On the Construction of Substitutes

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Abstract

Gross substitutability is a central concept in Economics and is connected to important notions in Discrete Convex Analysis, Number Theory and the analysis of Greedy algorithms in Computer Science. Many different characterizations are known for this class, but providing a constructive description remains a major open problem. The construction problem asks how to construct all gross substitutes from a class of simpler functions using a set of operations. Since gross substitutes are a natural generalization of matroids to real-valued functions, matroid rank functions form a desirable such class of simpler functions.

Shioura proved that a rich class of gross substitutes can be expressed as sums of matroid rank functions, but it is open whether all gross substitutes can be constructed this way. Our main result is a negative answer showing that some gross substitutes cannot be expressed as positive linear combinations of matroid rank functions. En route, we provide necessary and sufficient conditions for the sum to preserve substitutability, uncover a new operation preserving substitutability and fully describe all substitutes with at most 4 items.

1 Introduction

The concept of gross substitutes (GS) occupies a central place in different areas such as economics, discrete mathematics, number theory and has been rediscovered in many contexts. In Economics it was proposed by Kelso and Crawford [17] as a sufficient (and in some sense necessary [11]) condition for Walrasian equilibrium to exist in economies with indivisible goods. The notion also appears in the existence of stable matchings in two sided markets [13, 27], to design combinatorial auctions [3], in the study of trading networks [14, 16], among others. In fact, Hatfield and Kominers [12] show that many tractable classes of preferences with complementarities have a hidden substitutability structure. The phenomenon of substitutes being embedded in more complex settings is also present in [30, 23].

In discrete mathematics, Murota and Shioura [21] define the concept of $M^\natural$-concave function which ports the concept of convex functions from continuous domains to the discrete lattice, carrying over various strong (Fenchel-type) duality properties (see [20] for a recent survey on discrete convex analysis). In number theory, Dress and Wenzel [7] defined the concept of valuated matroids to generalize the Grassmann-Plücker relations in p-adic analysis. Those correspond to the same class of functions as shown by Fujishige and Yang [10].

Finally, gross substitutes have a special role in computer science, since they correspond to the class of set function $v : 2^{[n]} \to \mathbb{R}$ for which the optimization problem $\max_{S\subseteq[n]} v(S) - \sum_{i\in S} p_i$ can be solved by the natural greedy algorithm for all $p \in \mathbb{R}^n$.

Given the central role that GS plays in many different fields, understanding its structure is an important problem. There have been many equivalent characterizations of GS through time: Kelso and Crawford [17], Murota [19], Murota and Shioura [21], Dress and Terhalle [9, 8], Ausubel and Milgrom [3], Reijnierse et al. [25], Lehmann et al. [18], Ben-Zwi [4]. All the previous characterizations define GS as the class of functions satisfying a certain property. Yet, providing a constructive description of GS remains an elusive open problem. This is in sharp contrast with submodular functions, a more complex class in many respects, but one that has a simple constructive description.

The construction problem. A constructive description consists of a base class of simpler functions (e.g. unit demand or matroid rank functions) together with a set of operations (e.g. sum, convolution, endowment, affine transformations) such that all GS functions can be constructed from the base class by applying such operations.

The first version of this question was asked by Hatfield and Milgrom [13]. They noted that most examples of substitutes arising in practical applications could be described as valuations that are built from assignment valuations (which are convolutions of unit-demand valuations) and the endowment operation. They called this class *Endowed Assignment Valuations* (EAV) and asked whether EAV exhaust all gross substitutes. Ostrovsky and Paes Leme [22] provided a negative answer to this question showing that some matroid rank functions cannot be constructed using those operations. They do so by adapting a result of Brualdi [6] on the structure of transversal matroids to the theory of gross substitutes. The main insight in [22] is that unit-demand valuations are not strong enough as a base class. They propose a class called *Matroid Based Valuations* (MBV) which are constructed from weighted matroid rank functions. It is still unknown whether MBV exhausts the whole class of substitutes or not.

