The Schwarzian Derivative in 2D CFT

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September 25, 2019

These are the handwritten lecture notes for a talk I gave to the Harvard grad student theory seminar on Monday September 23, 2019. I offered to give the intro to 2D CFT lecture, but then as a joke I said I wouldn't give it, instead giving a talk titled "This is (not) an Introduction to 2D CFT." I tried to avoid all the standard material usually contained in an intro to 2D CFT lecture.

I began by discussing projective space $\mathbb{R}P^1$ ("modding out" by the scaling transformation so as to avoid CFT) which naturally lead to a discussion of $SL(2,\mathbb{R})$ transformations and the cross ratio. I then discussed the $SL(2,\mathbb{C})$ Möbius transformations on $\mathbb{C}P^1$, which are isomorphic to the Lorentz transformations $SO(3,1)$ because $\mathbb{C}P^1$ can be thought of as the "celestial sphere." While thinking about special relativity, I introduced the 1+1 D conformal group as the "causality preserving" coordinate transformations, and then discussed the 2+0 D conformal group for good measure, which are just the holomorphic functions. (However, the only globally defined ones are the Möbius transformations.)

I then introduced the main topic of discussion, the Schwarzian derivative $S(f)$, as an infinitesimal change in the cross ratio, meaning it measures how much f deviates from being a Möbius transformation. I then gave three ways to understand the Schwarzian intuitively, (1) as the rate of change of the best approximating Möbius transformation to f at z, (2) as measuring how small circles around z get mapped to small ellipses, and (3) as a "ridge system" or "finger print" arising from the fact that the Schwarzian is best understood as a quadratic differential operator.

I then briefly discussed the Witt algebra as holomorphic vector fields, although in these notes I also discuss them as vector fields on $\mathbb{R}P^1 \cong S^1$. I then introduced the central extension, making the Virasoro algebra. (For compact Lie groups, these are given by the second de Rham cohomology of the group, which provides fun visualizations for some groups but is not relevant to the Virasoro algebra.) My main point was that the central extension $\frac{c}{12}(n^3-n)\delta_{m+n,0}$ can be thought of as the infinitessimal Schwarzian derivative. The CFT stress energy tensor $T_{zz}(z)$ transforms with a Schwarzian because of this term.

I always thought the appearance of $S(f)$ in the transformation of T was very bizarre, and wanted to find a physical reason for this. I think I gave a somewhat satisfying explanation in that the second derivative of the two point function $\langle\phi(z_1)\phi(z_2)\rangle = -\ln(|z_1-z_2|^2)$ gives an infinitesimal cross ratio, and therefore the appearance of $S(f)$ is natural. The Schwarzian appears as a result of normal ordering with respect to a certain coordinate system.

Throughout the notes I list "Noah's personal confusions" (or NPCs) as avenues for further thought on the topics enclosed.

In preparing these notes I used multiple sources, listed in order of importance here:

1. "Zippers and univalent functions." William P. Thurston, 1986.

2. `mathoverflow.net/a/38125/125254`
   Bill Thurston's answer to "Is there an underlying explanation for the magical powers of the Schwarzian derivative?"

3. *Visual Complex Analysis.* Tristan Needham, 1997. (Especially chapter 3.)

4. *Projective differential geometry old and new: from Schwarzian derivative to cohomology of diffeomorphism groups.* V. Ovsienko, S. Tabachnikov, 2005.

5. "The topology of ridge systems." Roger Penrose, 1979.

6. *A Mathematical Introduction to Conformal Field Theory.* Martin Schottenloher, 2008.

7. "EPFL Lectures on Conformal Field Theory in $D >= 3$ Dimensions." Slava Rychkov, 2016. `arxiv.org/abs/1601.05000`

8. *SPINORS AND SPACE-TIME Volume 2: Spinor and twistor methods in space-time geometry.* Roger Penrose, Wolfgang Rindler, 1986.

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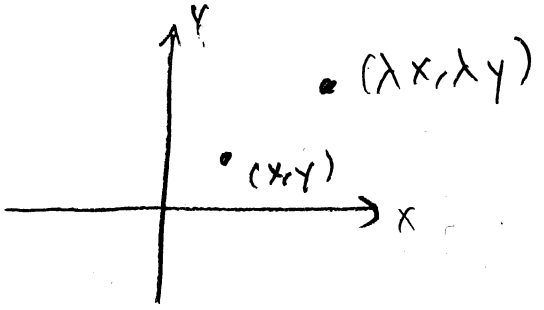
This is (not) an Introduction to 2D CFT

Noah Miller
September 2019

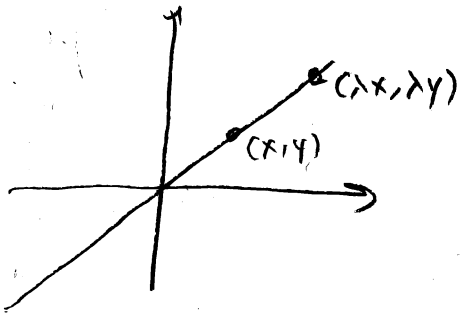
Consider scaling transformations of the 2D plane.



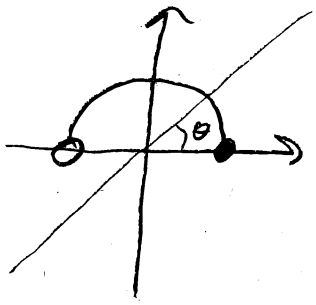
Because this is not an introduction to 2D CFT, lets rid ourselves of this awful transformation. We accomplish this by modding out by scaling.

$(x,y) \sim (\lambda x, \lambda y)$ if $\lambda \neq 0$, $(x,y) \neq (0,0)$ is an equivalence relation.

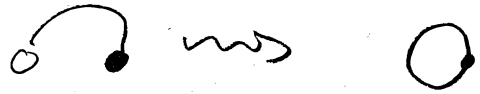
$\mathbb{RP}^1 \equiv \mathbb{R}^2 - \{0\}/\sim$ is "real projective space."



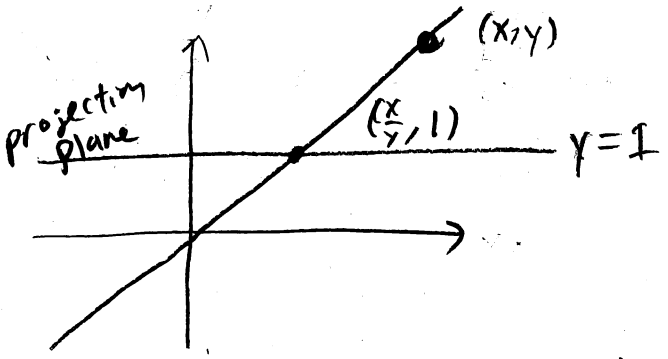
It is the space of lines through the origin.



Every line passes through the semi-circle on the left exactly once. Note we should glue together the ends of it, which are $\theta \in [0, \pi)$.



So topologically $\mathbb{RP}^1 \simeq S^1$, the circle.



The line passing through (x,y) intercepts the $y=1$ axis at $\frac{1}{y}(x,y) = (\frac{x}{y}, 1)$.

Note that the line $(1,0)$ never intercepts this line. Therefore $(1,0)$ is the "point at infinity."

We can act on these lines by 2×2 matrices.

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \sim \begin{pmatrix} \frac{ax+by}{cx+dy} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a\frac{x}{y}+b}{c\frac{x}{y}+d} \\ 1 \end{pmatrix}$$

We can also think of this as the action

$$x/y \longrightarrow \frac{a\,x/y + b}{c\,x/y + d}.$$

Note that an overall scaling of the matrix has no effect on this transformation, so we might as well take $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. This is the group $SL(2,\mathbb{R})$, and it has a natural action on $\mathbb{RP}^1$.

It has 3 degrees of freedom. We now prove that we can always find an $SL(2,\mathbb{R})$ matrix that sends any three points in $SL(2,\mathbb{R})$ to any other three points.

Proof: Choose any three distinct vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which span the first set of lines in $\mathbb{RP}^1$. (They are not defined up to scaling.) Say we want to map them to the lines spanned by $\vec{v}_1', \vec{v}_2', \vec{v}_3'$.

Because $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are distinct, we can find $w, z \neq 0$ such that

$$\vec{v}_3 = w\vec{v}_1 + z\vec{v}_2.$$

Likewise, we can find $w', z' \neq 0$ such that

$$\vec{v}_3' = w'\vec{v}_1' + z'\vec{v}_2'$$

Now rescale $\vec{v}_1, \vec{v}_2, \vec{v}_1'$ and $\vec{v}_2'$ such that

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2$$

$$\vec{v}_3' = \vec{v}_1' + \vec{v}_2'.$$

Next, note that there is always exactly one matrix which sends

$$\vec{v}_1 \mapsto \vec{v}_1' \quad \text{and} \quad \vec{v}_2 \mapsto \vec{v}_2'.$$

This matrix also sends

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 \mapsto \vec{v}_1' + \vec{v}_2' = \vec{v}_3'.$$

□

How can we picture these transformations?

The x-coordinate of the intersection of the line spanned by $\begin{bmatrix} x' \\ y' \end{bmatrix}$ with the $y=1$ axis is found by solving for x in the equation

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \end{bmatrix}.$$ (we don't care what λ is)

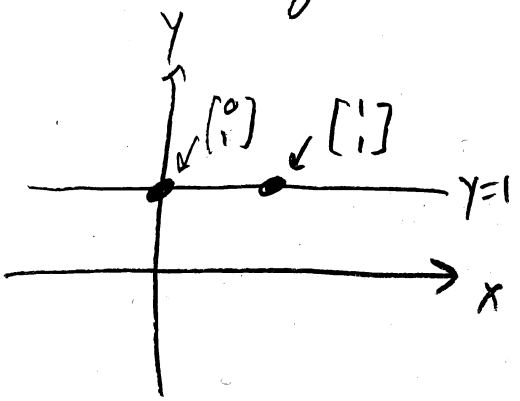
When we act on $\begin{bmatrix} x' \\ y' \end{bmatrix}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ this becomes

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

However, this is also just

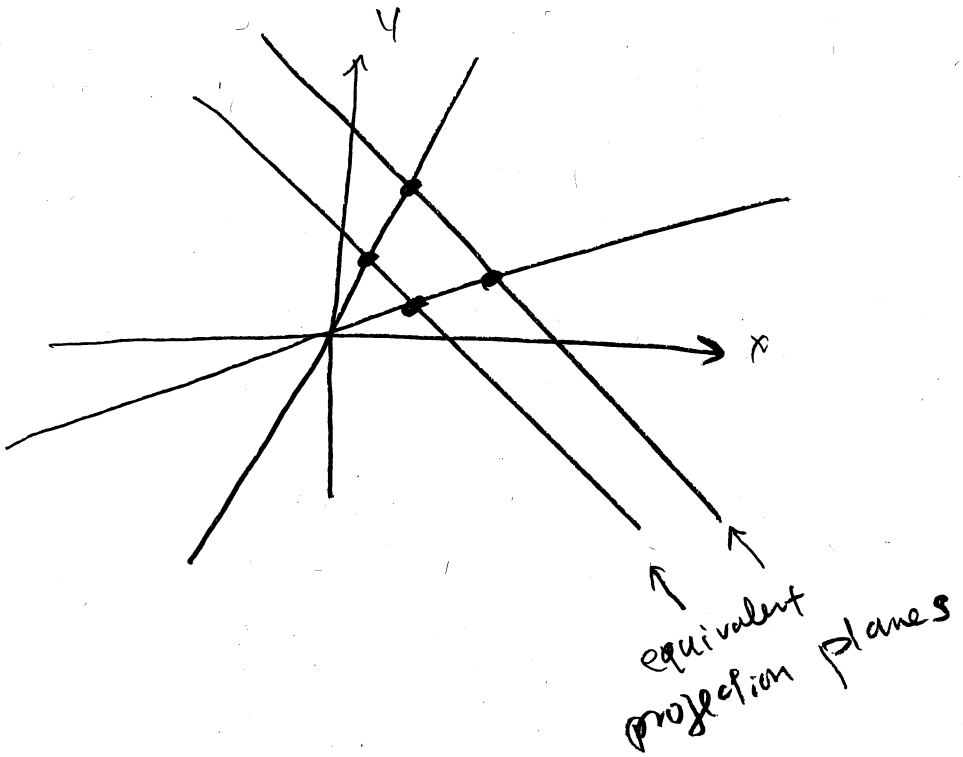
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \end{bmatrix}$$

So we can imagine acting the <u>inverse matrix</u> on $\begin{bmatrix} x \\ 1 \end{bmatrix}$. In other words, these transformations can be thought of as acting on the $y=1$ axis.



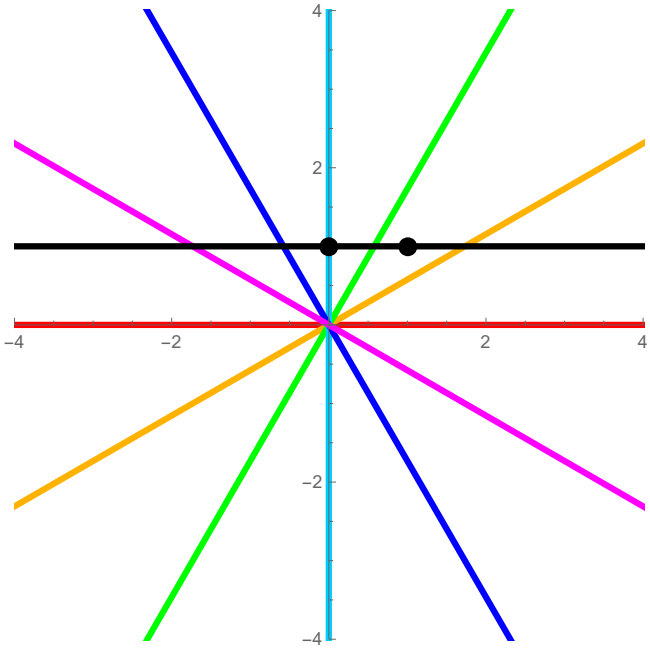
In the pictures on the following pages I denote the points $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with black dots.

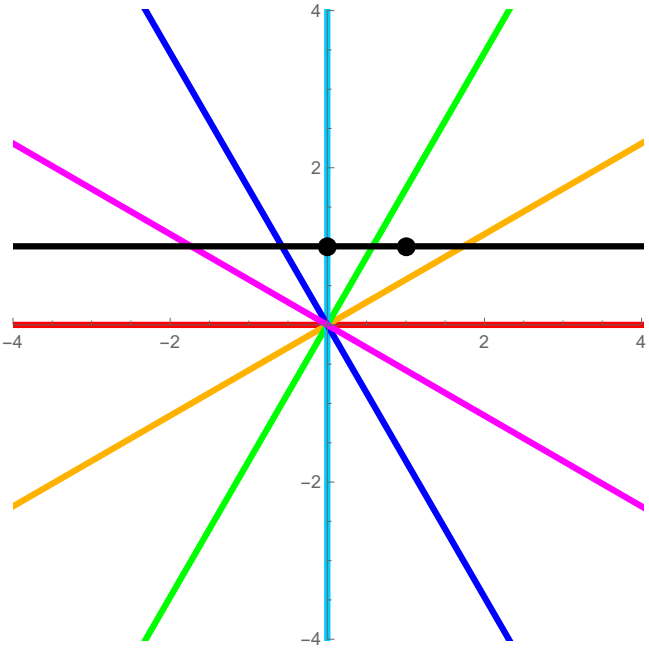
We can therefore think about "grabbing" the two black dots and moving around the "projection plane" passing through them. This suggests we have 4 degrees of freedom, not 3. The discrepancy comes from the fact that there is a scaling equivalence of projection planes.



Identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

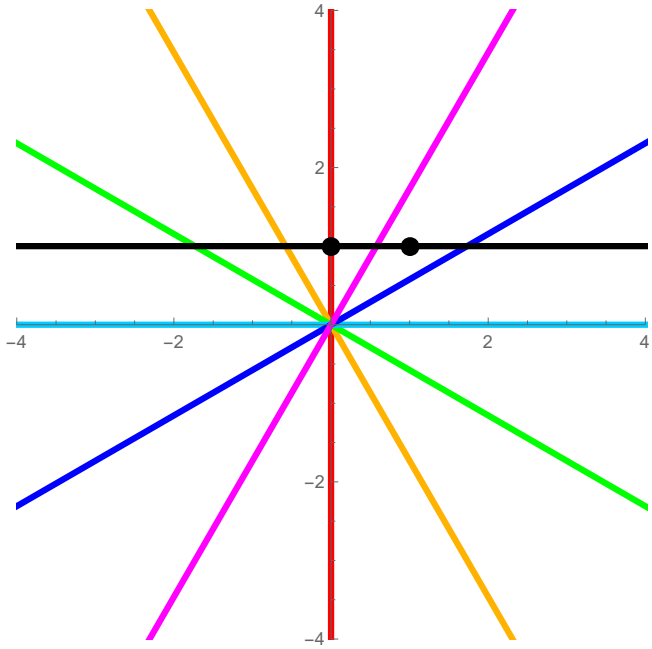


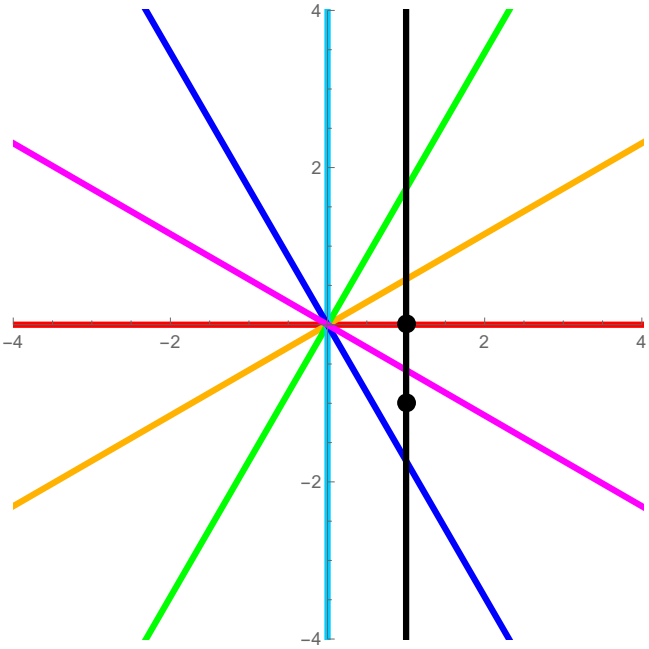


For pictures on the left we act on the lines directly. On the right, we act on the line $y = 1$ by the inverse matrix. The red line is the "point at infinity" ($x = \infty$).

