A STROLL THROUGH PROJECTIVE GEOMETRY

Noah Miller, 2019

A series of easily digestible subjects with numerous visualizations.

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Sources

- Projective Differential Geometry Old and New: From the Schwarzian Derivative to the Cohomology of Diffeomorphism Groups, V. Ovsienko, S. Tabachnikov
- The Four Pillars of Geometry, John Stillwell
- www.cut-the-knot.org, Alexander Bogomolny
- Conversations with Alek Bedroya

$\mathbb{RP}^1$ and $SL(2,\mathbb{R})$

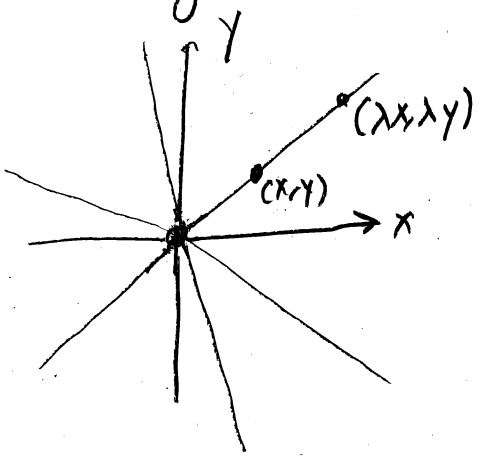
Define an equivalence relation on $\mathbb{R}^2 - \{0\}$ by

$$(x,y) \sim (\lambda x, \lambda y) \quad \forall \lambda \in \mathbb{R},\ \lambda \neq 0$$

Then we define 1-dimensional "projective space" as

$$\mathbb{RP}^1 \equiv \mathbb{R}^2 - \{0\} / \sim .$$

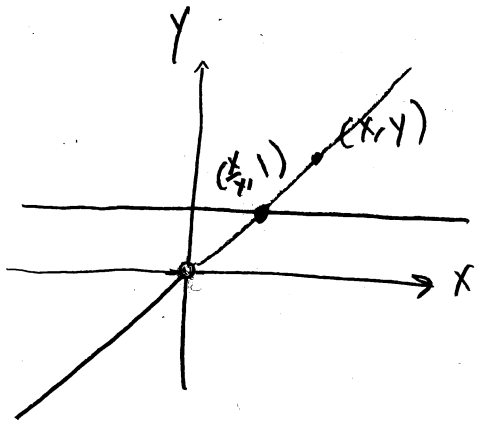
Each point in $\mathbb{RP}^1$ is a line in $\mathbb{R}^2$ passing through the origin, i.e. a dimension 1 subspace.



we can write each line using "homogeneous coordinates," denoted with a colon : and are defined to be equivalent up to scaling.

$$(x:y) = (\lambda x : \lambda y).$$

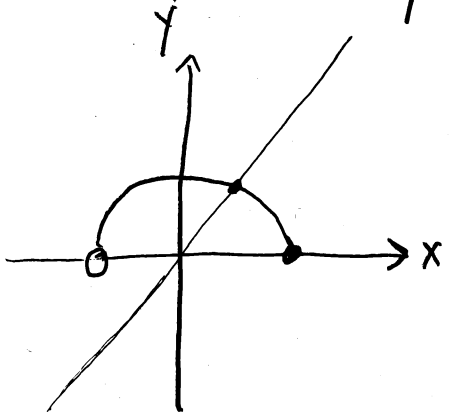
Therefore if $y \neq 0$, then $(x:y) = (\frac{x}{y} : 1)$. This is the intersection of the line with the $y=1$ axis.



The line with $y=0$, i.e. the x-axis, does not intersect the $y=1$ axis. It is often called the "point at infinity."

However, as long as $x \neq 0$, $(x;y) = (1 : \frac{y}{x})$ so a different set of coordinates can accomodate this "point at infinity" just fine.

Note that every line intersects the following semi-circle once:



So topologically, $\mathbb{RP}^1 =$ $=$,

so $\mathbb{RP}^1$ is just the circle, often called S^1.

We will now discuss an important group of transformations which acts on $\mathbb{RP}^1$: $SL(2,\mathbb{R})$.

Linear transformations (A.K.A. matrices) send dimension 1 subspaces to other dimension 1 subspaces, A.K.A. lines to lines.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \sim \begin{pmatrix} \lambda(ax+by) \\ \lambda(cx+dy) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

Note that for $\mu \neq 0$, $\mu\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives the same exact action as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ when acting on projective space, so we might as well only consider matrices for which

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1.$$

These are called "special linear transformations in two dimensions", i.e. $SL(2,\mathbb{R})$. "Special" just means the determinant equals 1.

In homogeneous coordinates, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts on $(x:y)$ as

$$(x:y) \longmapsto (ax+by : cx+dy) = \left(\frac{ax+by}{cx+dy} : 1\right)$$

assuming $cx+dy \neq 0$. This can also be written as

$$\left(\frac{x}{y} : 1\right) \longmapsto \left(\frac{a\frac{x}{y}+b}{c\frac{x}{y}+d} : 1\right)$$

Often people just write this action as

$$x \longmapsto \frac{ax+b}{cx+d}$$

relabelling $\frac{x}{y}$ as just x.

There is actually one extra wrinkle: note $\det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1$, but it sends $(x:y) \longmapsto (-x:-y) = (x:y)$.

so on projective space, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has the same action as the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. To rid ourselves of this redundancy, we often talk about the group

$$PSL(2,\mathbb{R}) \equiv SL(2,\mathbb{R}) \Big/ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

where "P" stands for "projective". It is just $SL(2,\mathbb{R})$ "modded out" by a new equivalence relation on matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.

We will however continue to talk about $SL(2,\mathbb{R})$ for simplicity.

Let us now build some intuition for how $SL(2,\mathbb{R})$ acts on $\mathbb{RP}^1$.

Claim: You can send any three distinct points in $\mathbb{RP}^1$ to any other three distinct points using a matrix in $SL(2,\mathbb{R})$.

Proof: For each of the six points in question, choose an arbitrary vector in $\mathbb{R}^2$ which lies on the line in $\mathbb{R}^2$ corresponding to the point in $\mathbb{RP}^1$. Call the first three $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and the second three $\vec{v}_1', \vec{v}_2', \vec{v}_3'$.

Because $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are distinct vectors, there are some $w, z \in \mathbb{R}$, $w, z \neq 0$ such that

$$\vec{v}_3 = w\vec{v}_1 + z\vec{v}_2.$$

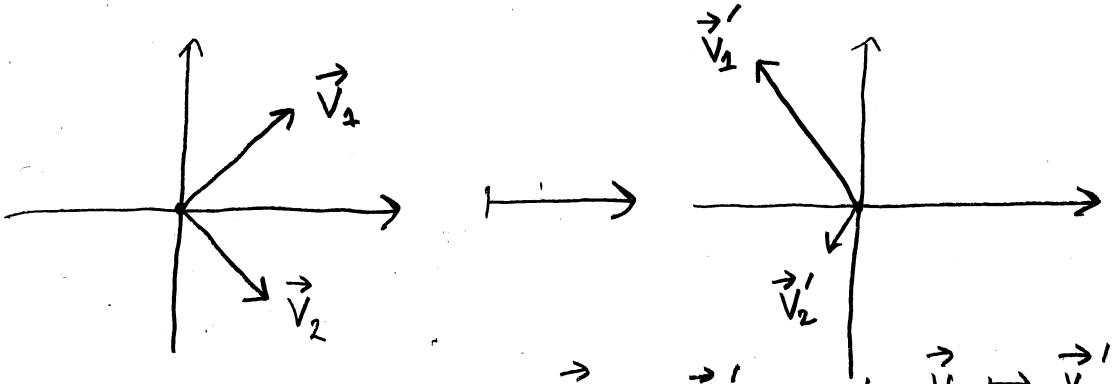
Likewise, there are some $w', z' \in \mathbb{R}$, $w', z' \neq 0$ such that

$$\vec{v}_3' = w'\vec{v}_1' + z'\vec{v}_2'.$$

Next, note that if you rescale a vector, it corresponds to the same point in $\mathbb{RP}^1$. Use this freedom to rescale $\vec{v}_1, \vec{v}_2, \vec{v}_1', \vec{v}_2'$ such that

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2$$
$$\vec{v}_3' = \vec{v}_1' + \vec{v}_2'.$$

Next, note that there always is a unique matrix (with non-zero determinant) that sends any two distinct (non-zero) vectors to any other two distinct (non-zero) vectors.



↖ There is always a unique matrix that can do this

If this matrix sends $\vec{v}_1 \mapsto \vec{v}_1'$ and $\vec{v}_2 \mapsto \vec{v}_2'$, then it also sends

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 \longmapsto \vec{v}_1' + \vec{v}_2' = \vec{v}_3'.$$

Therefore there is always a matrix that sends any three points in $\mathbb{RP}^1$ to any three other points. □

Note that this proof would not work for four points.

Furthermore, we have used all the freedom available to us to find this matrix, so there is only <u>one</u> matrix that can send any particular three points to any particular other three points.

We can see $SL(2,\mathbb{R})$ has three degrees of freedom.

Let's look at some specific transformations and see how they act.

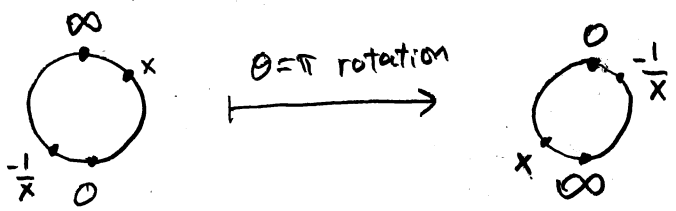
Scaling: $\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} : x \mapsto \frac{ax}{1/a} = a^2 x$ This scales the line by a constant.

Inversion: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : x \mapsto -\frac{1}{x}$ This flips 0 and ∞. (The -1 preserves orientation.)

Rotation: $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} : x \mapsto \frac{\cos(\frac{\theta}{2})x - \sin(\frac{\theta}{2})}{\sin(\frac{\theta}{2})x + \cos(\frac{\theta}{2})}$

A rotation in $\mathbb{R}^2$ sends $0 \mapsto -\tan(\theta/2)$, $\infty \mapsto \cot(\theta/2)$.
(The reason I wrote $\theta/2$ and not θ is because $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as the identity. I used $\theta/2$ so a $\theta = \pi$ rotation is not the identity, but a $\theta = 2\pi$ rotation is.)

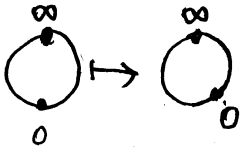
Note that an inversion is a rotation with $\theta = \pi$.



Shear 1: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : x \to x+b$ This is a constant translation along the line. Note it maps $\infty \mapsto \infty$, so ∞ is fixed.

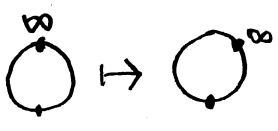
Shear 2: $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} : x \to \frac{x}{cx+1}$ This is like a translation that translates ∞ instead of 0. It maps $0 \mapsto 0$ and $\infty \to \frac{1}{c}$. (I will give a better way to think about this later.)

Shear 1:



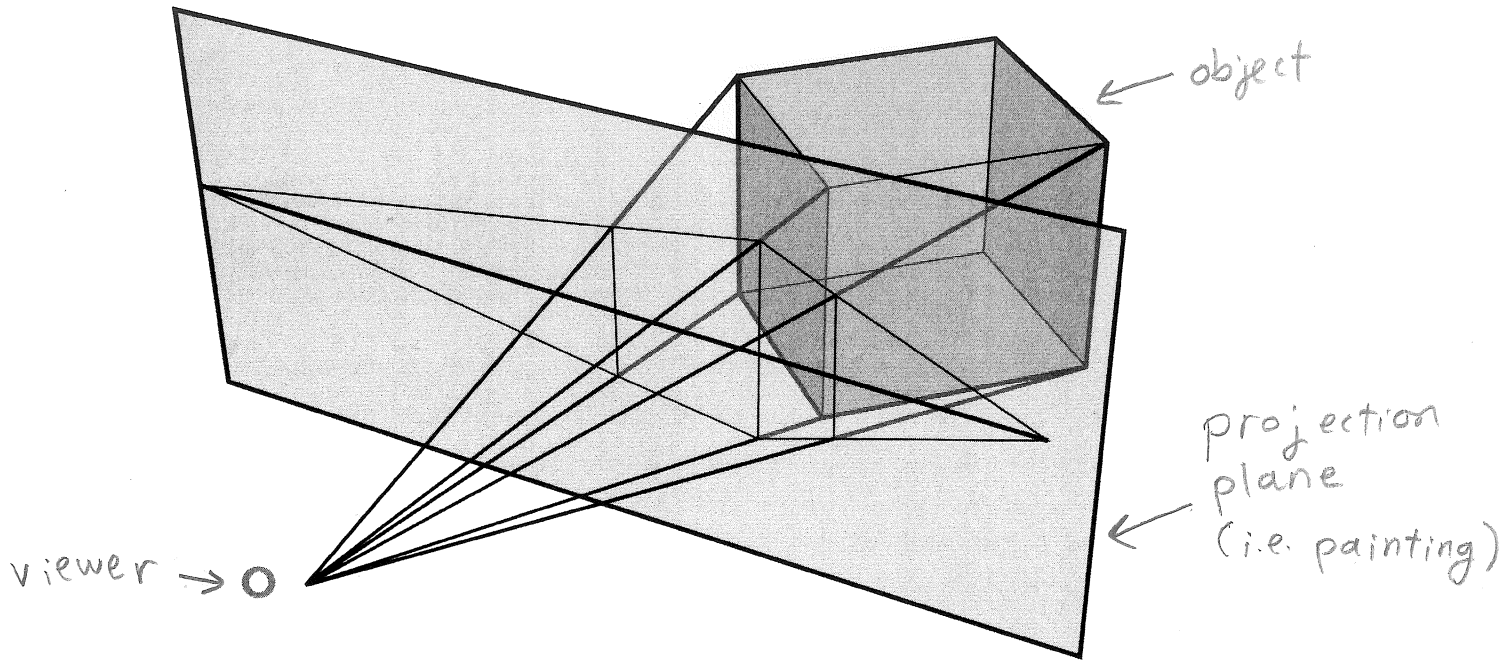
(0 goes to b)

Shear 2:

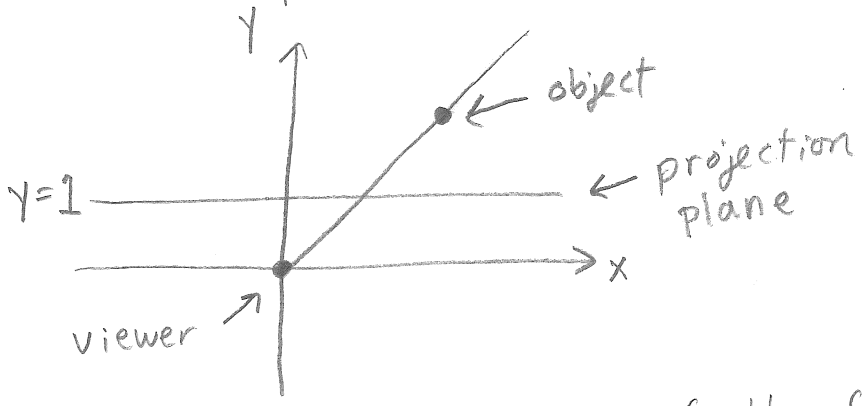


(∞ goes to $\frac{1}{c}$)

Let me now motivate another way to think about these $SL(2,\mathbb{R})$ transformations. The reason why people initially became interested in projective geometry hundreds of years ago is because they were interested in perspective art.



For us, the projection plane has been the line $y=1$.



Every point on the "projection plane" is of the form $(x,1)$. If the "object" has coordinates (x',y'), then for some x and λ, we have

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

The point $\begin{bmatrix} x \\ 1 \end{bmatrix}$ is where the projection plane intersects with the "line of sight" to the object.

When we act by an $SL(2,\mathbb{R})$ transformation, instead of solving the equation $\begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \end{bmatrix}$ to find x we instead solve

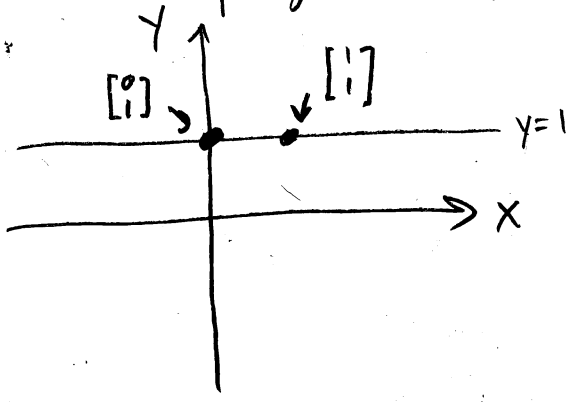
$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Multiplying both sides by the inverse matrix $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, we see

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \end{bmatrix}$$

and the value of x will be the same. So you can see that $SL(2,\mathbb{R})$ transformations can be understood as different positionings of the projection plane!