Another important construction is due to Shioura [29], who provides a construction of a rich class of valuations called *Matroid Rank Sums* (MRS) and shows that this class contains many important examples of gross substitutes. Matroid rank sums are positive linear combinations of matroid rank functions such that the matroids satisfy a *strong quotient* property. In general positive linear combinations of matroids are not GS but the strong quotient property provides a sufficient condition for this to be true.

GS and matroids. Gross substitutes are similar to matroids in many respects. First, matroids can be described as the subset systems that can be optimized via greedy algorithms while GS is the collection of real-valued set functions that can be optimized using these same greedy algorithms. Another similarity is that both classes can be described via the exchange property. Finally, when we restrict our attention to GS functions where marginals are in $\{0, 1\}$ we obtain exactly the set of matroid rank functions. These similarities, among others, explain why GS are seen as the natural extension of matroids to real-valued functions.

The lack of constructive characterization for GS, combined with GS naturally generalizing matroids, begs the following question.

Can gross substitutes be constructed from matroids?

In other words, do all GS functions look like matroids or does the lift from subset systems to real-valued functions produce significantly different functions? Similarly as for Shioura's construction, we focus on positive linear combinations and ask the question of whether all gross substitutes are positive linear combinations of matroids.

Our results. Our main result is a negative answer to the question above. We show that not all gross substitute functions can be constructed via positive linear combinations from matroid rank functions. This implies in particular that MRS does not exhaust the class of GS functions.

The proof consists of exhibiting a GS valuation function over 5 elements that cannot be expressed as a positive linear combination of matroids. We note that there are 406 matroids over 5 elements (38 up to isomorphism) and that a GS function over 5 elements is defined by 40 non-linear conditions. The search space is huge, continuous and non-convex, so solving it by enumeration is infeasible even for 5 elements.

Our techniques involve building a combinatorial and polyhedral understanding of GS functions. In fact, we prove that for 3 and 4 elements, all GS functions can be written as convex combinations of matroids. In this process, we provide a polyhedral understanding of the set of GS functions and give necessary and sufficient conditions for the positive combination of two GS functions to be GS. We obtain the counter-example for $n = 5$ by carefully understanding where the techniques used for proving the $n = 3$ and 4 fail.

Upon obtaining an example for which the proof technique fails, we still need to argue that it is not in the convex combination of matroids. One way to do it is to enumerate over the 406 matroids over 5 elements and solve a large linear program. That would be a valid approach, but one that would make verifying correctness a much more complicated task. Instead, we use a combination of linear algebra and combinatorial facts about matroids to provide a complete mathematical proof that our counter-example is indeed not a convex combination of matroids.

Paper organization. We begin with preliminaries in Section 2. In Section 3, we discuss the main question of obtaining a constructive characterization of GS. We prove our main result in Section 4, that there exists a GS function that is not a positive linear combination of matroid rank functions. In Section 5, we present the tree-concordant-sum operation, show that it preserves substitutability, use it to show a positive answer to our main question for $n \leq 4$, and discuss how it helped in finding the negative instance for the main result. The conclusion is in Section 6.

2 Preliminaries

Valuation functions A valuation function is a map $v : 2^{[n]} \to \mathbb{R}$. We restrict ourselves to functions defined on the hypercube $2^{[n]}$ although the notions studied here generalize to the integer lattice $\mathbb{Z}^n$. Given a vector $p \in \mathbb{R}^n$ we define v_p as the function

$$v_p(S) = v(S) - \sum_{i \in S} p_i.$$

Affine transformations and normalized valuations We say that a valuation $\tilde{v}$ is an *affine transformation* of v if there is a vector $p \in \mathbb{R}^n$ and a constant $c \in \mathbb{R}$ such that $\tilde{v} = c + v_p$. We say that a valuation function v is normalized if $v(\emptyset) = 0$ and $v(\{i\}) = 0$ for every $i \in [n]$. Every valuation function can be obtained from a normalized valuation function via an affine transformation. Unless otherwise specified, we consider matroid rank functions in their normalized form.

Marginals and discrete derivatives Given sets $S, T \subseteq [n]$ we define the marginal contribution of S to T as $v(S|T) = v(S \cup T) - v(T)$. We omit parenthesis when clear from context and often replace $v(\{i, j\}|S)$ and $S \cup \{j\}$ by $v(i, j|S)$ and $S \cup j$ respectively.