Inversion:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

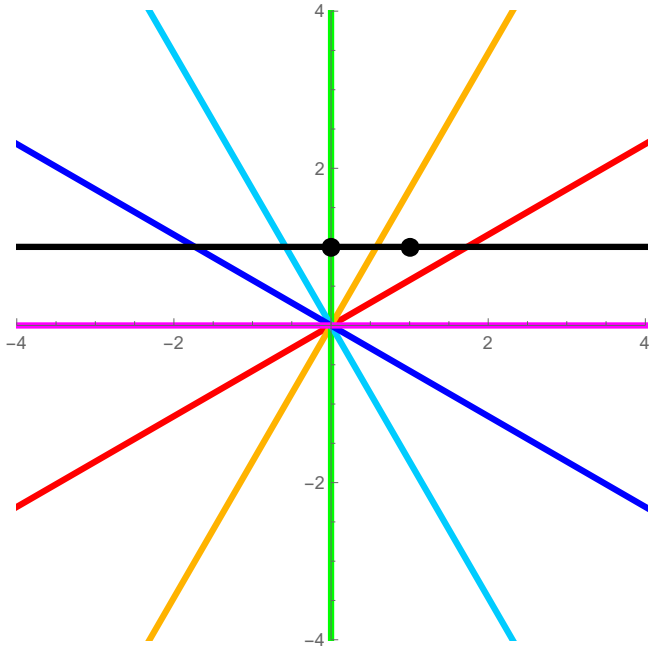


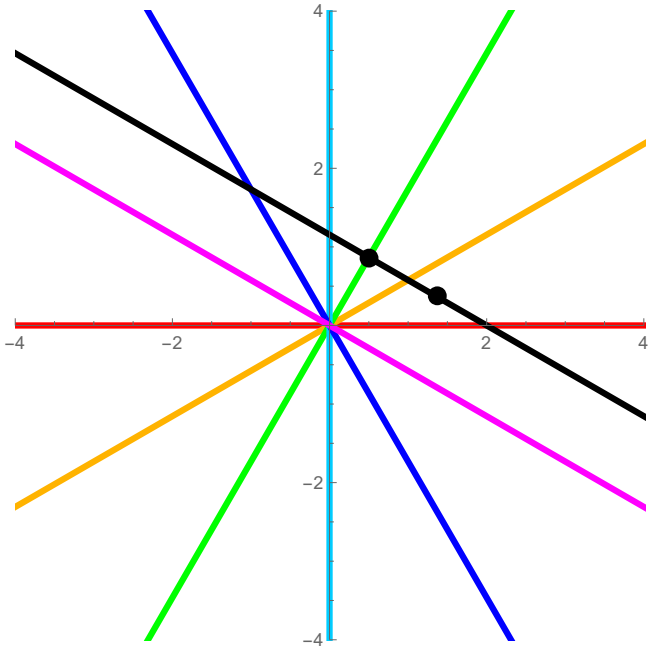


Note that inversion flips the cyan and red lines, A.K.A. 0 and ∞.

Rotation:

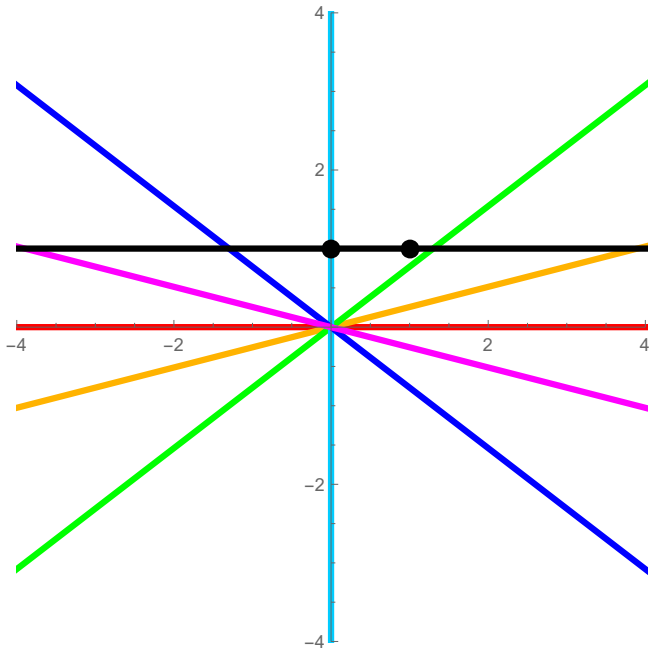
$$\begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix}$$

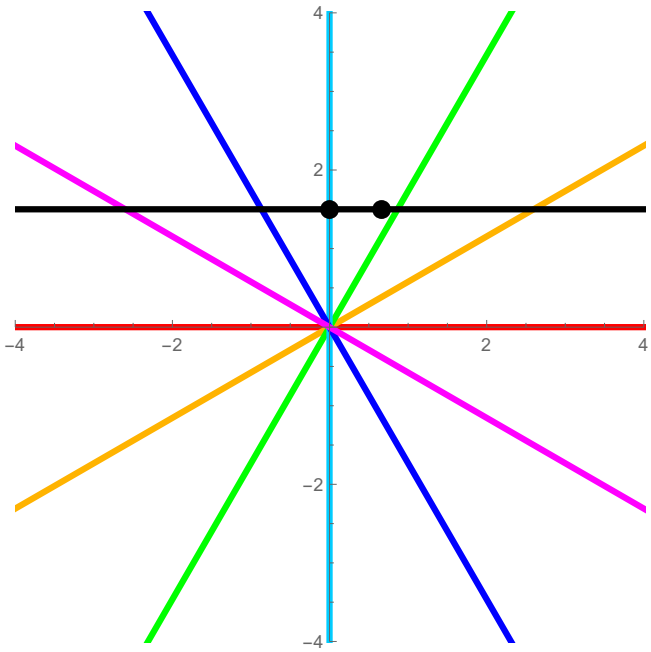




Scale:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1/1.5 \end{pmatrix}$$

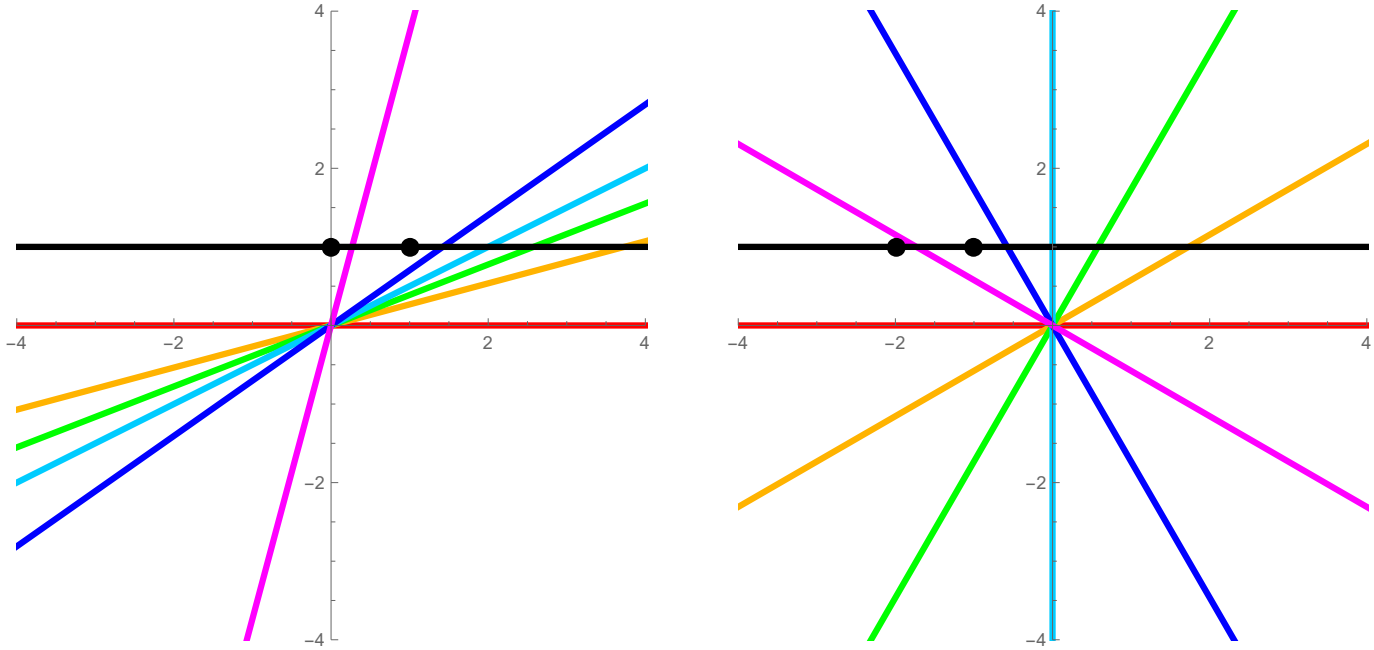




Scaling changes the distance of the projection plane $y = 1$ from the origin and changes the distance between the $x = 0$ dot and the $x = 1$ dot.

Shear 1:

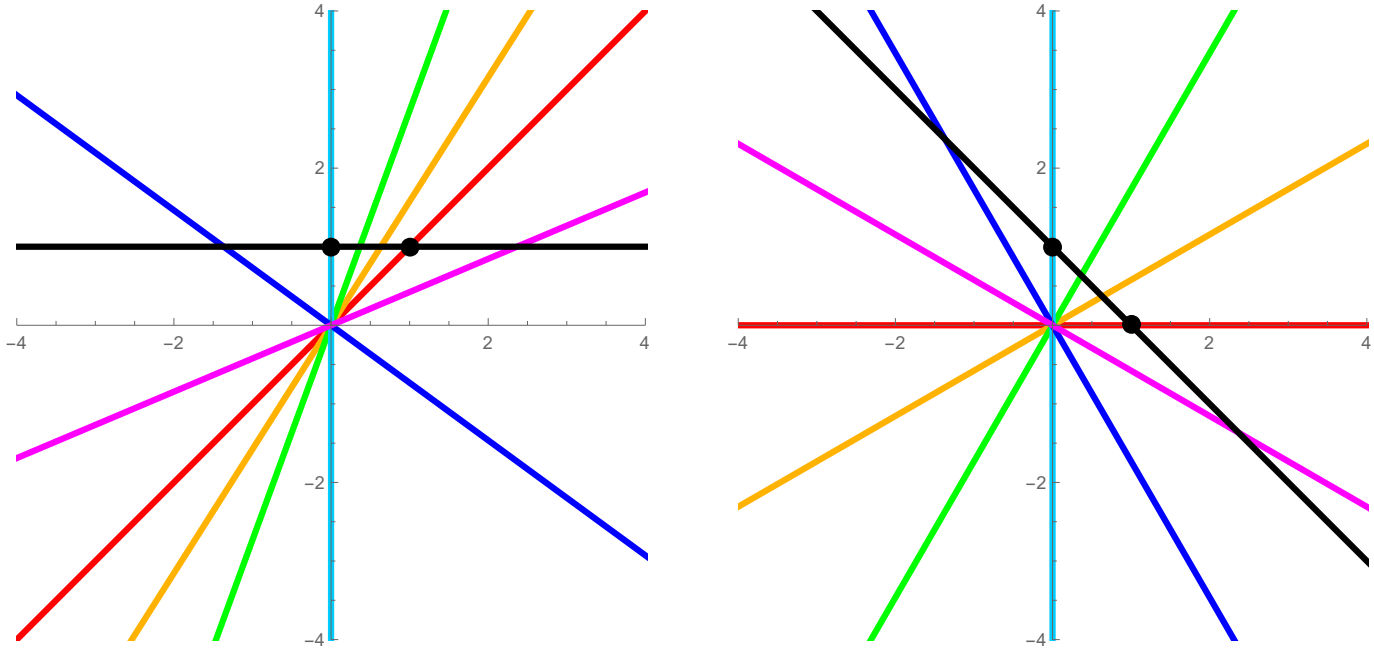
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$



This type of shearing translates the projection plane. Note that the red line, $x = \infty$, is fixed.

Shear 2:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



This sort of shering translates the y coordinates of the $x = 1$ dot. Note that the cyan line ($x = 0$) is fixed but the red line ($x = \infty$) is not.

The cross ratio

The cross ratio of four points x_1, x_2, x_3, x_4 (which should be thought of as the lines passing through $\begin{bmatrix} x_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ 1 \end{bmatrix}, \begin{bmatrix} x_3 \\ 1 \end{bmatrix}, \begin{bmatrix} x_4 \\ 1 \end{bmatrix}$) is

$$[x_1, x_2; x_3, x_4] = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)}$$

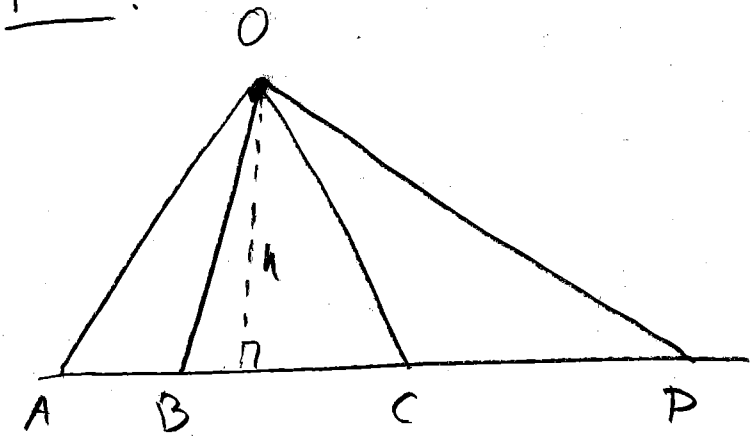
(other conventions are just functions of each other)

Miraculously, if $M(x) = \frac{ax+b}{cx+d}$

$$[M(x_1), M(x_2); M(x_3), M(x_4)] = [x_1, x_2; x_3, x_4]$$

so it is a projective invariant.

Proof:



Area of $\Delta AOC = \frac{1}{2} h \cdot AC$

Area of $\Delta BOD = \frac{1}{2} h \cdot BD$

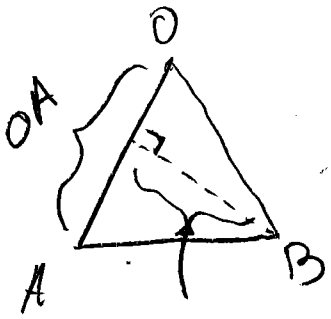
Area of $\Delta AOB = \frac{1}{2} h \cdot AB$

Area of $\Delta COD = \frac{1}{2} h \cdot CD$

$$\text{Cross ratio} = \frac{AC \cdot BD}{AB \cdot CD} = \frac{\text{Area of } \Delta AOC \cdot \text{Area of } \Delta BOD}{\text{Area of } \Delta AOB \cdot \text{Area of } \Delta COD}$$

However, there is another way to rewrite the areas

For instance



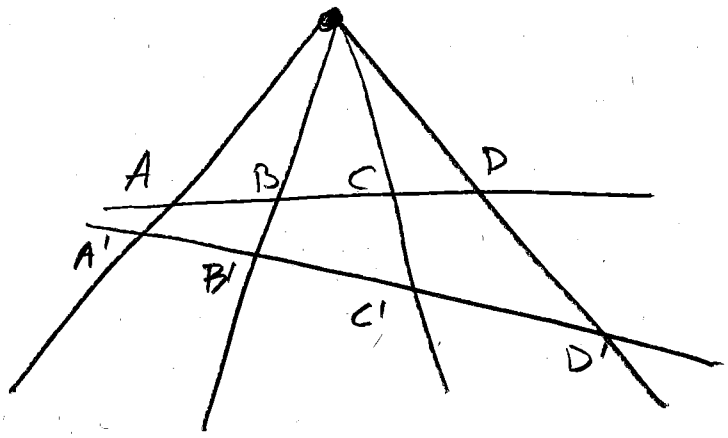
$OB \cdot \sin(\angle AOB)$

Area of $\Delta AOB = \frac{1}{2} OA \cdot OB \cdot \sin(\angle AOB)$

So

$$\text{Cross ratio} = \frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle AOB)\sin(\angle COD)} \underbrace{\frac{\frac{1}{2}OA\cdot OC \cdot \frac{1}{2}OB\cdot OD}{\frac{1}{2}OA\cdot OB \cdot \frac{1}{2}OC\cdot OD}}_{1}$$

So the cross ratio only depends on the angles between the lines intersecting O and not the fifth line! Moving around the fifth line as we please (i.e. an $SL(2,\mathbb{R})$ transformation) we see that



$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}.$$

□

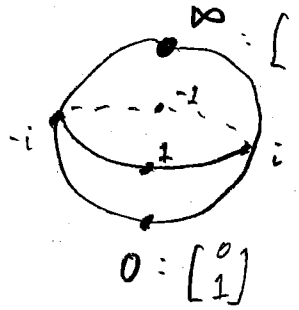
$\mathbb{C}P^1$

What happens if we use complex numbers instead of real numbers?

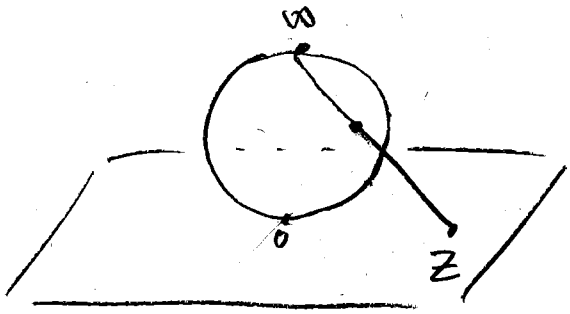
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim \lambda \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad (z_1, z_2) \in \mathbb{C}^2 - \{0\}$$
$$\lambda \in \mathbb{C}, \quad \lambda \neq 0$$

$$\mathbb{C}P^1 \equiv \mathbb{C}^2 - \{0\} / \sim$$

Topologically, $\mathbb{C}P^1 \cong S^2$ "the Riemann Sphere"



we can map it to $\mathbb{C}$ by a stereographic projection.



★ Noah's Personal confusion 1 (NPC 1): Is there a way to understand the stereographic projection as being on the same footing as the "projection plane" $Y=1$ in the real RP^1 case?

