and Gervais (2002) find that capital taxes are indeed zero if preferences are homothetic over consumption at different ages and separable from labor, as the log-log utility function is. However, their model features no uncertainty, and so ignores transitory desires to redistribute between generations stemming from shocks hitting some generations harder than others.

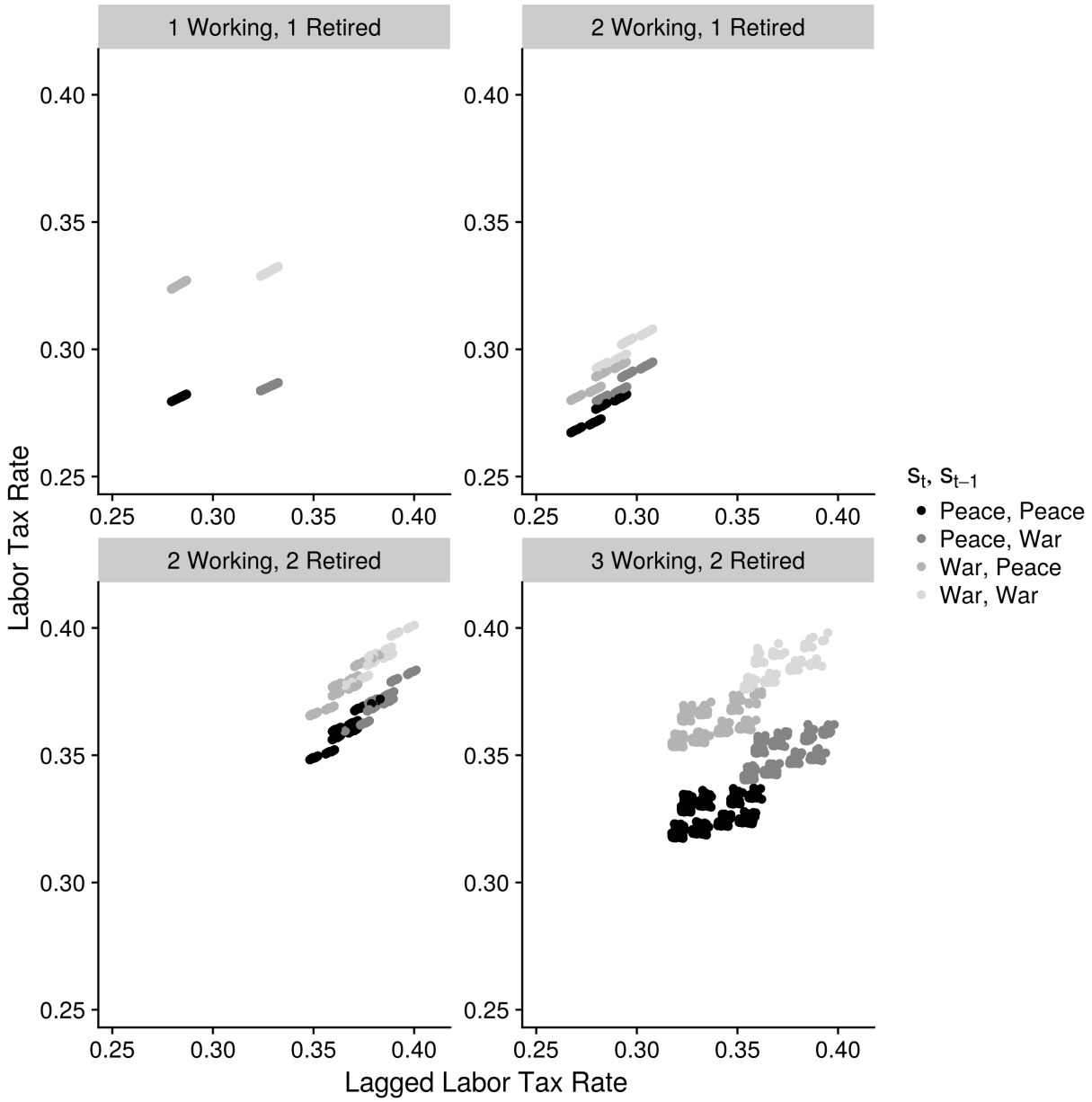


Figure 17: This figure presents the results of a random simulation of the complete markets model of stochastic government expenditure when utility is log-log and the planner is utilitarian, with the number of generations ranging from 2 to 5. The model is simulated over 10000 periods (after a 100-period startup period that is excluded). The lagged labor tax rate is on the horizontal axis, and the current labor tax rate is on the vertical axis. The different shades of gray represent different combinations of the state s and lagged state s_{-1}.

7 Conclusion

This paper has developed a framework for analyzing optimal policy in a macroeconomic model with overlapping generations, aggregate shocks, and a planner who has available only linear taxes on labor and capital. This setup complicates a standard Ramsey taxation model by adding a redistributive motive across generations—and, therefore, a tradeoff between equity and efficiency that is ubiquitous in the public finance literature—as well as removing Ricardian equivalence. We saw analytically that optimal policy in this model requires that the government equate *total* costs of taxation over time, rather than purely focusing on *distortionary* costs.

These total costs fall into four distinct categories: First is the standard substitution effect (the distortionary cost). Second is a mechanical effect of income loss, which are equal across time in a standard dynastic economy with homogeneous agents, but cannot be assumed so in a model with redistributive motive. Third is the static income effect in which higher taxes reduce income which in turn encourages labor supply and helps the government budget constraint. Fourth is a dynamic income effect which results from the effect of a proposed policy perturbation on asset prices and, therefore, perceived wealth. These two income effects can safely be ignored in the standard dynastic economy, since they are felt today even if they occur in the future; the same cannot be said of an OLG economy. Finally, these costs must be weighted by a social marginal welfare weight that captures redistributive preferences.

After the analytical investigation, I numerically applied the model to three distinct policy problems, which highlighted different aspects of these above findings. In the first, the government must decide how to optimally finance a stochastic sequence of required expenditure; optimal policy exhibits *partial* rather than perfect tax smoothing, with smoother taxes for higher IESs and more concave welfare functions. In the second, the government seeks to intergenerationally distribute the risk associated with a sequence of productivity shocks; optimal policy uses procyclical Social Security payments, if available, to ensure that retirees share in these productivity shocks. Finally, I add trend productivity growth, which creates

a desire to systematically redistribute from later generations to earlier ones; this is achieved through a Social Security-like system that involves high government debt and high labor taxes—all the more so when the social welfare function is more concave. These numerical results qualitatively match how developed countries actually behave, suggesting that policy is on the right track.

Finally, we saw that these results are qualitatively robust to the addition of further generations and associated shortening of the period, at least for a benchmark case.

7.1 Future Work

This paper suggests numerous directions for future research. Perhaps most importantly, policy recommendations could become more numerically realistic in a model with a substantially shorter period and, as a result, substantially more generations and a richer distribution of possible shocks. Unfortunately, the difficulty of solving such a model normally grows exponentially in the number of generations, as each generation adds dimensions to the state space of the recursive representation; my sparse-grid method reduces this growth to cubic, though that still becomes quite rapid. A likely solution to this curse of dimensionality would be a switch to continuous time and an adoption of a framework akin to the perpetual youth model of Blanchard (1985). While this is not straightforward in a model with elastic labor supply, it seems more promising than the gradual addition of generations in a discrete framework. Alternatively, one might keep a discrete set of generations with different life expectancies, but solve the model approximately using distributions as state variables, similar to Krusell and Smith, Jr. (1998).

A second obvious extension of the work regards intra-cohort heterogeneity and, therefore, a motivation for redistribution within a single cohort. Such assumptions would allow dropping the linear taxation assumption and, instead, a switch to a Mirrleesian framework. Ideally, such a study would examine an economy facing both aggregate *and* idiosyncratic uncertainty and begin to address the issue that redistribution may be most desirable precisely

when it is least affordable.

Third, one could consider other applications of the model. These might include fertility shocks, shocks to asset returns that do not affect labor productivity, or innovations to the growth rates of variables rather than transitory shocks to levels.

Fourth, the model presented in this paper, together with the characterization of optimal policy, raises empirical questions. While the response of taxable income to contemporaneous labor tax rates is a heavily studied topic,[36] the response of asset prices to labor tax rates is poorly understood. Nonetheless, such elasticities play an important role in the formulas developed in Sections 4.1 and 4.2, suggesting further empirical investigations would be worthwhile. This could could be dovetailed with a sufficient-statistic, rather than structural, approach to this sort of policy problem.

Finally, the present model assumes full commitment of the social planner. The various political economy constraints a government may face—especially in the context of an OLG economy—could lead to substantial changes to optimal policy and would be worth investigating seriously.

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[36]See, for example, Feldstein (1995), Saez (2010), and, for a nice review, Saez, Slemrod and Giertz (2012)

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A Proofs from Section 2

A.1 Formal Definition of Equilibrium

Equilibrium definitions are very similar for the two models, so I explicitly define it only for complete markets.