At this point I will start drawing two points with black dots: the points $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which show the "midpoint" and "scale" of the projection plane.

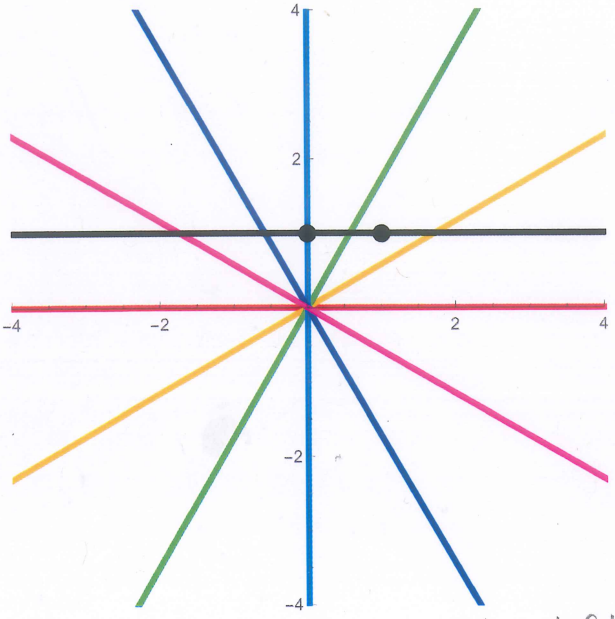


We can visualize $SL(2,\mathbb{R})$ transformations by acting with the inverse matrix on these two points.

The following pages contain colorful images showing how $SL(2,\mathbb{R})$ transformations can be understood either by their action on lines in $\mathbb{R}^2$ or conversely by their action on the projection plane.

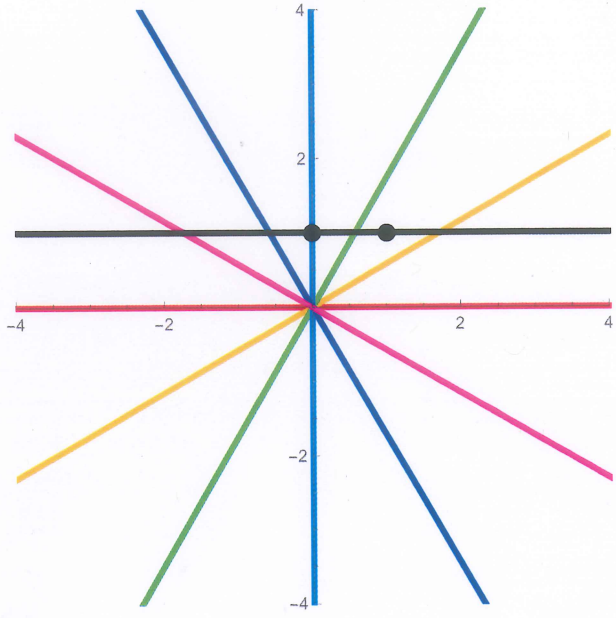
Identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



For the pictures on the left we will be acting the matrix on the lines directly.

cyan line is (0:1) A.K.A. $x=0$

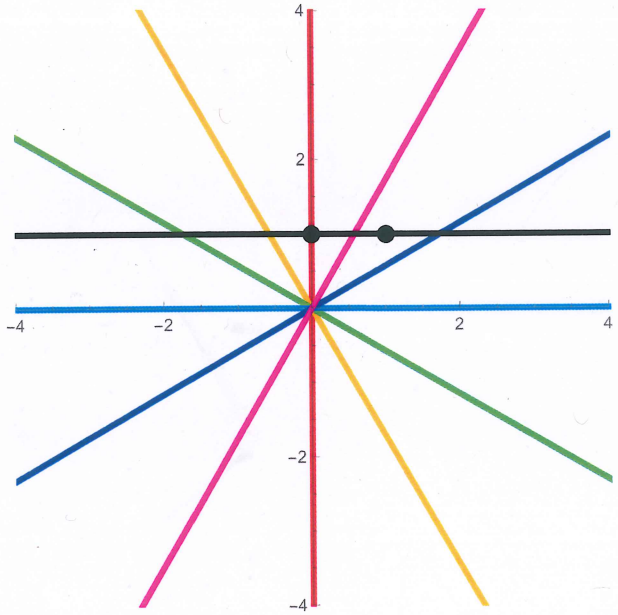


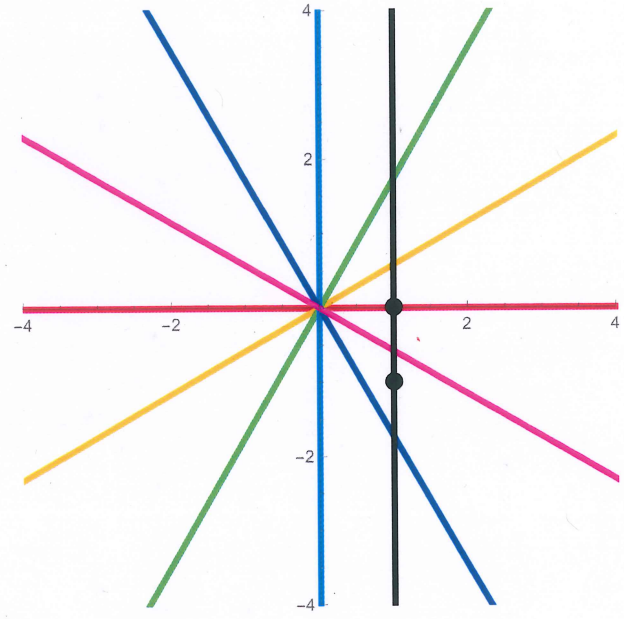
red line is (1:0) A.K.A $x=\infty$

For the pictures on the right we will instead act the inverse matrix on the "projection plane" $y=1$.

Inversion:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

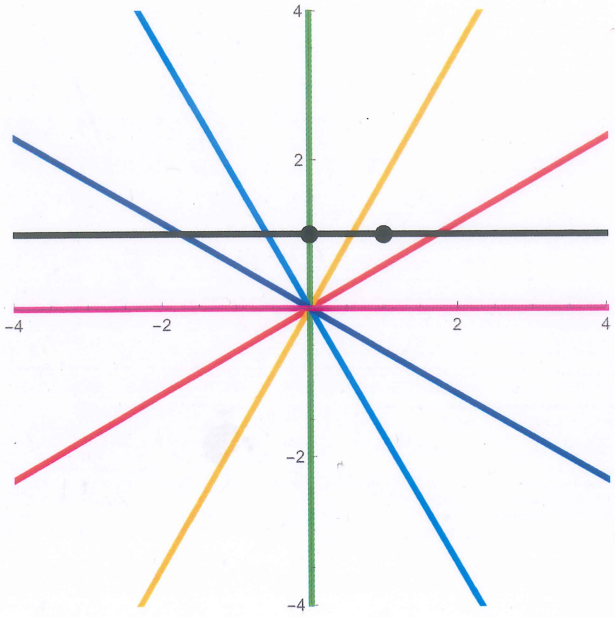


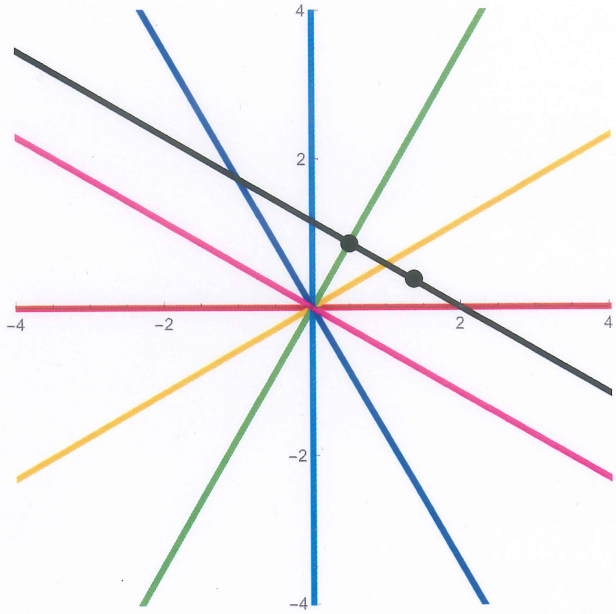


Note that inversion flips the red and cyan lines, A.K.A. 0 and ∞.

Rotation:

$$\begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix}$$

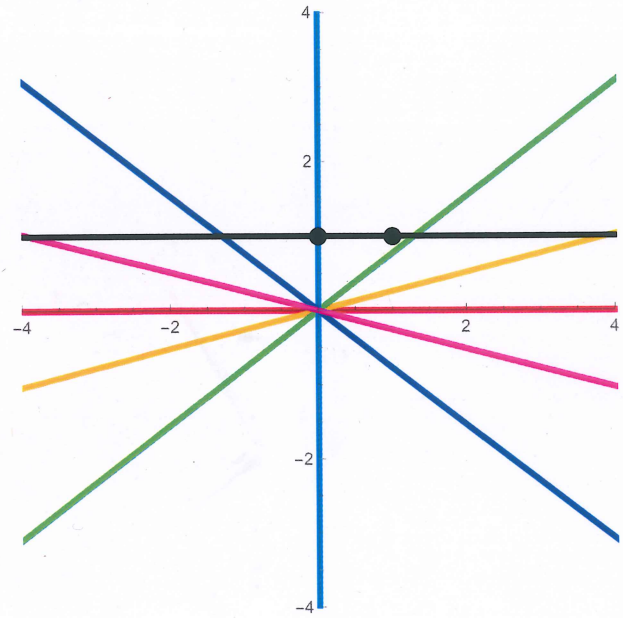


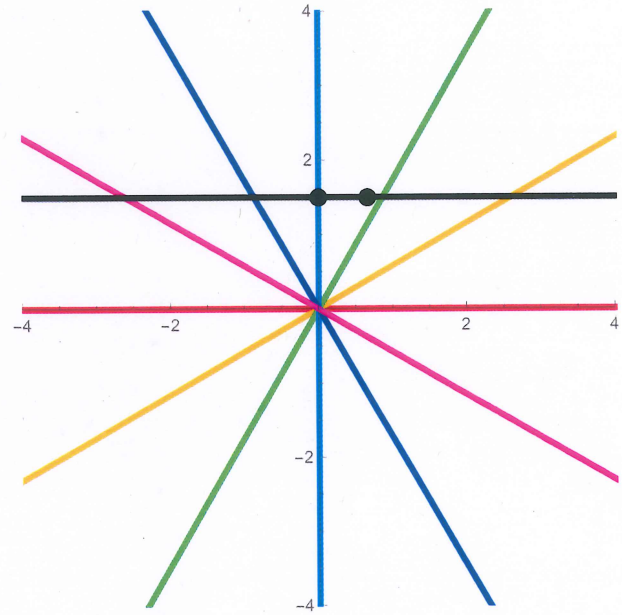


Rotation simply rotates the projection plane around the origin.

Scale:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1/1.5 \end{pmatrix}$$

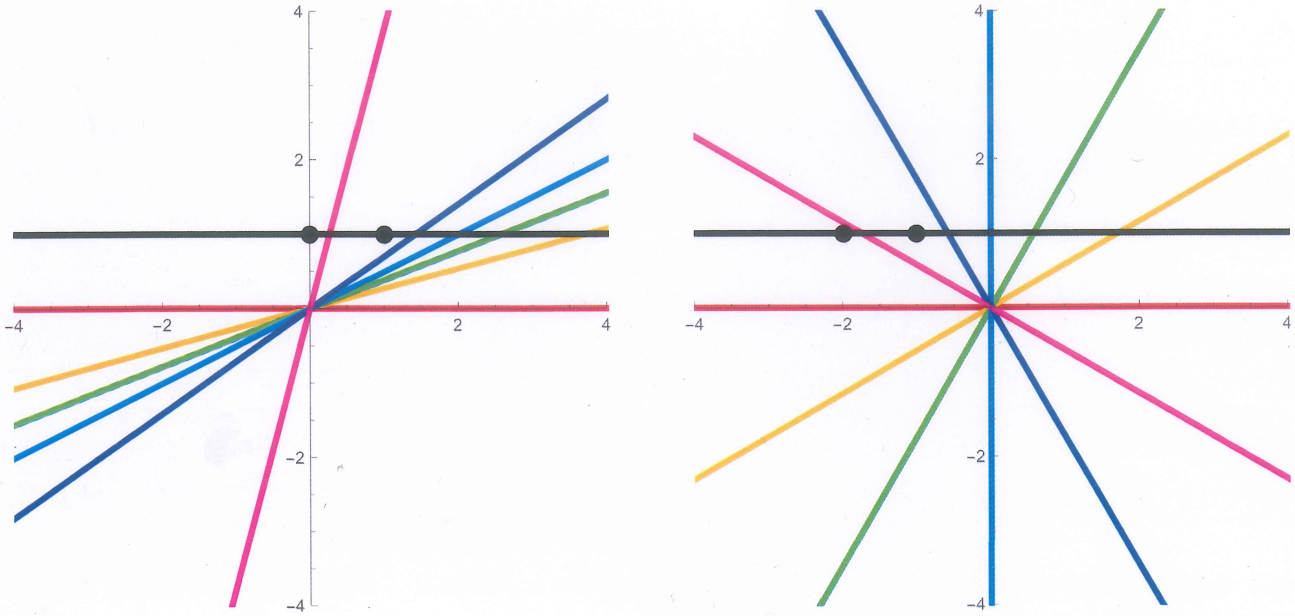




Scaling changes the distance of the projection plane from the origin (and changes the distance between the x=0 dot and the x=1 dot).

Shear 1:

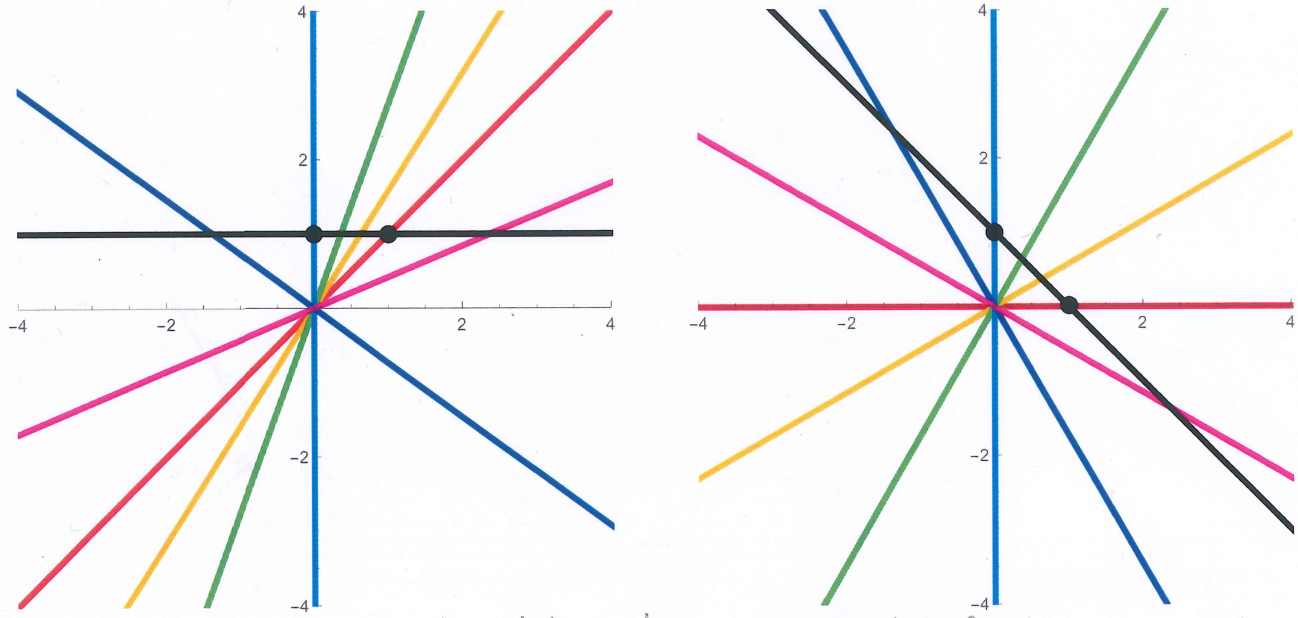
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$



This type of shearing just translates the projection plane. Note that the red line, A.K.A. $x=\infty$, is fixed.

