Location estimation for symmetric log-concave densities

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$$\mathcal{I}_f = \int_{f>0} \frac{f'(x)^2}{f(x)} dx.$$

- We want to find an estimator $\widehat{\theta}_n$ of θ without *tuning parameters* such that

$$\sqrt{n}(\widehat{\theta}_n - \theta) \rightarrow_d N(0, \mathcal{I}_f{}^{-1}).$$

Outline

Past Literature

- Consider the larger model

$$\mathcal{P}_s = \{f | f(x;\theta) = f_0(x-\theta),\ \theta \in \mathbb{R},\ f_0 \in \mathcal{S}\}$$

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- Stone (1975) considered a one-step estimator.

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- Let $\bar{\theta}_n$ be an initial estimator of θ.
- Then we can define a one-step estimator:

$$\widehat{\theta}_n = \bar{\theta}_n - \frac{\displaystyle\int_{-\xi_n+\bar{\theta}_n}^{\xi_n+\bar{\theta}_n} \widehat{\phi}'_n(x) d\mathbb{F}_n(x)}{\widehat{\mathcal{I}}_n}$$

where ξ_n is a truncation parameter.

Stone (1975)'s estimator

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- Stone (1975) let the truncation parameter and the scale parameter (for Gaussian kernel) approach ∞ and 0 respectively with n.
- Stone (1975) showed that

$$\sqrt{n}(\widehat{\theta}_n - \theta) \to_d N(0, \mathcal{I}_f{}^{-1})$$

if the **rates of the tuning parameters satisfy some condition.**

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- **Her procedure involves tuning parameters for choosing a fraction of the data, partitioning the sample space to estimate scores etc.**

Outline

Why log-concave

- MLE exists for log-concave class and is easily computable (Dümbgen and Rufibach, 2011).

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- Many standard continuous distributions are log-concave (e.g. Gaussian, Beta(a, b) with $a, b \geq 1$, Gamma distribution with shape parameter greater than 1, Laplace, logistic, Gumbel etc.)
- Log-concave densities are unimodal, closed under convolution and affine transformations. They also have exponentially decaying tails.

Maximum likelihood estimator (MLE) of a log-concave density

- Let $X_1, \ldots, X_n \overset{iid}{\sim} f$ where f is a log-concave density. Let $\widehat{f}_n$ denote the log-concave MLE of f.

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- Dümbgen and Rufibach (2009) showed that $\widehat{\phi}_n = \log \widehat{f}_n$ is a concave affine function with knots belonging to the set $\mathcal{S}_n(\widehat{\phi}_n) \subseteq \{X_1, \ldots, X_n\}$.

Plot of the MLE $\widehat{f}_n$ of a log-concave density

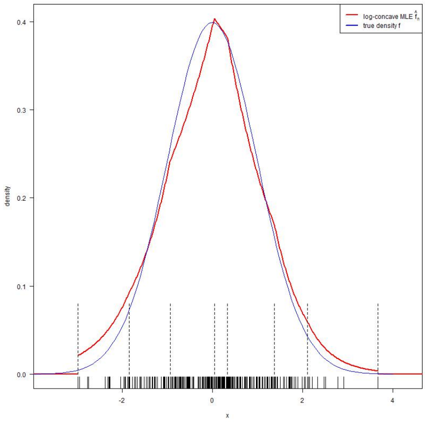

Figure: Plot of $\widehat{f}_n$ for a sample of size 300 generated from standard normal distribution; the blue line and the red line represent the standard normal density and $\widehat{f}_n$ respectively where black ticks and dashed line segments represent data points and the knots.

Plot of $\widehat{\phi}_n = \log \widehat{f}_n$

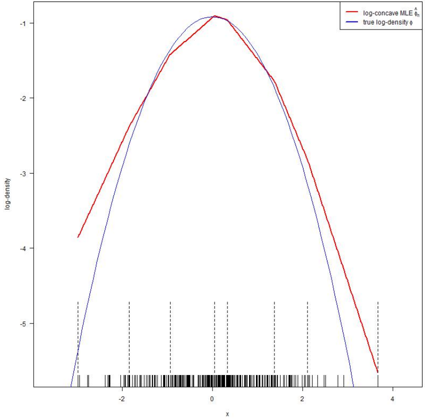

Figure: Plot of $\widehat{\phi}_n$ for a sample of size 300 generated from standard normal distribution; the blue line and the red line represent the logarithm of a standard normal density and $\widehat{\phi}_n$ respectively.