Definition A.1 (Policy Equilibrium—Complete Markets) *A policy equilibrium in the complete markets model is a collection of*

- *policies* $T_0^o, \{T(s^t), \tau^L(s^t), \tau^K(s^{t+1})\}_{t\geq 0}$
- *prices* $\{q(s^{t+1}), w(s^t), R^K(s^t)\}_{t\geq 0}$

- *an allocation* $\{c^o(s^t), c^y(s^t), \ell(s^t), k(s^t), z(s^t)\}_{t\geq 0}$
- *and government debt* $\{b(s^{t+1})\}_{t\geq 0}$

such that at all histories s^t, $t \geq 0$

- *The resource constraint is satisfied:*

$$c^y(s^t) + c^o(s^t) + g(s_t, t) + k(s^t) \leq F(k(s^{t-1}), \ell(s^t); A(s_t, t)) \tag{26}$$

- *The government's budget constraint (1) is satisfied*
- *The household's budget constraint (5) and bequest constraint (3)are satisfied*
- *Households optimize subject to their budget constraint, taking prices and policies as given*
- *Firms optimize, taking prices as given*
- *The state-contingent asset markets clear:*

$$b(s^t) + R^K(s^t)k(s^{t-1})(1 - \tau^K(s^t)) = c^o(s^t) + z(s^t)$$

- *The markets for capital and labor clear*
- *The no-arbitrage condition between capital and state-contingent assets holds:*

$$\sum_{s^{t+1}\succeq s^t} q(s^{t+1})R^K(s^{t+1})(1 - \tau^K(s^{t+1})) = 1$$

- *The initial old consume or bequeath their untaxed assets plus any transfer:*

$$c^o(s_0) + z(s_0) = (1 - \tau^K(s_0))R^K(s_0)k_{-1} + b(s_0) + T^o(s_0)$$

The only changes for the incomplete markets model are that households and the government face different budget constraints, and we lose the no-arbitrage condition (there are no redundant assets) and the asset market clearing condition (which is found in the household's old-age budget constraint).

A.2 Proof of Proposition 2.1

I begin by proving the "if" direction; that is, any allocation satisfying the conditions is implementable. I prove this by construction.

First, define prices

$$\begin{aligned} q(s^{t+1}) &= \beta Pr(s_{t+1}|s^t)\frac{u_c^o(s^{t+1})}{u_c^y(s^t)} \\ w(s^t) &= F_L(s^t) \\ R^K(s^t) &= F_K(s^t) \end{aligned}$$

and government debt

$$b(s^t) = c(s^t) + z(s^t) - (1 - \tau^K(s^t))R^K(s^t)k(s^{t-1}).$$

Then define policies

$$\begin{aligned}
\tau^L(s^t) &= 1 + \frac{u_\ell^y(s^t)}{w(s^t)u_c^y(s^t)}\ell(s^t) + \beta\mathbb{E}_t\left[\frac{u_c^o(s^{t+1})}{u_c^y(s^t)}(c^o(s^{t+1}) + z(s^{t+1})\right] \\
T_0^o &= c^o(s_0) + z(s_0) - b(s_0) \\
\tau^K(s^{t+1}) &= 1 - \frac{u_c^y(s^t)}{\beta\mathbb{E}_t[R^K(s^{t+1})u_c^o(s^{t+1})]} \\
\tau^K(s_0) &= 1
\end{aligned}$$

Several conditions of equilibrium have already been satisfied: The resource constraint is satisfied by assumption, firms are satisfying their first order conditions at market-clearing levels of capital and labor, and the state-contingent asset markets clear by construction. Additionally, the budget constraint of the initial old is satisfied by construction. Next, the no-arbitrage condition is clearly satisfied:

$$\sum_{s_{t+1}} \beta Pr(s_{t+1}|s^t)\frac{u_c^o(s^{t+1})}{u_c^y(s^t)}R^K(s^{t+1})\left[\frac{u_c^y(s^t)}{\beta\mathbb{E}_t[R^K(s^{t+1})u_c^o(s^{t+1})]}\right] = 1$$

It remains to be verified that households and the government are satisfying their budget constraints, and that households are optimizing. The household budget constraint is easily verified by dividing the implementability condition by $u_c^y(s^t)$ and substituting in the prices and policies as defined. Subtracting the household budget constraint (which is satisfied with equality) from the resource constraint then yields the government budget constraint. The asset prices imply that the household's intertemporal first order conditions (Euler equations) are satisfied, while the net-of-tax wage implies that the household's intratemporal first order condition (labor-leisure tradeoff) is satisfied. Finally, the consumption-bequest tradeoff is satisfied since either the first order condition between the two is satisfied at equality,[37] or consumption is more valued than bequests, but the bequest is already at its minimum—a complementary slackness condition.

Next I continue to the "only if" direction; that is, any implementable allocation must satisfy these conditions. This is easier to show. By definition, implementable allocations must satisfy the resource constraint and the constraint on the initial old. Prices $q(s^t)$ and net-of-tax wages $(1 - \tau^L(s^t))w(s^t)$ must be defined as above for households to satisfy their first order conditions. Substituting those into the household budget constraint and dividing by $u_c^y(s^t)$ yields the implementability condition. Likewise, the optimal bequest conditions come directly from the first order condition with respect to $z(s^t)$ in the household's problem.

[37] To see this, take the envelope condition from (4)

A.3 Proof of Proposition 2.2

The proof proceeds similarly to the complete markets case, substituting prices out of budget constraints using first order conditions, and is thus omitted.

B Proofs From Section 3

B.1 Proof of Proposition 3.1

All of these equations are obtained through straightforward rearrangement of the planner's first order conditions.

B.2 Proof of Proposition 3.2

The implementability condition on the generation born at date t takes the form

$$1 + \beta - v'(\ell_t)\ell_t \geq 0.$$

So long as the optimum does not feature lump-sum transfers, this implementability condition holds with equality, which means that labor is fixed at ℓ^* satisfying

$$v'(\ell^*)\ell^* = 1 + \beta.$$

As a result, the planner faces

$$\begin{aligned} \max_{c_t^y,\, c_t^o} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} [\log c_t^y + \log c_t^o] \\ \text{s.t.} \quad & c_t^y + c_t^o \leq A_t \ell^* - A_t g_t \end{aligned}$$

which is merely a static endowment allocation problem, with an obvious solution: $c_t^y = c_t^o = \frac{A_t}{2}(\ell^* - g_t)$.[38] Using the individual's first order condition, we find taxes as

$$\begin{aligned} v'(\ell^*)c_t^y &= A_t(1 - \tau_t^L) \\ \frac{1}{2}v'(\ell^*)(\ell^* - g_t) &= 1 - \tau_t^L \\ \frac{1+\beta}{2}\left(1 - \frac{g_t}{\ell^*}\right) &= 1 - \tau_t^L \end{aligned}$$

which is a function of only the current state.

To show that the distortionary cost is also a function only of the current state, consider

[38] I've multiplied g_t by A_t for convenience; it is without loss of generality.

the unaltered planner's problem:

$$\begin{aligned}\max_{c_t^y,\, c_t^o} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} [\log c_t^y + \log c_t^o - v(\ell_t)] \\ \text{s.t.} \quad & c_t^y + c_t^o + A_t g_t \leq A_t \ell_t \\ & 1 + \beta - v'(\ell_t)\ell_t \geq 0\end{aligned}$$

The first order condition with respect to ℓ_t is

$$A_t \psi_t = v'(\ell_t) + [v''(\ell_t)\ell_t + v'(\ell_t)]\mu_t$$

while the first order condition with respect to c_t^y is $\psi_t = 1/c_t^y$. Substituting in the values from above for ℓ_t and c_t^y and rearranging yields

$$\tilde{\mu}_t = \frac{\frac{2}{\ell^* - g_t} - v'(\ell^*)}{v''(\ell^*)\ell^* + v'(\ell^*)}$$

and so the distortionary cost of taxation depends only on the current state of the world and is not intertemporally linked in any way.

C Proofs from Section 4

C.1 Proof of Proposition 4.1

This expression combines the first order condition for $z(s^{t+1})$ with the household's Euler equation for the asset paying off at history s^{t+1}.

C.2 Proof of Proposition 4.2

This expression combines the first order conditions with respect to c_{t+1}^y and c_{t+1}^o with the household's Euler equation for the asset paying off at history s^{t+1}.

C.3 Proof of Proposition 4.3

This combines the first order conditions with respect to c_t^y, c_{t+1}^o, and k_t, then employs the no-arbitrage condition between state-contingent assets and capital.

C.4 Proof of Corollary 4.4

This follows directly from differentiation of the utility function, rearranging, and employing the no-arbitrage condition.

C.5 Proof of Proposition 4.5

This expression combines the first order conditions for $z(s^{t+1})$ and $b(s^t)$ with the household's Euler equation for risk free debt issued at s^t.