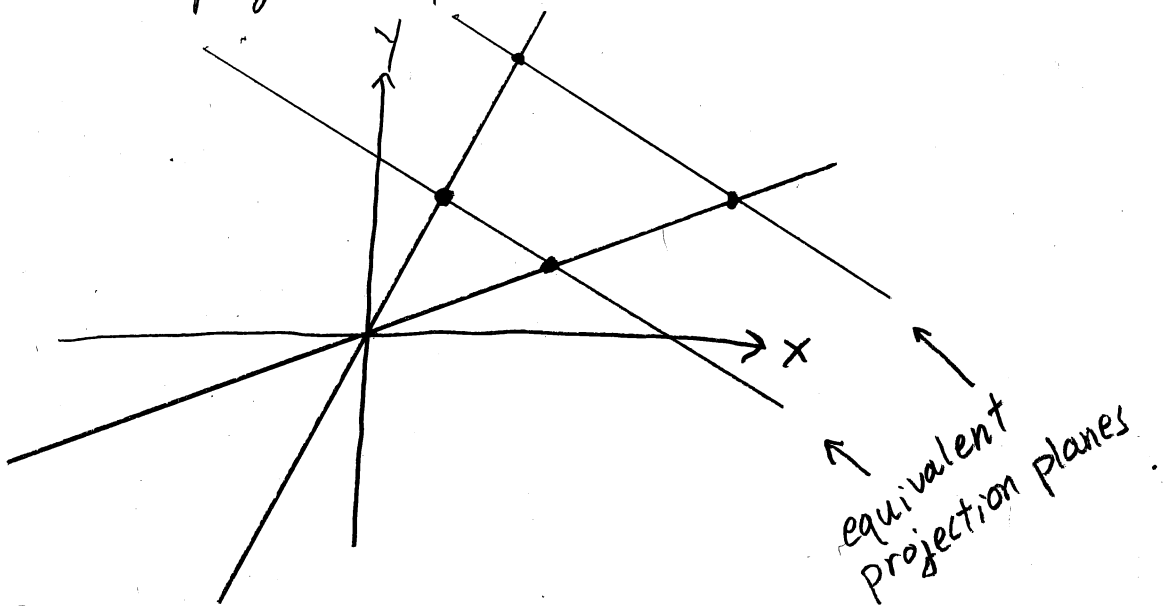
Shear 2:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



This sort of shearing translates the y-coordinate of the $x=1$ dot. Note that the cyan line ($x=0$) is fixed but the red line ($x=\infty$) is not. I think this is the easiest way to picture the action of $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$.

$SL(2,\mathbb{R})$ transformations correspond to the positions of the two black dots in $\mathbb{R}^2$. This might be surprising, because $SL(2,\mathbb{R})$ has three degrees of freedom, while the positions of the two dots in $\mathbb{R}^2$ has four. The reason for the apparent discrepancy is because you can scale both dots by a constant and recieve an equivalent projection plane.



Subtracting off one degree of freedom for this scaling redundancy resolves the discrepancy.

The Cross Ratio

You might say it was a triumph of algebra to invent this quantity that turns out to be so valuable and could not be imagined geometrically. Or if you are a geometer at heart, you may say it is an invention of the devil and hate it all your life.

Robin Hartshorne, *Geometry: Euclid and Beyond*, p. 341.

Say you have four points in $\mathbb{RP}^1$ with coordinates $(x_1 : 1)$, $(x_2 : 1)$, $(x_3 : 1)$, and $(x_4 : 1)$. The "cross ratio" of these four points is defined to be

$$[x_1, x_2; x_3, x_4] \equiv \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)}.$$

Miraculously, this quantity is invariant under any $SL(2,\mathbb{R})$ transformation.

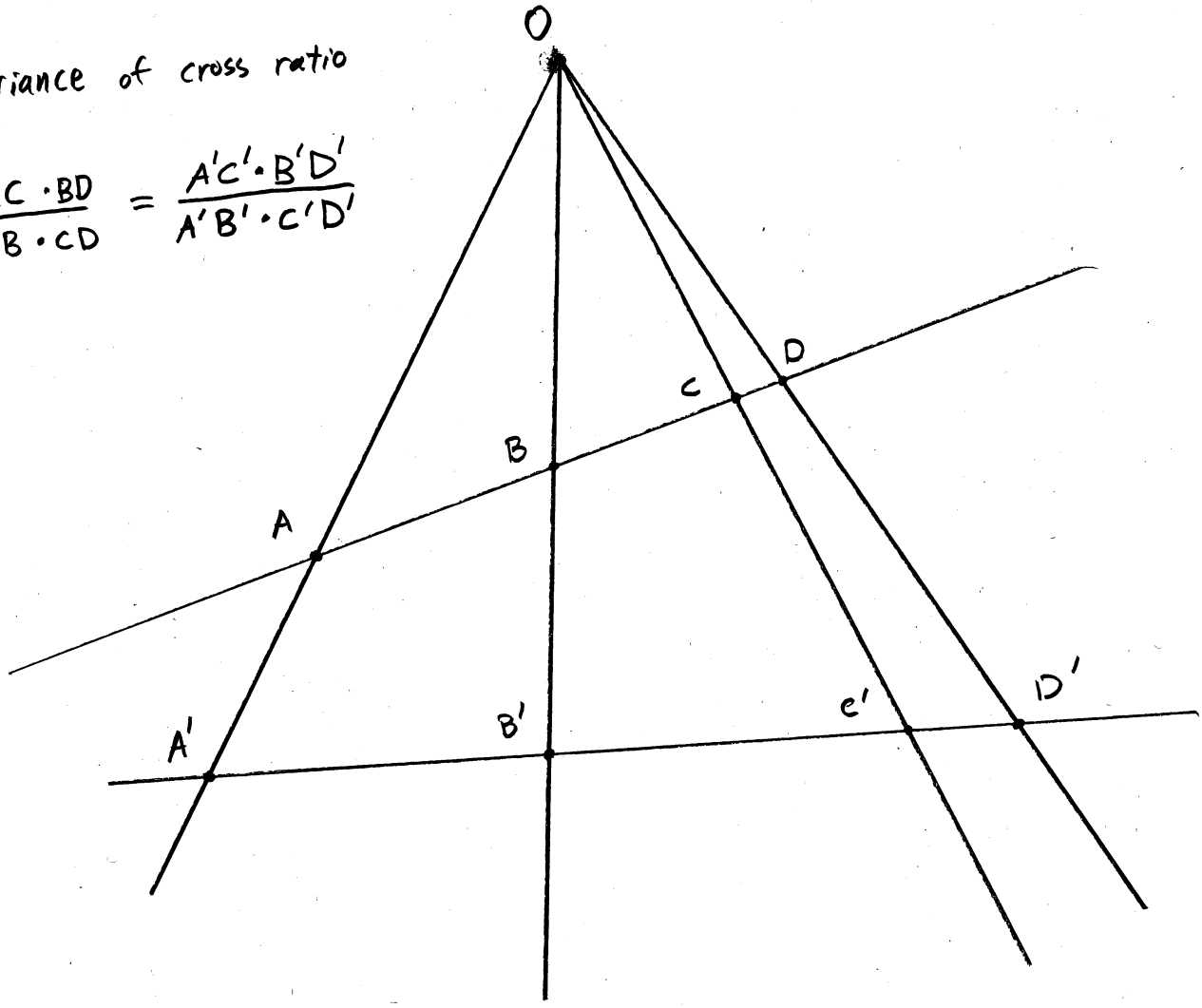
```
In[1]:= FullSimplify[ (a*x1+b)/(c*x1+d) - (a*x3+b)/(c*x3+d)   (a*x2+b)/(c*x2+d) - (a*x4+b)/(c*x4+d)
                      ------------------------------------ / ------------------------------------ ]
                      (a*x1+b)/(c*x1+d) - (a*x2+b)/(c*x2+d)   (a*x3+b)/(c*x3+d) - (a*x4+b)/(c*x4+d)

Out[1]= (x1 - x3) (x2 - x4)
        -------------------
        (x1 - x2) (x3 - x4)
```

The interpretation of the cross ratio is rather opaque. I have not been able to come up with a killer visualization or argument that makes the existence of this simple yet magical quantity seem inevitable. (This does not mean that such a visualization does not exist.) However, it is not entirely true that the cross ratio "could not be imagined geometrically." I will now present two geometric arguments for the invariance of the cross ratio which you can hold onto for dear life.

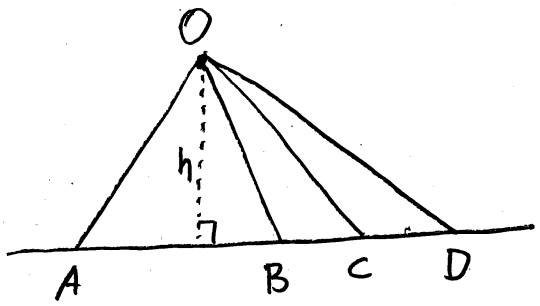
Invariance of cross ratio

$$\Rightarrow \frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}$$



Argument 1, from www.cut-the-knot.org, by Alexander Bogomolny:

Draw a perpendicular line connecting point O with the intersecting line. Label the height as "h."



we can compute the areas of the triangles ΔAOC, ΔBOD, ΔAOB and ΔCOD as follows:

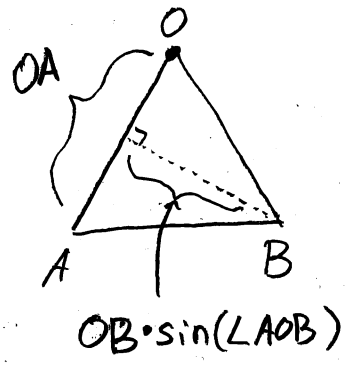
$$\text{Area of } \Delta AOC = \frac{1}{2} h \cdot AC$$
$$\text{Area of } \Delta BOD = \frac{1}{2} h \cdot BD$$
$$\text{Area of } \Delta AOB = \frac{1}{2} h \cdot AB$$
$$\text{Area of } \Delta COD = \frac{1}{2} h \cdot CD$$

Note that

$$\text{Cross ratio} = \frac{AC \cdot BD}{AB \cdot CD} = \frac{\text{Area of } \Delta AOC \cdot \text{Area of } \Delta BOD}{\text{Area of } \Delta AOB \cdot \text{Area of } \Delta COD}.$$

However, there is another way to express the area of a triangle.

For instance,



$$\text{Area of } \Delta AOB = \frac{1}{2} OA \cdot OB \cdot \sin(\angle AOB).$$

So

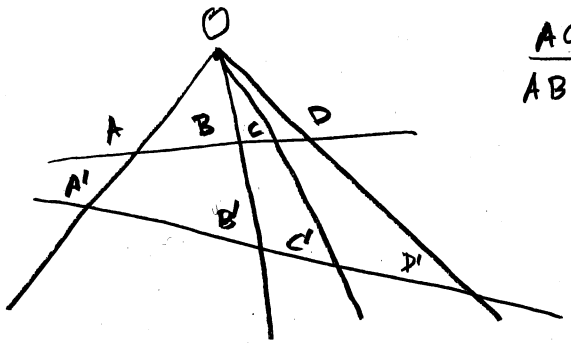
$$\frac{\text{Area of } \Delta AOC \cdot \text{Area of } \Delta BOD}{\text{Area of } \Delta AOB \cdot \text{Area of } \Delta COD}$$

$$= \frac{\frac{1}{2} OA \cdot OC \cdot \sin(\angle AOC) \cdot \frac{1}{2} \cdot OB \cdot OD \cdot \sin(\angle BOD)}{\frac{1}{2} OA \cdot OB \cdot \sin(\angle AOB) \cdot \frac{1}{2} \cdot OC \cdot OD \cdot \sin(\angle COD)} = \frac{\sin(\angle AOC) \cdot \sin(\angle BOD)}{\sin(\angle AOB) \cdot \sin(\angle COD)}$$

This shows that the cross ratio only depends on the angles between the four concurrent lines and not on the fifth intersecting line!

$$\text{Cross ratio} = \frac{\sin(\angle AOC)\cdot\sin(\angle BOD)}{\sin(\angle AOB)\cdot\sin(\angle COD)}$$

Therefore, we can see that the cross ratio will be the same no matter what fifth line we draw.



$$\frac{AC\cdot BD}{AB\cdot CD} = \frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle AOB)\sin(\angle COD)}$$

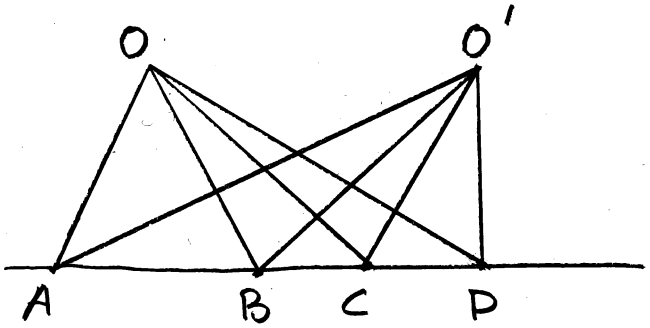
$$= \frac{\sin(\angle A'OC')\sin(\angle B'OD')}{\sin(\angle A'OB')\sin(\angle C'OD')}$$

$$= \frac{A'C'\cdot B'D'}{A'B'\cdot C'D'}$$

□

This argument took advantage of an interesting fact: the cross ratio has two equivalent definitions, one for four collinear lines and one for four concurrent points.* The equality of these definitions demonstrates the invariance of the cross ratio.

A modified version of this argument could show that these two groups of four concurrent lines have the same cross ratio:

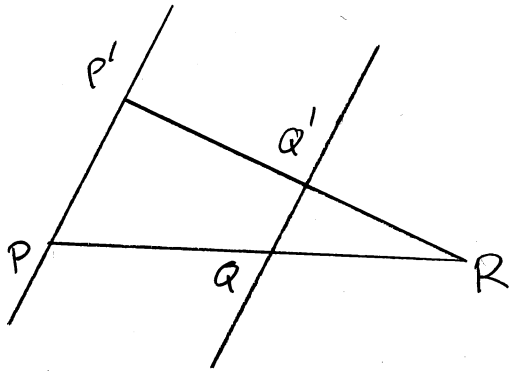


$$\frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle AOB)\sin(\angle COD)} = \frac{\sin(\angle AO'C)\sin(\angle BO'D)}{\sin(\angle AO'B)\sin(\angle CO'D)}$$

* This is related to something called "projective duality."

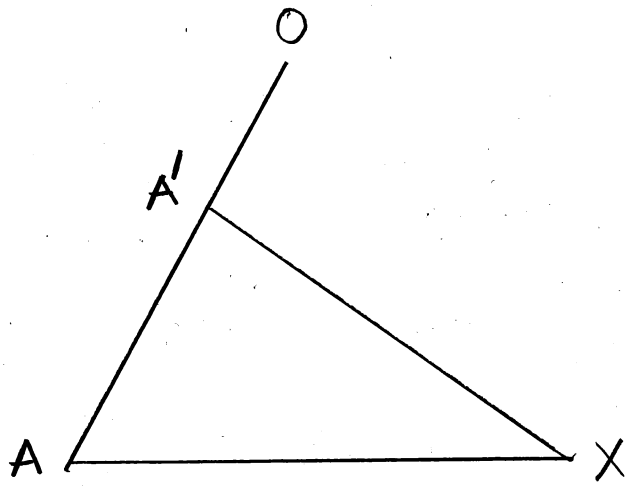
Argument 2, from my friend Alek Bedroya:

This proof relies on Thales' theorem, which states that, in the diagram to the right, if PP' is parallel to QQ', then we have

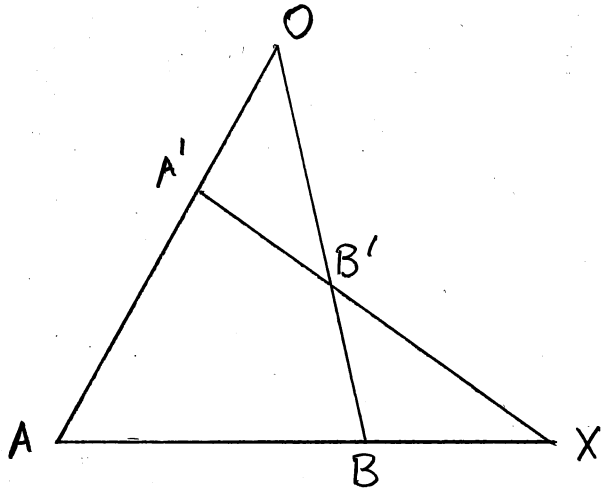


$$\frac{PR}{QR} = \frac{P'R}{Q'R} \quad \text{and} \quad \frac{QR}{QP} = \frac{Q'R}{Q'P'}.$$

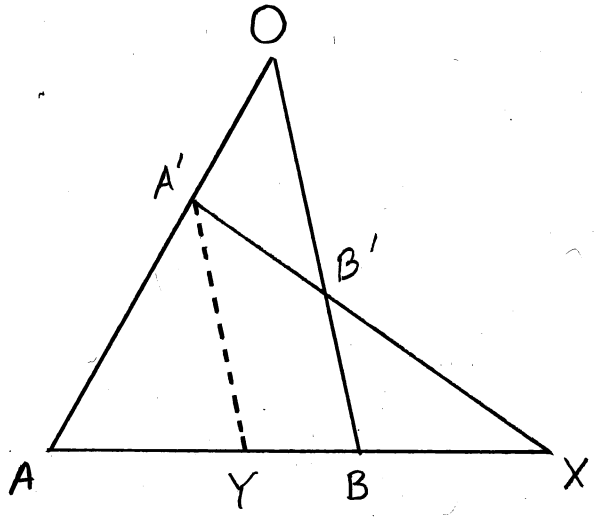
Consider the following diagram:



Then pick a point B on AX. Draw the line BO and call the intersection with $A'X$ point B'.