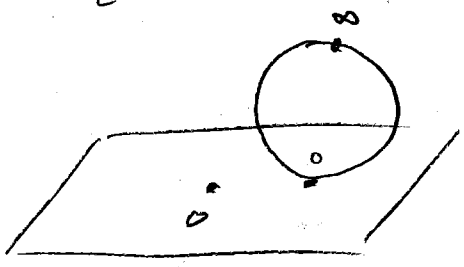
The group $SL(2,\mathbb{C})$ give Möbius transformations

$$z \longmapsto \frac{az+b}{cz+d}.$$

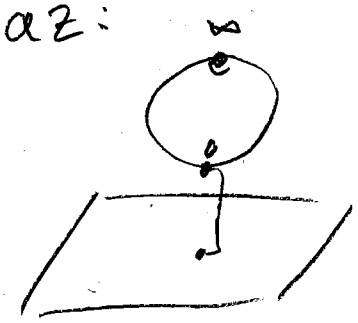
They can be understood in one way by moving around the Riemann sphere before projecting onto the plane:

6 d.o.f. of rigid motions!

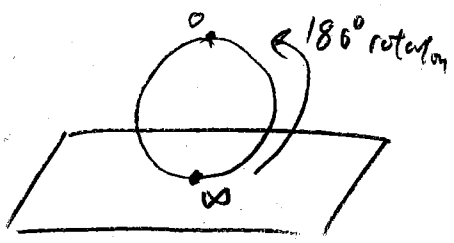
$z+c$:



az:



$-\frac{1}{z}$:



* NPC 2: Can these be understood straight from the definition of $\mathbb{CP}^1$ like we did for $\mathbb{RP}^1$, where $SL(2,\mathbb{R})$ moved around the projection plane?

We can also take the cross ratio of four points in $\mathbb{CP}^1$ just like we did in $\mathbb{RP}^1$.

Here's a great way to think about the cross ratio:

The cross ratio $[z, s; q, r)$ is the unique möbius transformation that sends

$q \mapsto 0$, $r \mapsto 1$, $s \mapsto \infty$

$$[z, s; q, r] = \frac{(z-q)(s-r)}{(z-s)(q-r)} = \frac{az+b}{cz+d} \quad \text{for some } a, b, c, d.$$

This makes the miraculous property

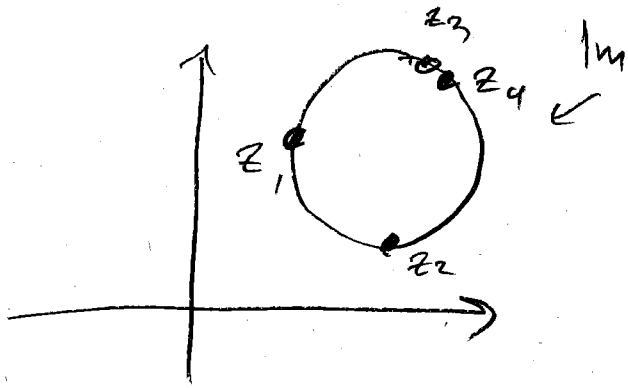
$$[M(z_1), M(z_2); M(z_3), M(z_4)] = [z_1, z_2; z_3, z_4]$$

seem natural.

- Furthermore, it can be (easily) proved that there is always exactly one Möbius transformation that sends any four points in $\mathbb{CP}^1$ to any other four points if and only if both groups of points have the same cross ratio.

- There are a lot of fun things that are easy to prove with the cross ratio.

For example, if the cross ratio of four pts is real (no imaginary part) then it maps the four points to $0, 1, \infty$, and another point on the real axis. It turns out that Möbius transformations map circles on $\mathbb{C}$ to other circles. (stereographic projection maps circles on the sphere to circles on the plane.) Therefore, four points in $\mathbb{C}$ all lie on a circle or a line (infinitely big circle) if and only if their cross ratio is real.



How can we picture generic möbius transformations? Note that the equation

$$\frac{az+b}{cz+d} = z$$

is a quadratic equation in z and has 2 solutions.

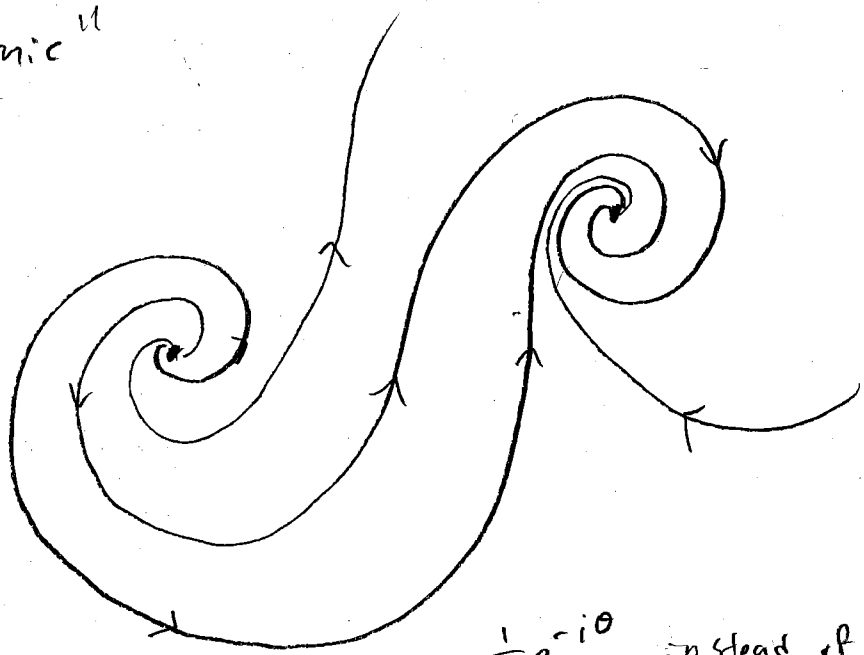
Thurlow Mobius transformations have 2 fixed points. The placement of these two fixed points accounts for four real degrees of freedom.

Around a fixed point z_0, a function looks like

$$f(z) \approx z_0 + \underbrace{re^{i\theta}}_{f'(z_0)} (z - z_0) + \dots$$

r and θ account for the final two degrees of freedom. Locally, around a fixed point, a mobius transformation acts as a scaling and a rotation.

"Loxodromic"

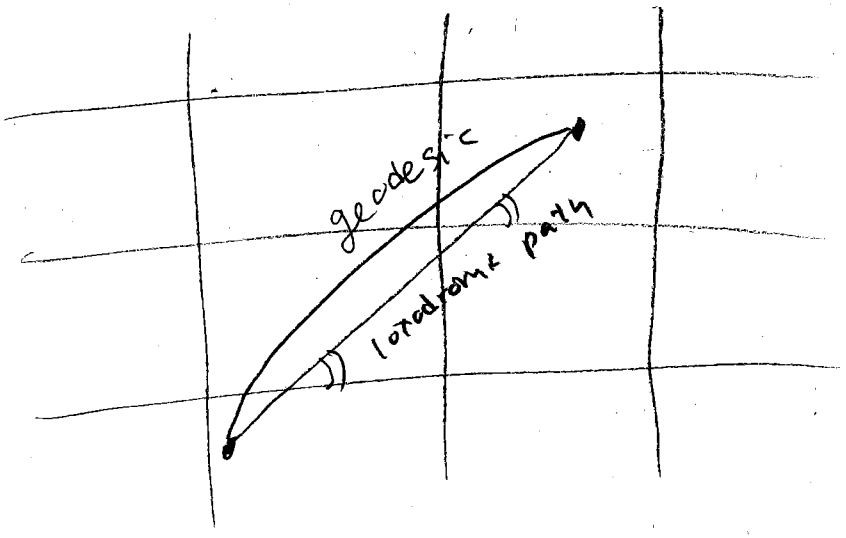


The other fixed point has $\frac{1}{r}e^{-i\theta}$ instead of $re^{i\theta}$.

Also $re^{i\theta}$ is the (squared) eigenvalue of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The name "loxodromic" comes from the method of loxodromic navigation, where you always travel at a constant angle with respect to lines of latitude

Mercator projection



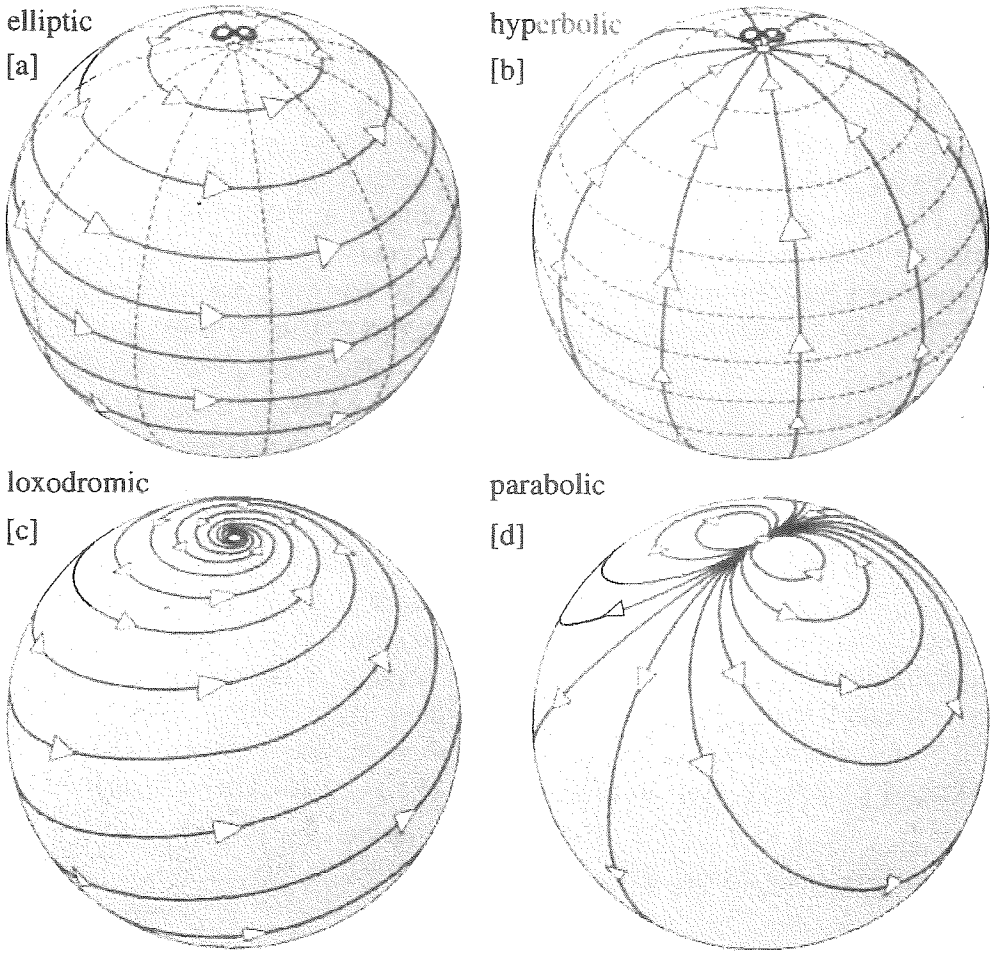


Figure [26]

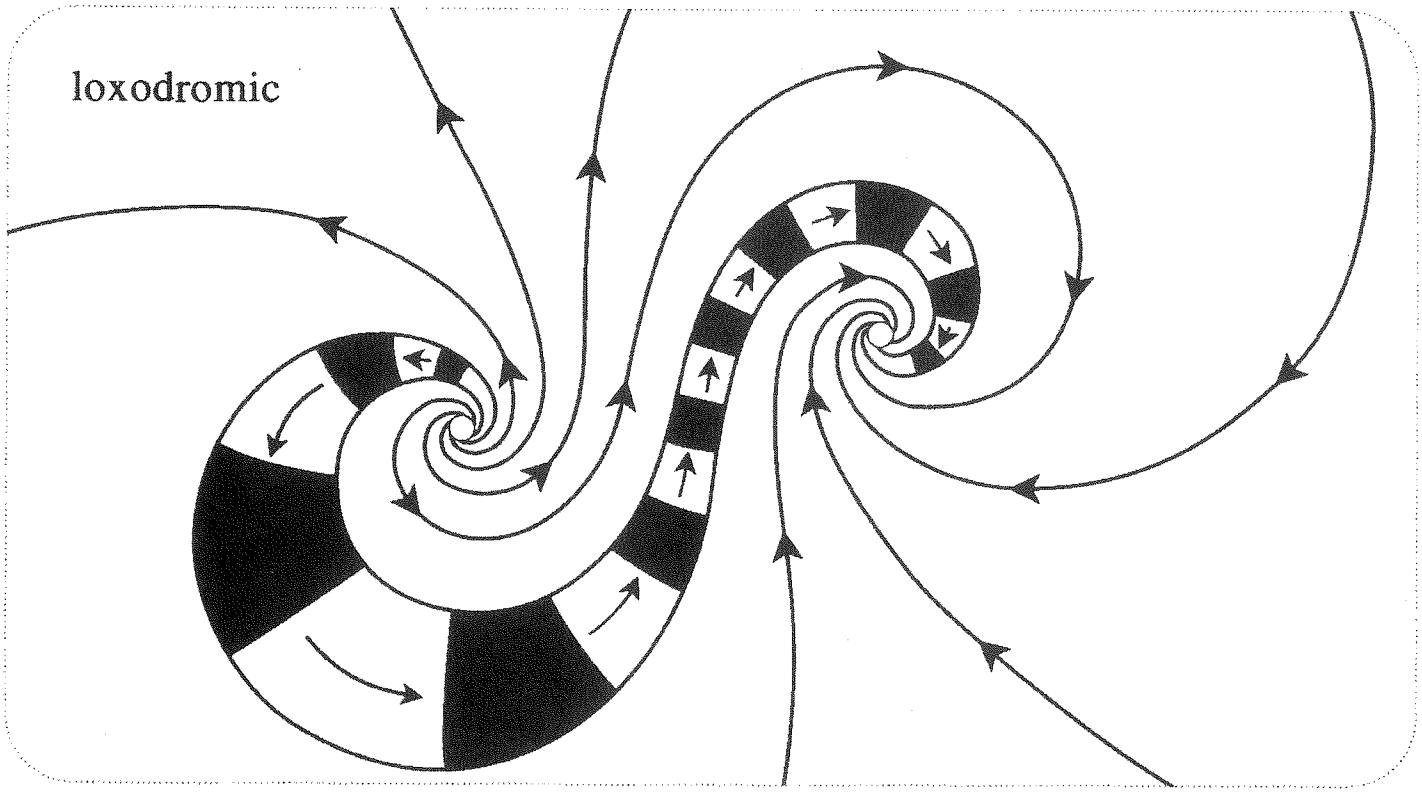
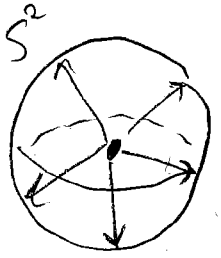


Figure [32]

$SL(2,\mathbb{C})$ and $SO(1,3)$

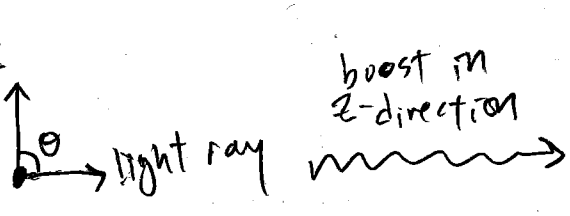
Note that Möbius transformations have 6 real degrees of freedom, just like Lorentz transformations in 3+1 dim! This is because Möbius transformations *are* Lorentz transformations!



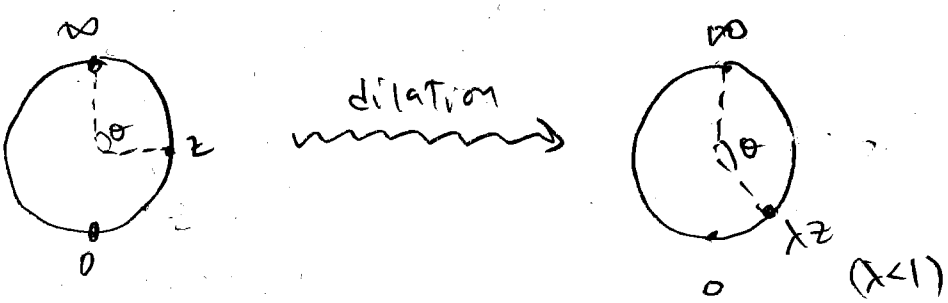
Imagine a flash of light at the origin releasing light in all directions. The light makes a sphere, with each photon defined by its angle on the sphere.

Under a Lorentz transformation, each light ray will be sent to another light ray on the sphere. After a stereographic projection, these are the Möbius transformations! For example, rotations are given by matrices where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary. (This is the Bloch sphere.)

Lets now investigate boosts. Intuitively, under a boost, the angle that light rays make with the z-axis will change.



This is just like a "dilation" $z \to \lambda z$ on the stereographic plane!



Let's look at this numerically. A boost with tiny "rapidity" ε along the z-direction acts as

$$\begin{pmatrix} 1 & 0 & 0 & \varepsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \varepsilon & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} 1+\varepsilon\cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta+\varepsilon \end{pmatrix}$$

Next we multiply vector by $\frac{1}{1+\varepsilon\cos\theta} = 1-\varepsilon\cos\theta$ to set $t=1$

mult by $1-\varepsilon\cos\theta$ $\rightsquigarrow$

$$\begin{pmatrix} 1 \\ (1-\varepsilon\cos\theta)\sin\theta\cos\phi \\ (1-\varepsilon\cos\theta)\sin\theta\sin\phi \\ (1-\varepsilon\cos\theta)(\cos\theta+\varepsilon) \end{pmatrix}$$

So $\sin\theta \mapsto \sin\theta(1-\varepsilon\cos\theta) = \sin\theta - \varepsilon\sin\theta\cos\theta$

$\cos\theta \mapsto (1-\varepsilon\cos\theta)(\cos\theta+\varepsilon) = \cos\theta + \varepsilon(1-\cos^2\theta)$

If $\theta \mapsto \theta+\delta$, then

$\sin\theta \mapsto \sin(\theta+\delta) = \sin\theta + \delta\cos\theta$

$\cos\theta \mapsto \cos(\theta+\delta) = \cos\theta - \delta\sin\theta$

$\Rightarrow \quad \delta = -\varepsilon\sin\theta \quad (*)$

Now lets look at the stereographic projection. A dilation is $z \mapsto e^{\lambda} z$, where $z = e^{i\phi}\cot(\theta/2)$. For tiny λ,

$$\cot(\theta/2) \mapsto (1+\lambda)\cot(\theta/2) = \cot(\theta/2) + \lambda\cot(\theta/2)$$

If $\theta \mapsto \theta+\delta$, then

$$\cot(\theta/2) \mapsto \cot(\theta/2) + \delta\frac{-1/2}{\sin^2(\theta/2)}$$

If $\lambda\cot(\theta/2) = \delta\frac{-1/2}{\sin^2(\theta/2)}$, $\quad \delta = -\lambda\sin\theta \quad (**)$

Using $(*)$ and $(**)$, we can see $\boxed{\varepsilon=\lambda}$, so for boosts along z-direction,

$$z \mapsto e^{\lambda} z = e^{(\text{rapidity})} z. \quad \checkmark$$

Actually, This correspondence between the conformal group and Lorentz transformations generalizes to other dimensions.