C.6 Proof of Proposition 4.6

The first order condition with respect to b_t when debt limits do not bind is

$$u^y_{c,t}\mu^y_t = \beta R^f_t \mathbb{E}_t \mu^o_{t+1}.$$

The first order condition with respect to c^y_t when bequest constraints bind (and so $\tilde{\xi}^\delta_t = 0$) is

$$\begin{aligned}
\psi_t =& \tilde{w}_t + \mu^y_t \left[u^y_{c,t} + u^y_{cc,t}(c^y_t + k_t + b_t - z_t) + u^y_{c\ell,t}\ell_t\right] \\
&- \beta \mathbb{E}_t \left\{ \mu^o_{t+1} \left[\frac{u^y_{cc,t}}{\beta \mathbb{E}_t u^o_{c,t+1}} b_t + \frac{u^y_{cc,t} F_{K,t+1}}{\beta \mathbb{E}_t [F_{K,t+1} u^o_{c,t+1}]} k_t \right] \right\} \\
=& \tilde{w}_t \left\{ 1 + \tilde{\mu}_t \left[\eta_t - \sigma^y_t \frac{k_t + b_t}{c^y_t} \right] \right\} + \sigma^y_t \frac{b_t}{c^y_t} \beta R^f_t \mathbb{E}_t \mu^o_{t+1} + \sigma^y_t \frac{k_t}{c^y_t} \beta (1 - \tau^K_t) \mathbb{E}_t [F_{K,t+1} \mu^o_{t+1}] \\
=& \tilde{w}_t \left[1 + \tilde{\mu}_t \eta_t\right] + \sigma^y_t \frac{k_t}{c^y_t} \left\{ w_t \beta (1 - \tau^k_t) \mathbb{E}_t [F_{K,t+1} u^o_{c,t+1} \hat{\mu}_{t+1}] - \tilde{w}_t \tilde{\mu}_t \right\} \\
=& \tilde{w}_t \left[1 + \tilde{\mu}_t \eta_t\right] + \sigma^y_t \frac{k_t}{c^y_t} \left\{ \tilde{w}_t (1 - \tau^k_t) \mathbb{E}_t \left[F_{K,t+1} \frac{\beta u^o_{c,t+1}}{u^y_{c,t}} \hat{\mu}_{t+1} \right] - \tilde{w}_t \tilde{\mu}_t \right\} \\
=& \tilde{w}_t \left\{ 1 + \tilde{\mu}_t \eta_t + \frac{\sigma^y_t (1 - \tau^K_t)}{c^y_t} \left(\sum_{s_{t+1}} [q_{t+1} F_{K,t+1} k_t \hat{\mu}_{t+1}] - \frac{\tilde{\mu}_t k_t}{1 - \tau^K_t} \right) \right\}
\end{aligned}$$

The first order condition with respect to c^o_t when bequest constraints bind (and so $\tilde{\xi}^\delta_t = 0$) is

$$\begin{aligned}
\frac{\Delta}{\beta}\psi_t =& w_{t-1} u^o_{c,t} + \mu^o_t + \frac{u^y_{c,t-1} u^o_{cc,t}}{\beta (\mathbb{E}_{t-1} u^o_{c,t})^2} b_{t-1} \mathbb{E}_{t-1} \mu^o_t + \frac{u^y_{c,t-1} F_{K,t} u^o_{cc,t}}{\beta (\mathbb{E}_{t-1} [F_{K,t} u^o_{c,t}])^2} k_{t-1} \mathbb{E}_{t-1} [\mu^o_t F_{K,t}] \\
& w_{t-1} u^o_{c,t} + \mu^o_t + R^f_{t-1} \frac{u^o_{cc,t}}{\mathbb{E}_{t-1} u^o_{c,t}} b_{t-1} \mathbb{E}_{t-1} \mu^o_t + (1 - \tau^K_{t-1}) \frac{F_{K,t} u^o_{cc,t}}{\mathbb{E}_{t-1} [F_{K,t} u^o_{c,t}]} k_{t-1} \mathbb{E}_{t-1} [\mu^o_t F_{K,t}]
\end{aligned}$$

Now take expectations of both sides at date $t-1$:

$$\begin{aligned}
\frac{\Delta}{\beta}\mathbb{E}_{t-1}\psi_t =& w_{t-1} \mathbb{E}_{t-1} u^o_{c,t} + \mathbb{E}_{t-1} \mu^o_t + R^f_{t-1} \frac{\mathbb{E}_{t-1} u^o_{cc,t}}{\mathbb{E}_{t-1} u^o_{c,t}} b_{t-1} \mathbb{E}_{t-1} \mu^o_{t-1} \\
&+ (1 - \tau^K_{t-1}) \frac{\mathbb{E}_{t-1} [F_{K,t} u^o_{cc,t}]}{\mathbb{E}_{t-1} [F_{K,t} u^o_{c,t}]} k_{t-1} \mathbb{E}_{t-1} [\mu^o_t F_{K,t}]
\end{aligned}$$

Using the first order condition for b_{t-1} and the definition of $\epsilon_{b_t}^{R_t^f}$ developed before leaves

$$\Delta\mathbb{E}_{t-1}\psi_t = \frac{\tilde{w}_{t-1}}{R_{t-1}^f}\left[1+\tilde{\mu}_{t-1}\left(1-\epsilon_{b_{t-1}}^{R_{t-1}^f}\right)\right] + (1-\tau_{t-1}^K)\frac{\mathbb{E}_{t-1}[F_{K,t}u_{cc,t}^o]}{\mathbb{E}_{t-1}[F_{K,t}u_{c,t}^o]}k_{t-1}\mathbb{E}_{t-1}[\mu_t^o F_{K,t}]$$

Now differentiate

$$u_{c,t}^y = \beta(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u_{c,t+1}^o]$$

with respect to c_{t+1}^o, holding fixed $u_{c,t}^y$ and k_t, to obtain

$$0 = \beta(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u_{cc,t+1}^o dc_{t+1}^o] - d\tau_t^K\mathbb{E}_t[F_{K,t+1}u_{c,t+1}^o]$$

Considering a perturbation of c_{t+1}^o that involves only an increase in b_t, and therefore a rise in c_{t+1}^o by R_t^f across all states, yields

$$\frac{\partial\tau_t^K}{\partial b_t} = (1-\tau_t^K)R_t^f\frac{\mathbb{E}_t[F_{K,t+1}u_{cc,t+1}^o]}{\mathbb{E}_t[F_{K,t+1}u_{c,t+1}^o]}.$$

Substituting this back in yields

$$\Delta\mathbb{E}_{t-1}\psi_t = \frac{\tilde{w}_{t-1}}{R_{t-1}^f}\left[1+\tilde{\mu}_{t-1}\left(1-\epsilon_{b_{t-1}}^{R_{t-1}^f}\right)\right] + \frac{1}{R_{t-1}^f}\frac{\partial\tau_t^K}{\partial b_t}k_{t-1}\mathbb{E}_{t-1}[F_{K,t}\mu_t^o].$$

Replacing $\mathbb{E}_{t-1}[F_{K,t}\mu_t^o]$ as before yields

$$\Delta\mathbb{E}_{t-1}\psi_t = \frac{\tilde{w}_{t-1}}{R_{t-1}^f}\left[1+\tilde{\mu}_{t-1}\left(1-\epsilon_{b_{t-1}}^{R_{t-1}^f}\right)\right] + \frac{1}{R_{t-1}^f}\frac{\partial\tau_t^K}{\partial b_t}k_{t-1}\tilde{w}_{t-1}\sum_{s_t}[q_t F_{K,t}\hat{\mu}_t]$$

Combining this with the expectation at date $t-1$ of the first order condition with respect to young consumption at date t, and then advancing the time index yields the desired result.