Given a function $v : 2^{[n]} \to \mathbb{R}$ and $i \in [n]$ we define the derivative with respect to element i as the function $\partial_i v : 2^{[n] \setminus i} \to \mathbb{R}$ where $\partial_i v(S) = v(S \cup i) - v(S)$. The first derivative is simply the marginal $v(i|S)$. Applying the operator twice we obtain the second derivative:

$$\partial_{ij} v(S) = \partial_j[\partial_i v(S)] = \partial_i v(S \cup j) - \partial_i v(S) = v(S \cup ij) - v(S \cup i) - v(S \cup j) + v(S)$$

There is a nice economic interpretation of the second derivatives as a measure of the degree of substitutability of two goods. $\partial_{ij} v(S)$ represents the difference between the value of the bundle $\{i, j\}$ and sum of values of the two goods i and j separately. For example, if $\partial_{ij} v(S) = 0$, it means that having good i does not affect the value for good j conditioned on having a bundle S.

Functions as vectors We often view valuation functions $v : 2^{[n]} \to \mathbb{R}$ as a vector in $\mathbb{R}^{2^n}$ with coordinates indexed by $S \subseteq [n]$. This allows us to view a class of valuation functions as a subset of $\mathbb{R}^{2^n}$. We define the inner product between two valuations in the usual way:

$$\langle v, w \rangle = \sum_{S \subseteq [n]} v(S) w(S).$$

2.1 Substitutability

There are several equivalent ways to define gross substitutes, often also called GS, discrete concave functions or simply substitutes. The definition that is most convenient to work with for our theorems is the definition via discrete derivatives due to Reijnierse et al. [25] . The set $\mathbf{G}^n$ of gross substitutes is defined by:

$$\mathbf{G}^n = \{v : 2^{[n]} \to \mathbb{R};\ \partial_{ij} v(S) \leq \max[\partial_{ik} v(S), \partial_{jk} v(S)] \leq 0,\ \forall S \subseteq [n], \forall i, j, k \notin S\}$$

We note that the definition does not require monotonicity.

We also state some of the most common ways to define substitutes below. We refer to [24] for a proof that they are equivalent to the definition above as well as other formulations.

- **No price complementarity.** In economics, substitutes was originally formulated as the condition that an increase in price for a certain good, cannot decrease the demand for other goods. Formally, given a vector $p \in R^n$, let $D(v; p) = \text{argmax}_{S \subseteq [n]} v_p(S)$ be the demand correspondence. Then $v \in \mathbf{G}^n$ iff for all vectors $p \leq p'$ and $S \in D(v; p)$ there is $S' \in D(v; p')$ such that $S \cap \{j; p_j = p'_j\} \subseteq S'$.

- **Discrete concavity.** In discrete mathematics, substitutes are a natural notion of concavity for functions defined in the hypercube. We say that a function over the reals $f : \mathbb{R}^n \to \mathbb{R}$ is concave if for every vector $p \in \mathbb{R}^n$, every local minimum of $f_p(x) = f(x) - \sum_i p_i x_i$ is also a global minimum[1]. This definition naturally extends to the hypercube: $v \in \mathbf{G}^n$ iff for every $p \in \mathbb{R}^n$, if S is a local minimum of v_p, i.e.

$$v_p(S) \geq v_p(S \cup i) \quad v_p(S) \geq v_p(S \setminus j) \quad v_p(S) \geq v_p(S \cup i \setminus j), \forall i \notin S, j \in S$$

  then S is also a global minimum, i.e. $S \in D(v; p)$.

[1] That is perhaps not the most common definition of concavity but it is completely equivalent to the condition $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y), \forall t \in [0, 1]$.

- **Matroidality.** Substitutes can also be defined in terms of greedy algorithms: $v \in \mathbf{G}^n$ iff for every $p \in \mathbb{R}^n$, the greedy algorithm always computes the maximum of v_p, i.e., if we start with $S = \emptyset$ and keeps adding the element $i \in \operatorname{argmax}_i v_p(i|S)$ with largest marginal contribution while it is positive, we obtain $S \in D(v;p)$.