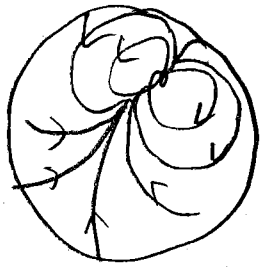
As shown in Chapter 2 of

"EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions" by Slava Rychkov, 1601.05000,

conformal transformations in signature (r, s) correspond to $SO(r+1, s+1)$. This can be seen as a jumping off point for Andy's program on Celestial correlators.

(Maybe the fact that $(2,0)$-signature CFT is special means $3+1$ bulk dimensions are special!)

Note that the parabolic Möbius Transformations



are the Lorentz transformations that fix one light ray. They are the "little group" transformations of that light ray.

Note that a "translation" $z+b$ is a parabolic transformation that fixes ∞ while special conformal transformations $\frac{az}{cz+d}$ are parabolic transformations that fix 0 (and send $\infty \mapsto \frac{a}{c}$)

Conformal Transformations (ah shit)

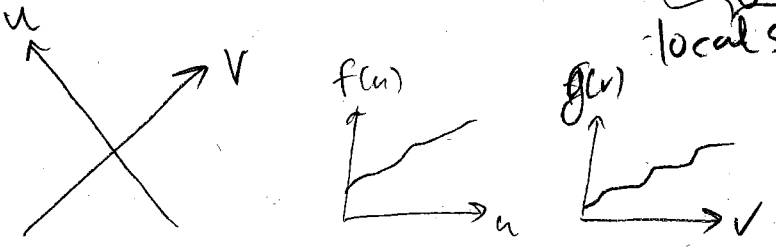
Lorentz transformations preserve causality. What are the most general transformations that preserve causality? They must preserve the sign of the metric, but can have an overall scaling. Lets see what happens in 1+1:

$$ds^2 = dt^2 - dx^2 \qquad u = t - x$$
$$v = t + x$$
$$= du\,dv$$

If we send $u \mapsto u' = f(u)$, where $\frac{\partial}{\partial v} f(u) = 0$
$v \mapsto v' = g(v)$ $\frac{\partial}{\partial u} g(v) = 0$

Then $ds^2 \mapsto df(u)\,dg(v) = f'(u)g'(v)\,du\,dv$

local scaling



(These act as two "accordion" squishes of u and v that preserve lightcones, 45° angles.)

So we have an infinite dimensional group of transformations, what about 2+0 signature?

$$ds^2 = dx^2 + dy^2 \qquad z = x + iy$$
$$= d\bar{z}\,dz \qquad \bar{z} = x - iy$$

If we send $z \mapsto z' = f(z)$ where $\frac{\partial}{\partial \bar{z}} f(z) = 0$
$\bar{z} \mapsto \bar{z}' = f(\bar{z})$ $\frac{\partial}{\partial z} f(\bar{z}) = 0$

Then $ds^2 \mapsto df(\bar{z})\,df(z)$
$= |f'(z)|^2\,d\bar{z}\,dz$

The reason why the conformal group is infinite dimensional in 1+1 and 2+0 is because this trick we used only works in 2 dim because the metric is a bilinear form!

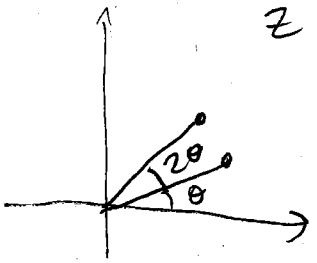
$\frac{\partial}{\partial \bar{z}} f(z)$ means f is holomorphic. This can be pictured as follows:

conformal transformations preserve angles but not lengths. Also note

$$f(z+\epsilon) = f(z) + \epsilon \underbrace{f'(z)}_{re^{i\theta}}$$

where $re^{i\theta}$ scales the tiny vector and rotates it, meaning it is conformal, preserving angles but not necessarily lengths! ✓

However, note that $f(z) = z^2$, is not conformal at $z=0$. This is because it doubles angles, $\theta \mapsto 2\theta$. So at critical points, $f'(z)=0$, f is not conformal.



(There is a difference between doubling the angle at $z=0$ and doubling the angle at the basepoint at any other z, which still preserves relative angles.)

What about the point at infinity in $\mathbb{CP}^1$?
Which conformal transformations are well defined at $z=\infty$?

To find out, call $w = \frac{1}{z}$.

$\delta w = -\frac{\delta z}{z^2}$. Therefore for δw to not blow up as $z \to \infty$,

$$\delta z = \alpha + \beta z + \gamma z^2$$

are the only terms you may have. This is a vector field with 6 real parameters. In fact they are infinitessimal Möbius transformations

$$\begin{pmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1+\varepsilon d \end{pmatrix} : z \mapsto z + \varepsilon(b + (a-d)z - c z^2)$$

so it's the Möbius transformations that are defined everywhere! Note $\frac{d}{dz}\frac{az+b}{cz-d} = \frac{ad-bc}{(cz+d)^2} \neq 0$. ✓

NPC 3: what happens with Wick rotation? How does a $1+1$ dim conf. trans. get mapped to a $2+0$ dim conf. trans.?

Schwarzian Derivative

For any holomorphic function f, take the cross ratio of $f(z)$, $f(z+\varepsilon)$, $f(z+2\varepsilon)$, $f(z+3\varepsilon)$. There is no $\mathcal{O}(\varepsilon)$ term, but there is a $\mathcal{O}(\varepsilon^2)$ term

$$[f(z), f(z+\varepsilon), f(z+2\varepsilon), f(z+3\varepsilon)]$$

← This is our definition of $S(f)$

$$= \underbrace{[z, z+\varepsilon, z+2\varepsilon, z+3\varepsilon]}_{4} - 2\varepsilon^2 S(f)(z)$$

where

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

$$= \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

$$= (\ln(f'))'' - \frac{1}{2}((\ln(f'))')^2$$

Is called the Schwarzian Derivative.

(To derive this you must taylor expand f to the third order)

Based on its definition we can immediately see

$$S(g\circ f) = (f')^2 S(g)\circ f + S(f)$$

This means that $S(f)$ is best thought of as a quadratic differential $S(f) = \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)(dz)^2$ in order to see the transformation property clearly.

Also from the definition, we can see

$$S(f) = 0 \quad \text{if and only if} \quad f = \frac{az+b}{cz+d}.$$

In other words, $S(f)$ measures the extent to which f deviates from being a Möbius transformation.

For an infinitessmal diffeomorphism $f(z) = z + \varepsilon h(z)$,

$$S(f) = \varepsilon h'''$$

and so will vanish for the tiny Möbius transformations given by $h = \alpha + \beta z + \gamma z^2$.

I will now give three ways to understand the Schwarzian derivative.

First way: $S(f)(z)$ is the rate of change of the best approximating Möbius transformation at z.

To find this best approx. Möbius transformation, note that

$$\partial_z \left(\frac{az+b}{cz+d} \right) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

$$\partial_z^2 \left(\frac{az+b}{cz+d} \right) = \frac{-2c(ad-bc)}{(cz+d)^3} = \frac{-2c}{(cz+d)^3}$$

Define $M(f,w)(z)$ to be the Möbius transformation that has the same value, first derivative, and second derivative as $f(z)$ at $z=w$. We then have four unknowns, a,b,c,d, and four equations, as $M(f,w)(z) = \frac{az+b}{cz+d}$.

(1) $ad-bc=1$, (2) $M(f,w)(w)=f(w)$, (3) $(\partial_z M(f,w))(w)=f'(w)$,

(4) $(\partial_z^2 M(f,w))(w)=f''(w)$

Solving this specifies $M(f,w)$. (This is done in Mathematica on the next page.)

Note that for any Möbius transformation $A(z)$,

$$M(A\circ f,w) = A\circ M(f,w).$$

We now define the "rate of change" of the best approximating Möbius transformation to be

$$\frac{d}{dt} M(f,w)^{-1}\circ M(f,w+t).$$

(NPC 4: Is this a connection in any way?)

We define it this way so that if $f\mapsto A\circ f$, our above quantity does not change, as

$$\frac{d}{dt} M(A\circ f,w)^{-1}\circ M(A\circ f,w+t) = \frac{d}{dt} M(f,w)^{-1}\circ \cancel{A^{-1}\circ A}\circ M(f,w+t)$$

This relates to the Schwarzian derivative as follows:

$$\frac{d}{dt} M(f,w)^{-1}\circ M(f,w+t) = \frac{1}{2}(z-w)^2 S(f)(z)$$

four equations to solve for
a, b, c, d

↓

In[1]:= `Solve[a*d - b*c == 1 && f[w] == (a*w + b)/(c*w + d) && f'[w] == (a*d - b*c)/(c*w + d)^2 && f''[w] == (-2*c (a*d - b*c))/(c*w + d)^3, {a, b, c, d}]`

Out[1]=
$$\left\{\left\{a \to \frac{2f'[w]^2 - f[w]\,f''[w]}{2f'[w]^{3/2}},\ b \to \frac{2f[w]\,f'[w] - 2w\,f'[w]^2 + w\,f[w]\,f''[w]}{2f'[w]^{3/2}},\ c \to -\frac{f''[w]}{2f'[w]^{3/2}},\ d \to \frac{2f'[w] + w\,f''[w]}{2f'[w]^{3/2}}\right\},\right.$$
$$\left.\left\{a \to \frac{-2f'[w]^2 + f[w]\,f''[w]}{2f'[w]^{3/2}},\ b \to \frac{-2f[w]\,f'[w] + 2w\,f'[w]^2 - w\,f[w]\,f''[w]}{2f'[w]^{3/2}},\ c \to \frac{f''[w]}{2f'[w]^{3/2}},\ d \to \frac{-2f'[w] - w\,f''[w]}{2f'[w]^{3/2}}\right\}\right\}$$

In[2]:= `M[w_, z_] := ((2f'[w]^2 - f[w] f''[w])/(2f'[w]^(3/2)) * z + (2f[w] f'[w] - 2w f'[w]^2 + w f[w] f''[w])/(2f'[w]^(3/2))) / (-(f''[w])/(2f'[w]^(3/2)) * z + (2f'[w] + w f''[w])/(2f'[w]^(3/2)))`

$M = \frac{az+b}{cz+d}$

In[3]:= `Mi[w_, z_] := ((2f'[w] + w f''[w])/(2f'[w]^(3/2)) * z - (2f[w] f'[w] - 2w f'[w]^2 + w f[w] f''[w])/(2f'[w]^(3/2))) / (-(-f''[w])/(2f'[w]^(3/2)) * z + (2f'[w]^2 - f[w] f''[w])/(2f'[w]^(3/2)))`

$M^{-1} = \frac{dz-b}{-cz+a}$ is inverse Möbius transformation

In[4]:= `FullSimplify[Mi[w, M[w, z]]]` ← Yup, works as it should ✓

Out[4]= z

In[5]:= `FullSimplify[D[Mi[w, M[w + t, z]], t] /. t → 0]`

Out[5]=
$$\frac{(w-z)^2\,(-3f''[w]^2 + 2f'[w]\,f^{(3)}[w])}{4f'[w]^2} = \frac{1}{2}(z-w)^2 S(f)$$ ✓

I read about this in a paper called

"Zippers and Univalent functions" by Bill Thurston and have attatched the first four pages to the end of these notes.

NPC 5: What does the rate of change of $M(f,\omega)$ have to do with our definition given by $[f(z), f(z+\epsilon), f(z+2\epsilon), f(z+3\epsilon)]$?

Second Way: The Schwarzian derivative tells you how a small circle in $\mathbb{C}$ gets mapped to an ellipse. (This interpretation also comes from Bill Thurston's paper.)

It is a well known fact that Möbius transformations map any circle in $\mathbb{C}$ to another circle. Therefore if $f(z)$ is not Möbius, it will map a small circle to an ellipse. This is a third order effect.

For example, consider the function $f(z) = z + \frac{1}{3}z^3$ near $z=0$. A small circle around $z=0$ will be mapped to a small (warped) circle around $z=0$.

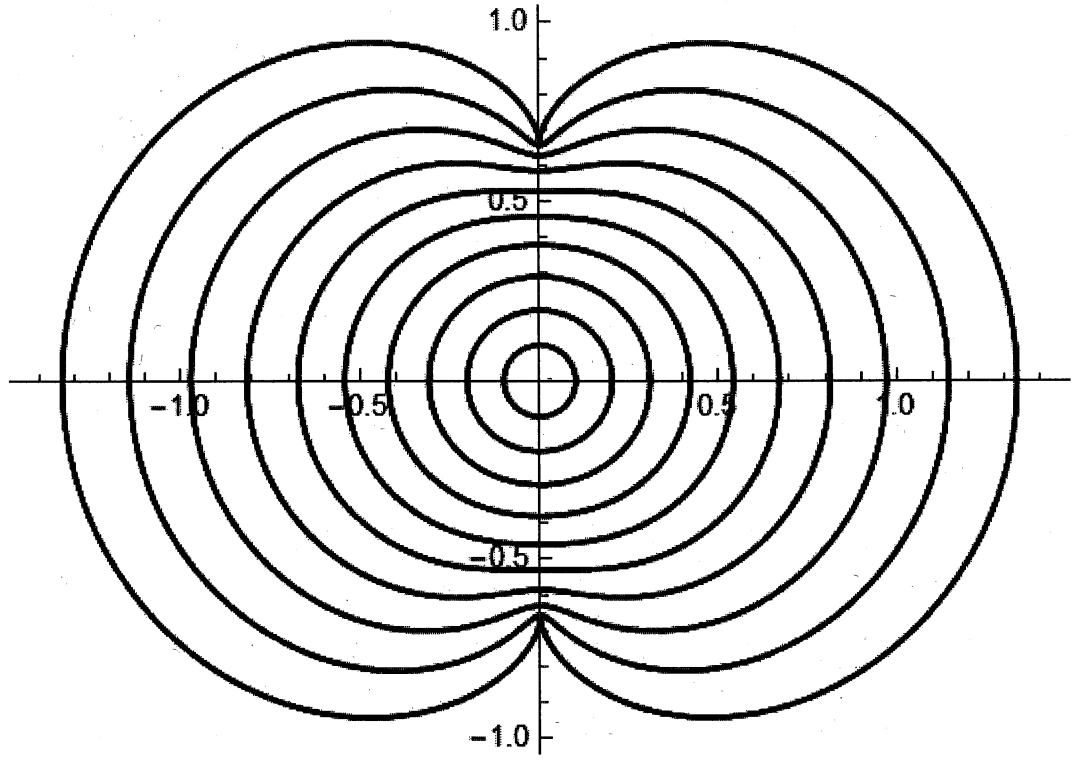
Here I provide a parametric plot of circles $z = re^{it}$ for $r = 0.1, 0.2, \ldots, 1.0$ under the map $f(z) = z + \frac{1}{3}z^3$. Notice how the circles become slightly elliptical, like

. When r is no longer small it becomes a weird shape.

```
In[1]:= P[t_, r_] = ReIm[r * Exp[I * t] + 1/3 * (r * Exp[I * t])^3];

        ParametricPlot[{P[t, 0.1], P[t, 0.2], P[t, 0.3], P[t, 0.4],
          P[t, 0.5], P[t, 0.6], P[t, 0.7], P[t, 0.8], P[t, 0.9], P[t, 1]},
         {t, 0, 2 * Pi}, PlotStyle → Black]
```

Out[2]=



The Schwarzian derivative measures this ellipticalness. Note that at $z=0$, $S(f) = 2$ for $f(z) = z + \frac{1}{3}z^3$. This f should be thought of as the prototypical example, because we can always find a Möbius transformation $A(z)$ such that for <u>any</u> f, we have

$$A(f(z)) = 0 + 1\cdot z + 0\cdot z^2 + \dots$$

In other words, its value, first derivative, and second derivative match the identity $f(z) = z$. Only the third derivative of $A\circ f$ will then specify $S(f)$ at $z=0$.

Recall that, based on its properties under composition, the schwarzian derivative is best thought of as a "quadratic differential"

$$S(f) = \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)(dz)^2.$$

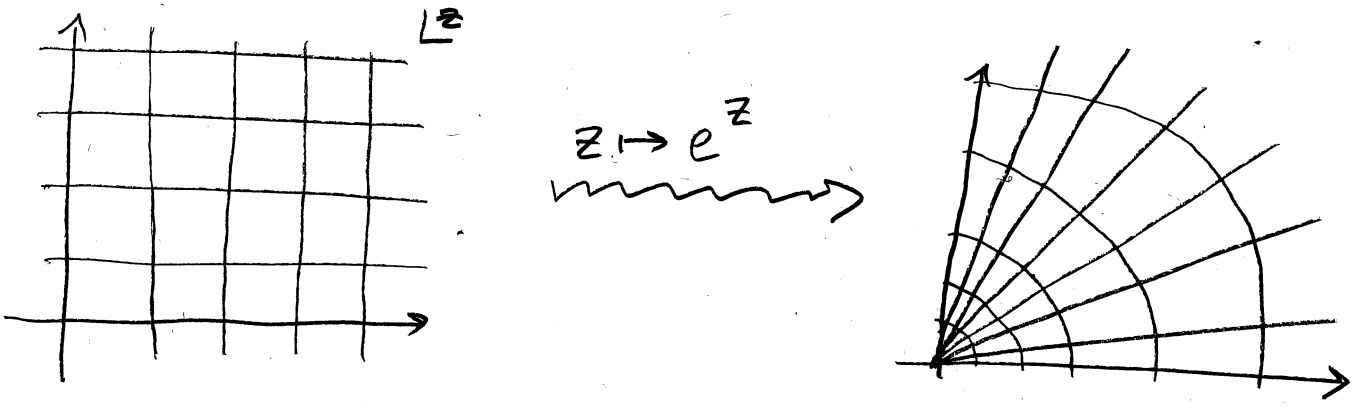
Here, $(dz)^2$ takes in a tangent vector and squares it. So if $dz = \varepsilon$, where ε is real, $(dz)^2 = \varepsilon^2 > 0$. If $dz = i\varepsilon$, $(dz)^2 = -\varepsilon^2 < 0$.