Imagine varying B, causing B′ to change. Let us try to find an invariant ratio that does not change as B changes. To do this, draw a line parallel to OB passing through A′. Call the intersection of this new line with AX point Y.



By Thales' theorem, we have

$$(1)\quad \frac{A'O}{AO} = \frac{YB}{AB} \qquad \text{and} \qquad (2)\quad \frac{YB}{BX} = \frac{A'B'}{B'X}.$$

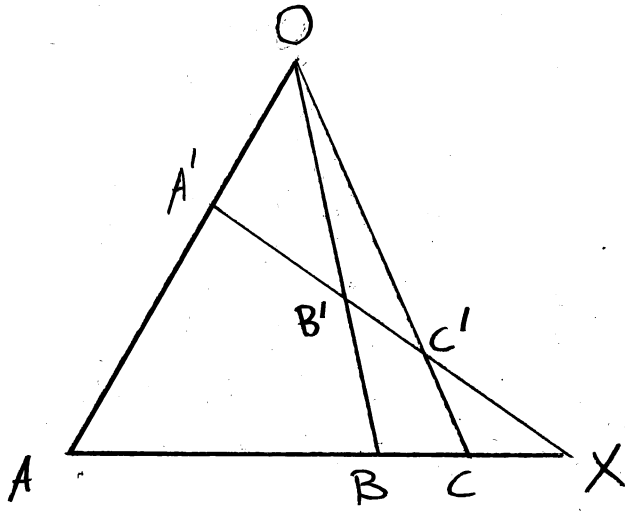
Note that the RHS of (1) depends on B but the LHS does not. This gives us the invariant ratio we were looking for.

Trivially dividing the numerator and denominator of (1) by BX,

$$\frac{A'O}{AO} = \frac{\frac{YB}{BX}}{\frac{AB}{BX}}$$

then using (2), $\dfrac{A'O}{AO} = \dfrac{\frac{A'B'}{B'X}}{\frac{AB}{BX}}$.

Now imagine changing B to a new point, call it C.



Simply replacing B with C and B' with C' in the equation

$$\frac{A'O}{AO} = \frac{\frac{A'B'}{B'X}}{\frac{AB}{BX}}$$

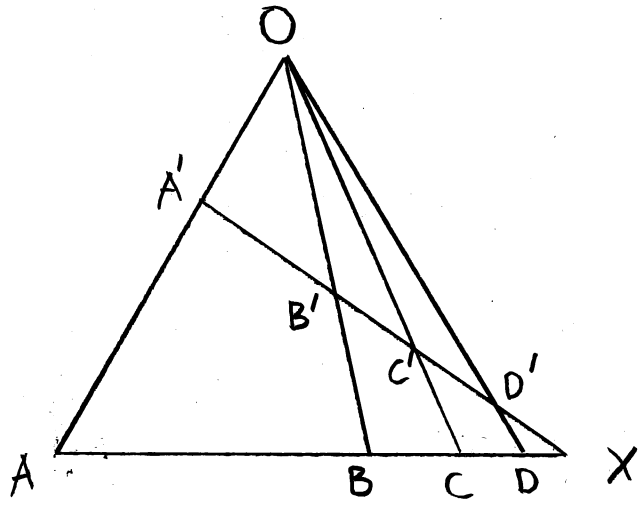
we get

$$\frac{A'O}{AO} = \frac{\frac{A'C'}{C'X}}{\frac{AC}{CX}}$$

implying

$$\frac{\frac{A'B'}{B'X}}{\frac{AB}{BX}} = \frac{\frac{A'C'}{C'X}}{\frac{AC}{CX}} \Rightarrow \frac{AC \cdot BX}{AB \cdot CX} = \frac{A'C' \cdot B'X}{A'B' \cdot C'X}.$$

Now draw another new point called D.



All of our reasoning so far remains valid if you permute

$$A \to D \qquad A' \to D'$$
$$B \to C \qquad B' \to C'$$
$$C \to B \qquad C' \to B'$$

which changes

$$\frac{AC \cdot BX}{AB \cdot CX} = \frac{A'C' \cdot B'X}{A'B' \cdot C'X} \longrightarrow \frac{DB \cdot CX}{DC \cdot BX} = \frac{D'B' \cdot C'X}{D'C' \cdot B'X}.$$

Multiplying the two above equations together, we get

$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}.$$

□

Claim: $[f(x_1), f(x_2); f(x_3), f(x_4)] = [x_1, x_2; x_3, x_4]$ if and only if $f(x) = \frac{ax+b}{cx+d}$.

Proof: We already know that $f(x) = \frac{ax+b}{cx+d}$ leaves the cross ratio invariant, so we only need to prove that all functions which leave the cross ratio invariant are of this form.

Assume $f(x)$ is any function that leaves the cross ratio invariant. We have already proven that there is some $g(x) = \frac{ax+b}{cx+d}$ for which

$$g(x_1) = f(x_1),\ g(x_2) = f(x_2),\ g(x_3) = f(x_3)$$

because there is always a $SL(2,\mathbb{R})$ transformation that sends any three points in $\mathbb{RP}^1$ to any other three points. We then know that

$$\frac{(f(x_1)-f(x_3))(f(x_2)-f(x_4))}{(f(x_1)-f(x_2))(f(x_3)-f(x_4))} = \frac{(g(x_1)-g(x_3))(g(x_2)-g(x_4))}{(g(x_1)-g(x_2))(g(x_3)-g(x_4))}$$

but this just reduces to

$$\frac{f(x_2)-f(x_4)}{f(x_3)-f(x_4)} = \frac{f(x_2)-g(x_4)}{f(x_3)-g(x_4)}$$

$$\Rightarrow \big(f(x_2)-f(x_4)\big)\big(f(x_3)-g(x_4)\big) = \big(f(x_2)-g(x_4)\big)\big(f(x_3)-f(x_4)\big)$$

$$\Rightarrow \quad -f(x_4)f(x_3) - f(x_2)g(x_4) = -g(x_4)f(x_3) - f(x_2)f(x_4)$$

$$\Rightarrow \quad \big(f(x_2)-f(x_3)\big)f(x_4) = \big(f(x_2)-f(x_3)\big)g(x_4)$$

$$\Rightarrow \quad f(x_4) = g(x_4)$$

x_4 can be any point in $\mathbb{RP}^1$, so $f(x)=g(x)$. □

Claim: If you pick three points in $\mathbb{RP}^1$, $x_1, x_2, x_3 \in \mathbb{RP}^1$, and a number C, there is a unique point $x_4 \in \mathbb{RP}^1$ such that

$$C = [x_1, x_2; x_3, x_4].$$

Proof: You can just solve for x_4 in the equation

$$C = \frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_2)(x_3-x_4)}.$$

□

Claim: Any projective invariant of four points, i.e. a function $I(x_1, x_2, x_3, x_4)$ that satisfies

$$I(x_1, x_2, x_3, x_4) = I(f(x_1), f(x_2), f(x_3), f(x_4))$$

for any $f(x) = \frac{ax+b}{cx+d}$, is just a function of the cross ratio:

$$I(x_1, x_2, x_3, x_4) = F([x_1, x_2; x_3, x_4]).$$

Proof: Say $[x_1, x_2; x_3, x_4] = [y_1, y_2; y_3, y_4]$. There is some $f(x) = \frac{ax+b}{cx+d}$ for which

$$y_1 = f(x_1), \quad y_2 = f(x_2), \quad y_3 = f(x_3).$$

Furthermore

$$[y_1, y_2; y_3, f(x_4)] = [y_1, y_2; y_3, y_4].$$

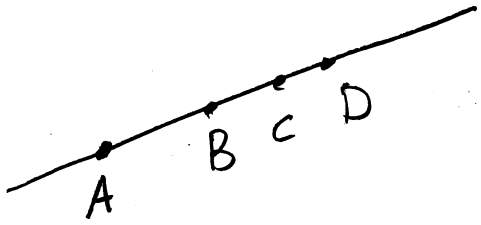
But from the previous claim, we know

$$y_4 = f(x_4)$$

as well. Therefore $I(x_1, x_2, x_3, x_4) = I(y_1, y_2, y_3, y_4)$ and the projective invariant I must take the same value on all quadruples of points with the same cross ratio.

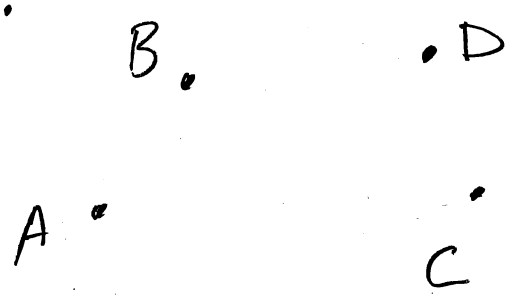
□

So far we have defined the cross ratio for four collinear points:



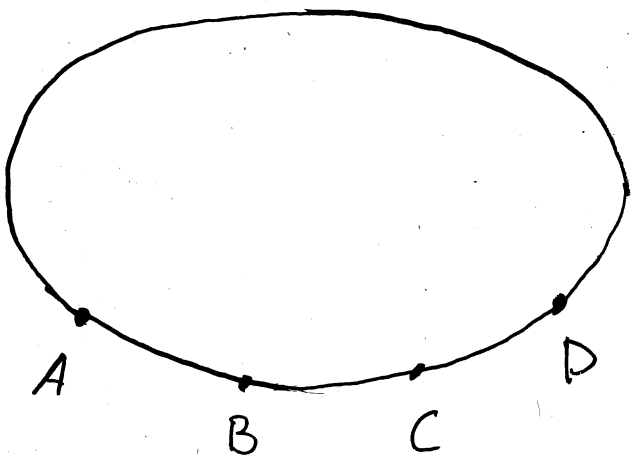
← Cross ratio is well defined

You can't evaluate the cross ratio of any four random points.



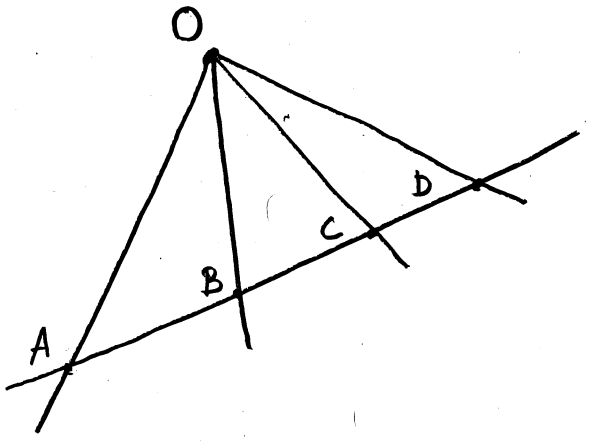
← No cross ratio for these points.

However, there is still a more general notion of the cross ratio as long as your four points lie on a <u>conic</u>.



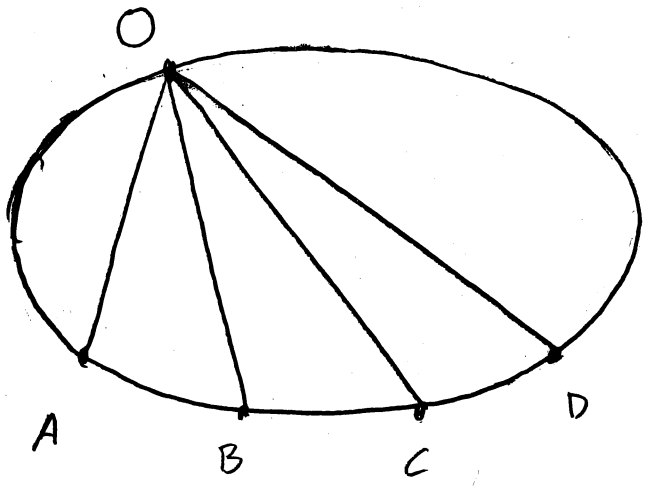
← Cross ratio is well defined.

How is this more general cross ratio defined? First, recall that, for four points on a line, we can express the cross ratio in terms of the angles between the lines these points make with an arbitrary fifth point:



$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle AOB)\sin(\angle COD)}$$

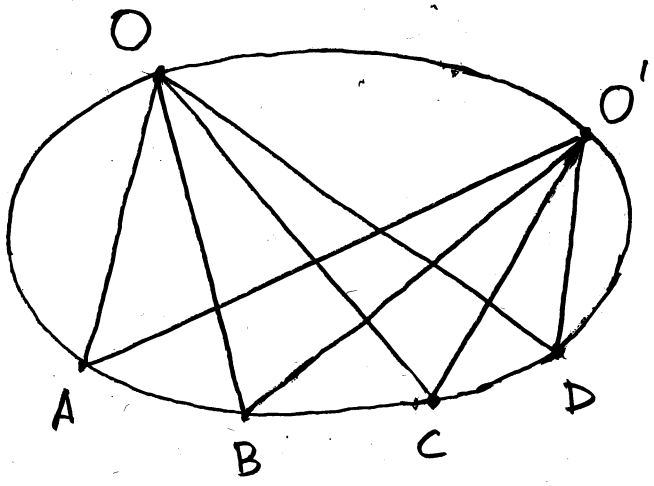
In this spirit, pick an arbitrary fifth point "O" which is also on the conic, and define the cross ratio in terms of the four lines OA, OB, OC, OD.



$$\text{Cross ratio} = \frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle AOB)\sin(\angle COD)}$$

However, in order to show that this new definition of the cross ratio is well defined, we must show that it does not depend on the choice of O.

Steiner's Theorem:

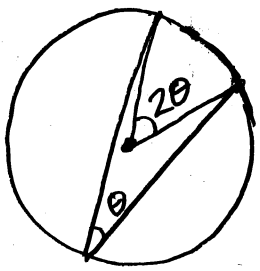


$$\frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle AOB)\sin(\angle COD)} = \frac{\sin(\angle AO'C)\sin(\angle BO'D)}{\sin(\angle AO'B)\sin(\angle CO'D)}$$

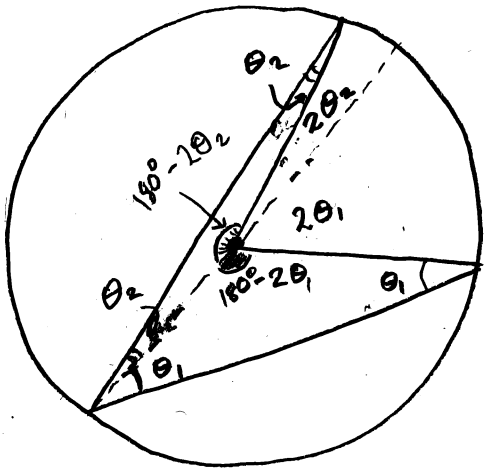
Proof (from Alek Bedroya):

The first step is to prove it for when the conic is a circle.