C.7 Proof of Proposition 4.7

In addition to the previous first order conditions, take the first order condition with respect to k_t:

$$\begin{aligned}\psi_t =& \Delta\mathbb{E}_t[F_{K,t+1}\psi_{t+1}] + \mu_t^y u_{c,t}^y - u_{c,t}^y\frac{\mathbb{E}_t[F_{K,t+1}\mu_{t+1}^o + F_{KK,t+1}k_t\mu_{t+1}^o]}{\mathbb{E}_t[F_{K,t+1}u_{c,t+1}^o]}\\ &+ u_{c,t}^y\frac{\mathbb{E}_t[k_t F_{K,t+1}\mu_{t+1}^o]}{(\mathbb{E}_t[F_{K,t+1}u_{c,t+1}^o])^2}\mathbb{E}_t[F_{KK,t+1}u_{c,t+1}^o]\end{aligned}$$

The last two terms will clearly give rise to the "hedging" term of Farhi (2010) and the quasilinear example, so I'll start by simplifying just those. The last term reduces to

$$\beta w_t(1-\tau_t^K)\frac{\mathbb{E}_t[F_{KK,t+1}u_{c,t+1}^o]}{\mathbb{E}_t[F_{K,t+1}u_{c,t+1}^o]}\left\{k_t\mathbb{E}_t[F_{K,t+1}u_{c,t+1}^o]\mathbb{E}_t\hat{\mu}_{t+1} + \mathrm{Cov}_t[\hat{\mu}_{t+1}, k_t F_{K,t+1}u_{c,t+1}^o]\right\}$$

The penultimate term reduces to

$$-\beta w_t(1-\tau_t^K)\left\{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]+\mathbb{E}_t[F_{KK,t+1}k_t u^o_{c,t+1}]\mathbb{E}_t\hat{\mu}_{t+1}+\mathrm{Cov}_t[\hat{\mu}_{t+1},k_tF_{KK,t+1}u^o_{c,t+1}]\right\}$$

Summing them yields

$$\beta w_t(1-\tau_t^K)\left\{\frac{\mathbb{E}_t[F_{KK,t+1}u^o_{c,t+1}]}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]}\mathrm{Cov}_t[\hat{\mu}_{t+1},k_tF_{K,t+1}u^o_{c,t+1}]-\mathrm{Cov}_t[\hat{\mu}_{t+1},k_tF_{KK,t+1}u^o_{c,t+1}]\right.$$
$$\left.-\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]\right\}.$$

I summarize the first two of these terms as $-\beta w_t(1-\tau_t^K)H.T.$ (hedging terms) and simplify the first order condition to

$$\psi_t=\Delta\mathbb{E}_t[F_{K,t+1}\psi_{t+1}]+\mu_t^y u^y_{c,t}-\beta w_t(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]-\beta w_t(1-\tau_t^K)H.T.$$

Now substitute in for ψ_t using the first order condition with respect to c_t^y and for ψ_{t+1} using the first order condition with respect to c^o_{t+1}:

$$w_tu^y_{c,t}\left\{1+\tilde{\mu}_t\eta_t+\frac{\sigma_t^y(1-\tau_t^K)}{c_t^y}\left(\sum_{s_{t+1}}[q_{t+1}F_{K,t+1}k_t\hat{\mu}_{t+1}]-\frac{\tilde{\mu}_tk_t}{1-\tau_t^K}\right)\right\}$$
$$=\beta\mathbb{E}_t\left\{F_{K,t+1}\left[w_tu^o_{c,t+1}+\mu^o_{t+1}+R_t^f\frac{u^o_{cc,t+1}}{\mathbb{E}_tu^o_{c,t+1}}b_t\mathbb{E}_t\mu^o_{t+1}+(1-\tau_t^K)\frac{F_{K,t+1}u^o_{cc,t+1}}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]}k_t\mathbb{E}_t[\mu^o_{t+1}F_{K,t+1}]\right]\right\}$$
$$+\mu_t^yu^y_{c,t}-\beta w_t(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]-\beta w_t(1-\tau_t^K)H.T.$$

Next, subtract $\mu_t^yu^y_{c,t}$ from both sides and expand q_{t+1} and $\mu^o_{t+1}=w_tu^o_{c,t+1}\hat{\mu}_{t+1}$:

$$w_t\beta(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}][1+\tilde{\mu}_t(\eta_t-1)]+w_t(1-\tau_t^K)\sigma_t^y\frac{k_t}{c_t^y}\mathbb{E}_t[\beta u^o_{c,t+1}F_{K,t+1}\hat{\mu}_{t+1}]$$
$$-w_t\beta(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]\sigma_t^y\tilde{\mu}_t\frac{k_t}{c_t^y}$$
$$=\beta w_t\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]+\beta w_t\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]+\beta w_tR_t^f\tilde{\mu}_tb_t\mathbb{E}_t[F_{K,t+1}u^o_{cc,t+1}]$$
$$+\beta w_t(1-\tau_t^K)\frac{\mathbb{E}_t[(F_{K,t+1})^2u^o_{cc,t+1}]}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]}k_t\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]$$
$$-\beta w_t(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]-\beta w_t(1-\tau_t^K)H.T.$$

Divide through by βw_t and subtract $\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]$ from both sides:

$$
\begin{aligned}
&-\tau_t^K\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}] + (1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]\tilde{\mu}_t(\eta_t - 1)\\
&+(1-\tau_t^K)\sigma_t^y\frac{k_t}{c_t^y}\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}(\hat{\mu}_{t+1}-\tilde{\mu}_t)]\\
=&\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}] + R_t^f\tilde{\mu}_t b_t\mathbb{E}_t[F_{K,t+1}u^o_{cc,t+1}]\\
&+(1-\tau_t^K)\frac{\mathbb{E}_t[(F_{K,t+1})^2u^o_{cc,t+1}]}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]}k_t\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}]\\
&-(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}\hat{\mu}_{t+1}] - (1-\tau_t^K)H.T.
\end{aligned}
$$

Next define two concepts analogous to those defined in the last proof. Start with the household Euler equation with respect to risk free bonds:

$$u^y_{c,t} = \beta R_t^f\mathbb{E}_t[u^o_{c,t+1}]$$

Consider the differential form holding c_t^y constant:

$$0 = R_t^f\mathbb{E}_t[u^o_{cc,t+1}dc^o_{t+1}] + dR_t^f\mathbb{E}_t[u^o_{c,t+1}]$$

Now suppose that $dc^o_{t+1} = (1-\tau_t^K)F_{K,t+1}dk_t$, as if accumulation of k_t were increased. Then write

$$\frac{\partial R_t^f}{\partial k_t} = -\frac{R_t^f(1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{cc,t+1}]}{\mathbb{E}_t u^o_{c,t+1}}$$

which immediately yields

$$\epsilon_{k_t}^{R_t^f} = -(1-\tau_t^K)\frac{\mathbb{E}_t[F_{K,t+1}u^o_{cc,t+1}]}{\mathbb{E}_t u^o_{c,t+1}}k_t$$

Likewise consider the household Euler equation with respect to capital in differential form, holding c_t^y constant:

$$0 = (1-\tau_t^K)\mathbb{E}_t[F_{K,t+1}u^o_{cc,t+1}dc^o_{t+1}] - d\tau_t^K\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]$$

Supposing again that $dc^o_{t+1} = (1-\tau_t^K)F_{K,t+1}dk_t$ we have

$$\frac{\partial\tau_t^K}{\partial k_t} = (1-\tau_t^K)^2\frac{\mathbb{E}_t[(F_{K,t+1})^2u^o_{cc,t+1}]}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]}$$

which yields

$$\epsilon_{k_t}^{1-\tau_t^K} = -(1-\tau^K)\frac{\mathbb{E}_t[(F_{K,t+1})^2u^o_{cc,t+1}]}{\mathbb{E}_t[F_{K,t+1}u^o_{c,t+1}]}k_t$$

Substituting in these two expressions into the main equation gives

$$
\begin{aligned}
&- \tau_t^K \mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}] + (1-\tau_t^K)\mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}]\tilde{\mu}_t(\eta_t - 1) \\
&+ (1-\tau_t^K)\sigma_t^y \frac{k_t}{c_t^y}\mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}(\hat{\mu}_{t+1} - \tilde{\mu}_t)] \\
=&\tau_t^K \mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}\hat{\mu}_{t+1}] - \frac{R_t^f}{1-\tau_t^K}\tilde{\mu}_t \frac{b_t}{k_t}\mathbb{E}_t[u^o_{c,t+1}]\epsilon_{k_t}^{R_t^f} \\
&- \epsilon_{k_t}^{1-\tau_t^K}\mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}\hat{\mu}_{t+1}] - (1-\tau_t^K) H.T.
\end{aligned}
$$

Note that $R_t^f \mathbb{E}_t[u^o_{c,t+1}] = (1-\tau_t^K)\mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}]$ and combine terms:

$$
\frac{\tau_t^K}{1-\tau_t^K} = \frac{\left\{ \begin{array}{c} \mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}]\tilde{\mu}_t(\eta_t - 1) + \sigma_t^y \frac{k_t}{c_t^y}\mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}(\hat{\mu}_{t+1} - \tilde{\mu}_t)] \\ + \frac{1}{1-\tau_t^K}\mathbb{E}_t\left\{ F_{K,t+1} u^o_{c,t+1}\left[\epsilon_{k_t}^{R_t^f}\frac{b_t}{k_t}\tilde{\mu}_t + \epsilon_{k_t}^{1-\tau_t^K}\hat{\mu}_{t+1}\right]\right\} + H.T. \end{array} \right\}}{\mathbb{E}_t[F_{K,t+1} u^o_{c,t+1}(1+\hat{\mu}_{t+1})]}
$$

Reordering gives the desired result.

D Additional Details about Numerical Simulations

D.1 Recursive Formulations of Problems 2.1 and 2.2

In all cases, the initial period is different, and will be treated separately. I focus only on an "OLG" case with no bequest motive and a zero bequest limit, or a "Ramsey" case with perfect bequest motives ($\delta = \Delta = \beta$) and no bequest limits, since these are simpler to specify and are the only cases I simulate.