Asymptotic properties of the MLE

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- Cule and Samworth (2010) showed that we have

$$\int |\widehat{f}_n(x) - f(x)|dx \rightarrow_{a.s.} 0 \tag{1}$$

and if f is continuous (which is true in our case)

$$\sup_{x \in \mathbb{R}} |\widehat{f}_n(x) - f(x)| \rightarrow_{a.s.} 0. \tag{2}$$

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- Dümbgen and Rufibach (2009) showed that for any compact interval $C \subset \text{int}(dom(\phi))$,

$$\sup_{x\in C} |\widehat{\phi}_n(x) - \phi(x)| \rightarrow_{a.s.} 0. \tag{3}$$

Data determined smoothing

Data determined smoothing

Define

$$\widehat{\sigma}_n^2 = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \int z^2 d\mathbb{F}_n(z) - \left(\int z d\mathbb{F}_n(z) \right)^2. \qquad (4)$$

and

$$\tilde{\sigma}_n^2 = \int z^2 \widehat{f}_n(z) dz - \left(\int z \widehat{f}_n(z) dz \right)^2. \qquad (5)$$

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Let

$$b_n^2 = \widehat{\sigma}_n^2 - \tilde{\sigma}_n^2.$$

It follows (Dümbgen and Rufibach, 2009; Cule and Samworth, 2010) that $b_n > 0$ and as $n \to \infty$,

$$b_n \to_{a.s.} 0.$$

Data determined smoothing: continued

Definition 1

The smoothed density estimator $\widehat{f}_n^{sm}$ is defined as

$$\widehat{f}_n^{sm}(x) = \frac{1}{b_n} \int_{-\infty}^{\infty} \widehat{f}_n(t)\ \psi\left(\frac{x-t}{b_n}\right) dt$$

where ψ is the standard Gaussian density function.

Plot of the smoothed log-concave MLE

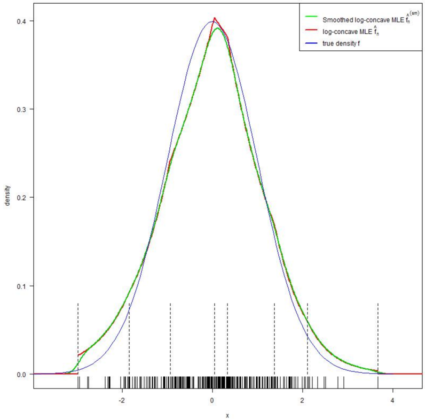

Figure: The green line represents the smoothed log-concave MLE computed from 300 standard Gaussian observations; the unsmoothed log-concave MLE $\widehat{f}_n$ and the true density are drawn in red and blue lines respectively.

Symmetric log-concave MLE

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- Xu and Samworth (2017), Doss and Wellner (2016) showed that the maximum likelihood estimator of f among all symmetric log-concave functions with a known center of symmetry can be obtained.
- The symmetric log-concave MLE enjoys many asymptotic properties of the unconstrained log-concave MLE if the model assumptions hold.

Plot of symmetric log-concave MLE

Standard normal distribution, $n = 200$

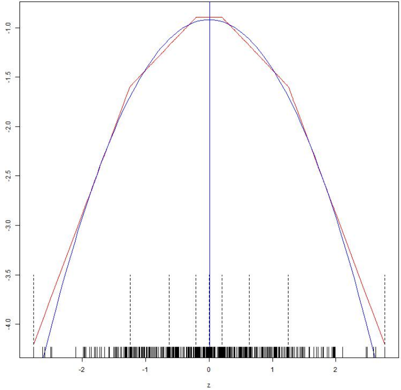

Figure: In this case the MLE (in red line) is computed among all log-concave densities symmetric about the sample mean (value= 0.007, the vertical line). The true density is drawn in blue.

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Notations

Let us fix some notations first:

- Assume that $X_1, \dots, X_n \sim f \in \mathcal{P}_0$ where $f = f_0(\cdot - \theta)$ with f_0 being symmetric about 0.
- Let $\mathbb{F}_n$ stand for the empirical distribution function of the X_i's.
- Write F and $\widehat{F}_n$ for the distribution function of the density f and its log-concave estimate $\widehat{f}_n$.
- Denote the initial estimator as $\bar{\theta}_n$.
- Let $b_n^2 = \widehat{\sigma}_n^2 - \tilde{\sigma}_n^2$ where $\widehat{\sigma}_n^2$ and $\tilde{\sigma}_n^2$ were defined in (4) and (5).
- We continue using the notations $\phi(x) = \log f(x)$ and $\phi'(x) = f'(x)/f(x)$, $\widehat{\phi}_n(x) = \log \widehat{f}_n(x)$ and $\widehat{\phi}_n'(x) = \widehat{f}_n'(x)/\widehat{f}_n(x)$.
- Denote the standard normal density by ψ.