2.2 Matroids

Many of those definitions strikingly resemble the definition of matroids. In fact some of the early appearances of gross substitutes were attempts to generalize matroids from collections of subsets to real-valued functions. Dress and Terhalle [9, 8] called their definition *matroidal maps* and Dress and Wenzel called their notion *valuated matroids*.

A subset collection over $[n]$ is simply a subset $\mathcal{M} \subseteq 2^{[n]}$. This subset collection is a matroid if it satisfies one of the following equivalent properties:

- **Greedy optimization.** The collection $\mathcal{M}$ is a matroid if for every vector $p \in \mathbb{R}^n$, the set $S \in \mathcal{M}$ maximizing $\sum_{i \in S} p_i$ can be obtained by the greedy algorithm that starts with the empty set $S = \emptyset$ and keeps adding $i \in \operatorname{argmax}_{i; S \cup i \in \mathcal{M}} p_i$ to S while p_i is positive.

- **Exchange property.** We say that collection $\mathcal{M}$ is a matroid if $T \subseteq S \in \mathcal{M}$ then $T \in \mathcal{M}$ and for every $S, T \in \mathcal{M}$ with $|S| < |T|$, there is $i \in T \setminus S$ such that $S \cup i \in \mathcal{M}$.

Given any subset system $\mathcal{M}$, we can define its rank function $r_{\mathcal{M}} : 2^{[n]} \to \mathbb{Z}_+$ as $r_{\mathcal{M}}(S) = \max\{|T|\,; T \subseteq S \text{ and } T \in \mathcal{M}\}$. This allows us to define the set of matroid rank functions as:

$$\mathbf{M}^n = \{r_{\mathcal{M}}; \mathcal{M} \text{ is matroid over } [n]\}$$

When we translate the subset system $\mathcal{M}$ to a rank function $r_{\mathcal{M}}$, the exchange property becomes exactly the discrete differential equation in the definition of $\mathbf{G}^n$. This observation implies that matroid rank functions are exactly the gross substitutes functions with $\{0, 1\}$-marginals:

$$\mathbf{M}^n = \{v \in \mathbf{G}^n; v(\emptyset) = 0; \partial_i v(S) \in \{0, 1\}, \forall S \subseteq [n], i \notin S\}$$

For completeness, we provide a proof of this result in Theorem 19 in the appendix.

2.3 Relation between classes of functions

Another class that is important for us is submodular functions:

$$\mathbf{S}^n = \{v : 2^{[n]} \to \mathbb{R}; \partial_{ij} v(S) \le 0, \forall S \subseteq [n], i, j \notin S\}$$

The following relation between the classes hold: $\mathbf{M}^n \subseteq \mathbf{G}^n \subseteq \mathbf{S}^n$. The classes $\mathbf{G}^n$ and $\mathbf{S}^n$ are defined in terms of second order discrete derivatives, which are invariant under affine transformations, i.e, if $\tilde{v}$ is obtained from v via an affine transformation then $\partial_{ij} v(S) = \partial_{ij} \tilde{v}(S)$. In particular this means that gross substitutes is invariant under affine transformations. If we want to understand $\mathbf{G}^n$ it is enough to understand the class $\mathbf{G}^n_0$ of normalized gross substitutes:

$$\mathbf{G}^n_0 = \{v \in \mathbf{G}^n; v(\emptyset) = v(i) = 0, \ \forall i \in [n]\}$$

since we can describe $\mathbf{G}^n = \mathbf{G}^n_0 + \mathbf{E}^n$ where $\mathbf{E}^n$ is the class of affine valuations functions:

$$\mathbf{E}^n = \{v : 2^{[n]} \to \mathbb{R}; v(S) = c + \sum_{i \in S} p_i; c, p_i \in \mathbb{R}\}$$

It will also be convenient to define the notion of normalized matroid rank functions:

$$\mathbf{M}^n_0 = \{v : 2^{[n]} \to \mathbb{R}; \exists \mathcal{M} \text{ matroid s.t. } v(S) = r_{\mathcal{M}}(S) - \sum_{i \in S} r_{\mathcal{M}}(i)\},$$

which are exactly the normalized gross substitutes functions $\mathbf{G}^n_0$ with $\{-1, 0\}$-marginals.