I allow for trend growth a in the following manner. For $x \in \{A, g, c^y, c\}$, I redefine $x_t = \exp(at)x$. I redefine $c_t^o = \exp(a(t-1))c_t^o$ and $k_t = \exp(a(t+1))k_t$. Then, if $u(\bullet)$ is log-log and $W(\bullet)$ is CARA, we can substitute $\ell = \ell^*$ as the result of the implementability condition since labor is constant[39], and we have

Problem D.1 (OLG Complete Markets, Log-Log) *Define*

$$
\begin{aligned}
V(\theta, s_-) = \max_{c^y, \boldsymbol{c}_s^o, k'} \quad & W(u(c^y, \ell^*) + \beta\mathbb{E}[u(c_s^o, 0)|s_-]) \\
& + \exp(-\zeta(1-\gamma)a)\Delta\mathbb{E}\left[V\left(F(k', \ell^*; A_s) - \frac{c_s^o}{\exp(a)} - g_s, s\right)|s_-\right] \\
\text{s.t.} \quad & c^y + \exp(a)k' \leq \theta
\end{aligned}
$$

[39] assuming it binds, which is does in all of my simulations

Then the planner solves

$$\max_{c^o_-, c^y, \boldsymbol{c}^o_s, k'} \quad \exp(\zeta(1-\gamma)a)\Delta^{-1}W(u(c^y_-, \ell_-) + \beta u(c^o_-, 0)) + W(u(c^y, \ell) + \beta\mathbb{E}[u(c^o_s, 0)|s_-])$$
$$+ \exp(-\zeta(1-\gamma)a)\Delta\mathbb{E}\left[V\left(F(k', \ell^*; A_s) - \frac{c^o_s}{\exp(a)} - g_s, s\right)|s_-\right]$$
$$\text{s.t.} \quad c^y + \frac{c^o_-}{\exp(a)} + g_{s_-} + \exp(a)k' \leq F(k, \ell^*; A_{s_-})$$
$$c^o_- \geq F_K(k, \ell^*; A_{s_-})k + b_0$$

If $u(\bullet)$ is isoelastic and $W(\bullet)$ is CRRA, then we have

Problem D.2 (OLG Complete Markets, Isoelastic) *Define*

$$V(k, c^o, s_-) = \max_{c^y, \ell, \boldsymbol{c}^o_s, k'} \quad W(u(c^y, \ell) + \beta\mathbb{E}[u(c^o_s, 0)|s_-]) + \exp((1-\zeta)(1-\sigma)a)\Delta\mathbb{E}\left[V(k', c^o_s, s)\right]$$
$$\text{s.t.} \quad c^y + \frac{c^o}{\exp(a)} + g_{s_-} + \exp(a)k' \leq F(k, \ell; A_{s_-})$$
$$u_c(c^y, \ell)c^y + u_\ell(c^y, \ell)\ell + \beta\mathbb{E}[u_c(c^o_s, 0)c^o_s|s_-] \geq 0$$

Then the planner solves

$$\max_{c^o, c^y, \ell, \boldsymbol{c}^o_s, k'} \quad \exp((1-\zeta)(1-\sigma)a)\Delta^{-1}W(u(c^y_-, \ell_-) + \beta u(c^o_-, 0)) + W(u(c^y, \ell) + \beta\mathbb{E}[u(c^o_s, 0)|s_-])$$
$$+ \exp((1-\zeta)(1-\sigma)a)\Delta\mathbb{E}V(k', c^o_s, s)$$
$$\text{s.t.} \quad c^y + \frac{c^o_-}{\exp(a)} + g(s_-) + \exp(a)k' \leq F(k, \ell; A_{s_-})$$
$$c^o_- \geq F_K(k, \ell; A)k + b_0$$
$$u_c(c^y, \ell)c^y + u_\ell(c^y, \ell)\ell + \beta\mathbb{E}[u_c(c^o_s, 0)c^o_s|s_-] \geq 0$$

For Ramsey models, I assume there is no trend growth[40] and that utility is separable between labor and consumption. This ensures that $c^y(s^t) = c^o(s^t) \equiv c(s^t 0)$.

Problem D.3 (Ramsey Complete Markets) *Define*

$$V(k, \theta, s_-) \quad \max_{c, \ell, k', \boldsymbol{\theta}'_s} \quad u(c, \ell) + u(c, 0) + \Delta\mathbb{E}[V(k', \theta'_s, s)|s_-]$$
$$\text{s.t.} \quad 2c + g_{s_-} + k' \leq F(k, \ell; A_{s_-})$$
$$u_c(c, \ell)c + u_\ell(c, \ell)\ell + u_c(c, 0)c + \beta\mathbb{E}[\theta'_s|s_-] \geq \theta$$

[40] All models with trend growth that I simulate have an obvious first best implementation in a Ramsey model.

Then the planner solves

$$\begin{aligned}
\max_{c,\ell,k',\boldsymbol{\theta}'_s} \quad & u(c,\ell) + u(c,0) + \Delta\mathbb{E}[V(k',\theta'_s,s)|s_-] \\
\text{s.t.} \quad & 2c + g_{s_-} + k' \leq F(k,\ell;A_{s_-}) \\
& u_c(c,\ell)c + u_\ell(c,\ell)\ell + u_c(c^o,0) + \beta\mathbb{E}[\theta'_s|s_-] \geq u_c(c,\ell)[F_K(k,\ell;A_{s_-})k + b_0] \\
& u_c(c^y,\ell) = u_c(c^o,0).
\end{aligned}$$

If utility is log-log, the incomplete markets model simplifies substantially—especially the old implementability condition.

Problem D.4 (OLG Incomplete Markets, Log-Log) *Define*

$$\begin{aligned}
V(\theta,l,s_-) = \max_{c^y,\boldsymbol{c}^o_s,k',b',\boldsymbol{\theta}'_s,\boldsymbol{\ell}'_s} \quad & W(u(c^y,\ell) + \beta\mathbb{E}[u(c^o_s,0)|s_-]) \\
& + \exp(-\zeta(1-\gamma)a)\Delta\mathbb{E}\left[V\left(F(k',\ell_s;A_s) - \frac{c^o_s}{\exp(a)} - g_s,\ell'_s,s\right)|s_-\right] \\
\text{s.t.} \quad & c^y + \exp(a)k' \leq \theta \\
& u_c(c^y,\ell)(c^y + b' + \exp(a)k') + u_\ell(c^y,\ell)\ell \geq 0 \\
& c^y \geq \frac{1/c^o_s}{\beta\mathbb{E}[1/c^o_s|s_-]}b' + \frac{F_K(k',\ell'_s;A_s)/c^o_s}{\beta\mathbb{E}[F_K(k',\ell'_s;A_s)/c^o_s|s_-]}\exp(a)k' \\
& \underline{b} \leq b' \leq \bar{b}
\end{aligned}$$

Then the planner solves

$$\begin{aligned}
\max_{c^o,\theta',\ell'} \quad & W(u(c^y_-,\ell_-) + \beta u(c^o,0)) + \exp(-\zeta(1-\gamma)a)\Delta V\left(F(k_-,\ell';A_{s_-}) - \frac{c^o}{\exp(a)} - g_{s_-},\ell',s_-\right) \\
\text{s.t.} \quad & c^o \geq R^f_- b_- + (1-\tau^K_-)F_K(k_-,\ell';A_{s_-})k_-
\end{aligned}$$

In principle, this formulation could be used almost identically for other IESs. Unfortunately, in practice, finding a rectangular state space that encloses the ergodic set is impossible. Thus, I change the recursive formulation for isoelastic utility to one that is defined *ex interim*:

Problem D.5 (OLG Incomplete Markets, Isoelastic) *Define*

$$
\begin{aligned}
V(k,\tilde{a},\tilde{b},v,s_-) = \max_{\boldsymbol{c}_s^y,\boldsymbol{c}_s^o,\boldsymbol{\ell}_s,\boldsymbol{k}_s',\tilde{\boldsymbol{a}}_s',\tilde{\boldsymbol{b}}_s'} \quad & W(v+\beta\mathbb{E}[u(c_s^o,0)|s_-]) \\
& + \exp((1-\zeta)(1-\sigma)a)\Delta\mathbb{E}\left[V(k_s',\tilde{a}_s',\tilde{b}_s',u(c_s^y,\ell_s),s)|s_-\right] \\
& \text{s.t.} \\
& c_s^y + \frac{c_s^o}{\exp(a)} + g_s + \exp(a)k_s' \leq F(k,\ell_s;A_s) && \forall s\in\mathcal{S} \\
& u_c(c_s^y,\ell_s)c_s^y + u_\ell(c_s^y,\ell_s)\ell_s + \tilde{a}_s' \geq 0 && \forall s\in\mathcal{S} \\
& c_s^o \geq \frac{\tilde{b}}{\beta\mathbb{E}[u_c(c_s^o,0)|s_-]} + \frac{F_K(k,\ell_s;A_s)(\tilde{a}-\tilde{b})}{\beta\mathbb{E}[F_K(k,\ell_s;A_s)u_c(c_s^o,0)|s_-]} && \forall s\in\mathcal{S} \\
& \underline{b} \leq \frac{\tilde{b}_s'}{u_c(c_s^y,\ell_s)} \leq \overline{b} && \forall s\in\mathcal{S} \\
& \tilde{a}_s' - \tilde{b}_s' = \exp(a)k_s'u_c(c_s^y,\ell_s) && \forall s\in\mathcal{S}
\end{aligned}
$$