Outline

Symmetrized Logconcave MLE estimator $\widehat{f}_n^{symm}$

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- $\tilde{f}_n(z) = \widehat{f}_n(z + \bar{\theta}_n)$ is an estimator of $f_0(z)$ since $f_0(z) = f(z + \theta)$ for $z \in \mathbb{R}$.

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- $\dfrac{\tilde{f}_n(z) + \tilde{f}_n(-z)}{2}$ is symmetric.

Definition 2

$$\widehat{f}_n^{symm}(z) = \frac{1}{2}(\widehat{f}_n(\bar{\theta}_n + z) + \widehat{f}_n(\bar{\theta}_n - z)), \ z \in \mathbb{R}. \tag{6}$$

Smoothed symmetrized Logconcave MLE estimator $(\widehat{f}_n^{symm})^{sm}$

We can consider a smoothed version of $\widehat{f}_n^{symm}$.

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Definition 3

For $z \in \mathbb{R}$, define

$$(\widehat{f}_n^{symm})^{sm}(z) = \int \frac{1}{b_n} \widehat{f}_n^{symm}(t) \psi\left(\frac{z-t}{b_n}\right) dt. \tag{7}$$

Symmetrized logconcave MLE estimator geometric mean type - $\widehat{f}_n^{geo,symm}$

- $\tilde{f}_n(z) = \widehat{f}_n(z + \bar{\theta}_n), \; z \in \mathbb{R}.$

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Definition 4

For $z \in \mathbb{R}$ define

$$\widehat{f}_n^{geo,symm}(z) = K\sqrt{\widehat{f}_n(\bar{\theta}_n + z)\widehat{f}_n(\bar{\theta}_n - z)} \tag{8}$$

where K is a random normalizing constant.

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where K is a random normalizing constant.

$\widehat{f}_n^{geo,symm}$ is log-concave. However, it has a shorter support $[-R, R]$ where $R = \min(|X_{(1)}|, |X_{(n)}|)$. Others have support $[-R, R]$ with $R = \max(|X_{(1)}|, |X_{(n)}|)$.

Symmetric logconcave MLE estimator, $\widehat{f}_n^{MLE,symm}$

For $\theta' \in \mathbb{R}$, let $\tilde{f}_n^{\theta'}$ denote the maximum likelihood estimator among the log-concave densities symmetric around θ'.

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Definition 5

$$\widehat{f}_n^{MLE,symm}(z) = \tilde{f}_n^{\bar{\theta}_n}(z), \ \ z \in \mathbb{R}. \tag{9}$$

Plot of the estimators of ϕ_0

Standard normal distribution, sample size $n = 100$

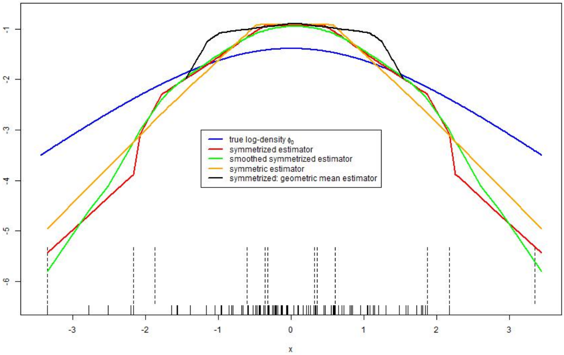

Figure: $\bar{\theta}_n$ is taken as the mean; ϕ_0(blue line) is the true log-density and $\log \widehat{f}_n^{symm}$. $\log(\widehat{f}_n^{symm})^{sm}$, $\log \widehat{f}_n^{MLE,symm}$, $\log \widehat{f}_n^{geo,symm}$ are plotted in red, green, orange and black lines respectively.

Outline

One-step estimator for θ: untruncated

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- Let $\tilde{\phi}'_n = \tilde{h}'_n / \tilde{h}_n$.
- Ideally one would calculate $\widehat{\mathcal{I}}_n = \int_{-\infty}^{\infty} \tilde{\phi}'_n(x - \bar{\theta}_n)^2 d\mathbb{F}_n(x)$ and set

$$\widehat{\theta}_n = \bar{\theta}_n - \int_{-\infty}^{\infty} \frac{\tilde{\phi}'_n(x - \bar{\theta}_n)}{\widehat{\mathcal{I}}_n} d\mathbb{F}_n(x). \tag{10}$$

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- $\tilde{\phi}'_n$ is unbounded if ϕ' is so. In that case, controlling the tail behavior becomes difficult.
- Hence we will use a truncated version of (10) (a similiar treatment can be found in Hendrickx and Groeneboom (2017) for current status regression model).