Recall the "Inscribed Angle Theorem" for a circle:

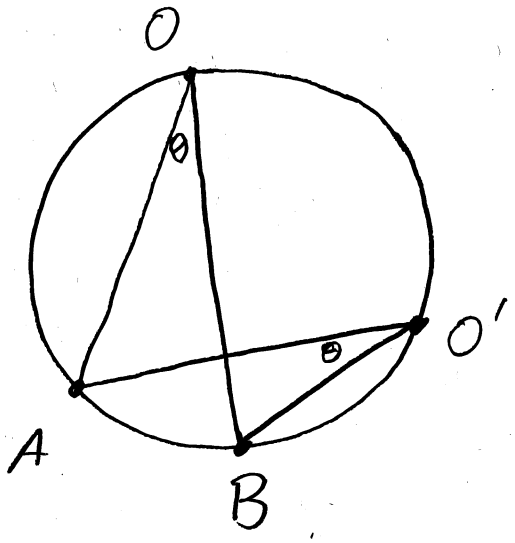


(The proof of the Inscribed Angle Theorem is given by angle chasing:)



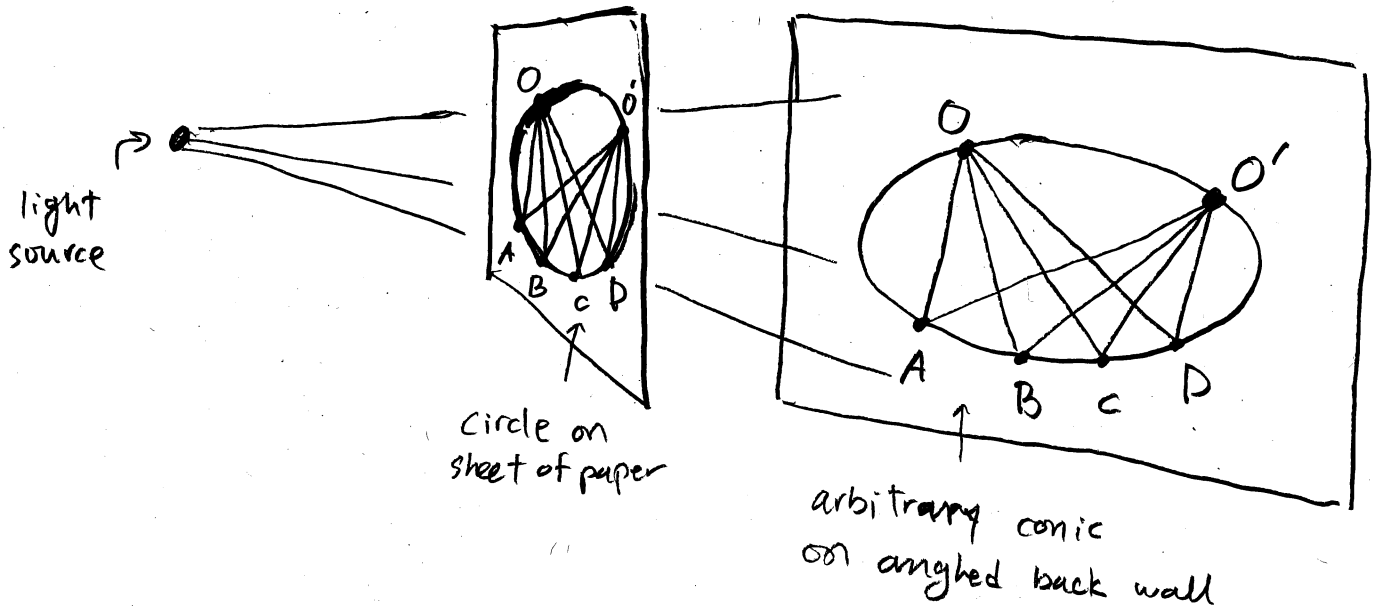
$\theta = \theta_1 + \theta_2$

From the Inscribed Angle Theorem, we can see that the angles between the lines does not depend on the choice of O:

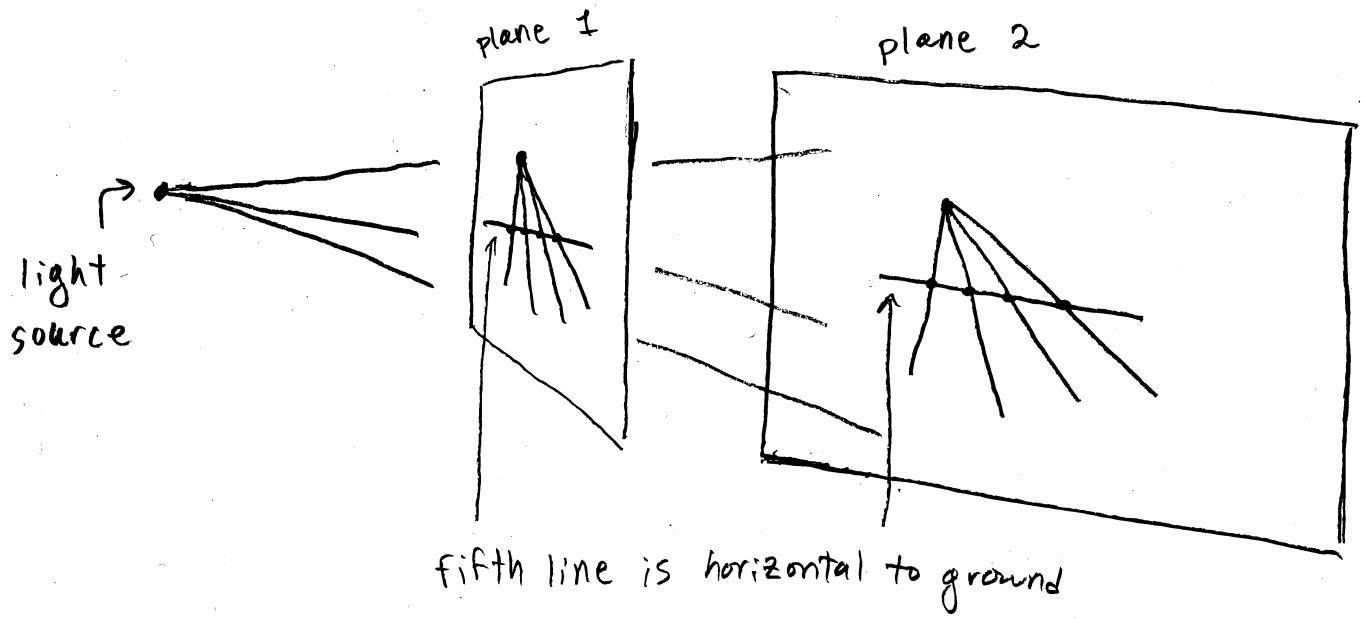


This completes the first step.

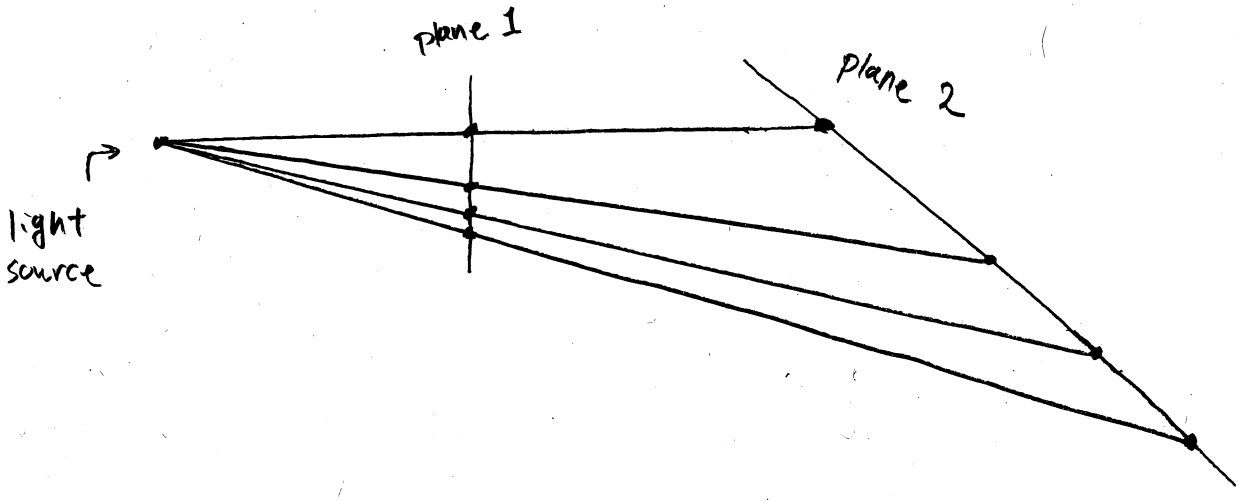
The second step is to prove it for all conics. This step is rather clever and involves the third dimension. Imagine picking up the diagrams from the previous pages and shining a point source of light through the page and onto an angled back wall.



The circle can become any desired conic, the points on the circle will map to points on the conic, and straight lines remain straight lines. The only thing we must show is that cross ratios of four concurrent lines remain invariant under this projection. To do this, draw a fifth line intersecting all four lines horizontal with respect to the ground:



From a bird's eye view, each plane and its four intersection points look like this:



However, we can clearly see that the cross ratio of the four points on plane 1 are equal to the cross ratio of the four points on plane 2. Therefore the cross ratios of the four lines must be invariant under this projection as well, completing the proof.

□

$\mathbb{RP}^2$

Define an equivalence relation on $\mathbb{R}^3 - \{0\}$ by

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad \forall\ \lambda \in \mathbb{R}, \lambda \neq 0$$

Then we define 2-dimensional projective space as

$$\mathbb{RP}^2 \equiv \mathbb{R}^3 - \{0\} / \sim .$$

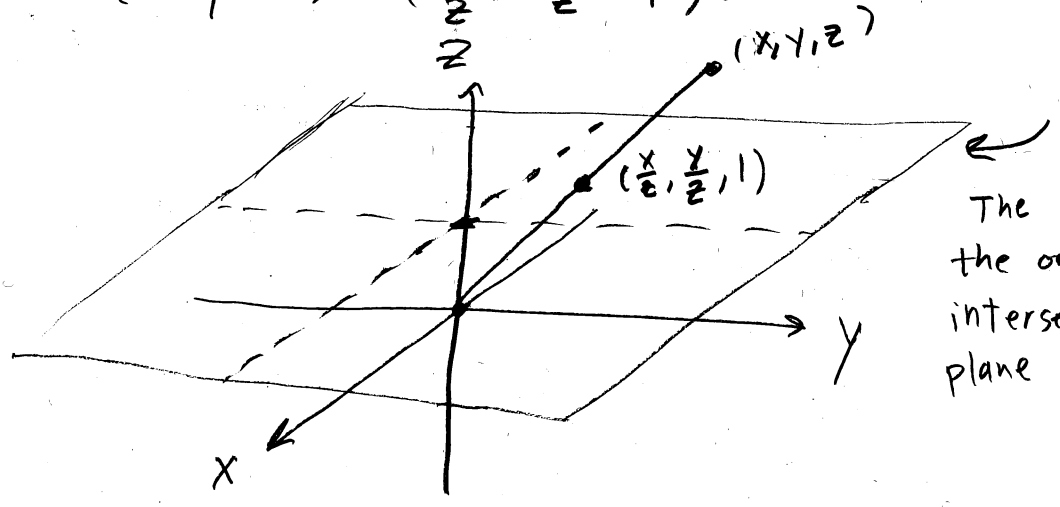
Each point in $\mathbb{RP}^2$ is a line in $\mathbb{R}^3$ passing through the origin, i.e. a dimension 1 subspace.

Once again, we can write points in $\mathbb{RP}^2$ using "homogeneous coordinates"

$$(x : y : z) = (\lambda x : \lambda y : \lambda z) \quad \lambda \neq 0.$$

Therefore, if $z \neq 0$,

$$(x : y : z) = \left(\frac{x}{z} : \frac{y}{z} : 1\right).$$



The line passing through the origin and (x, y, z) intersects the $z=1$ plane at $\left(\frac{x}{z}, \frac{y}{z}, 1\right)$.

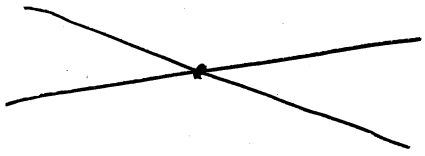
All the points that do not intersect the $z=1$ plane are of the form

$$(\cos\theta : \sin\theta : 0).$$

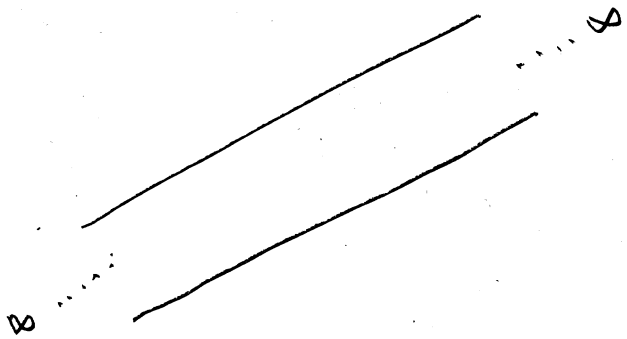
Note that

$$(\cos\theta : \sin\theta : 0) = (-\cos\theta : -\sin\theta : 0).$$

Therefore there is a "circle of points at infinity" that are antipodally identified. This means that all "lines" in $\mathbb{RP}^2$ intersect at exactly one point, no exceptions! If they are "parallel", then they intersect at a point at infinity.



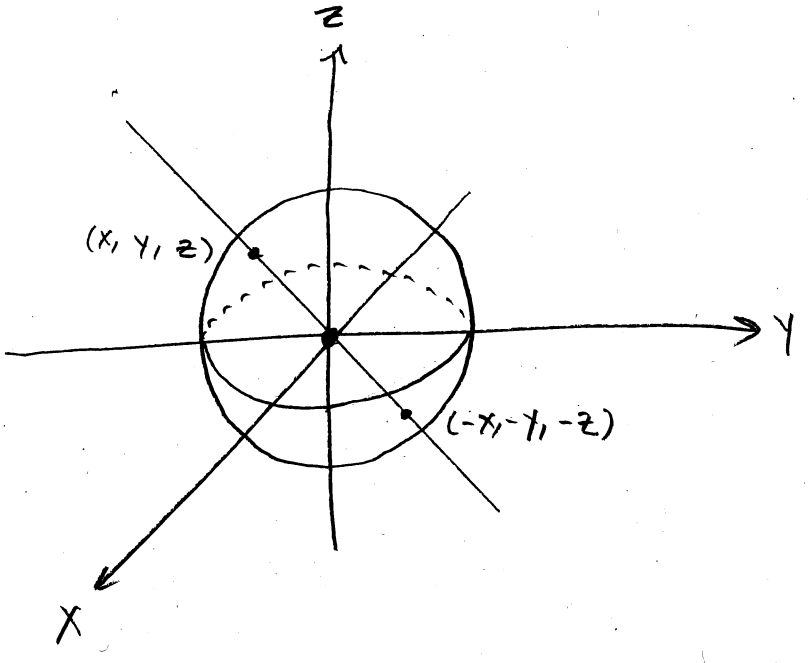
These lines clearly intersect at one point.



These seemingly parallel lines intersect at one point at infinity.

(Note that "lines" in $\mathbb{RP}^2$ can be thought of as planes in $\mathbb{R}^3$ passing through the origin.)

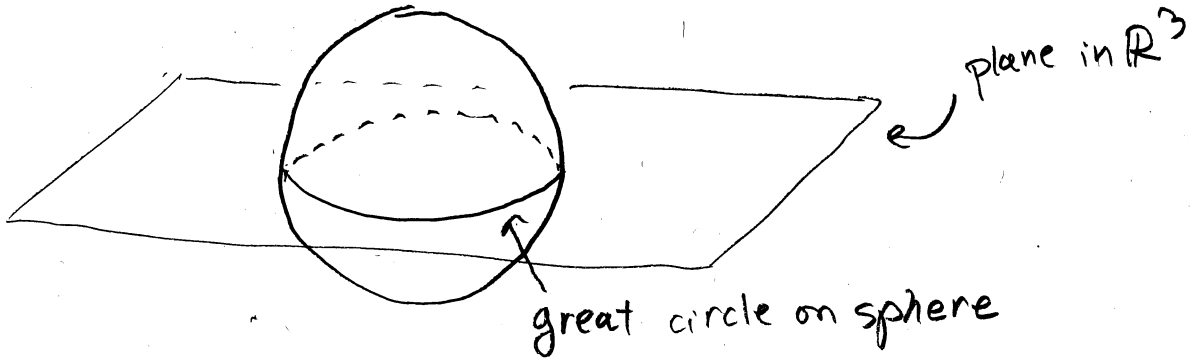
There is another way to think about $\mathbb{RP}^2$. Consider the sphere of radius 1 in $\mathbb{R}^3$ centered at the origin. Every line in $\mathbb{R}^3$ passing through the origin intersects this sphere at two antipodal points.



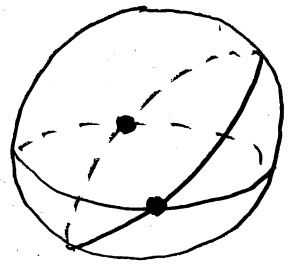
We can therefore see that $\mathbb{RP}^2$ is topologically the 2-sphere (denoted S^2) where antipodal points are identified.

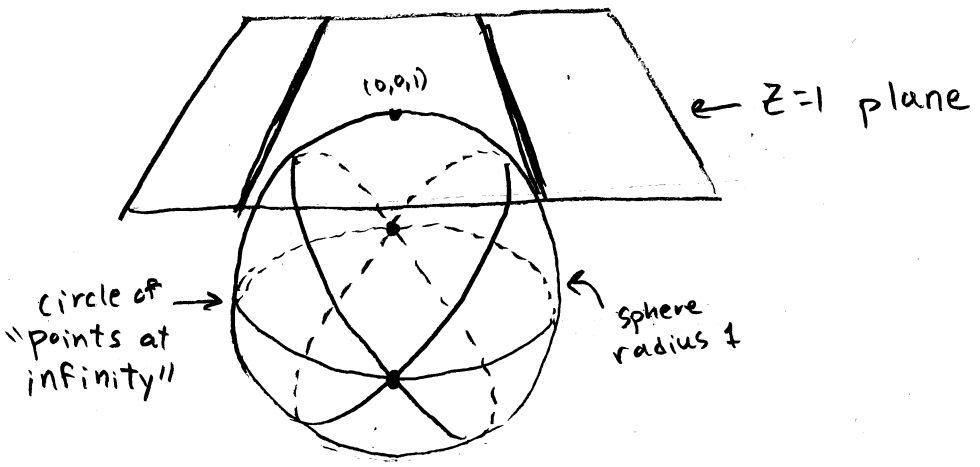
$$\mathbb{RP}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} \Big/ (x,y,z) \sim (-x,-y,-z)$$

Lines in $\mathbb{RP}^2$ (planes in $\mathbb{R}^3$) are "great circles" on this sphere.