For our $f(z) = z + \frac{1}{3}z^3$, at $z=0$

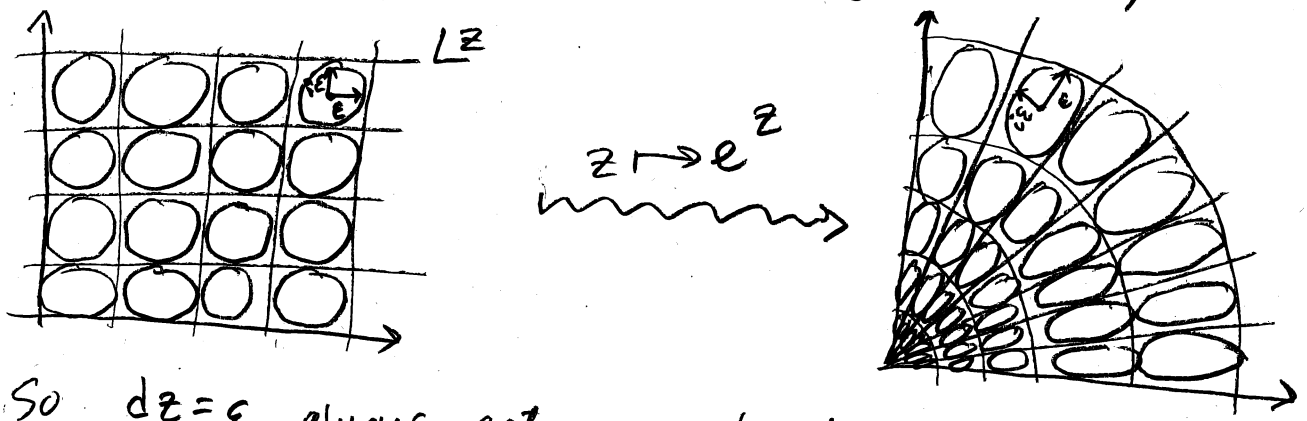
$$S(f)(0) = 2(dz)^2.$$

Note that $dz = \pm\varepsilon$ gets mapped to the <u>major axis</u> of the tiny ellipse while $dz = \pm i\varepsilon$ gets mapped to the <u>minor axis</u> of the ellipse. So in the direction of the major axis $S(f) > 0$ and in the direction of the minor axis $S(f) < 0$.

Actually, this gives a good way to understand why $S(f) = \text{const}$ when $f(z) = e^z$, which is an important fact in string theory.
You can visualize the e^z map as follows:



Putting little circles in the grid lines,



So $dz = \varepsilon$ always gets mapped along the major axis of the tiny ellipse while $dz = i\varepsilon$ always gets mapped along the minor axis, so $S(f)$ is a positive real number everywhere.
Furthermore, the taylor expansion "looks the same everywhere"

$$e^z = e^{z_0}(1 + (z - z_0) + \tfrac{1}{2}(z - z_0)^2 + \tfrac{1}{3}(z - z_0)^3 + \dots)$$

which means the möbius transformation needed to put it in "standard form" looks the same everywhere as well. It is

$$A(z) = \frac{-z + 1}{-\frac{1}{2}z - \frac{1}{2}}, \quad \text{as} \quad A \cdot f = 0 + 1 \cdot z + 0 \cdot z^2 - \frac{z^3}{4} + \dots .$$

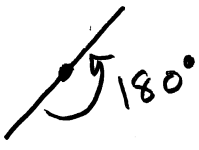
Third Way: The third way was given by Bill Thurston's Math overflow answer to the question "Is there an underlying explaination to the magical powers of the Schwarzian derivative?" (In some ways this document is just an exigesis of that thread.)

But first, lets talk about finger prints!

(This comes straight from a wonderful and highly readable biology paper Roger Penrose wrote in 1979 called "The Topology of Ridge Systems.")

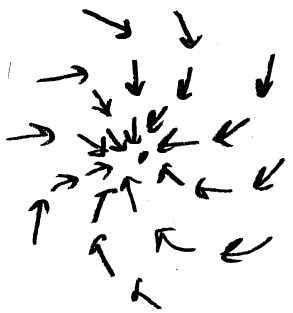
Imagine a 2D vector. It has 360° rotational symmetry.

A "ridge" is an unoriented line. It has 180° symmetry.

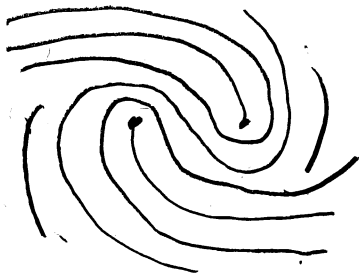


A "ridge system" is just like a "vector field of ridges."

Example of vector field:

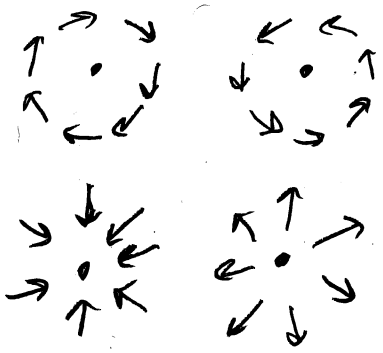


Example of ridge system:

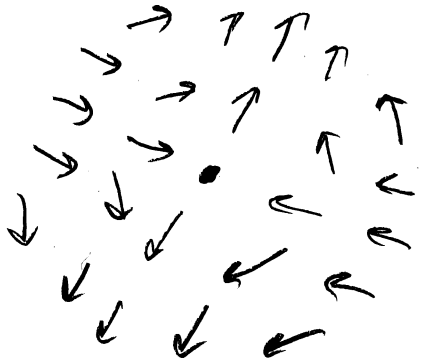


Let's review what the singularities of vector fields are like, where $\vec{v} = 0$. They come in two types: +1 and -1:

+1:



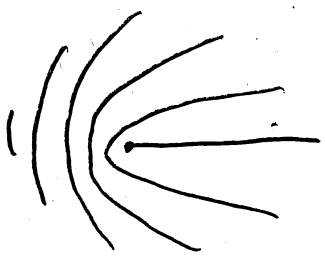
-1:



If you make a clockwise circle around a +1 singularity, $\vec{v}$ will do a clockwise 360° rotation. If you make a clockwise circle around a -1 singularity, $\vec{v}$ will make a counter clockwise 360° rotation. Visually, a -1 singularity looks like a "hyperbola." Note that the four examples of +1 singularities I drew are local rotations of each other, where each $\vec{v}$ is rotated without rotating the base point.

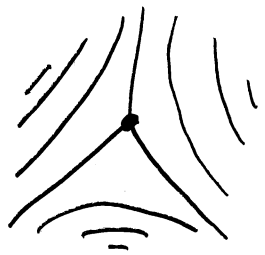
Because ridge systems have an extra symmetry that vector fields do not, 180° rotational symmetry, they have singularities that look unlike the vector field singularities. They are:

+1:



"loop"

-1:



"triradius"

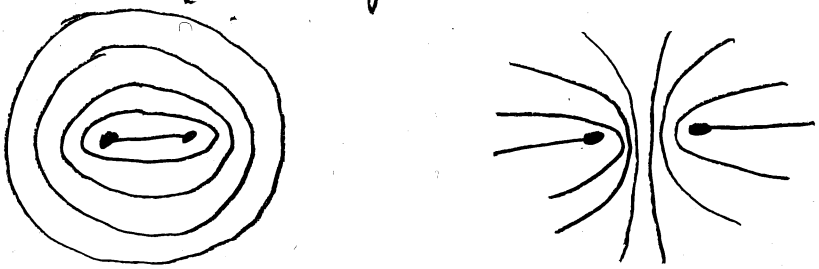
You can find these things all over your palm prints and footprints! In fact, if you trust Roger Penrose, you should have 4 more triradii on each palm than loops for topological reasons.

These are the fundamental singularities of ridge systems and can not be oriented consistently:

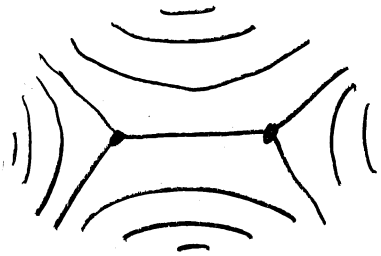
If you make a clockwise circle around a loop, the ridge on the circle will rotate 180° clockwise. Around a triradius, it will rotate 180° counterclockwise.

Therefore, two loops make a +1 vector field singularity and two triradii make a -1 vector field singularity.

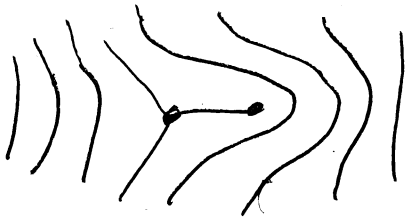
Examples of two loops:



Two triradii:



Note that a loop cancels out a triradius:



There are many examples of this on your fingerprints.

Anything with 180° rotational symmetry can make a ridge system. For example, masses in general relativity warp tiny circles of light into ellipses, which can be used to make ridge systems on the celestial sphere. (see the book "Spinors and Space-Time, Vol. 2" by Penrose and Rindler, Chapter 8, section "The fingerprint of the Weyl Tensor!") In fact, the graviton is a spin 2 particle and has 180° symmetry in general.

However, what we will be interested in here is quadratic differentials, which also have 180° symmetry. A general one is of the form

$$f(z)(dz)^2.$$

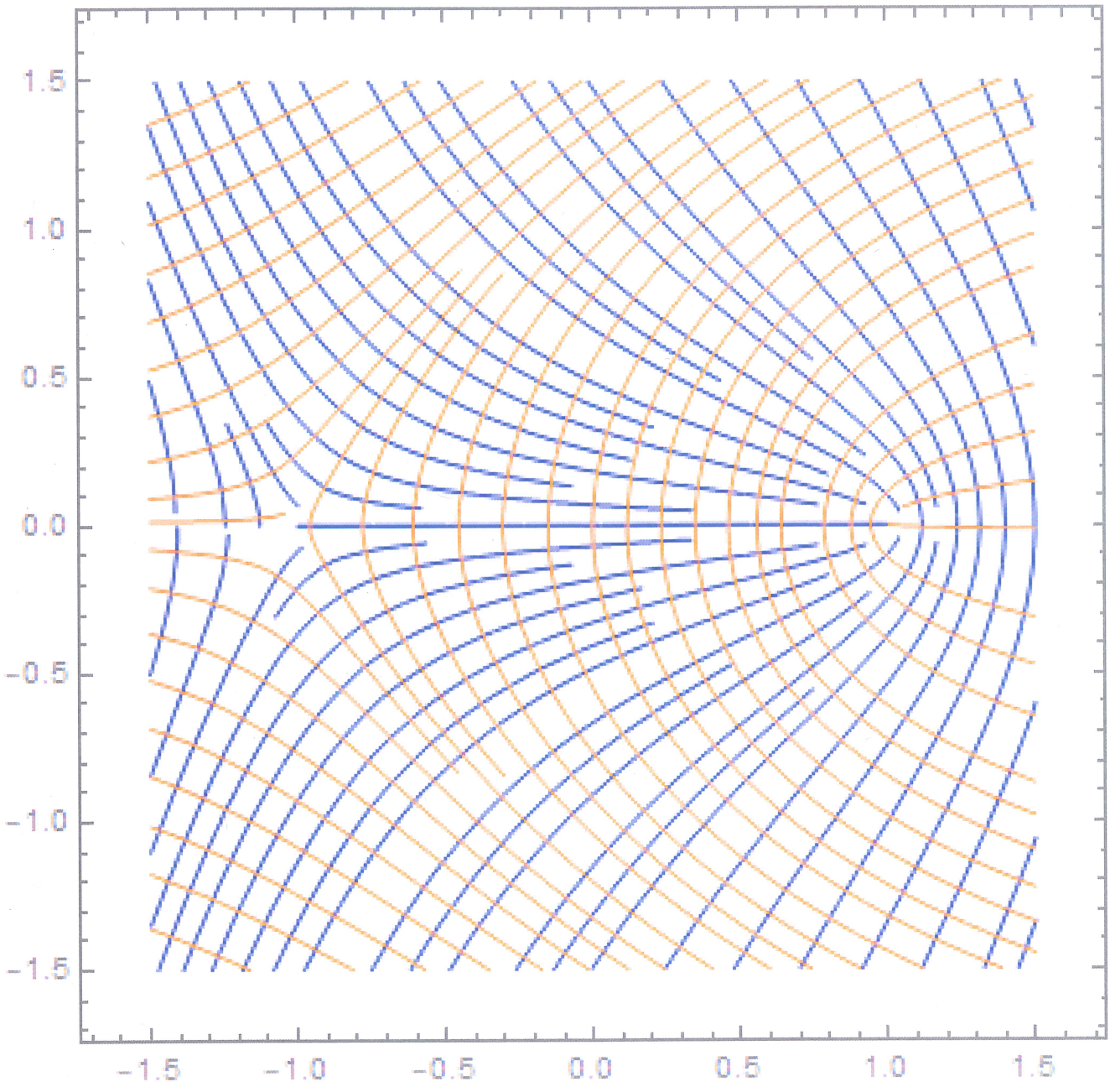
The $(dz)^2$ gives the 180° symmetry. We will depict quadratic differentials with orthogonal ridge systems. The blue lines will give the direction in which $f(z)(dz)^2$ is a positive real number and the orange lines give the direction in which $f(z)(dz)^2$ is a negative real number. An example is on the next page.

In[1]:=
$$f[z_] := \frac{1+z}{1-z};$$

quadratic differential is $f(z)(dz)^2$

```
xl = -1.5; xu = 1.5; yl = -1.5; yu = 1.5;
bplot = StreamPlot[ReIm[1 / Sqrt[f[x + I y]]],
   {x, xl, xu}, {y, yl, yu}, StreamMarkers → "Segment",
   StreamStyle → Blue];
oplot = StreamPlot[ReIm[I / Sqrt[f[x + I y]]],
   {x, xl, xu}, {y, yl, yu}, StreamMarkers → "Segment",
   StreamStyle → Orange];
Show[bplot, oplot]
```

Out[5]=



(How does the code work? Which tangent vectors dz give $f(z)(dz)^2 > 0$? If $f(z) = re^{i\theta}$, then $dz = \pm \varepsilon e^{-i\theta/2}$ where ε is a real number. This points in the direction of $1/\sqrt{f(z)}$. Because the square root has two branches, the streamline breaks at some points. For $f(z)(dz)^2 < 0$, $dz = \pm i e^{-i\theta/2}$, which points in the direction of $i/\sqrt{f(z)}$.)

Note that <u>poles</u> of f give <u>loops</u> and <u>zeros</u> of f give triradii.

Lets now turn to our actual object of interest, the Schwarzian derivative $S(f)$.

$$S(f) = \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \right)(dz)^2$$

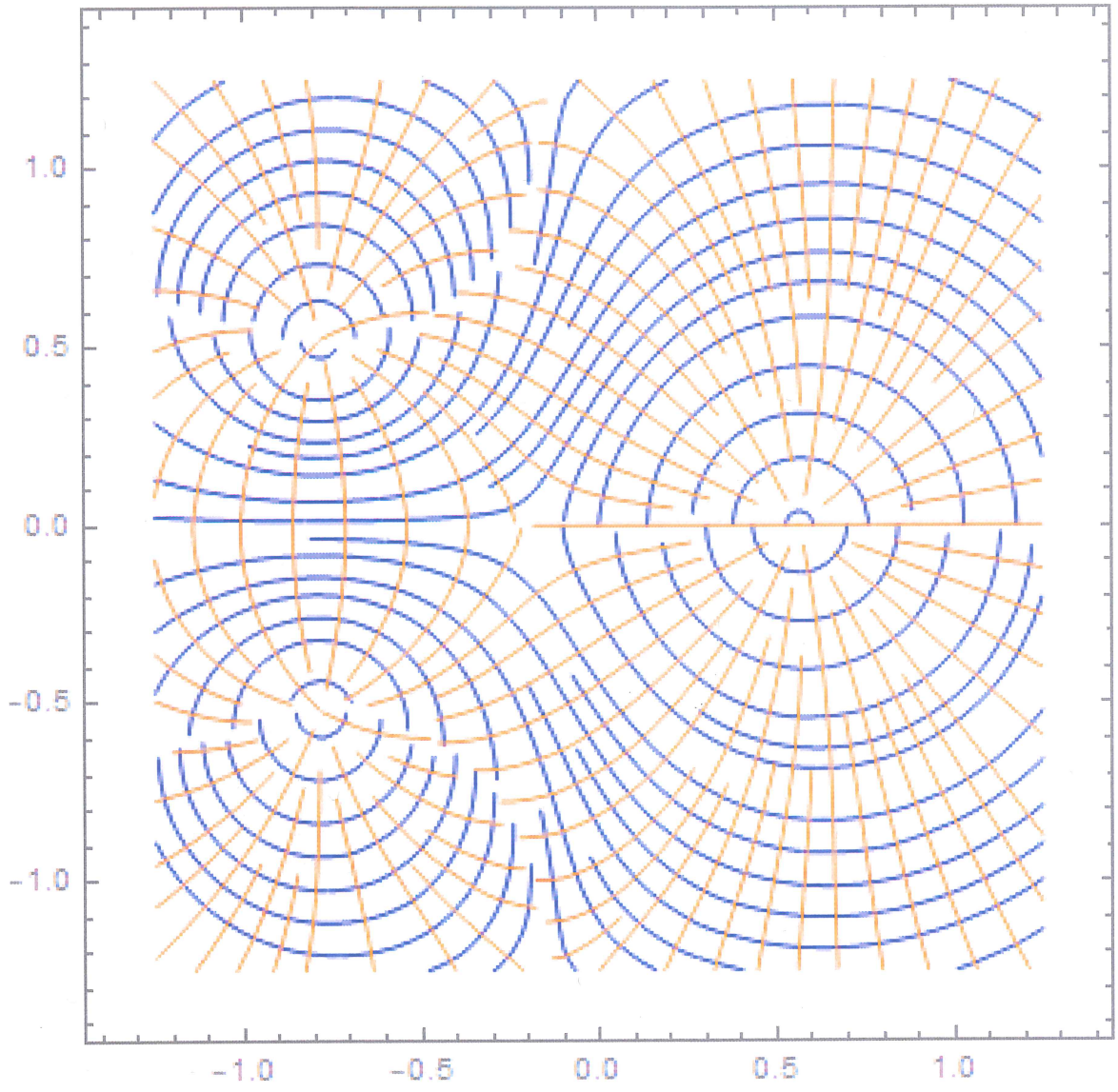
Poles and zeros of f are of no particular significance to $S(f)$, as they are just like any other points in $\mathbb{CP}^1$. However, critical points where $f'(z) = 0$ give double poles.