One-step estimator for θ: truncated

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Let $\xi_n = \tilde{H}_n^{-1}(1-\eta)$ where $\eta \in (0, 1/2)$. Since $\tilde{h}_n$ is symmetric, $\tilde{H}_n^{-1}(\eta) = -\xi_n < 0$.

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Definition 6

Let

$$\widehat{\mathcal{I}}_n^{\eta} = \int_{-\xi_n+\bar{\theta}_n}^{\xi_n+\bar{\theta}_n} \tilde{\phi}_n'(x-\bar{\theta}_n)^2 d\mathbb{F}_n(x). \tag{11}$$

Define

$$\widehat{\theta}_n = \bar{\theta}_n - \int_{-\xi_n+\bar{\theta}_n}^{\xi_n+\bar{\theta}_n} \frac{\tilde{\phi}_n'(x-\bar{\theta}_n)}{\widehat{\mathcal{I}}_n^{\eta}} d\mathbb{F}_n(x). \tag{12}$$

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Profile likelihood method

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- The maximum likelihood estimator of θ is

$$\hat{\theta}_{MLE} = \mathrm{argmax}_{\theta' \in \mathbb{R}} \sum_{i=1}^{n} \log \tilde{f}_n^{\theta'}(X_i).$$

Theorem 7

Let $(\hat{\theta}_{MLE}, \hat{f}_{0,MLE})$ *be the joint MLE of* (θ, f_0)*. Let* $\hat{\phi}_{0,MLE} = \log \hat{f}_{0,MLE}$*. Then*

$$\frac{1}{n}\sum_{i=1}^{n} \hat{\phi}'_{0,MLE}(X_i - \hat{\theta}_{MLE}) = 0.$$

MLE of θ

Standard normal distribution, $n = 100$

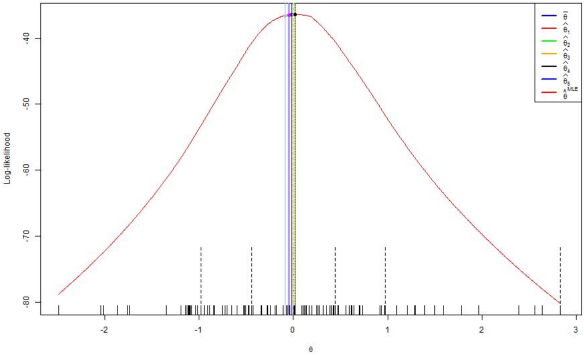

Figure: $\bar{\theta}_n$ is the sample-mean, $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, $\hat{\theta}_4$, $\hat{\theta}_5$ are the one-step estimators obtained using $\widehat{f}_n$, $\widehat{f}_n^{symm}$, $(\widehat{f}_n^{symm})^{sm}$, $\widehat{f}_n^{MLE,symm}$, $\widehat{f}_n^{geo,symm}$ repectively in (12). The log-likelihood function is drawn in red.

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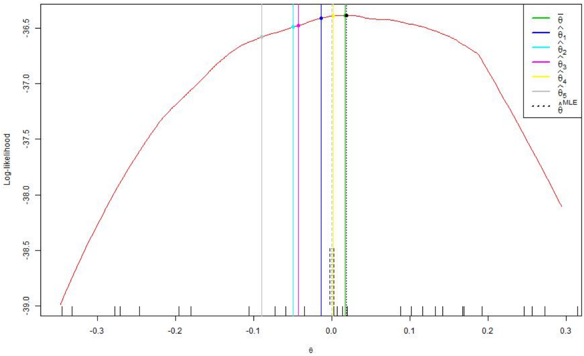

Figure: $\bar{\theta}_n$ is the sample-mean, $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, $\hat{\theta}_4$, $\hat{\theta}_5$ are the one-step estimators obtained using $\widehat{f}_n$, $\widehat{f}_n^{symm}$, $(\widehat{f}_n^{symm})^{sm}$, $\widehat{f}_n^{MLE,symm}$, $\widehat{f}_n^{geo,symm}$ repectively in (12). The log-likelihood function is drawn in red.