Note that any two great circles intersect at two antipodal points on the sphere. Therefore, we can see again that any two lines in $\mathbb{RP}^2$ always intersect at exactly one point!

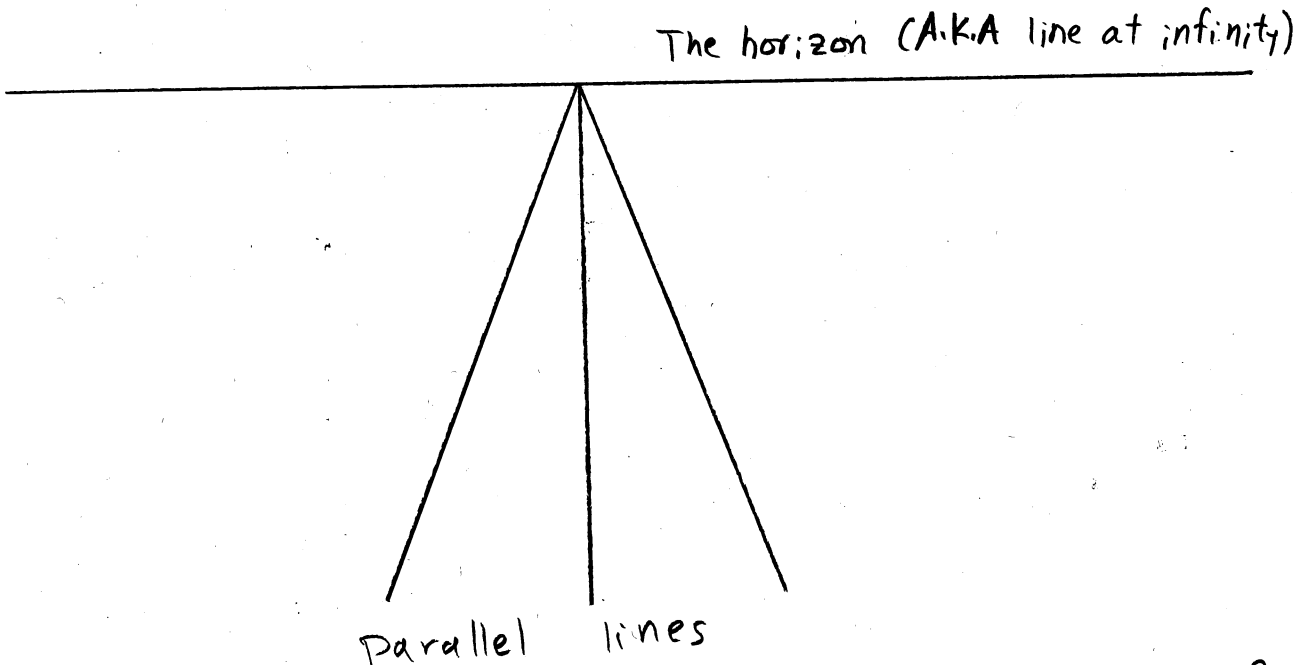




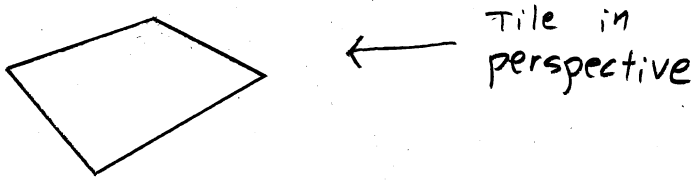
Here we draw two "parallel lines" on the $z=1$ plane and their corresponding great circles on the sphere. These great circles intersect on the great circle of points "at infinity" A.K.A. the "horizon" or the "line at infinity."

The Horizon

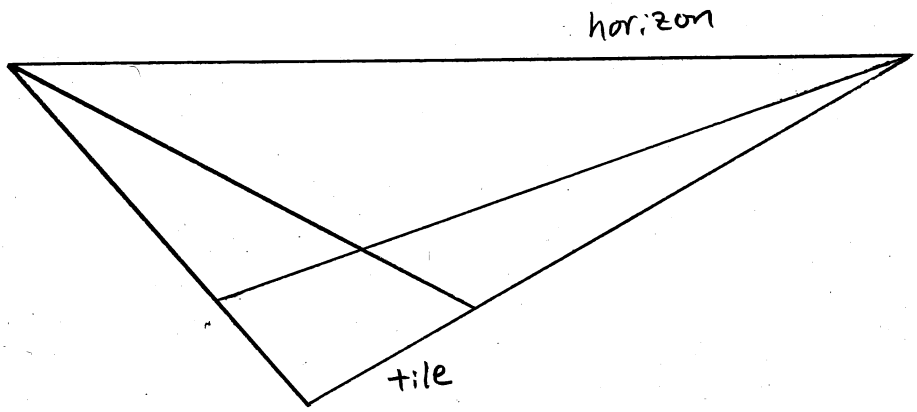
In middle school, your art teacher told you all parallel lines meet at infinity.



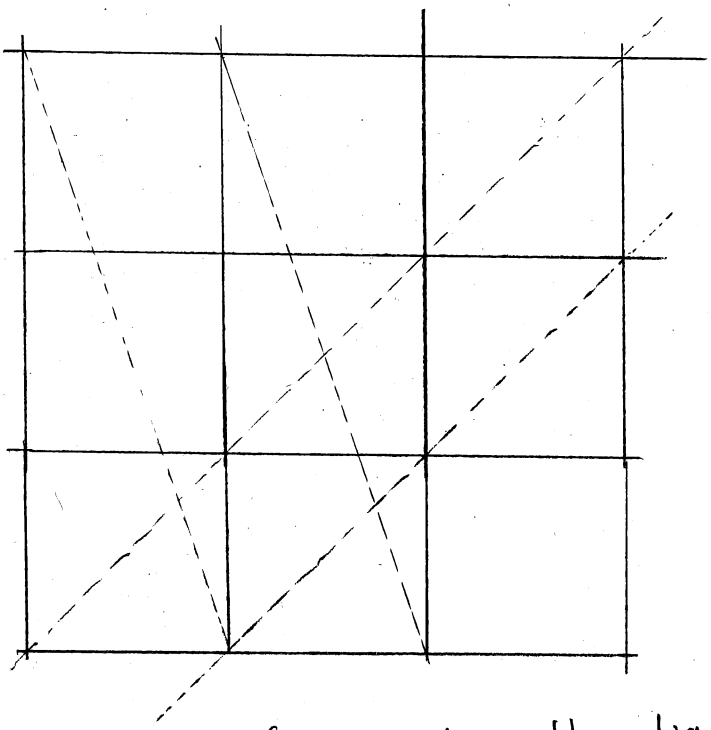
This seemingly intuitive fact can be used to prove non-trivial statements. But first, let's review how to draw an infinite plaza of rectangular tiles in perspective. Start by drawing the first tile in perspective.



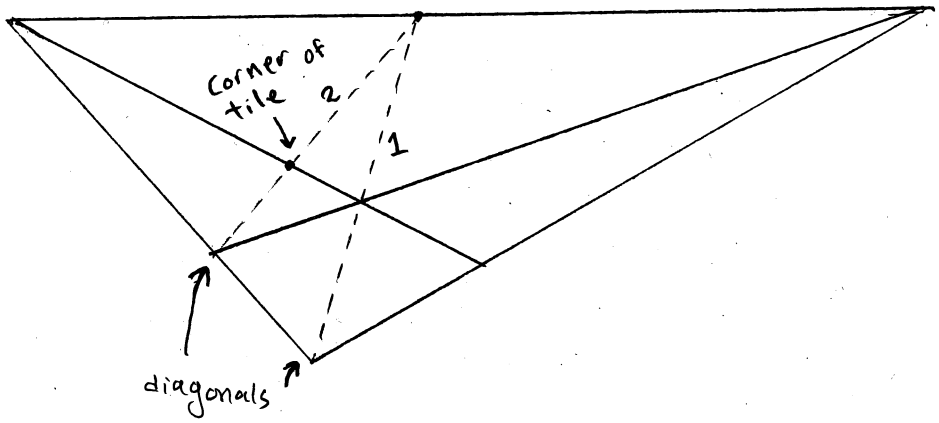
Each pair of opposite sides are parallel and therefore meet at the horizon. Extend them and draw the horizon.



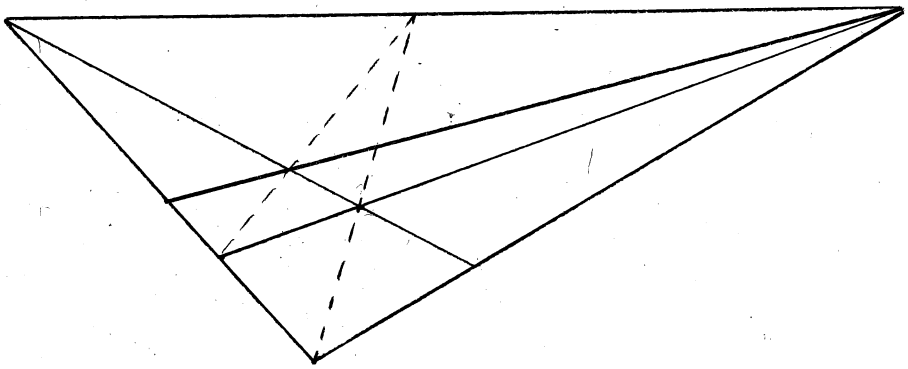
Our next task is to draw a second tile. Imagine looking straight down on our plaza.



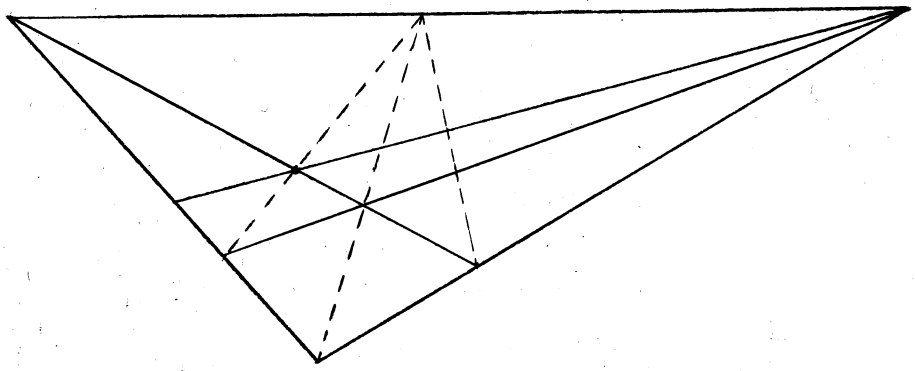
In addition to the sides of our tiles, the <u>diagonals</u> which intersect the tile corners will also be parallel to each other! We can use these diagonals to find the corners of the other tiles.



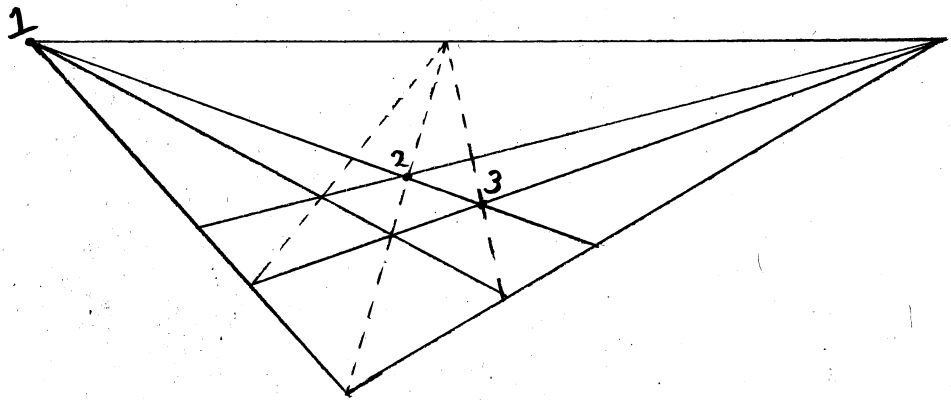
First draw line 1 above, which passes between the two opposite corners of the firs tile. Stop when you hit the horizon. Then draw line 2, connecting that horizon point to another corner of the first tile. Where line 2 intersects another line is the corner of our second tile.



* We will soon repeat this procedure and draw a third tile on the plane. Before we do that, though, let me discuss coincidences. A line is defined by two points. Therefore, if a line happens to pass through three pre-defined points, that is a "coincidence." In general, if three points lie on the same line, we say this "coincidence" is a non-trivial mathematical fact requiring a proof.



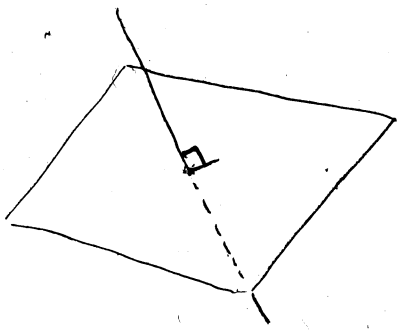
It is at this step that we reach our coincidence: three points in this diagram are collinear.



In the context of projective geometry, it is obvious that these three points should be collinear. However, if we view the diagram above as just some random construction in Euclidean geometry, this collinearity is not obvious at all. (It follows from something called Desargues' Theorem.) So we can see that thinking about the horizon helps us draw plazas as well as prove non-trivial statements.

Projective Duality

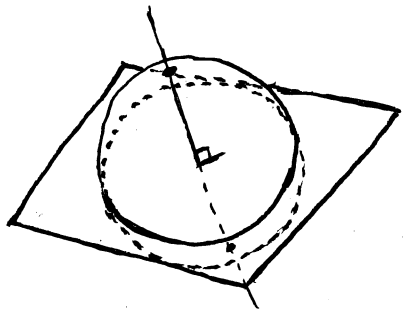
In $\mathbb{R}^3$, lines passing through the origin are "dual" to planes passing through the origin. A line and a plane are said to be dual if the line is normal to the plane.



Every line has one dual plane and every plane has one dual line. Also, the dual of the dual is just the original line/plane.

A line in $\mathbb{R}^3$ is a point in $\mathbb{RP}^2$ and a plane in $\mathbb{R}^3$ is a line in $\mathbb{RP}^2$. Therefore points and lines are dual in $\mathbb{RP}^2$.

Thinking of $\mathbb{RP}^2$ as S^2 with antipodal points identified, we can see that the equator is the line dual to the north/south pole.