On the next page is the ridge system for $S(f)$ when $f(z) = \dfrac{z^3 - 3z - 1}{z^3 + 1}$.

In[1]:=
$$f[z_] := \frac{z^3 - 3*z - 1}{z^3 + 1}; \quad S[z_] := \frac{f'''[z]}{f'[z]} - \frac{3}{2}\left(\frac{f''[z]}{f'[z]}\right)^2;$$

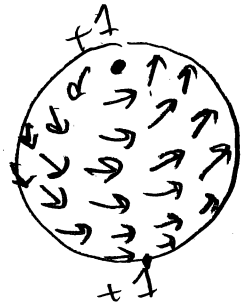
```
xl = -1.25; xu = 1.25;
yl = -1.25; yu = 1.25;
bplot = StreamPlot[ReIm[1 / Sqrt[S[x + I y]]], {x, xl, xu},
   {y, yl, yu}, StreamMarkers → "Segment", StreamStyle → Blue];
oplot = StreamPlot[ReIm[I / Sqrt[S[x + I y]]], {x, xl, xu},
   {y, yl, yu}, StreamMarkers → "Segment", StreamStyle → Orange];
Show[bplot, oplot]
```

Out[6]=



We have a circle (two loops) around each critical point and a triradius at the zeros of $S(f)$.

Let us now briefly discuss topology. It is a well known fact that any vector field on the sphere S^2 has overall +2 singularities. Therefore, any ridge system on $\mathbb{CP}^1$ will have overall +4 singularities (four more loops than triradii).



This Topological fact allows us to prove an interesting statement about the Schwarzian derivative generically.

$$2 \cdot (\#\text{ of crit. pts. of } f) - (\#\text{ of zeros of } S(f)) = 4$$

We'll verify this for our old example on the next page!

Tangent: You might wonder if this proof is too powerful. By considering the quadratic differential $g(z)(dz)^2$, can't you prove that

$$(\#\text{ poles of } g) - (\#\text{ zeros of } g) = 4$$

which is incorrect? The thing is you must remember how $(dz)^2$ transforms. If $w = \frac{1}{z}$, $dw = -\frac{dz}{z^2}$, $(dw)^2 = \frac{(dz)^2}{z^4}$, $(dz)^2 = \frac{(dw)^2}{w^4}$,

so $g(z)(dz)^2 = g(1/w)\frac{(dw)^4}{w^4}$, and w^4 adds an order 4 pole, so really we have

$$(\#\text{ poles of } g) + 4 - (\#\text{ zeros of } g) = 4$$

which is correct! Tangent over.

So $(+2)\times 4 + (-1)\times 4 = 4$ ✓

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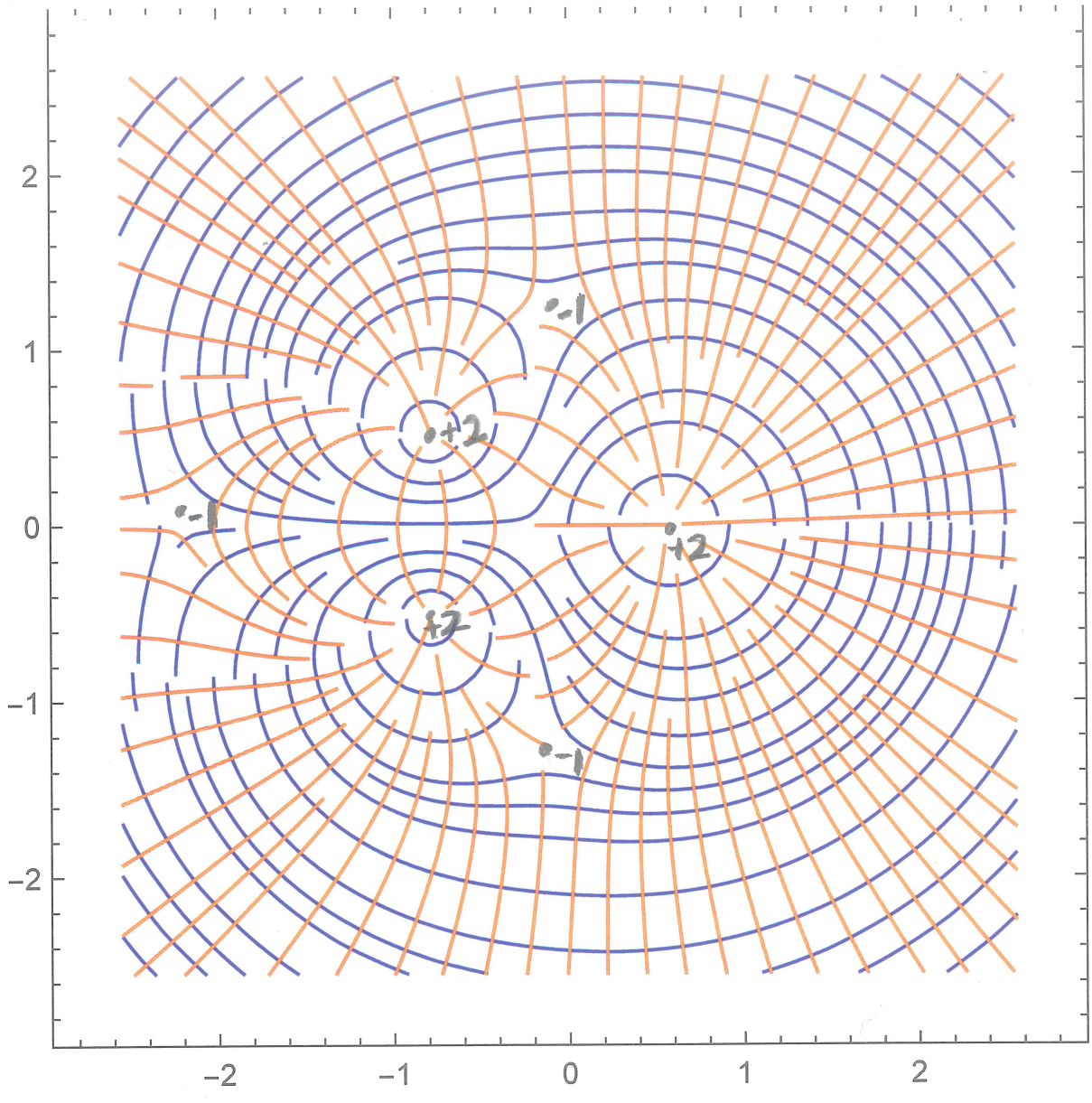


+2 at $z=\infty$

Don't forget the double pole at infinity!

Representation Theory

Lets think about the Lie algebra of infinitessimal holomorphic diffeomorphisms. They are generated by vector fields of the form

$$\delta z = \varepsilon z^{n+1} \qquad \delta \bar{z} = \varepsilon \bar{z}^{n+1}.$$

which we call

$$L_n = -z^{n+1}\partial_z \qquad \bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}.$$

Note that we must have "real" combinations of the two in order to get actual vector fields on $\mathbb{C}$, so built out of $L_n + \bar{L}_n$ and $i(L_n - \bar{L}_n)$.

Note that those vector fields generate the infinitessimal Möbius transformations when $n = -1, 0, 1$. This is a closed sub algebra of the whole Lie algebra.

The commutation relations are

$$\begin{aligned}
[L_m, L_n] &= z^{m+1}(\partial_z z^{n+1})\partial_z - z^{n+1}\partial_z(z^{m+1})\partial_z \\
&= (n+1)z^{m+n+1}\partial_z - (m+1)z^{n+m+1}\partial_z \\
&= (m-n)z^{m+n+1}\partial_z \\
&= (m-n)L_{m+n}
\end{aligned}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}$$

$$[L_m, \bar{L}_n] = 0.$$

This is the Witt Algebra.

Witt Vector Fields

Noah Miller

June 2019

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \qquad \frac{\mathrm{d}}{\mathrm{d}\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

$$L_n = z^{1-n}\frac{\mathrm{d}}{\mathrm{d}z} \qquad \bar{L}_n = \bar{z}^{1-n}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}$$

$$L_0 + \bar{L}_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad i(L_0 - \bar{L}_0) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$L_1 + \bar{L}_1 = \frac{\partial}{\partial x} \qquad i(L_1 - \bar{L}_1) = \frac{\partial}{\partial y}$$

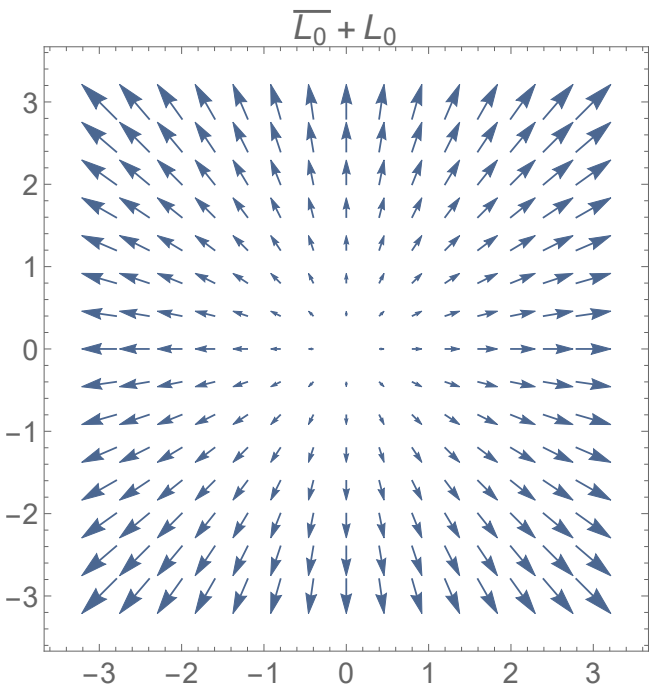
$$L_{-1} + \bar{L}_{-1} = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} \qquad i(L_{-1} - \bar{L}_{-1}) = -2xy\frac{\partial}{\partial x} + (x^2 - y^2)\frac{\partial}{\partial y}$$

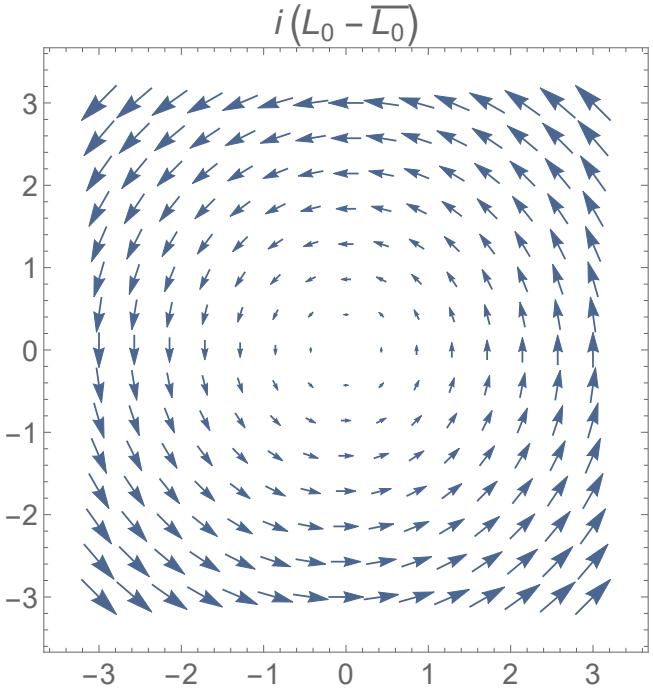
$$L_2 + \bar{L}_2 = \frac{x}{x^2+y^2}\frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial y} \qquad i(L_2 - \bar{L}_2) = \frac{y}{x^2+y^2}\frac{\partial}{\partial x} + \frac{x}{x^2+y^2}\frac{\partial}{\partial y}$$

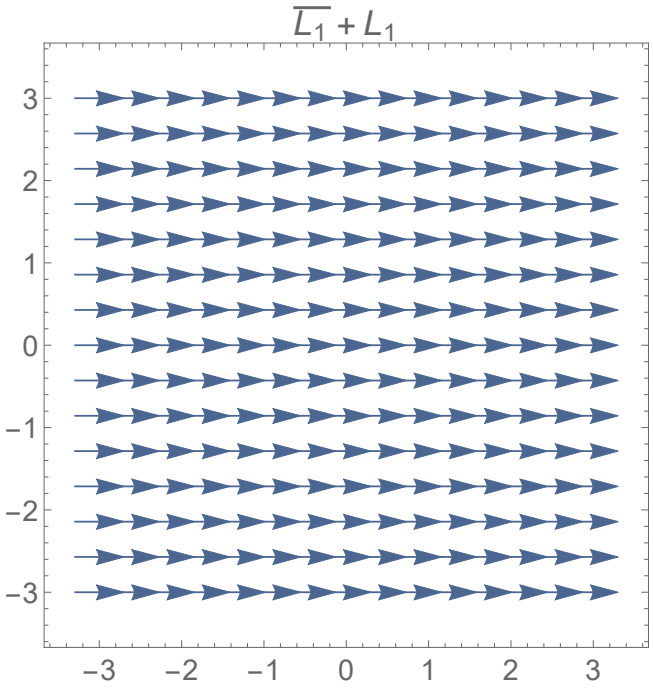
$$L_{-2} + \bar{L}_{-2} = (x^3 - 3xy^2)\frac{\partial}{\partial x} + +(3x^2y - y^3)\frac{\partial}{\partial y} \qquad i(L_{-2} - \bar{L}_{-2}) = (-3x^2y + y^2)\frac{\partial}{\partial x} + (x^3 - 3xy^2)\frac{\partial}{\partial y}$$

$$L_3 + \bar{L}_3 = \frac{x^2-y^2}{(x^2+y^2)^2}\frac{\partial}{\partial x} + \frac{-2xy}{(x^2+y^2)^2}\frac{\partial}{\partial y} \qquad i(L_3 - \bar{L}_3) = \frac{2xy}{(x^2+y^2)^2}\frac{\partial}{\partial x} + \frac{x^2-y^2}{(x^2+y^2)^2}\frac{\partial}{\partial y}$$

$$L_{-3} + \bar{L}_{-3} = (x^4 - 6x^2y^2 + y^4)\frac{\partial}{\partial x} + 4xy(x^2 - y^2)\frac{\partial}{\partial y} \qquad i(L_{-3} - \bar{L}_{-3}) = -4xy(x^2 - y^2)\frac{\partial}{\partial x} + (x^4 - 6x^2y^2 + y^4)\frac{\partial}{\partial y}$$