3 What are the building blocks of Substitutes ?

A major open question in the theory of gross substitutes is how to find a constructive description of the class. A constructive description has two parts: a base class of simpler functions and a set of operations that allow us to build complex functions from simpler ones. There are a number of operations that are known to preserve substitutability. We mention the two that are most relevant for this paper here and discuss additional operations in Section 6.

- **Affine transformations.** If $v : 2^{[n]} \to \mathbb{R}$ satisfies gross substitutes and $p \in \mathbb{R}^{[n]}$ is a vector and $u_0 \in \mathbb{R}$ then we can build the affine transformation $\tilde{v} : 2^{[n]} \to \mathbb{R}$ as $\tilde{v}(S) = v(S) + \sum_{i \in S} p_i + u_0$.
- **Strong Quotient Sum [29].** Give two valuations $v, w : 2^{[n]} \to \mathbb{R}$ and $\alpha_1, \alpha_2 \geq 0$, we say that v is a strong quotient of w if $v(S|T) \leq w(S|T)$ for all $S, T \subseteq [n]$. Given two valuations v and w such that v is a strong quotient of w and w is a matroid rank function, we define the strong quotient sum $\tilde{v} : 2^{[n]} \to \mathbb{R}$ as $\tilde{v}(S) = \alpha_1 v(S) + \alpha_2 w(S)$.

Those operations are known to preserve gross substitutability. A major open question is whether all substitutes can be built from matroid rank functions using those operations (or perhaps a larger class of simpler operations). We focus here on affine transformations and positive linear combinations (which is a strict generalization of the strong quotient sum operation).

Main Question. *Are all gross substitutes positive combinations of matroid rank functions modulo an affine transformation? Formally, given $v \in \mathbf{G}^n$, is there an affine transformation $\tilde{v} \in \mathbf{G}^n$, matroid rank functions $r_i \in \mathbf{M}^n$ and positive constants $\alpha_i \in \mathbb{R}_+$ such that:*

$$\tilde{v} = \sum_i \alpha_i r_i.$$

An equivalent way to ask this question is via the normalized classes: this allows us to ignore the affine transformations. Given $v \in \mathbf{G}^n_0$, are there $r_i \in \mathbf{M}^n_0$ and $\alpha_i \geq 0$ such that $v = \sum_i \alpha_i r_i$?

3.1 Building blocks for submodular functions

Before we go into our results, we would like to mention a simple constructive description for submodular functions $\mathbf{S}^n$ having the set of matroid rank functions as base. This will serve as a warm up for the study of substitute valuations. Besides affine transformations, the following operations preserve submodularity:[2]

[2] We note that even though positive linear combination and item grouping preserve submodularity, neither of them preserves substitutability in general.

- **Positive linear combination.** If $v_1, v_2 : 2^{[n]} \to \mathbb{R}$ are submodular and $\alpha_1, \alpha_2 \geq 0$ then $\tilde{v} = \alpha_1 v_1 + \alpha_2 v_2$ is also submodular.
- **Item grouping.** If $v : 2^{[n]} \to \mathbb{R}$ is submodular and $S_1, \ldots, S_k$ is a partition of $[n]$ then the function $w : 2^{[k]} \to \mathbb{R}$ defined as $w(T) = v(\cup_{t \in T} S_t)$ is also submodular.

It turns out those operations are sufficient to build any submodular function starting from the set of matroid rank functions. A proof is provided in Appendix B.

Theorem 1. *Any submodular function can be obtained starting from the set of matroid rank functions and applying the operations of affine transformations, positive linear combination and item grouping.*

4 GS is not in the cone of matroids

In this section, we exhibit a specific GS function and show that it cannot be expressed as a positive linear combination of matroid rank functions. This section is devoted to a (non-computational) proof of that fact. We defer to Section 5 a discussion on how we found such function.