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Asymptotic properties of the one-step estimator

Let

$$\mathcal{I}_f(\eta) = \int_{F^{-1}(\eta)}^{F^{-1}(1-\eta)} (\phi'(x))^2 dF(x). \tag{13}$$

Theorem 8

Assume $f \in \mathcal{P}_0$ and suppose that $\bar{\theta}_n$ is a $\sqrt{n}$-consistent estimator of the center of symmetry of f, namely θ. Let $\eta \in (0, 1/2)$. Then for the estimator $\widehat{\mathcal{I}}_n^\eta$ in (11) we have as $n \to \infty$,

$$\widehat{\mathcal{I}}_n^\eta \to_p \mathcal{I}_f(\eta).$$

For the estimator $\widehat{\theta}_n$ of θ in (12) we have

$$\sqrt{n}(\widehat{\theta}_n - \theta) \to_d N(0, \mathcal{I}_f{}^{-1}(\eta))$$

We want to inspect the values of $\mathcal{I}_f(\eta)/\mathcal{I}_f$ for different $\eta \in (0, 1/2)$ for the following symmetric log-concave densities.

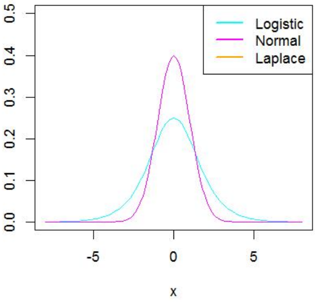

Figure: Density of Laplace, normal and logistic distribution

symmetrized beta distribution with parameter r

$$f(x) = \frac{\Gamma(\frac{3+r}{2})}{\sqrt{\pi r}\Gamma(1+r/2)}\left(1 - \frac{x^2}{r}\right)^{r/2} 1_{[-\sqrt{r},\sqrt{r}]}(x), \;\; r \geq 1. \qquad (14)$$

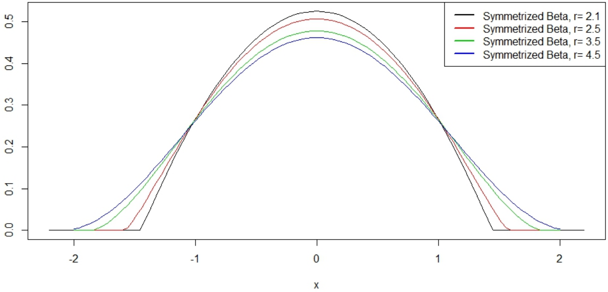

Figure: Density of symmetrized beta with $r = 2.1, 2.5, 3.5, 4.5$

Plot of Fisher's information of symmetrized beta with parameter r vs r

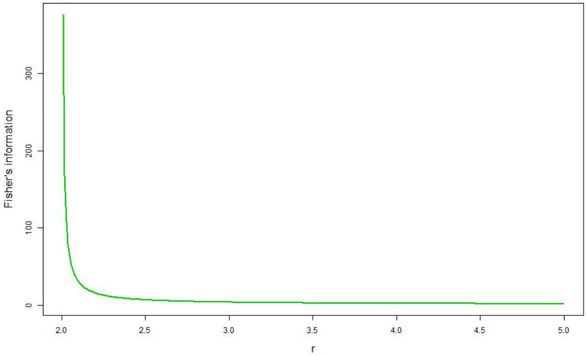

Figure: The Fisher's information $\mathcal{I}_f$ of symmetrized beta with parameter r plotted against r

Plot for $\mathcal{I}_f(\eta)/\mathcal{I}_f$ for some symmetric log-concave densities

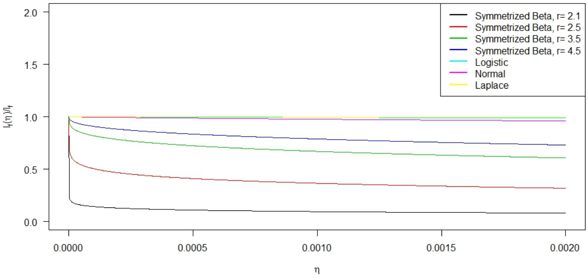

Figure: Plot of $\mathcal{I}_f(\eta)/\mathcal{I}_f$ vs η for different distributions

Conjecture: Truncation at knots

- Let $-\xi_n$ be the second knot of a density $\tilde{h}_n$ in (6)-(9).
- Compute

$$\widehat{\mathcal{I}}_n = \int_{-\xi_n+\bar{\theta}_n}^{\xi_n+\bar{\theta}_n} \tilde{\phi}_n'(x-\bar{\theta}_n)^2 d\mathbb{F}_n(x).$$

  Compute $\widehat{\theta}_n$ as in (12).