$\overline{L_1} + L_1$
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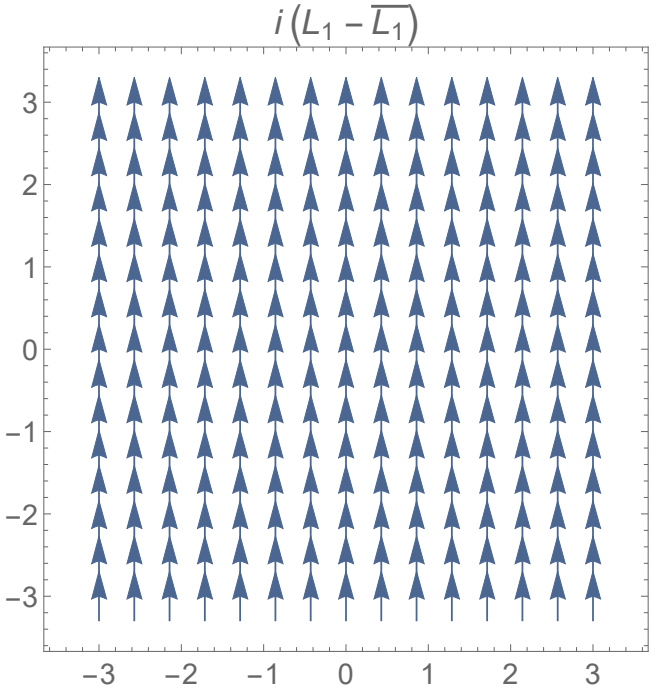
$i\left(L_1 - \overline{L_1}\right)$
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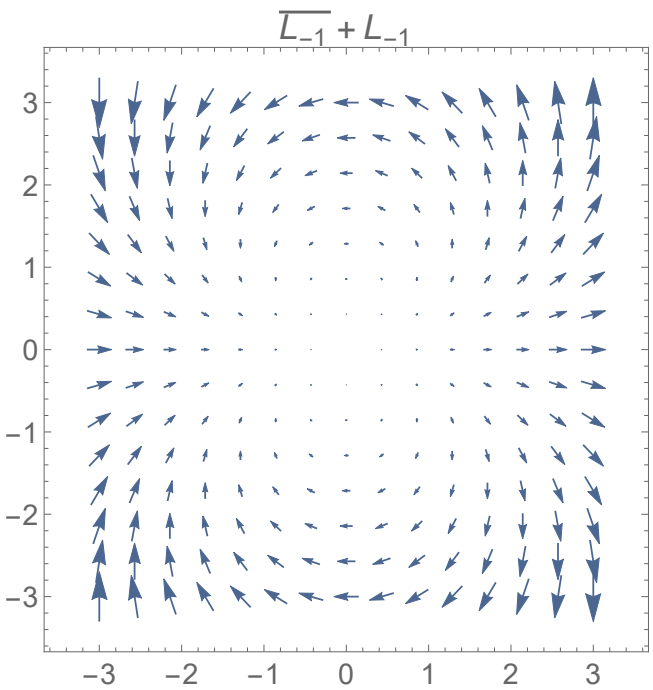
$\overline{L_{-1}} + L_{-1}$
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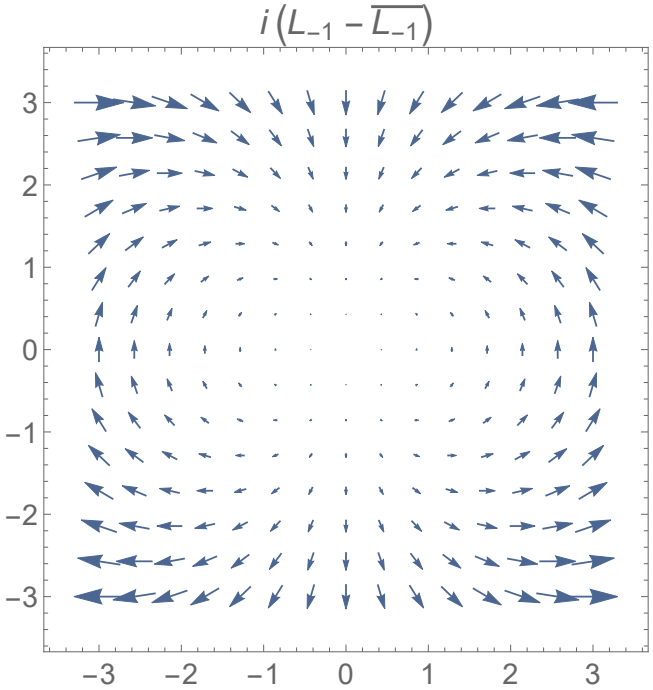
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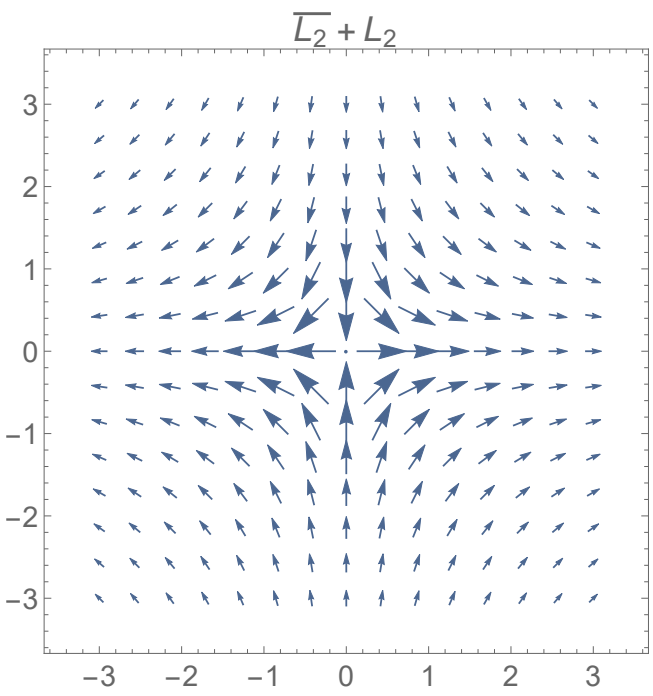
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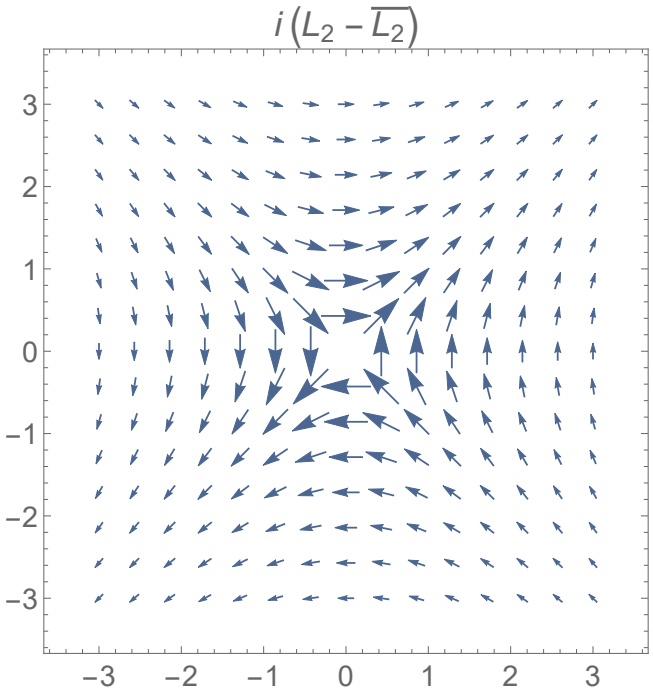
$i\left(L_2 - \overline{L_2}\right)$
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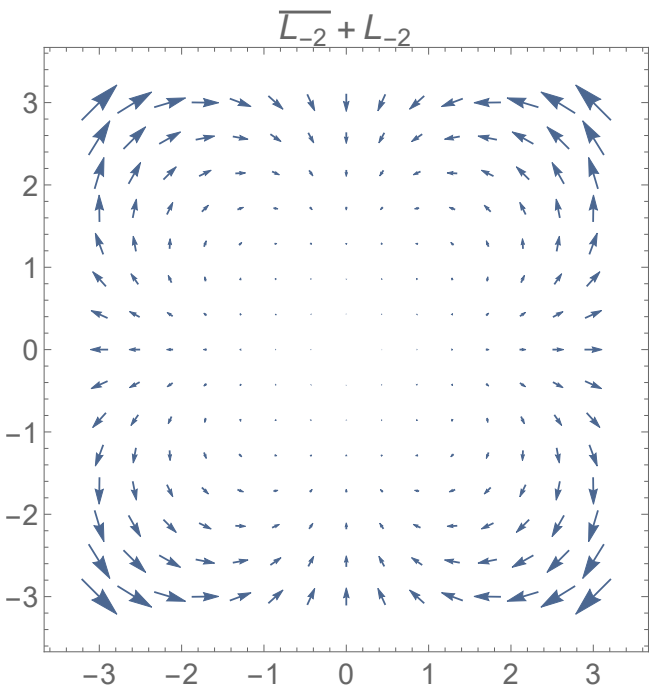
$\overline{L_{-2}} + L_{-2}$
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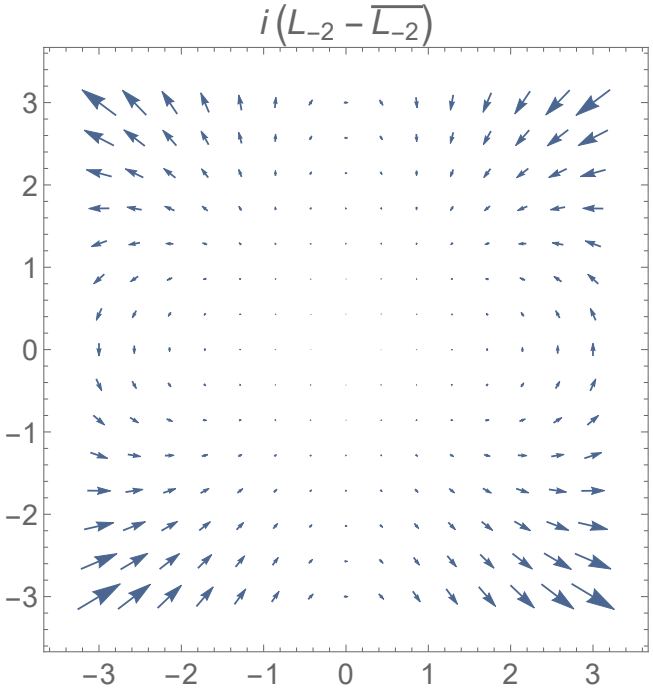
$i\left(L_{-2} - \overline{L_{-2}}\right)$
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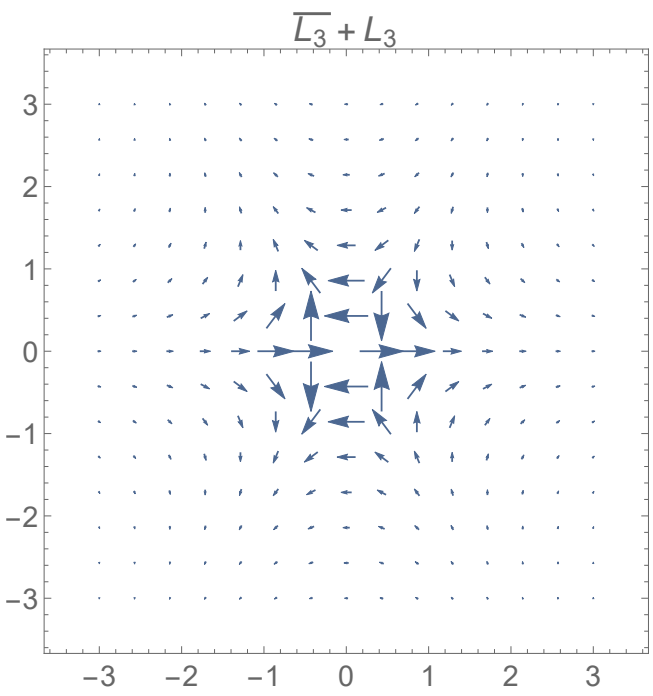
$\overline{L_3} + L_3$
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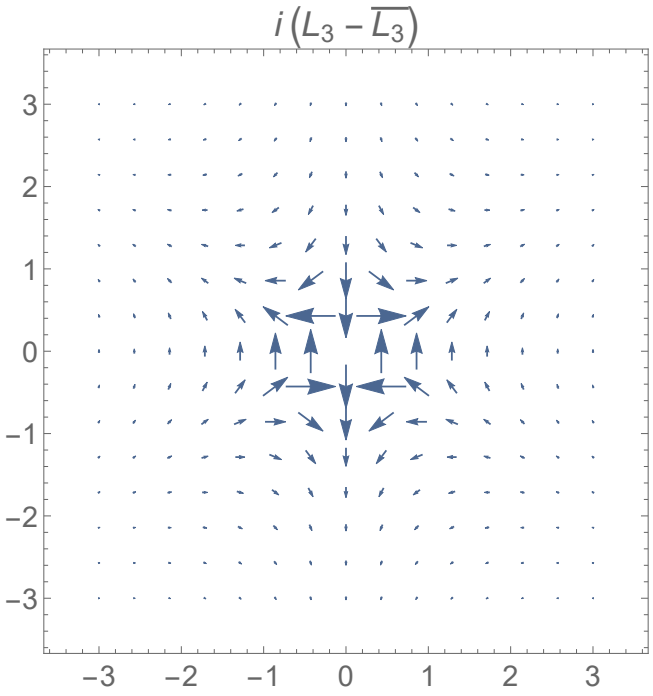
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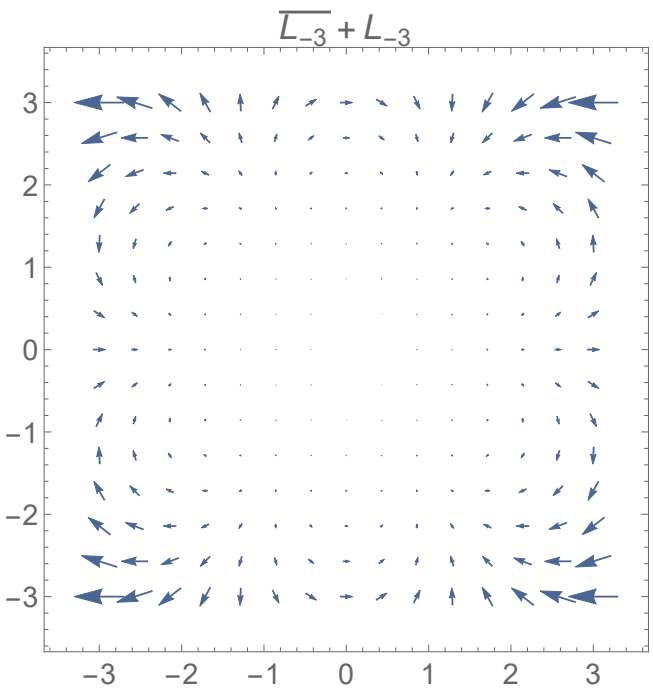
$\overline{L_{-3}} + L_{-3}$
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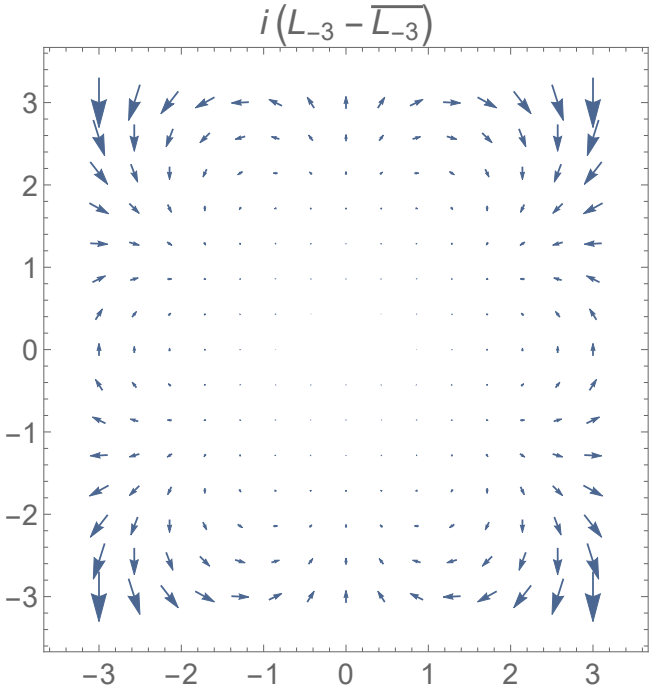
$i\left(L_{-3} - \overline{L_{-3}}\right)$
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−3 −2 −1 0 1 2 3

we can also think of the Witt algebra as the vector fields on $\mathbb{RP}^1 = S^1$, generated by

$$L_n = ie^{in\theta}\frac{d}{d\theta} \qquad \bar{L}_n = -ie^{-in\theta}\frac{d}{d\theta}.$$

Here, however, $L_{-n} = \bar{L}_n$.

A <u>central extension</u> of a Lie algebra $\mathfrak{g}$ is a map $c: \mathfrak{g}\times\mathfrak{g} \mapsto \mathbb{R}$ that satisfies

$$c(X,Y) = -c(Y,X) \quad \text{antisymmetric}$$

$$c(X, Y+Z) = c(X,Y) + c(X,Z) \quad \text{bilinear}$$

$$c(X,[Y,Z]) + c(Y,[Z,X]) + c(Z,[X,Y]) \quad \text{Jacobi Identity.}$$

For vector fields on S^1, which can be written as $h\frac{d}{d\theta}$, we have the following central extension:

$$c\left(h_1\frac{d}{d\theta}, h_2\frac{d}{d\theta}\right) = \int_{S^1} d\theta\, h_1 h_2'''.$$

Noting $\left[h_1\frac{d}{d\theta}, h_2\frac{d}{d\theta}\right] = (h_1h_2' - h_1'h_2)\frac{d}{d\theta}$, we can check the Jacobi identity:

$$c\left(h_1\tfrac{d}{d\theta}, \left[h_2\tfrac{d}{d\theta}, h_3\tfrac{d}{d\theta}\right]\right) + \text{perms} = \int_{S^1} d\theta\, h_1(h_2h_3' - h_2'h_3)''' + \text{perms}$$

$$= \int_{S^1} d\theta\, h_1''(h_2h_3' - h_2'h_3)' + \text{perms} = \int_{S^1} d\theta\, h_1''(h_2'h_3' + h_2h_3'' - h_2''h_3 - h_2'h_3') + \text{perms}$$

$$= \int_{S^1} d\theta\, h_1''(h_2h_3'' - h_2''h_3) + \text{perms} = 0 \quad \checkmark$$

We can also compute

$$c(L_m, L_n) = \int_{S^1} d\theta \; i e^{im\theta} (in)^3 e^{in\theta}$$

$$= n^3 \int_{S^1} d\theta \; e^{i(m+n)\theta}$$

$$= n^3 \delta_{m+n,0}$$

If we multiply this prefactor by an arbitrary constant "c" (different c) called "the central charge", this is just the Virasoro algebra! (A shift of $L_0 \mapsto L_0 - \frac{c}{24}$ will turn $\frac{c}{12} n^3$ into the familiar $\frac{c}{12}(n^3 - n)$.)

Recall that, for a tiny diffeomorphism $\theta + \epsilon h$, the Schwarzian is

$$h \frac{d}{d\theta} \longmapsto h''' (d\theta)^2.$$

One of the $d\theta$'s "eats" a $\frac{d}{d\theta}$ and the other can be integrated over:

$$h_1''' (d\theta)^2 \left(h_2 \frac{d}{d\theta} \right) = h_2 h_1''' d\theta,$$

So for the macroscopic adjoint representation of the group on this algebra, the finite diffeomorphism f acts on $h \frac{d}{d\theta}$ as $\int_{S^1} S(f)\left(h \frac{d}{d\theta}\right)$.

Let me tell you something cool about central extensions. If G is a compact Lie group, then the central extensions of $\mathfrak{g}$ are in 1-to-1 correspondence with elements of the second de Rham cohomology group $H^2_{dR}(G)$ of the manifold G. (This also holds for group cohomology.)

Take for example the trivial Lie algebra with two generators

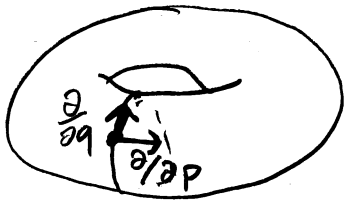
$$[q,p]=0.$$

This has a central extension

$$c(q,p)=1$$

which makes the Heisenberg Lie algebra.

The Lie group G corresponding to the trivial Lie algebra with two generators is the torus T^2. The Lie algebra elements are vector fields which we will call $\partial/\partial q$ and $\partial/\partial p$.



Note that

$c(q,p)$ is a lot like $\int (dq\wedge dp)\left(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right)=1.$

$dq\wedge dp$ is the area form of $H^2_{dR}(T^2)=\mathbb{R}$ and is the only non trivial de Rham cohomology element.

(Abstractly, integration is necessary to make any 2-form in H^2_{dR} invariant under the group action. Integration should be thought of as "averaging.")

However, this correspondance does not hold for non-compact Lie groups. Why? Well, you can't integrate a two form over an infinite volume and get a finite answer! And the group of diffeomorphisms of S^1 is very non-compact. So the correspondence does not work for the Virasoro algebra. Oh well.

In fact, $Diff(S^1)$ can be deformation retracted on $O(2)$, which has vanishing H^2_{dR}!

It turns out that the Virasoro algebra is the only central extension of the Witt Algebra:

> Any central extension that only depends on $[X,Y]$ is dumb and we don't care about it. Lets show that this is not the case for the Virasoro algebra. We want to show that
>
> $$c(X,Y) \neq \mu([X,Y])$$
>
> where μ is some linear function of g.

Proof: Suppose:

$$c(L_n, L_{-n}) \stackrel{?}{=} \mu([L_n, L_{-n}])$$

$$\frac{1}{12} n(n^2-1) = \mu((n-(-n))L_0)$$

$$\frac{1}{12} n(n^2-1) = 2n\mu(L_0)$$

$$\frac{1}{24}(n^2-1) = \mu(L_0)$$

This is true for all n, while $\mu(L_0)$ can only be one number, showing this is impossible. ✓

So the central extension (n^3-n) is non trivial. It is also unique. The proof for this is given in Martin Schottenloher's book "A mathematical Introduction to Conformal Field Theory" section 5.3.