Then the planner solves

$$
\begin{aligned}
\max_{c^y,c^o,\ell,k',\tilde{a}',\tilde{b}'} \quad & W(v+\beta u(c^o,0)) + \exp((1-\zeta)(1-\sigma)a)\Delta V(k',\tilde{a}',\tilde{b}',u(c^y,\ell),s) \\
\text{s.t.} \quad & c^y + \frac{c^o}{\exp(a)} + g_{s_-} + \exp(a)k' \leq F(k,\ell;A_{s_-}) \\
& u_c(c^y,\ell)c^y + u_\ell(c^y,\ell)\ell + \tilde{a}' \geq 0 \\
& c^o \geq R_-^f b_- + (1-\tau_-^K)F_K(k,\ell,A_{s_-})k \\
& \underline{b} \leq \frac{\tilde{b}'}{u_c(c,\ell)} \leq \overline{b}
\end{aligned}
$$

Problem D.6 (Ramsey Incomplete Markets) *Define*

$$
\begin{aligned}
V(k,\tilde{a},\tilde{b},s_-) = \max_{\boldsymbol{c}_s,\boldsymbol{\ell}_s,\boldsymbol{k}_s',\tilde{\boldsymbol{a}}_s',\tilde{\boldsymbol{b}}_s',} \quad & \mathbb{E}\left\{[u(c_s,\ell_s)+u(c_s,0)+\Delta V(k_s',\tilde{a}_s',\tilde{b}_s',s)]|s_-\right\} \\
\text{s.t.} \quad & \\
& 2c_s + g_s + k_s' \leq F(k,\ell_s;A_s) && \forall s\in\mathcal{S} \\
& 2u_c(c_s,\ell_s)c_s + u_\ell(c_s,\ell_s)\ell_s + \tilde{a}_s' \geq \frac{\tilde{b}u_c(c_s,\ell_s)}{\beta\mathbb{E}[u_c(c_s,\ell_s)|s_-]} \\
& \qquad + \frac{F_K(k,\ell_s;A_s)u_c(c_s,\ell_s)(\tilde{a}-\tilde{b})}{\beta\mathbb{E}[F_K(k,\ell_s;A_s)u_c(c_s,\ell_s)|s_-]} && \forall s\in\mathcal{S} \\
& \underline{b} \leq \frac{\tilde{b}_s'}{u_c(c_s,\ell_s)} \leq \overline{b} && \forall s \\
& \tilde{a}_s' - \tilde{b}_s' = k_s'u_c(c_s^y,\ell_s) && \forall s\in\mathcal{S}
\end{aligned}
$$

Then the planner solves

$$\max_{c,\ell,k',\tilde{a}',\tilde{b}'} \quad u(c,\ell) + u(c,0) + \Delta V(k',\tilde{a}',\tilde{b}',s_-)$$

$$\text{s.t.}$$

$$\begin{aligned} 2c + g_{s_-} + k' \leq & F(k,\ell;A_{s_-}) \\ 2u_c(c,\ell)c + u_\ell(c,\ell)\ell + \tilde{a}' \geq & u_c(c,\ell)[R^f_- b_- + (1-\tau^K_-)F_K(k,\ell;A_{s_-})k] \\ \underline{b} \leq \frac{\tilde{b}'}{u_c(c,\ell)} \leq & \bar{b} \end{aligned}$$

D.2 Further Details on Solution Method

As stated in the main text, my solution method was collocation using Chebyshev polynomials on a sparse grid; in the footnotes, I list a few references that explain the method in extensive detail, along with the reasons why various choices are made. Here, I merely give a brief explanation of the method in case the reader is unfamiliar. It differs from standard value function iteration in three major respects—the approximation function to the value function, the spacing of collocation points on the grid, and how the procedure scales with the number of state space dimensions.

In "standard" value function iteration, the value function is approximated using splines between equally spaced points. The simplest example would be linear splines, in which the value function between grid points is linearly interpolated using the two nearest points[41] on the grid; cubic splines, which require four points, are quite commonly used as well. By nature of spline interpolation, the interpolation function is local—it is different in different regions of the state space.

In contrast, the method I use approximates the value function with the same weighted sum of functions across the entire state space. For simplicity, assume that this space is $[0,1]$, though scaling and translating it is a trivial procedure. Specifically, an Nth degree approximation in a single dimension is as follows:

$$V(x) \approx \sum_{n=1}^{N} \theta_n T_n(x)$$

where $\boldsymbol{\theta}$ is some set of coefficients to be solved for and $T_n(x)$ is the nth Chebyshev polynomial of the first kind, defined recursively by $T_1(x) = 1$, $T_2(x) = x$, $T_{n+1}(x) = 2x$, $T_n(x) - T_{n-1}(x)$. This approximation method exploits the fact that most value functions encountered in economics—such as the ones I work with in the present paper—are globally smooth, in order to vastly reduce the number of necessary grid points.

Instead of spline approximation, which suggests using collocation points—the points at which to enforce that the Bellman equation holds exactly—on the boundaries of the splines, Chebyshev approximation suggests using Chebyshev nodes (either zeros or extrema of the Chebyshev polynomials) as collocation points. In accordance with Krueger and Kubler

[41] in each dimension

(2004), I choose the extrema:

$$x_n = -\cos\left(\frac{n-1}{N-1}\right) \quad n = 1, 2, ..., N$$

The procedure in one dimension is then quite similar to value function iteration—to progressively better approximate the value function. In the case of Chebyshev polynomials, the equation that enforces that the Bellman equation holds perfectly at the collocation points is $\Phi\boldsymbol{\theta} = \boldsymbol{v}(\boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is the vector of coefficients; $\boldsymbol{v}(\boldsymbol{\theta})$ is the vector of results of the Bellman maximization problems; with typical element $v_j(\boldsymbol{\theta})$ the result of the maximization problem at point x_j; and Φ is the collocation matrix with typical element $\phi_{ij} = T_j(x_i)$, which is constant. The update procedure, suggested by Miranda and Fackler (1997), is then $\boldsymbol{\theta} := \Phi^{-1}\boldsymbol{v}(\boldsymbol{\theta})$, repeated until convergence, measured according to either the Bellman equation error at the collocation points, or the change in $\boldsymbol{\theta}$.

The final issue to consider is how to scale this procedure to multiple dimensions; I follow Judd et al. (2014).[42] The most straightforward way would be to use a tensor product of the T_n and x_n—that is, take the Cartesian product of the x_n, and take the Cartesian product, multiplied together, of the T_n. However, this procedure scales exponentially—there are N^d points and coefficients for d dimensions—making it a poor choice for larger-dimensional problems. Instead, we choose a grid which is denser near the steady state (the center of the state space) and the edges of the state space and sparser in between, in a way that scales polynomially in the number of dimensions d.

Define a sequence of sets in a single dimension as follows. Let $S_i \equiv \bigcup_{n=1}^{m(i)} \left[-\cos\left(\frac{n-1}{m(i)-1}\right)\right]$, where $m(i) \equiv 2^{i-1}+1$ for $i \geq 2$, and $m(1) \equiv 1$. Then the set of points in the grid of dimension d using approximation level μ is

$$\Theta_\mu \equiv \bigcup_{\boldsymbol{i}|d \leq \sum_{j=1}^{d} i_j \leq d+\mu} [S_{i_1} \otimes S_{i_2} \otimes ... \otimes S_{i_d}]$$

Then choose the corresponding set of Chebyshev polynomials to include in the value function approximation.

E Definitions and Proofs from Section 6

Definition E.1 (Policy Equilibrium—More than Two Generations) *Given a set of after-tax assets held at the beginning of time, $\{a^j(s_0)\}_{j=2}^{J_2}$, a policy equilibrium is a collection of*

- *policies* $\{T_0^j\}_{j\geq 2}, \{T(s^t), \tau^L(s^t), \tau^K(s^{t+1})\}_{t\geq 0}$
- *prices* $\{q(s^{t+1}), w(s^t), R^K(s^t)\}_{t\geq 0}$

[42] In fact, the reason I originally learned and implemented this method is that I intended to solve problems featuring many dimensions—additional generations, and additional types within each generation. I have since decided to defer such investigations to future papers, but I hope this investment will pay dividends at that time.