At a high level, the analysis uses duality and Farkas' lemma. Farkas's conditions require the existence of a certificate whose inner product with all matroid rank functions has non-negative sign and the inner product with the candidate function is strictly negative (Section 4.1). The core of the proof consists in showing in a non-computational manner that the certificate satisfies the desired conditions (Section 4.2). This is done with a simple lemma about the local structure of matroid rank functions and a non-trivial partition of the collection of sets into local groups that can be analyzed individually with that lemma.

4.1 The GS function and the certificate

The GS function that we consider for the remaining in this section is over five elements $[5] = \{1, 2, 3, 4, 5\}$. This function v is described in its *normalized form*[3] in Figure 1. Since the function is normalized, it is enough to define it for $|S| \geq 2$. Checking that the function satisfies the GS conditions ($v \in \mathbf{G}_0^5$) amounts to checking the inequalities: $\partial_{ij} v(S) \leq \max[\partial_{ik} v(S), \partial_{kj} v(S)] \leq 0$ for all $S \subseteq [5]$. There are 40 such inequalities. It is a tedious but short verification, which can be found in Appendix C.1.

Now that we established that $v \in \mathbf{G}_0^5$, we want to prove that there do not exist (normalized) matroid rank functions[4] $r_i \in \mathbf{M}_0^5$ and $\alpha_i \geq 0$ such that $v = \sum_{i=1}^n \alpha_i r_i$.

Lemma 2 (Farkas' Lemma)**.** *Let $M \in \mathbb{R}^{m \times n}$ and $v \in \mathbb{R}^m$. Then exactly one of the following two statements is true:*

- *There exists $\alpha \in \mathbb{R}^n$ such that $v = M\alpha$ and $\alpha \geq 0$*
- *There exists $y \in \mathbb{R}^m$ such that $M^\intercal y \geq 0$ and $\langle v, y \rangle < 0$.*

We immediately obtain the following corollary which gives two conditions such that, if satisfied, we obtain the desired negative result.

[3]Recall from Section 2 that a valuation function is normalized if $v(\emptyset) = v(i) = 0$ for all $i \in [5]$.
[4]Recall the definition of a normalized matroid rank function in Section 2.3.

Sets of size 2	value	Sets of size 3	value	Sets of size 4	value
{1,2}	-1	{1,2,3}	-2	{1,2,3,4}	-3
{1,3}	-1	{1,2,4}	-2	{1,2,3,5}	-3
{1,4}	0	{1,2,5}	-2	{1,2,4,5}	-3
{1,5}	0	{1,3,4}	-1	{1,3,4,5}	-2
{2,3}	-1	{1,3,5}	-1	{2,3,4,5}	-2
{2,4}	0	{1,4,5}	-1		
{2,5}	0	{2,3,4}	-1		
{3,4}	0	{2,3,5}	-1	Sets of size 5	value
{3,5}	0	{2,4,5}	-1	{1,2,3,4,5}	-4
{4,5}	0	{3,4,5}	-1		

Figure 1: The GS function v

Sets of size 2	value	Sets of size 3	value	Sets of size 4	value
{1,2}	-1	{1,2,3}	-1	{1,2,3,4}	1
{1,3}	1	{1,2,4}	1	{1,2,3,5}	1
{1,4}	-1	{1,2,5}	1	{1,2,4,5}	-1
{1,5}	-1	{1,3,4}	-1	{1,3,4,5}	-1
{2,3}	1	{1,3,5}	1	{2,3,4,5}	-1
{2,4}	-1	{1,4,5}	1		
{2,5}	-1	{2,3,4}	-1		
{3,4}	-1	{2,3,5}	1	Sets of size 5	value
{3,5}	-1	{2,4,5}	1	{1,2,3,4,5}	-1
{4,5}	-1	{3,4,5}	1		

Figure 2: The certificate y

Corollary 3. *Let v be a function over five elements. If there exists $y \in \mathbb{R}^{32}$ such that $\langle v, y\rangle < 0$ and $\langle r, y\rangle \geq 0$ for all normalized matroid rank functions r over five elements, then v cannot be expressed as a positive linear combination of normalized matroid rank functions.*

4.2 Farkas' conditions

We consider the certificate y given in Figure 2 and show that the two conditions for Corollary 3 hold for that particular certificate y and the gross substitute function v. The first and second conditions are shown in Lemma 4 and Lemma 6 respectively. The first condition is trivial to verify.