Let me now briefly discuss the transformation of the stress energy tensor $T_{zz}(z)$ in 2D CFT.

You may know it picks up this funky Schwarzian derivative term under a conformal transformation. This is very strange, but can be seen at the level of the Virasoro Algebra.

First, recall how $T(z)$ is supposed to transform under $z \mapsto z + \varepsilon h$:

$$T(z) \longmapsto \left[(z+\varepsilon h)'\right]^2 T(z+\varepsilon h) + S(z+\varepsilon h)$$

$$= T(z) + \varepsilon\left(2h'^2 T(z) + hT'(z) + \varepsilon h'''\right)$$

So this is the expression we want to recreate now.

In the Quantum theory, $T(z)$ can be written in terms of the Noether charges L_n as

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}.$$

Do not think of these L_m's as $-z^{n+1}\partial_z$ anymore, they are now their own operators for which z is constant with respect to.

We also have

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}(n^3-n)\delta_{m+n,0}$$

So
$$[L_n, T(z)] = \sum_{m=-\infty}^{\infty} \frac{(n-m)L_{m+n}}{z^{m+2}} + \frac{c}{12}(n^3-n)z^{n-2}$$
$$m \mapsto m-n$$
$$= z^n \sum_{m=-\infty}^{\infty} \frac{(2n-m)L_m}{z^{m+2}} + \frac{c}{12}(n^3-n)z^{n-2}$$

Note that h for the diffeomorphism corresponding to L_n is z^{n+1}. So

$$\delta T = 2h'T + hT' + \frac{c}{12}h'''$$
$$= 2(n+2)z^n \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} + z^{n+1} \sum \frac{-(m+2)}{z} \frac{L_m}{z^{m+2}} + \frac{c}{12}(n+1)n(n-1)z^{n-2}$$
$$= z^n \sum_{m=-\infty}^{\infty} (2n+2-m-2) \frac{L_m}{z^{m+2}} + \frac{c}{12}(n^3-n)z^{n-2}$$
$$= [L_n, T(z)] \quad \checkmark$$

As expected! Note that the (n^3-n) central extension is an infinitesimal Schwarzian Derivative. Soon we will give a physical reason for its appearance.

As a final remark, note $n^3-n=0$ for $n=-1,0,1$, the Möbius subalgebra. This is just because the Schwarzian is 0 for Möbius transformations. Also, $SO(1,3)$ has no central extensions.

2D CFT and Normal Ordering

Let's just think about the free 2D scalar CFT with $c=1$. The two point function (in euclidean signature)

$$\langle \phi(z_1)\phi(z_2)\rangle = -\ln|z_1-z_2|^2$$

is logarithmic because $D=2$. Otherwise it would be a power law. The logarithmic dependence is very important.

Note that it diverges as $z_1 \to z_2$. This makes it annoying to talk about local operators built out of many copies of $\phi(z)$. To solve this, we subtract off the divergent part then take the limit $z_1 \to z_2$.

So for example

$$T^{(z)}_{zz}(z_1) = \lim_{z_2\to z_1} :-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2):_z$$

you need to pick a coordinate system to normal order with respect to

$$= \lim_{z_2\to z_1} -\frac{1}{2}\partial_{z_1}\partial_{z_2}\left(\phi(z_1)\phi(z_2)+\ln|z_1-z_2|^2\right)$$

$$= \lim_{z_2\to z_1} -\frac{1}{2}\left(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)+\frac{1}{(z_1-z_2)^2}\right)$$

Which coordinates you normal order with respect to matter a lot! I have denoted them by $T^{(z)}$ and $:\ldots:_z$. If it weren't for the normal ordering,

There would be no Schwarzian derivative in the transformation of T_{zz}! If we normal order with respect to the coordinates $w(z)$, then we subtract $-\ln|w(z_1)-w(z_2)|^2$ instead of $-\ln|z_1-z_2|^2$.

This requires a short (incomplete) explaination. Conformal transformations should really be thought of as a diffeomorphism followed by a compensating Weyl rescaling of the metric. In the w coordinates, the metric should be thought of as $d\bar{w}dw$ just like it is $d\bar{z}dz$ in the z-coordinates. So the distance between $w(z_1)$ and $w(z_2)$ is just $|w(z_1)-w(z_2)|^2$.

Anyway, lets see what happens if we normal order with respect to the w coordinates.

$$T_{zz}^{(w)}(z_1) = \lim_{z_2\to z_1} \partial_{z_1}\partial_{z_2}\left(-\tfrac{1}{2}\phi(z_1)\phi(z_2) + \ln|w(z_1)-w(z_2)|^2\right)$$

Here is where something special happens, and where the Schwarzian derivative has been hiding physically.

Note that we have a log being hit by 2 derivatives. $2 \times 2 = 4$, which means we are evaluating this log at four different points! In fact, we are really taking the log of a cross ratio!

$$\lim_{z_1 \to z_2} \partial_{z_1} \partial_{z_2} \ln \left| w(z_1) - w(z_2) \right|^2$$

$$= \lim_{z_1 \to z_2} \partial_{z_1} \partial_{z_2} \ln \left(w(z_1) - w(z_2) \right) \quad (\text{no } \partial_{\bar{z}})$$

$$= \lim_{\substack{\varepsilon \to 0 \\ z_1 \to z_2}} \frac{1}{\varepsilon^2} \Big[\ln \left(w(z_1+\varepsilon) - w(z_2+\varepsilon) \right) - \ln \left(w(z_1) - w(z_2+\varepsilon) \right) - \ln \left(w(z_1+\varepsilon) - w(z_2) \right) + \ln \left(w(z_1) - w(z_2) \right) \Big]$$

Say $z_2 = z_1 + \delta$

$$= \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \frac{1}{\varepsilon^2} \ln \left(\frac{\left(w(z_1+\varepsilon) - w(z_1+\varepsilon+\delta) \right) \left(w(z_1) - w(z_1+\delta) \right)}{\left(w(z_1+\varepsilon) - w(z+\delta) \right) \left(w(z_1) - w(z_1+\delta) \right)} \right)$$

$$\left[w(z_1), w(z_1+\varepsilon); w(z_1+\delta), w(z_1+\varepsilon+\delta) \right] !$$

Because of the log, we have expressed this as an infinitessimal cross ratio of $z, z+\varepsilon, z+\delta, z+\varepsilon+\delta$!

Be careful! Because this is a ratio, you need to take the limits in the correct order, because taking $z+\varepsilon \to z$, $z+\varepsilon+\delta \to z+\delta$ is different than taking $z+\delta \to z$, $z+\varepsilon+\delta \to z+\varepsilon$.

The correct way to do it is to take the $\xi \to 0$ limit before the $\delta \to 0$ limit, as the $\xi \to 0$ limit corresponds to the derivatives $\partial_{z_1}, \partial_{z_2}$.

$$\lim_{\delta\to 0} \lim_{\xi \to 0} \frac{1}{\xi^2} \log\left[w(z_1), w(z_1+\xi); w(z_1+\delta), w(z_1+\xi+\delta) \right]$$

$$= \underbrace{\frac{1}{\delta^2}}_{\frac{1}{|z_1-z_2|^2}} + \frac{1}{6} \underbrace{\left(\frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \right)}_{S(w)!} + \dots$$

So that is the origin of the Schwarzian in 2D CFT, and where the cross ratio has been hiding.

The origin of the effect basically comes down to the fact that when subtracting off a divergent term to get a finite piece, there is a difference between subtracting off ∞ and $\infty + 1$! Because log of a scaled up argument gets an additive constant, there is a difference between $|z_1 - z_2|^2$ and $|w(z_1) - w(z_2)|^2$ when $z_1 \to z_2$ which is additive. This shifts the stress energy tensor by a constant.

NPC 6: Is there a way to understand the trace anomaly for $T_{z\bar{z}}$ in a similar way?

http://dx.doi.org/10.1090/surv/021/15

Zippers and Univalent Functions

WILLIAM P. THURSTON

1. Introduction. Every simply connected domain $\mathbf{C}$, with the exception of $\mathbf{C}$ itself, can be parametrized by a conformal map from the unit disk, that is, by a univalent function. Families of domains which vary continuously with parameters arise in several mathematical contexts, particularly in the theory of Kleinian groups and in the theory of other 1-dimensional complex dynamical systems. A family of domains gives rise to a family of univalent functions. There are various topologies on the set of univalent functions. One topology which has proven valuable for the theory of Kleinian groups is the topology of uniform convergence of the Schwarzian derivative (to be described presently), uniform as measured using the hyperbolic metric on the disk. In this note, we will construct simple examples of domains whose corresponding univalent functions are isolated points in this topology. These domains have complements which are arcs, in fact, quasi-intervals, and they are reminiscent of zippers.

The construction and result here are related to that of Gehring [1].

2. Preliminaries. For the benefit of people to whom the Schwarzian derivative may seem a mystery, we will set the stage by discussing the Schwarzian derivative. Actually, the Schwarzian derivative is not really essential to the current discussion. We will use it only for the definition of a topology on the set of univalent maps, and logically we really use an equivalent description of the topology (to be explained later) which makes no mention of Schwarzian derivatives. This is not meant to downplay the significance of the Schwarzian derivative; by analogy, many qualitative constructions in differential geometry can be done without ever mentioning curvature, even though curvature is central to differential geometry.

The Schwarzian derivative is very much like a kind of curvature: the various kinds of curvature in differential geometry measure deviation of curves or manifolds from being flat, while the Schwarzian derivative measures the deviation of a conformal map from being a Mœbius transformation. A Mœbius transformation is determined by its two-jet (its value and first two derivatives) at any point. The two-jet is arbitrary, except for the proviso that the first derivative be nonsingular. Thus, for any locally univalent f and any point w in the domain

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of f there is associated a unique Mœbius transformation $M(f, w)$ which agrees with f through second order. Note that if A is a Mœbius transformation, then

$$M(A \circ f, w) = A \circ M(f, w).$$

More generally, if f and g are two locally univalent functions, then

$$M(g \circ f, w) = M(g, f(w)) \circ M(f, w).$$

Conformal mappings up to postcomposition (composition on the left) with Mœbius transformations are therefore conveniently described by their three-jets, up to the action of Mœbius transformations on three-jets. The quotient space of this action, at any point z in the domain of the mapping, is isomorphic to $\mathbf{C}$. To see the isomorphism, normalize a map f by postcomposing with the Mœbius transformation $M^{\circ -1}(f, w)$ to obtain a new map having the two-jet of the identity at w. The information that remains in the three-jet is the third derivative, which is an arbitrary element of $\mathbf{C}$. Note that if the original map is a Mœbius transformation, it will have been normalized to be the identity, so its third derivative will be 0.

don't worry too much about this stuff

As z varies, this determines a complex vector bundle over the domain of the map whose fiber consists of the bundle of three-jets modulo the chain rule action of Mœbius transformations on three-jets. The linear structure in the fiber arises not because the action is linear (which it isn't), but because a linear structure is inherent in the complex structure of $\mathbf{C}$, once the origin is determined.

What vector bundle is it? A third derivative is a cubic mapping from the tangent space of the domain to the tangent space of the range. However, for a nonsingular mapping, the tangent space of the domain is identified with the tangent space of the range via the first derivative, so we may describe the third derivative in terms of a cubic map P from the tangent space of the domain to itself. We can form the quotient $P(w)/w$ to replace this cubic map by a quadratic map from the tangent bundle of the domain to $\mathbf{C}$ which contains the same information.

It follows that this invariant for locally univalent functions up to Mœbius transformations, called the Schwarzian derivative, is a holomorphic quadratic differential form, usually referred to as a quadratic differential.

Sketching the picture of a conformal map which has the two-jet of the identity at a point (Figure 2.1: jets of locally univalent mappings), one can see that the third derivative tends to make the image of a small circle around a point in the domain look elliptic in the range. Since Mœbius transformations preserve circles, the Schwarzian derivative may be interpreted as measuring the third-order distortion of the shape of the image of a small circle.

A formula for the Schwarzian derivative can be readily determined from the information above, or it may be looked up—someplace else. Like the formula for the curvature of a curve in the plane, the formula looks somewhat mystical at first, and in a qualitative discussion, the formula tends to be a distraction from the real issues.

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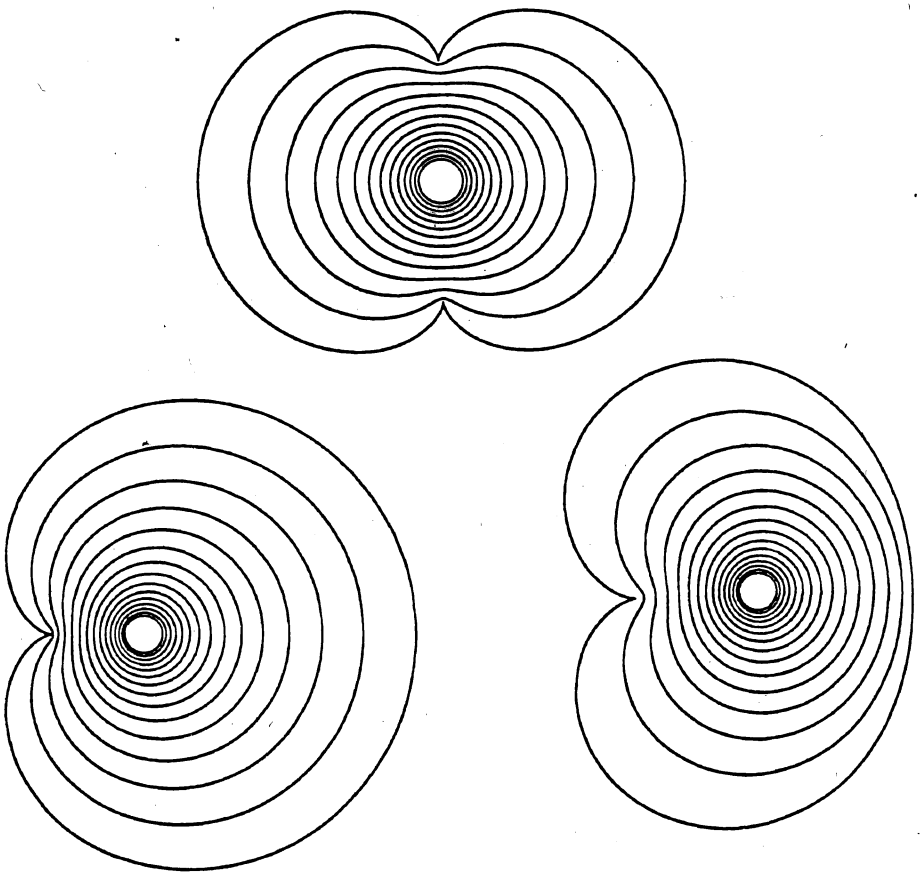

2.1. FIGURE: JETS OF LOCALLY UNIVALENT FUNCTIONS. Here are three univalent mappings illustrating the effects of second and third derivatives of a locally univalent mapping on a family of concentric circles. In (a) the mapping is $z \mapsto z + z^3/3$, and in (b), $z \mapsto z + z^2/2$. Thus, mapping (a) has the two-jet of the identity, and the third-order effect on the eccentricity of circles, as measured by the Schwarzian derivative, can be seen. Mapping (b) has the 1-jet but not the 2-jet of the identity, so even though its third derivative is 0, its Schwarzian derivative is not 0. The second derivative can be seen in the way that the centers of the images of circles are displaced toward the right. The 2-jet can be corrected by composing with the Mœbius transformation $z \mapsto z/(1-.5z)$. The composition is illustrated in (c). The third-order term in the Taylor expansion of the composition is $-z^3/4$, and again it can be seen in the third-order effect on eccentricity.

There's actually a sign error, should be $z \mapsto z/(1+.5z)$

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How does the Schwarzian derivative behave under composition? If f and g are two locally univalent mappings which have order two contact with the identity at a point z, then the third derivative of the composition is the sum of their third derivatives, so the Schwarzian derivative of the composition is the sum of the Schwarzian derivatives, at least in this case. The same formula holds in general, provided we are careful to treat the Schwarzian derivative as a quadratic form. This easily follows from the fact that the Schwarzian derivative is invariant under postcomposition by a Mœbius transformation, and it transforms as a quadratic form under precomposition by a Mœbius tranformation. More explicitly, for the Schwarzian derivative of an arbitrary composition $f \circ g$ evaluated at a point z, one can replace it by $(f \circ M_1) \circ (M_1^{\circ -1} \circ g \circ M_2)$, where M_1 and M_2 are the Mœbius transformations chosen so that the terms in parentheses have the 2-jet of the identity at $f \circ g(z)$.

The Schwarzian derivative together with the 2-jet of a locally univalent mapping determines the 3-jet, which is to say that the mapping satisfies a third-order differential equation in terms of its Schwarzian derivative. Since it is an ordinary differential equation, the equation can be locally integrated no matter what the holomorphic quadratic differential. Thus, every quadratic differential occurs, at least locally, as the Schwarzian derivative of a conformal map. Since Mœbius tranformations act transitively on nonsingular 2-jets, the three degrees of freedom for the solution are accounted for by compositions with Mœbius transformations. If the quadratic differential is defined over a simply connected domain, local solutions can be pasted together using Mœbius transformations to give a locally univalent mapping defined from the entire domain to the Riemann sphere $\hat{\mathbf{C}}$ and having the prescribed Schwarzian derivative.

Here is a slightly different way to think of the Schwarzian derivative. As a point w varies in the domain of f, the approximating Mœbius transformation $M(f, w)$ varies. The derivative of $M(f, w)$ with respect to w associates to any tangent vector V at w an infinitesimal Mœbius transformation, which is a holomorphic vector field on the Riemann sphere

$$X(f, w, V) = \frac{\partial}{\partial t} M^{\circ -1}(f, w) \circ M(f, w + tV).$$

The order of composition has been chosen so that $X(A \circ f, w, V) = X(f, w, V)$, so we may normalize f to have order 2 contact with the identity at w without changing X. Then it is clear that $X(f, w, V)$ vanishes through order 1 at w. X is thus determined by its second derivative, which is an element of $\hom(T \otimes T, T)$, where T is the tangent space to $\hat{\mathbf{C}}$ at w. Therefore we have as an invariant of f at w an element of $\hom(T, (T \otimes T, T))$, which is canonically isomorphic to $\hom(T \otimes T \otimes T, T)$, which in turn is isomorphic to $\hom(T \otimes T, \mathbf{C})$ (taking into account the fact that T is 1-dimensional).

This invariant is, of course, the same as the Schwarzian derivative of f, but it has perhaps a clearer interpretation as measuring the rate of change of the best approximating Mœbius transformation.

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