- *an allocation* $\{\{c^j(s^t)\}_{j=1}^{J_2}, \{\ell^j(s^t)\}_{j=1}^{J_2}, \{a^{j+1}(s^{t+1})\}_{j=1}^{J2-1}, k(s^t)\}_{t\geq 0}$
- *and government debt* $\{b(s^{t+1})\}_{t\geq 0}$

such that at all histories s^t, $t \geq 0$

- *The resource constraint is satisfied at all* $t \geq 0$*:*

$$\sum_{j=1}^{J_2} c^j(s^t) + g(s_t) + k(s^t) \leq F\left(k(s^{t-1}), \sum_{j=1}^{J_1} \ell(s^t)\right) \tag{27}$$

- *The government's budget constraint is satisfied at all* $t \geq 0$*:*

$$b(s^t) + g(s_t) + T(s^t) \leq \sum_{s_{t+1}} q(s^{t+1})b(s^{t+1}) + \tau^L(s^t)w(s^t)\sum_{j=1}^{J_1} \ell^j(s^t) + \tau^K(s^t)R^K(s^t)k(s^{t-1}) \tag{28}$$

- *Households' budget constraints are satisfied:*

$$\begin{aligned}
c^j(s_0) + \sum_{s_1} q(s^1)a^{j+1}(s^1) &\leq w(s_0)\ell^j(s_0) + a^j(s_0) + T_0^j \quad j = 2, ...J_1 \\
c^j(s_0) + \sum_{s_{t+1}} q(s^1)a^{j+1}(s^1) &\leq a^j(s_0) + T_0^j \quad j = J_1 + 1, ...J_2 - 1 \\
c^{J2}(s_0) &\leq a^{J2}(s_0) + T_0^{J2} \\
c^1(s^t) + \sum_{s_{t+1}} q(s^{t+1})a^2(s^{t+1}) &\leq w(s^t)\ell^1(s^t) + T(s^t) \\
c^j(s^t) + \sum_{s_{t+1}} q(s^{t+1})a^{j+1}(s^{t+1}) &\leq w(s^t)\ell^j(s^t) + a^j(s^t)\, j = 2, ...J_1 \\
c^j(s^t) + \sum_{s_{t+1}} q(s^{t+1})a^{j+1}(s^{t+1}) &\leq a^j(s^t)\, j = J_1 + 1, ...J_2 - 1 \\
c^{J2}(s^t) &\leq a^{J2}(s^t)
\end{aligned}$$

- *Households optimize subject to their budget constraint, taking prices and policies as given*
- *Firms optimize, taking prices as given*
- *The state-contingent asset markets clear:*

$$\sum_{j=2}^{J_2} a^j(s^{t+1}) = b(s^{t+1}) + (1 - \tau^K(s^{t+1}))R^K(s^{t+1})k(s^t)$$

- *The markets for capital and labor clear*

- *The no-arbitrage condition between capital and state-contingent assets holds:*

$$\sum_{s_{t+1}} q(s^{t+1})R^K(s^{t+1})(1-\tau^K(s^{t+1}))=1$$

Definition E.2 (Implementable Allocation—More than Two Generations) *An allocation is implementable if there exists a policy equilibrium of which it is a part. More importantly a welfare-relevant allocation—an allocation, ignoring $a^j(s^t)$—is implementable if there exists a policy equilibrium of which it is a part.*

Proposition E.1 (Implementability—More than Two Generations) *Given a set of after-tax assets held at the beginning of time, $\{a^j(s_0)\}_{j=2}^{J_2}$, a set $\{\{c^j(s^t)\}_{j=1}^{J_2}, \{\ell^j(s^t)\}_{j=1}^{J_2}, k(s^t)\}_{t\geq 0}$ is part of an implementable allocation if and only if*

- *the resource constraint is satisfied*
- *the implementability condition is satisfied for all generations not already alive*

$$\mathbb{E}_t\sum_{j=1}^{J_2}\beta^{j-1}u_c^j(s^{t+j-1})c^j(s^{t+j-1})+\mathbb{E}_t\sum_{j=1}^{J_1}\beta^{j-1}u_\ell^j(s^{t+j-1})\ell^j(s^{t+j-1})\geq 0 \quad \forall t\geq 0 \tag{29}$$

- *the implementability condition is satisfied for all generations already alive*

$$\mathbb{E}_t\sum_{j'=j}^{J_2}\beta^{j'-j}u_c^{j'}(s^{j'-j})c^{j'}(s^{j'-j})+\mathbb{E}_t\sum_{j'=j}^{J_1}\beta^{j'-j}u_\ell^{j'}(s^{j'-j})\ell^{j'}(s^{j'-j})\geq u_c^j(s_0)a^j(s_0) \quad \forall j=2,...,J_2 \tag{30}$$

- *there exists a set of market weights $\{\varphi(s^t)\}_{t=-J_2+1}^{\infty}$ such that, after defining $C(s^t)\equiv\sum_{j=1}^{J_2}c^j(s^t)$ and $L(s^t)\equiv\sum_{j=1}^{J_1}\ell^j(s^t)$:*

$$\{c^j(s^t)\}_{j=1}^{J_2},\{\ell^j(s^t)\}_{j=1}^{J_1}\in argmax\sum_{j=1}^{J_1}\varphi(s^{t-j+1})u(c_j,\ell_j)+\sum_{j=J_1+1}^{J_2}\varphi(s^{t-j+1})u(c_j,0)$$
$$s.t.\sum_{j=1}^{J_2}c_j\leq C(s^t)$$
$$\sum_{j=1}^{J_1}\ell_j\geq L(s^t)$$

Proof. I will prove the "if" direction first, by constructing a policy equilibrium. First define

prices as follows:

$$
\begin{aligned}
w(s^t) &= F_L\left(k(s^{t-1}), \sum_{j=1}^{J_1} \ell^j(s^t)\right) \\
R^K(s^t) &= F_K\left(k(s^{t-1}), \sum_{j=1}^{J_1} \ell^j(s^t)\right) \\
q(s^{t+1})u_c^1(s^t) &= \beta Pr(s^{t+1}|s^t)u_c^2(s^{t+1})
\end{aligned}
$$

Then define policies as follows:

$$
\begin{aligned}
w(s^t)(1-\tau^L(s^t)) &= -\frac{u_\ell^1(s^t)}{u_c^1(s^t)} \\
u_c^1(s^t) &= \beta(1-\kappa(s^t))\mathbb{E}_t[u_c^2(s^{t+1})R^K(s^{t+1}] \\
\tau^K(s^{t+1}) &= \kappa(s^t) \quad \forall s^{t+1} \succ s^t
\end{aligned}
$$

Most importantly, I must show that the market weights ensure that that other generations' (i.e., not the youngest's) first order conditions are satisfied with the proposed allocation and these prices. Define $\psi(s^t)$ and $\lambda(s^t)$ as the multipliers on the consumption and labor constraints from the intra-period allocation problem. Then for all $j = 1, ..., J_1$, we have

$$
-\frac{u_l^j(s^t)}{u_c^j(s^t)} = \frac{\lambda(s^t)/\varphi(s^{t-j+1})}{\psi(s^t)/\varphi(s^{t-j+1})} = \frac{\lambda(s^t)}{\psi(s^t)}
$$

which shows that all households' labor-leisure first order condition is satisfied at the same net-of-tax wage. Since we chose the net-of-tax wage to satisfy the youngest household's labor-leisure first order condition, all other households must be optimizing as well. Similarly, the Euler equation with respect to a state-contingent asset paying off at s^{t+1} for a household of age j is

$$
\begin{aligned}
q(s^{t+1})u_c^j(s^t) &= \beta Pr(s^{t+1}|s^t)u_c^{j+1}(s^{t+1}) \\
q(s^{t+1}) &= \frac{\beta Pr(s^{t+1}|s^t)u_c^{j+1}(s^{t+1})}{u_c^j(s^t)} \\
&= \frac{\beta Pr(s^{t+1}|s^t)\psi(s^{t+1})/\varphi(s^{t-j+1})}{\psi(s^t)/\varphi(s^{t-j+1})} \\
&= \beta Pr(s^{t+1})\frac{\psi(s^{t+1})}{\psi(s^t)}
\end{aligned}
$$

which again shows that all households' Euler equations are satisfied at the same asset price, and since we chose $q(s^{t+1})$ such that the youngest household's Euler equation is satisfied, so must all other households'. For households to be optimizing given their budget constraint, all that remains is to show that their budget constraints are satisfied. But this is enforced by the implementability conditions for an appropriately chosen $T(s^t)$.