Let us now quantitatively solve for the line dual to the point $(x_0 : y_0 : 1)$. This is the line in $\mathbb{R}^3$ given by $(\lambda x_0, \lambda y_0, \lambda)$ for $\lambda \in \mathbb{R}$. The plane dual to this line is given by all $(x,y,z) \in \mathbb{R}^3$ such that

$$(x,y,z) \cdot (x_0, y_0, 1) = 0.$$

Because we only care about projective space, we set $z=1$

$$(x,y,1) \cdot (x_0, y_0, 1) = 0$$

We can parameterize the plane $(x,y,1)$ with $\xi, \eta \in \mathbb{R}$ as

$$x = \frac{-\xi}{x_0 \xi + y_0 \eta} \qquad y = \frac{-\eta}{x_0 \xi + y_0 \eta}.$$

Our goal now is to write y as a function of x in $y = ax + b$ form. we rewrite the above equations as

$$x = \frac{-1}{x_0 + y_0 \frac{\eta}{\xi}} \qquad y = \frac{-1}{x_0 \frac{\xi}{\eta} + y_0}.$$

We use the first equation to solve for $\frac{\eta}{\xi}$.

$$\frac{\eta}{\xi} = \frac{-1}{Y_0}\left(\frac{1}{X} + X_0\right)$$

Plugging $\frac{\eta}{\xi}$ into the equation for y and simplifying,

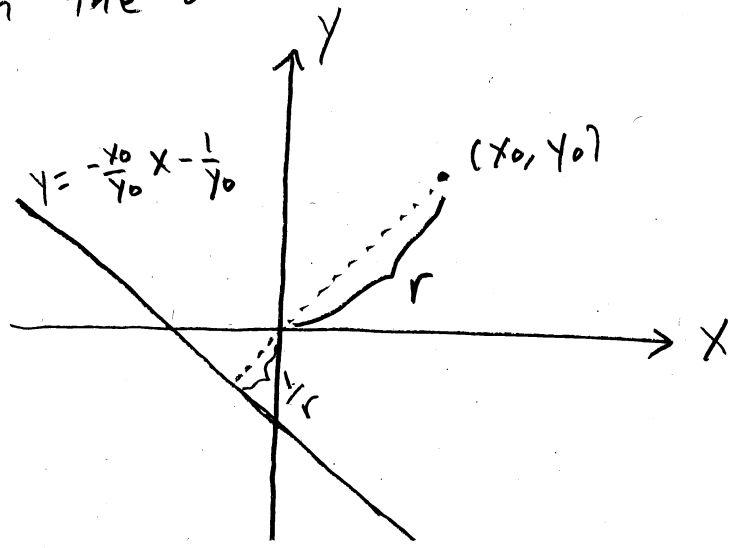
$$y = -\frac{X_0}{Y_0}X - \frac{1}{Y_0}.$$

So

$$(X_0 : Y_0 : 1) \xrightarrow{\text{dual line}} \left(X : -\frac{X_0}{Y_0}X - \frac{1}{Y_0} : 1\right)$$

$$(X : aX + b : 1) \xrightarrow{\text{dual point}} \left(\frac{a}{b} : -\frac{1}{b} : 1\right)$$

Here is how to visualize dual lines/points. Say (X_0, Y_0) has a distance of r from $(0,0)$. Then the dual line has a minimum distance of $1/r$ from the origin, and lies on the other side of the origin as the original point:



The really cool thing about projective duality is that it inverts <u>incidence relations</u>. We say a point and a line are "incident" if the point lies on the line. It turns out that if a point and line are incident, then the line dual to the point is incident with the point dual to the line!

line: $y = ax + b$ $\xrightarrow{\text{dual point}}$ $\left(\frac{a}{b}, -\frac{1}{b}\right)$

point: (x_0, y_0) $\xrightarrow{\text{dual line}}$ $y = -\frac{x_0}{y_0}x - \frac{1}{y_0}$

point on line:

$$y_0 = ax_0 + b$$

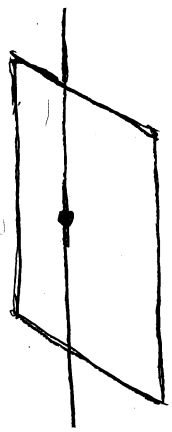
point on line!

$$-\frac{1}{b} = -\frac{x_0}{y_0}\frac{a}{b} - \frac{1}{y_0}$$

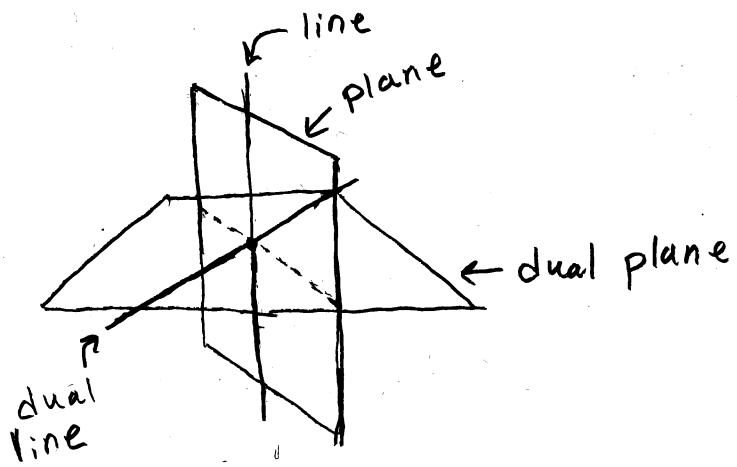
$$-\frac{1}{b} = -\frac{1}{y_0}\left(x_0\frac{a}{b} + 1\right)$$

$$y_0 = ax_0 + b \quad \checkmark$$

* There is another way to see that duality inverts incidence relations by thinking about lines and planes in $\mathbb{R}^3$. Picture a line lying on a plane.

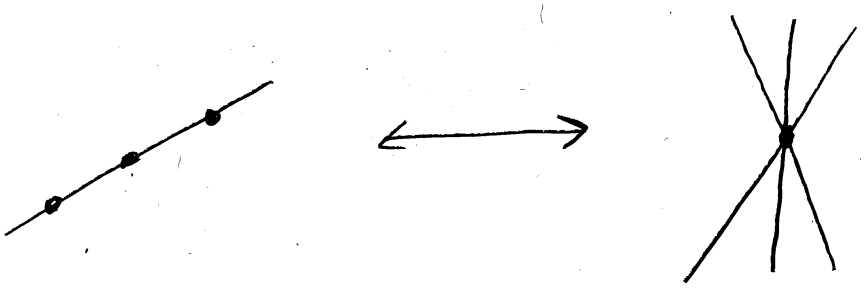


Now picture the dual plane and dual line.



If your visualization skills are developed enough you should be able to see that the dual line will have to lie on the dual plane.

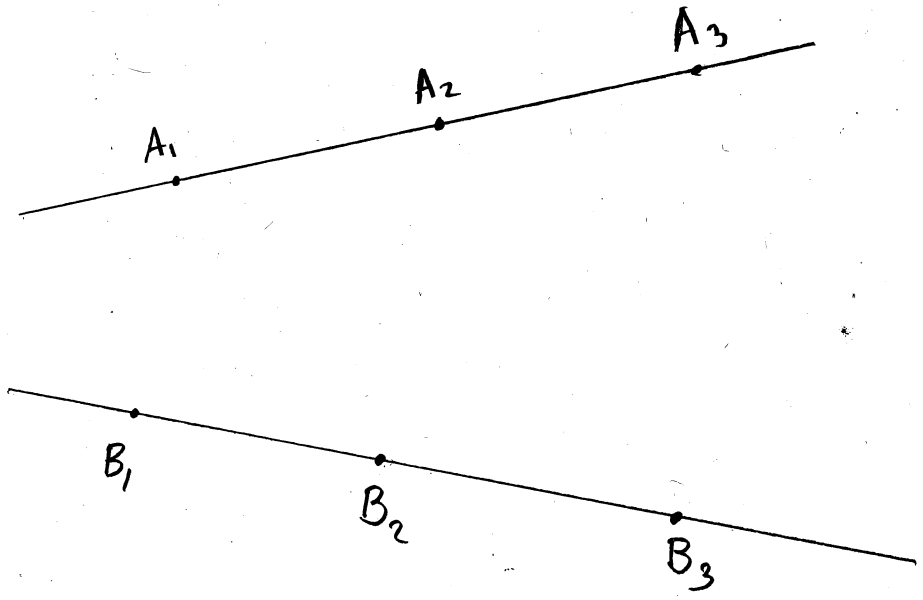
Finally, note that the dual of collinear points are concurrent lines.



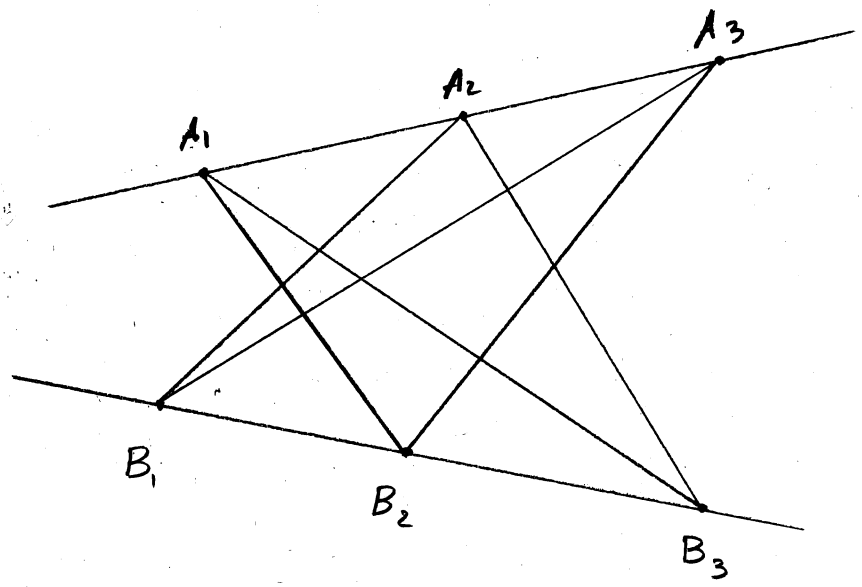
Dual Theorems

Projective duality allows you immediately dualize "incidence theorems," which are theorems of the form "If these points and these lines are incident then these other points are incident to these other lines." To dualize a theorem, simply replace all instances of the word "point" with "line" and vice versa. Statements like "for any two lines, there is one point incident to both" are dual to "for any two points, there is one line incident to both."

An example of an incidence theorem is Pappus' theorem. First, draw collinear points A_1, A_2, A_3 and B_1, B_2, B_3:



Then draw all lines $\overline{A_i B_j}$ for all $i \neq j$.



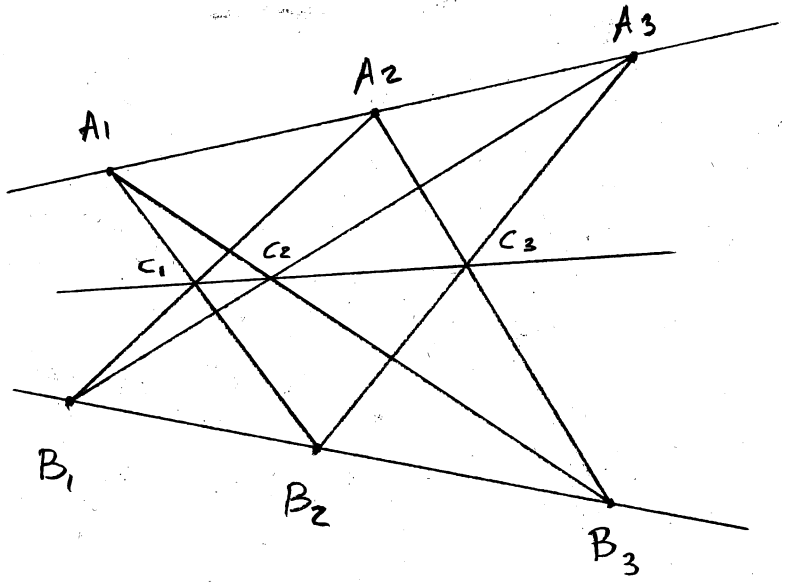
Name the three points

$$C_1 = \overline{A_2 B_3} \cap \overline{A_3 B_2}$$

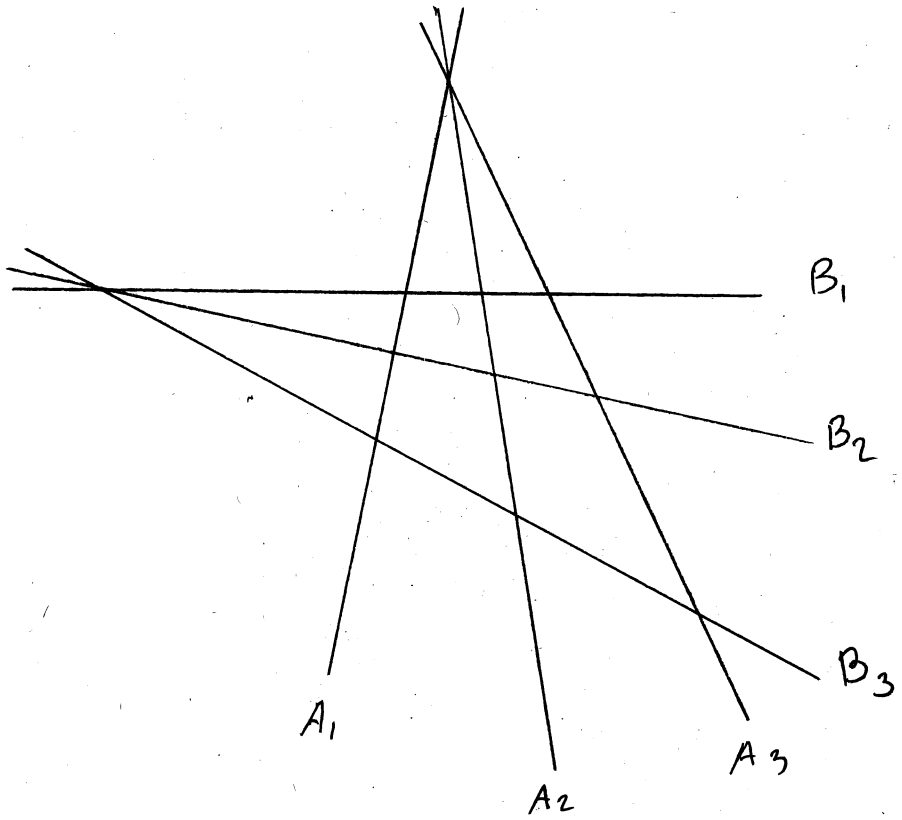
$$C_2 = \overline{A_1 B_3} \cap \overline{A_3 B_1}$$

$$C_3 = \overline{A_1 B_2} \cap \overline{A_2 B_1}.$$

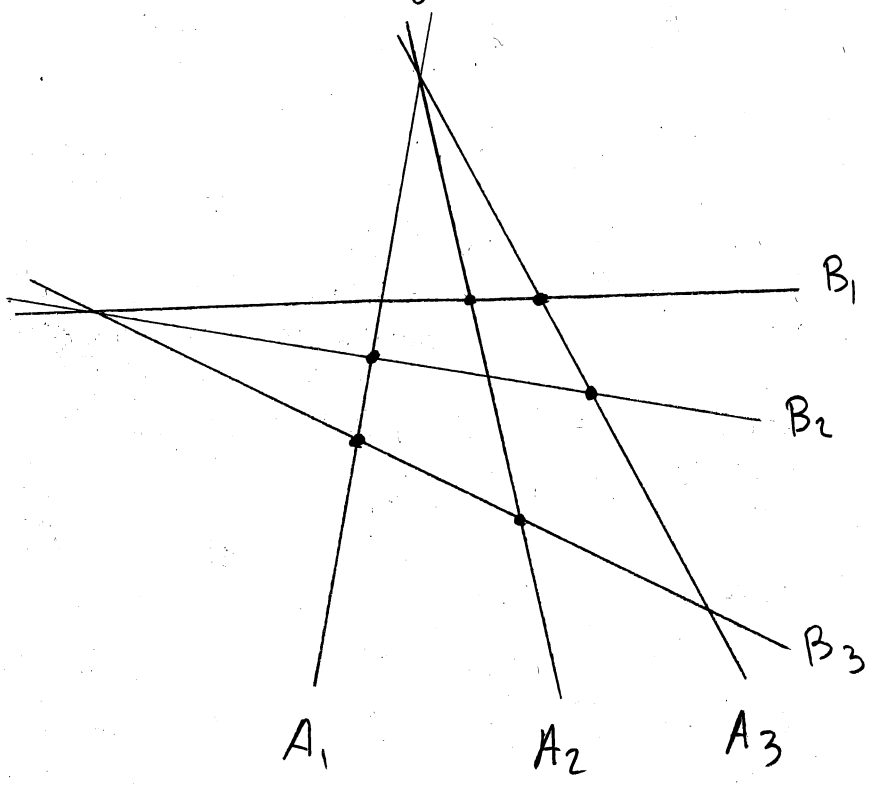
Pappus' Theorem states C_1, C_2, C_3 are collinear:



For the dual of Pappus' Theorem, draw concurrent lines A_1, A_2, A_3 and B_1, B_2, B_3:



Now label all points $A_i \cap B_j$ for $i \neq j$.