Conjecture 1

Assume $f \in \mathcal{P}_0$ and suppose that $\bar{\theta}_n$ is a $\sqrt{n}$-consistent estimator of the center of symmetry of f, namely θ. Then we have

$$\widehat{\mathcal{I}}_n \to_p \mathcal{I}_f$$

and

$$\sqrt{n}(\widehat{\theta}_n - \theta) \to_d N(0, \mathcal{I}_f^{-1}).$$

Conjecture 2

Assume $f \in \mathcal{P}_0$ with $\mathcal{I}_f < \infty$ and $\bar{\theta}_n$ is a $\sqrt{n}-$consistent estimator of the mean of f, namely θ. Then for the estimator $\widehat{\theta}_n$ of θ in (10) *we have*

$$\sqrt{n}(\widehat{\theta}_n - \theta) \to_d N(0, {\mathcal{I}_f}^{-1}).$$

Asymptotic properties of the MLE

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- Let $\hat{\phi}_{MLE} = \log \hat{f}_{MLE}$. On any compact set $K \subset \mathrm{int}(\mathrm{dom}(\phi))$ we have,

$$\sup_{x \in K} |\hat{\phi}_{MLE}(x) - \phi(x)| \to_{a.s.} 0.$$

  Suppose ϕ is differentiable at $x \in K$. Then

$$\hat{\phi}'_{MLE}(x) \to_{a.s.} \phi'(x).$$

Asymptotic properties of the MLE: continued

Hellinger distance between two densities f and g is defined as

$$H^2(f, g) = \frac{1}{2} \int_{-\infty}^{\infty} \left(\sqrt{f(x)} - \sqrt{g(x)} \right)^2 dx.$$

- $H(\hat{f}_{MLE}, f) \to_{a.s.} 0$.

Asymptotic properties of the MLE: continued

Hellinger distance between two densities f and g is defined as

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- $H(\hat{f}_{MLE}, f) \to_{a.s.} 0$.
- $H(\hat{f}_{MLE}, f) = O_p(n^{-2/5})$.

Conjecture 3

Assume $f \in \mathcal{P}_0$ with $\mathcal{I}_f < \infty$. Then maximum likelihood estimator $\hat{\theta}_{MLE}$ (via profile likelihood) of θ satisfies

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \to_d N(0, {\mathcal{I}_f}^{-1}).$$

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Set-up

- We simulate 3000 samples of size $n = 40, 100, 200, 500$ from some log-concave distributions symmetric about 0.
- The initial estimator $\bar{\theta}$ is considered to be either mean, median or the trimmed mean (12.5% from both end). However, for logistic distribution we have also considered the MLE as an initial estimator.
- We have computed both truncated and untruncated estimators of θ. We took $\eta = 0.002$.

Notations for the estimators

- Throughout this document we will use the following notations for the estimators of θ constructed using the following density estimators:
  $\hat{\theta}_1$: $\widehat{f}_n(\cdot + \bar{\theta}_n)$,
  $\hat{\theta}_2$: symmetrized logconcave MLE estimator $\widehat{f}_n^{symm}$,
  $\hat{\theta}_3$: smoothed symmetrized log-concave MLE estimator $(\widehat{f}_n^{symm})^{sm}$,
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- $\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{I}_4, \hat{I}_5$ refer to the corresponding estimators of $\mathcal{I}_f$ using the respective density estimates.
- Additionally let us define

$$E_i = \frac{1/(n\mathcal{I}_f)}{Var(\hat{\theta}_i)}, \ \ i = 1, \ldots, 4,$$

the efficiency of the estimator $\hat{\theta}_i$ for the untruncated case. We will call the corresponding quantity for the truncated case as Et_i. E_0 and Et_0 will stand for the efficiency of $\bar{\theta}_n$.

Outline

Comparison between truncated and untruncated estimators

- We observe from the simulations that there is not much difference between the truncated and untruncated estimators of θ.

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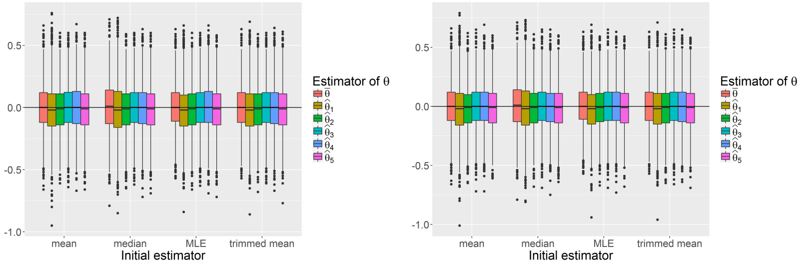
- Efficiency of the truncated estimators(Et) is slightly smaller than E, the untruncated ones.
- The untruncated estimators of $\mathcal{I}_f$ are slightly bigger.
- The difference decreases as n increases.