Finally, we must show that asset markets clear, the no arbitrage condition with respect

to capital is satisfied, and the government's budget constraint is satisfied. Given the capital tax chosen above, the no arbitrage condition follows immediately. Asset holdings can be defined as a forward looking variable. Specifically, set

$$u_c^j(s^t)a^j(s^t) = \mathbb{E}_t \sum_{j'=j}^{J_2} \beta^{j'-j} u_c^{j'}(s^{t+j'-j})c^{j'}(s^{t+j'-j}) + \mathbb{E}_t \sum_{j'=j}^{J_1} \beta^{j'-j} u_\ell^{j'}(s^{t+j'-j})\ell^{j'}(s^{t+j'-j}).$$

Substituting this definition into the right hand side leaves

$$u_c^j(s^t)a^j(s^t) = u_c^j(s^t)c^j(s^t) + \mathbb{1}_{j\leq J_1} u_\ell^j(s^t)\ell^j(s^t) + \mathbb{1}_{j<J_2}\beta \sum_{s_{t+1}} Pr(s^{t+1}|s^t)u_c^{j+1}(s^{t+1})a^{j+1}(s^{t+1})$$

Dividing through by $u_c^j(s^t)$ yields

$$a^j(s^t) = c^j(s^t) - \mathbb{1}_{j\leq J_1} w(s^t)(1-\tau^L(s^t))\ell^j(s^t) + \mathbb{1}_{j<J_2} \sum_{s_{t+1}} q(s^{t+1})a^{j+1}(s^{t+1}) \tag{31}$$

The equivalent expression for the youngest generation is

$$T(s^t) = c^1(s^t) - w(s^t)(1-\tau^L(s^t))\ell^1(s^t) + \sum_{s_{t+1}} q(s^{t+1})a^2(s^{t+1}) \tag{32}$$

Defining $b(s^t) = \sum_{j=2}^{J_2} a^j(s^t) - (1-\tau^K(s^t))R^K(s^t)k(s^{t-1})$, which ensures the that the asset

markets clear, and summing these last two equations across all j yields

$$
\begin{aligned}
T(s^t) + b(s^t) + (1-\tau^K(s^t))R^K(s^t)k(s^{t-1}) =& \sum_{j=1}^{J_2} c^j(s^t) - (1-\tau^L(s^t))w(s^t)\sum_{j=1}^{J_1}\ell^j(s^t) + \\
& \sum_{s_{t+1}} q(s^{t+1})(b(s^{t+1}) + (1-\tau^K(s^{t+1})R^K(s^{t+1})k(s^t))
\end{aligned}
$$

$$
\begin{aligned}
T(s^t) + b(s^t) + F\left(k(s^{t-1}), \sum_{j=1}^{J_1}\ell^j(s^t)\right) =& \tau^K(s^t)R^K(s^t)k(s^{t-1}) + \tau^L(s^t)w(s^t)\sum_{j=1}^{J_1}\ell^j(s^t) \\
& + \sum_{j=1}^{J_2} c^j(s^t) + k(s^t) + \sum_{s_{t+1}} q(s^{t+1})b(s^{t+1})
\end{aligned}
$$

$$
\begin{aligned}
T(s^t) + b(s^t) + g(s_t) =& \tau^K(s^t)R^K(s^t)k(s^{t-1}) + \tau^l(s^t)w(s^t)\sum_{j=1}^{J_1}\ell^j(s^t) \\
& + \sum_{j=1}^{J_2} c^j(s^t) + k(s^t) + \sum_{s_{t+1}} q(s^{t+1})b(s^{t+1}) \\
& + g(s_t) - F\left(k(s^{t-1}), \sum_{j=1}^{J_1}\ell^j(s^t)\right) \\
\leq& \tau^K(s^t)R^K(s^t)k(s^{t-1}) + \tau^L(s^t)w(s^t)\sum_{j=1}^{J_1}\ell^j(s^t) \\
& + \sum_{s_{t+1}} q(s^{t+1})b(s^{t+1})
\end{aligned}
$$

Thus, the "if" direction has been proven.

Proving the "only if" direction requires showing that every policy equilibrium satisfies the conditions. The resource constraint is direct, and the implementability conditions follow directly from the household's budgets and first order conditions. I must show that there exists a set of market weights with the properties specified. I do so by construction.

First define $\varphi(-J_2+1) = 1$. For any generation already alive, set $\varphi(-j+1)$ such that $\frac{u_c^j(s_0)}{u_c^{J_2}(s_0)} = \frac{\varphi(-J_2+1)}{\varphi(-j+1)}$. For any generation not already alive, set $\varphi(s^t)$ such that $\frac{u^1(s^t)}{u^2(s^t)} = \frac{\varphi(s^{t-1})}{\varphi(s^t)}$. This, by construction, enforces the first order conditions of the intra-period allocation problem with respect to consumption,

$$\varphi(s^{t-j+1})u_c^j(s^t) = \varphi(s^{t-j'+1})u_c^{j'}(s^t).$$

Multiplying both sides by $(1-\tau^L(s^t))w(s^t)$ for all pairs of young generations yields

$$-\varphi(s^{t-j+1})u_\ell^j(s^t) = -\varphi(s^{t-j'+1})u_\ell^{j'}(s^t)$$

which is the first order condition with respect to labor of the intra-period allocation problem.

Thus, with $C(s^t)$ and $L(s^t)$ suitably defined, the intra-period allocation problem indeed yields the actual allocation for these φs, and the "only if" direction is proven. ■

The planner's problem thus consists of choosing only aggregates, plus the Pareto weights. Any given period, he may only choose a single Pareto weight—that of the youngest generation; all other Pareto weights are simply moved up one "slot." The resulting recursive problem for a utilitarian planner is as follows, properly interpreting c^j and ℓ^j as functions of $C, L, \{\varphi_j\}_{j=1}^{J_2}$:

Problem E.1 (Planner's Problem—More than Two Generations) *Define*

$$
\begin{aligned}
V(k, \{\theta_j\}_{j=1}^{J_2-1}, \{\varphi_j\}_{j=1}^{J_2-1}) = \max_{C, L, k', \varphi_1', \{\theta_{s'}'^j\}_{j=1,s'}^{J_2-1}} & \sum_{j=1}^{J_1} u(c_j, \ell_j) + \sum_{j=J_1+1}^{J_2} u(c_j, 0) \\
& + \beta \mathbb{E} V(k', \{\theta_{s'}'^j\}_{j=1}^{J_2-1}, (\varphi_1', \varphi_1, ..., \varphi_{J2-2})) \\
& \text{s.t.} \\
& C + g(s_-) + k' \leq F(k, L) \\
& u_c(c_1, \ell_1)c_1 + u_\ell(c_1, \ell_1)\ell_1 + \beta \mathbb{E}[\theta_{s'}'^1 | s] \geq 0 \\
& u_c(c_j, \ell_j)c_j + u_\ell(c_j, \ell_j)\ell_j + \beta \mathbb{E}[\theta_{s'}'^j | s] \geq \theta^{j-1} \quad j = 2, ..., J_1 \\
& u_c(c_j, 0)c_j + \beta \mathbb{E}[\theta_{s'}'^j | s] \geq \theta^{j-1} \quad j = J_1 + 1, ..., J_2 - 1 \\
& u_c(c_{J_2}, 0)c_{J_2} \geq \theta^{J_2-1}
\end{aligned}
$$

Then the planner solves

$$
\begin{aligned}
\max_{C, L, k', \{\varphi_j\}_{j=1}^{J_2}, \{\theta_{s'}'^j\}_{j=1,s'}^{J_2-1}} & \sum_{j=1}^{J_1} u(c_j, \ell_j) + \sum_{j=J_1+1}^{J_2} u(c_j, 0) \\
& + \beta \mathbb{E} V(k', \{\theta_{s'}'^j\}_{j=1}^{J_2-1}, (\varphi_1, \varphi_2, ..., \varphi_{J2-1})) \\
& \text{s.t.} \\
& C + g(s_-) + k' \leq F(k, L) \\
& u_c(c_1, \ell_1)c_1 + u_\ell(c_1, \ell_1)\ell_1 + \beta \mathbb{E}[\theta_{s'}'^1 | s] \geq 0 \\
& u_c(c_j, \ell_j)c_j + u_\ell(c_j, \ell_j)\ell_j + \beta \mathbb{E}[\theta_{s'}'^j | s] \geq u_c(c_j, \ell_j)a_j \quad j = 2, ..., J_1 \\
& u_c(c_j, 0)c_j + \beta \mathbb{E}[\theta_{s'}'^j | s] \geq u_c(c_j, 0)a_j \quad j = J_1 + 1, ..., J_2 - 1 \\
& c_{J_2} \geq a_{J_2}
\end{aligned}
$$

Since the magnitude of $\{\varphi_j\}_{j=1}^{J_2}$ is irrelevant, the planner can always normalize $\phi_1 = 1$ before moving on to the next period, thus keeping the state space contained. An incomplete markets version, and/or a non-utilitarian version, can be defined similarly. However, both require substantially more state variables than the complete markets, utilitarian version, and so I will not simulate them.