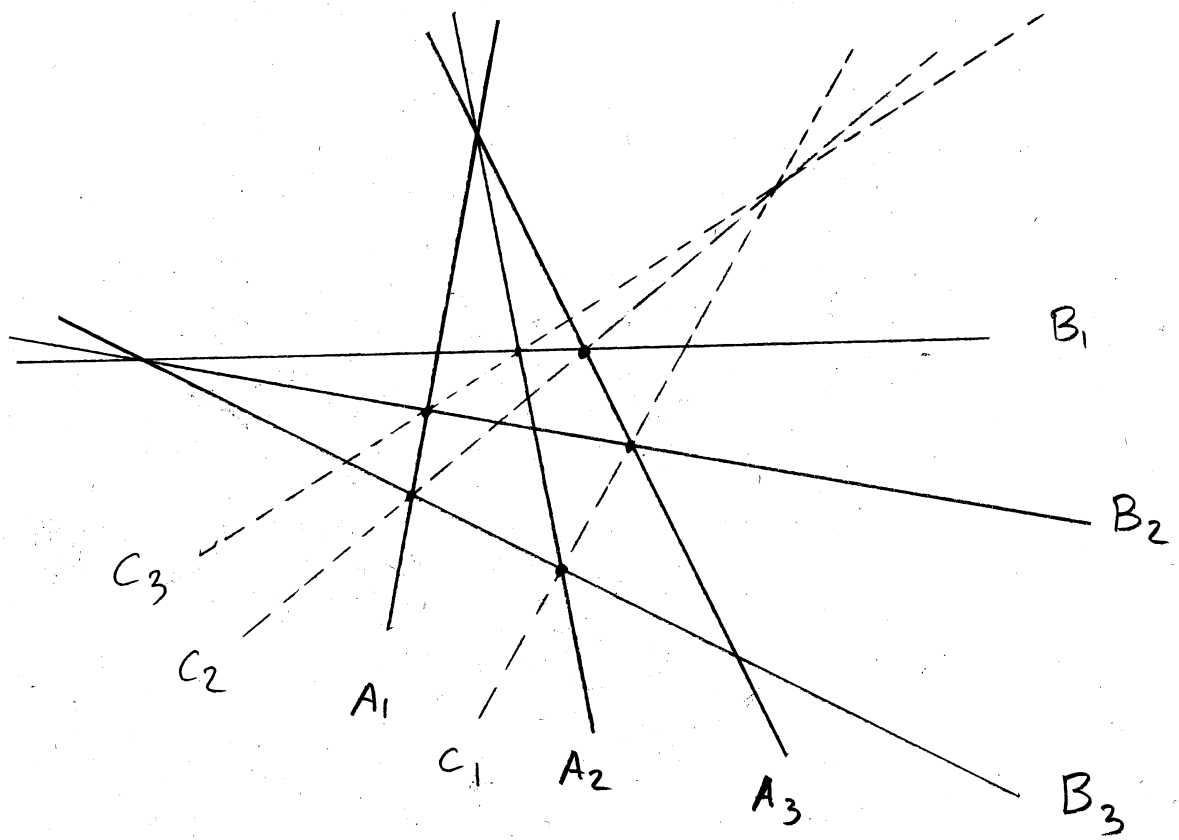
Draw the three lines

$$C_1 = \overline{(A_2 \cap B_3)(A_3 \cap B_2)}$$

$$C_2 = \overline{(A_1 \cap B_3)(A_3 \cap B_1)}$$

$$C_3 = \overline{(A_1 \cap B_2)(A_2 \cap B_1)}$$

The dual to Pappus' Theorem states these three lines are concurrent:

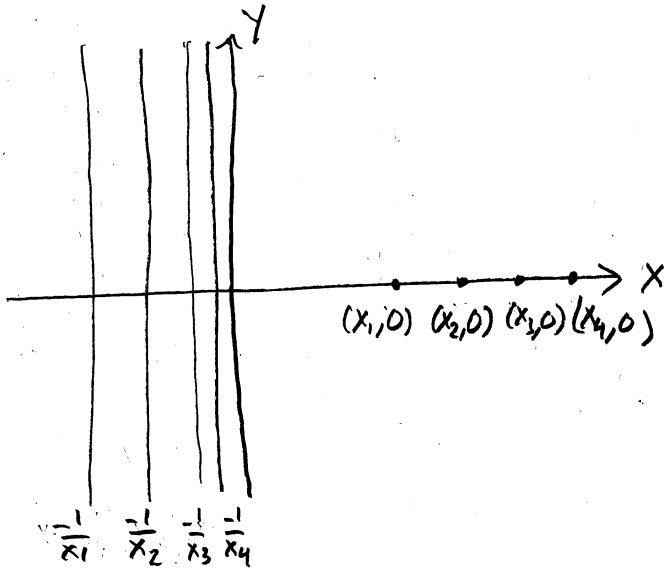


Another example of dual theorems are Pascal's Theorem and Brianchon's Theorem. Desargues' Theorem, mentioned earlier, is dual to its converse.

Projective Duality and the Cross Ratio

The cross ratio can be defined for four collinear points as well as for four concurrent lines. Perhaps you could guess that the cross ratio of four collinear points is equal to the cross ratio of the four dual concurrent lines.

Let us show this for an easy configuration of points. Say you have four points at $(x_1, 0), (x_2, 0), (x_3, 0), (x_4 0)$. The four dual lines are $x = -\frac{1}{x_1}$, $x = -\frac{1}{x_2}$, $x = -\frac{1}{x_3}$, $x = -\frac{1}{x_4}$.



(The four "parallel" lines are concurrent at infinity.)

$$\text{Cross ratio of points} = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)}$$

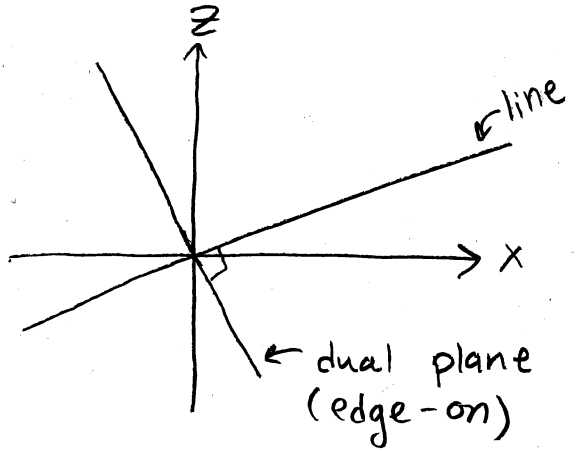
$$\text{Cross ratio of lines} = \frac{\left(\frac{-1}{x_1} - \frac{-1}{x_3}\right)\left(\frac{-1}{x_2} - \frac{-1}{x_4}\right)}{\left(\frac{-1}{x_1} - \frac{-1}{x_2}\right)\left(\frac{-1}{x_3} - \frac{-1}{x_4}\right)} = \frac{\left(\frac{1}{x_1} - \frac{1}{x_3}\right)\left(\frac{1}{x_2} - \frac{1}{x_4}\right)}{\left(\frac{1}{x_1} - \frac{1}{x_2}\right)\left(\frac{1}{x_3} - \frac{1}{x_4}\right)} \cdot \frac{x_1 x_2 x_3 x_4}{x_1 x_2 x_3 x_4}$$

$$= \text{Cross ratio of points} \checkmark$$

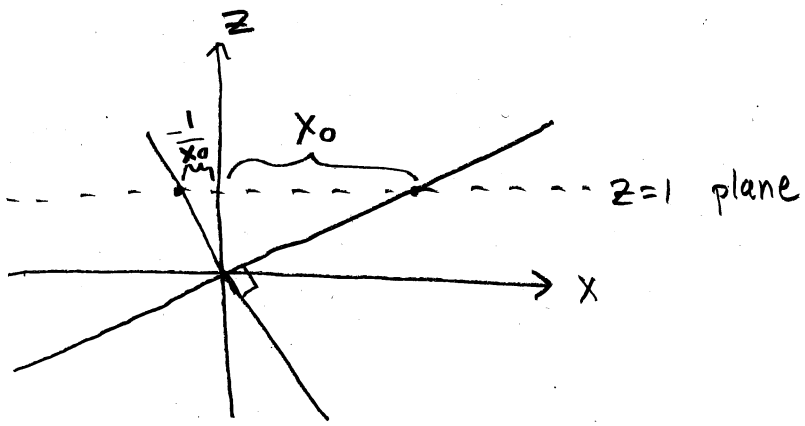
Actually, every configuration of four collinear points can be placed on the x-axis by some projective transformation, so proving it for this special case proves it in general.

There is another way to visualize this. Imagine a line in $\mathbb{R}^3$ and its dual plane.

Edge-on, it looks like this:



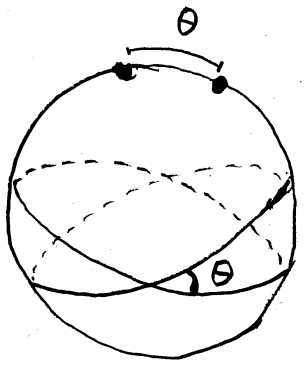
Intersecting with the $z=1$ plane, the line at $(x_0:0:1)$ can be seen to be dual to the plane $(\frac{-1}{x_0}:y:1)$.



But we can see that the 90° angle between the line and the plane is just the inversion map in $\mathbb{RP}^1$ from the beginning of these notes! Note that the map $x_0 \to -\frac{1}{x_0}$ is just an inversion. The cross ratio of the dual configurations are then equal because the cross ratio is invariant under inversion.

We can see that projective duality is closely related to inversion.

There is one more way to visualize why the cross ratio is invariant under projective duality, courtesy of Alek Bedroya. Simply note that the angle between two points on S^2 is equal to the angle between their dual great circles.

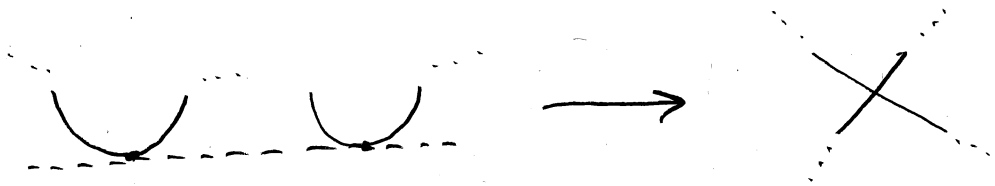


Therefore the cross ratio of four points is equal to the cross ratio of the four dual great circles.

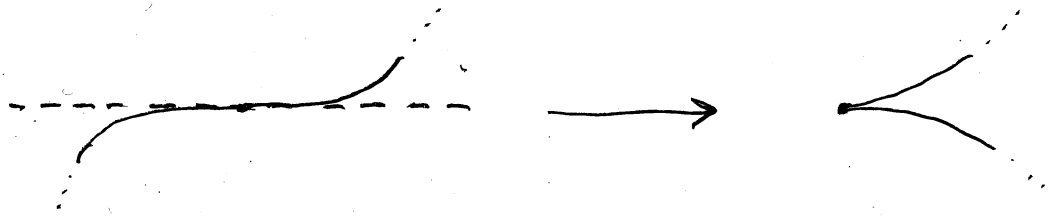
Dual Curves

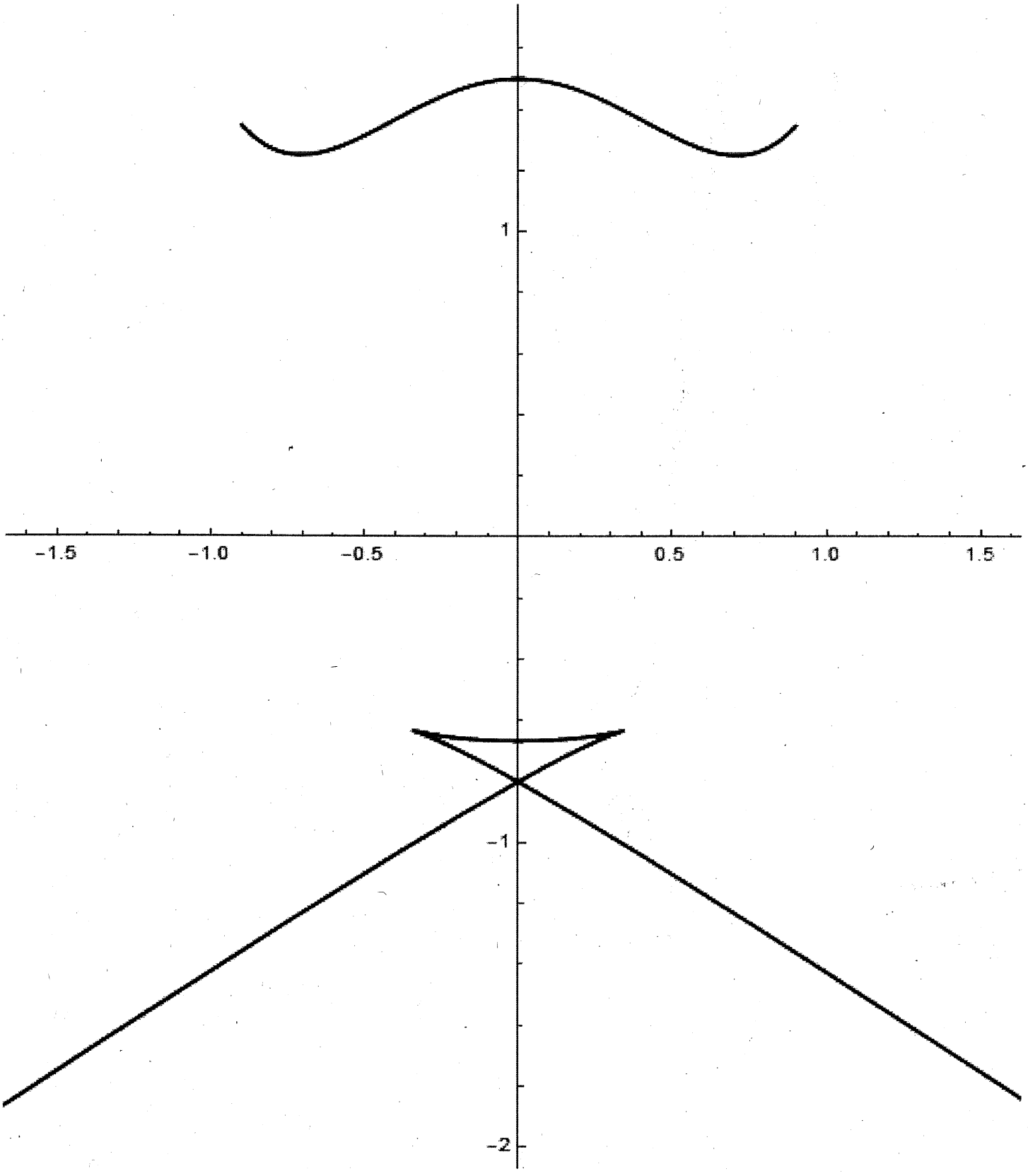
There is a more general notion of projective duality for any curve in $\mathbb{RP}^2$. At every point on the curve is a tangent line. The point dual to that tangent line is a point on the dual curve.

Dual curves in general look very different from each other. For example, a curve with a double tangent line gets mapped to a curve with a self intersection.



Furthermore, an inflection point gets mapped to a cusp.





Dual Curves

Let us find an expression for a dual curve in general. Say a curve is parameterized by

$$(X(t) : Y(t) : 1).$$

The line tangent to $(X(t), Y(t))$ is

$$Y = \frac{Y'(t)}{X'(t)}(X - X(t)) + Y(t)$$

$$= \frac{Y'(t)}{X'(t)} X + \left(Y(t) - X(t)\frac{Y'(t)}{X'(t)}\right).$$

Recall the point dual to $y = ax + b$ is $\left(\frac{a}{b} : -\frac{1}{b} : 1\right)$, which is in this case

$$\left(\frac{Y'(t)}{Y(t)X'(t) - X(t)Y'(t)} : \frac{-X'(t)}{Y(t)X'(t) - X(t)Y'(t)} : 1\right)$$

It is not at all obvious from the above expression, but the dual curve of a dual curve is the original curve. This is surprising because the dual curve of $(X(t) : Y(t) : 1)$ involves first and second derivatives $X'(t), Y'(t), X''(t), Y''(t)$, but miraculously these terms cancel out.

```
In[1]:= X[t_] := y'[t] / (y[t]*x'[t] - x[t]*y'[t])

In[2]:= Y[t_] := -x'[t] / (y[t]*x'[t] - x[t]*y'[t])

In[3]:= FullSimplify[Y'[t] / (Y[t]*X'[t] - X[t]*Y'[t])]

        FullSimplify[-X'[t] / (Y[t]*X'[t] - X[t]*Y'[t])]

Out[3]= x[t]

Out[4]= y[t]
```

I will now present a more enlightening proof of this claim.

Say $\vec{X}_1(t)$ is a curve in $\mathbb{R}^3$. Because we really only care about $\mathbb{RP}^2$, the magnitude $|\vec{X}_1(t)|$ is unimportant, as long as it is not zero.

$\vec{X}_1(t)$ and its velocity vector $\vec{X}_1'(t)$ span a plane in $\mathbb{R}^3$. This is called the "osculating plane," the plane which "kisses" the curve at the point $\vec{X}_1(t)$.

Let us call $\vec{X}_2(t)$ the curve dual to $\vec{X}_1(t)$. It is the unique curve (up to scaling) which is normal to the osculating plane.

Therefore, $\vec{X}_2(t)$ satisfies

$$\vec{X}_1(t) \cdot \vec{X}_2(t) = 0$$
$$\vec{X}_1'(t) \cdot \vec{X}_2(t) = 0 .$$

In fact, these are the defining equations of the dual curve $\vec{X}_2(t)$.

We will now show that $\vec{X}_1(t)$ is dual to $\vec{X}_2(t)$.

Simply differentiate the first equation:

$$\frac{d}{dt}\left(\vec{X}_1(t) \cdot \vec{X}_2(t)\right) = 0$$

$$\Rightarrow \underbrace{\vec{X}_1'(t) \cdot \vec{X}_2(t)}_{0} + \vec{X}_1(t) \cdot \vec{X}_2'(t) = 0$$

$$\Rightarrow \vec{X}_1(t) \cdot \vec{X}_2'(t) = 0$$

Because we also have $\vec{X}_1(t) \cdot \vec{X}_2(t) = 0$, this implies $\vec{X}_1(t)$ is the dual curve of $\vec{X}_2(t)$.

□