Comparison between truncated and untruncated $\hat{\theta}_i$-s

Logistic distribution, $n = 100$



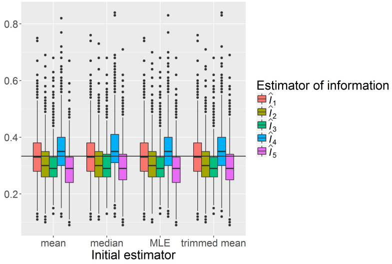
(a) Truncated estimators

(b) Untruncated estimators

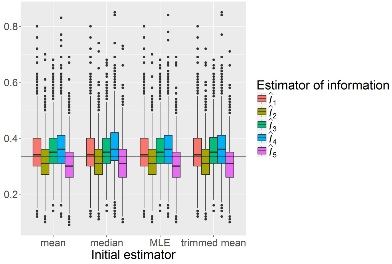
Figure: Box plot of the truncated and untruncated $\hat{\theta}_i$-s when $n = 100$

Comparison between truncated and untruncated estimators of $\mathcal{I}_f$

Logistic distribution, $n = 100$



(a) Truncated estimators



(b) Untruncated estimators

Figure: Box plot of the truncated and untruncated estimators of $\mathcal{I}_f$ for various initial estimators when $n = 100$

Comparison between efficiency

Normal distribution, $n = 500$

	mean	median	trimmed mean
Et0	1.00	0.66	0.84
Et1	0.92	0.54	0.70
Et2	0.81	0.82	0.81
Et3	0.91	0.91	0.91
Et4	0.87	0.87	0.87
Et5	0.88	0.88	0.88

	mean	median	trimmed mean
E0	1.00	0.66	0.84
E1	0.94	0.63	0.80
E2	0.93	0.94	0.94
E3	0.93	0.92	0.93
E4	0.91	0.91	0.91
E5	0.92	0.92	0.92

Table: Tables for efficiency of truncated and untruncated estimators of θ respectively for normal distribution when $n = 500$

Outline

Best estimator of θ: small samples

- We observe $\hat{\theta}_3$, estimator derived from smoothed symmetrized log-concave MLE estimator $(\widehat{f}_n^{symm})^{sm}$ has the highest efficiency for small samples for all distributions except Laplace.

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- $\hat{\theta}_4$, constructed using symmetric log-concave MLE estimator $\widehat{f}_n^{MLE,symm}$ has the highest efficiency for Laplace distribution and has second highest efficiency for all other distributions in small samples.
- $\hat{\theta}_1$, derived from unsymmetrized $\widehat{f}_n$ has the worst efficiency.

Plot of log-densities for Laplace distribution, $n = 40$

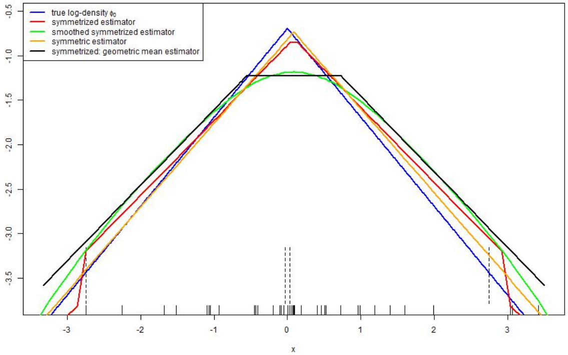

Figure: ϕ_0(blue line) is the true log-density and $\log \widehat{f}_n^{symm}$. $\log(\widehat{f}_n^{symm})^{sm}$, $\log \widehat{f}_n^{MLE,symm}$, $\log \widehat{f}_n^{geo,symm}$ are plotted in red, green, orange and black lines respectively; the sample mean was chosen as $\bar{\theta}_n$.

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- For normal, Laplace and logistic distribution, all estimators have quite similar efficiency in large samples. $\hat{\theta}_3$, $\hat{\theta}_4$ and $\hat{\theta}_5$ have highest efficiency.

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- For normal, Laplace and logistic distribution, all estimators have quite similar efficiency in large samples. $\hat{\theta}_3$, $\hat{\theta}_4$ and $\hat{\theta}_5$ have highest efficiency.
- For the symmetrized beta densities, $\hat{\theta}_3$ has higher efficiency than any other estimator for all sample sizes.

Comparison of efficiency

Symmetrized beta distribution with $r = 4.5$, $n = 500$

	mean	median	trimmed mean
Et0	0.75	0.40	0.53
Et1	0.52	0.23	0.32
Et2	0.50	0.49	0.49
Et3	0.68	0.68	0.68
Et4	0.63	0.62	0.63
Et5	0.57	0.56	0.56

	mean	median	trimmed mean
E0	0.75	0.40	0.53
E1	0.66	0.36	0.47
E2	0.75	0.75	0.75
E3	0.89	0.88	0.89
E4	0.75	0.74	0.76
E5	0.67	0.67	0.67

Table: Tables for efficiency of truncated and untruncated estimators of θ respectively for symmetrized beta distribution when $r = 4.5$ and $n = 500$

Outline

Estimation of $\mathcal{I}_f$

- When $\mathcal{I}_f$ is estimated well enough, efficiency is larger. For symmetrized beta distributions, $\mathcal{I}_f$ is underestimated and hence, efficiency increases very slowly with n.

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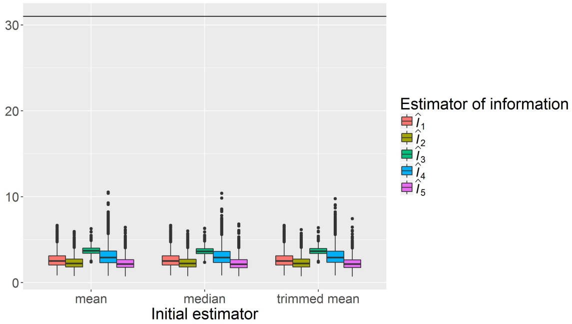


Figure: Box plot of the untruncated estimators of θ for symmetrized beta distribution $(r = 2.1)$ when $n = 500$, the black line indicates the true value of $\mathcal{I}_f$.

Symmetrized beta distribution with $r = 2.1$, $n = 500$

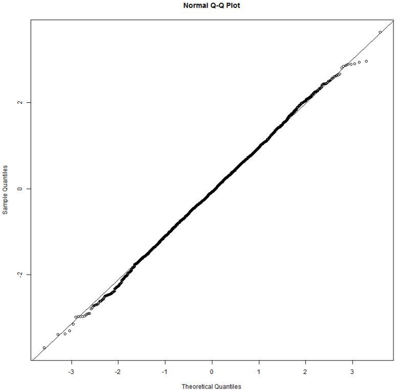
	mean	median	trimmed mean		mean	median	trimmed mean
Et0	0.08	0.04	0.05	Et0	0.08	0.04	0.05
Et1	0.04	0.02	0.02	Et1	0.06	0.03	0.04
Et2	0.05	0.05	0.04	Et2	0.09	0.09	0.094
Et3	0.09	0.10	0.09	Et3	0.14	0.13	0.14
Et4	0.07	0.07	0.07	Et4	0.10	0.10	0.10
Et5	0.05	0.05	0.05	Et5	0.07	0.07	0.07

Table: Efficiency of truncated and untruncated estimators of θ for symmetrized beta distribution with $r = 2.1$, $n = 500$

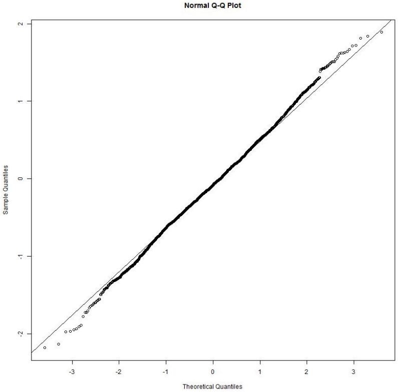
Outline

QQ plot of $\sqrt{n}(\hat{\theta}_2 - \theta)$

QQ-Plot of $\sqrt{n}(\hat{\theta}_2 - \theta)$ for symmetrized beta distribution ($r = 2.1$), $n = 500$ and $\bar{\theta}_n$ is the sample median



(a) QQ-plot for truncated case



(b) QQ-plot for untruncated case

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Future work

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- If $\mathcal{I}_f = \infty$ (e.g. symmetrized Beta density with $r = 2$), does the estimators of $\mathcal{I}_f$ approach ∞ as $n \to \infty$?

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  1. f symmetric, multimodal
  2. f symmetric, unimodal but heavier tails, i.e. Cauchy.

Thank you.

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