Lemma 4. *Let v be the gross substitute function given in Figure 1 and y be the certificate given in Figure 2, then $\langle y, v\rangle < 0$.*

Proof. Proof. This is a simple summation and we get $\langle y, v\rangle = -1 < 0$.

□

The interesting condition to show is $\langle y, r\rangle \geq 0$ for all matroid rank functions r. We first give a simple lemma about the local structure of matroid rank functions which will motivate the approach for the analysis of this second condition.

Group 1	value	Group 2	value	Group 4	value	Group 5	value
{3,4}	-1	{1,3}	1	{1,5}	-1	{1,2}	-1
{4,5}	-1	{1,4}	-1	{2,5}	-1	{1,2,4}	1
		{1,3,4}	-1	{3,5}	-1	{1,2,5}	1
				{1,4,5}	1	{1,2,4,5}	-1
		Group 3	value	{2,4,5}	1		
		{2,3}	1	{3,4,5}	1	Group 6	value
		{2,4}	-1	{2,3,5}	1	{1,2,3}	-1
		{2,3,4}	-1	{2,3,4,5}	-1	{1,2,3,4}	1
				{1,3,5}	1	{1,2,3,5}	1
				{1,3,4,5}	-1	{1,2,3,4,5}	-1

Figure 3: The partition of the sets into groups, with the corresponding values of y.

Lemma 5. *Let r be a matroid rank function. For any set S and elements $a_1, a_2 \notin S$,*

- *if $r(S \cup a_1) - r(S) = -1$ and $r(S \cup a_2) - r(S) = -1$, then $r(S \cup a_1, a_2) - r(S) = -2$;*
- *if $r(S \cup a_1) - r(S) = -1$ and $r(S \cup a_2) - r(S) = 0$, then $r(S \cup a_1, a_2) - r(S) = -1$.*

Proof. Proof. We first decompose the quantity of interest in two terms,

$$r(S \cup a_1, a_2) - r(S) = (r(S \cup a_1, a_2) - r(S \cup a_2)) + (r(S \cup a_2) - r(S)).$$

Since r is a matroid rank function, marginal contributions are either -1 or 0 and $r(S \cup a_1, a_2) - r(S \cup a_2) \in \{-1, 0\}$. Next, by submodularity and the assumption for both cases, we get $r(S \cup a_1, a_2) - r(S \cup a_2) \leq r(S \cup a_1) - r(S) = -1$. Thus, $r(S \cup a_1, a_2) - r(S \cup a_2) = -1$ and

$$r(S \cup a_1, a_2) - r(S) = -1 + r(S \cup a_2) - r(S),$$

which concludes the proof. □

The main idea to show that $\langle y, r \rangle \geq 0$ for all matroid rank functions r is to first partition the sets into six local groups $G_1, \ldots, G_6$, described in Figure 3, and then use Lemma 5 to argue about the value $\langle y_G, r_G \rangle$ for each local group G, where v_G denotes the subvector of a vector v of length $|G|$ induced by the indices corresponding to sets in group G.

Lemma 6. *Let y be the certificate. Then for all matroid rank functions r, $\langle y, r \rangle \geq 0$.*

Proof. Proof. We first show that for all G_i such that $i \neq 4$, $\langle y_{G_i}, r_{G_i} \rangle \geq 0$ for all matroid rank functions r. Then, we show that $\langle y_{G_4}, r_{G_4} \rangle \geq -1$ and that if $\langle y_{G_4}, r_{G_4} \rangle = -1$, then for at least one other group G, $\langle y_G, r_G \rangle \geq 1$. Since $\langle y, r \rangle = \sum_{i=1}^{6} \langle y_{G_i}, r_{G_i} \rangle$, we then obtain $\langle y, r \rangle \geq 0$ (recall that the empty set and singletons are normalized to have value 0 and that function values are nonpositive for all subsets).

We first show that for all G_i such that $i \neq 4$, $\langle y_{G_i}, r_{G_i} \rangle \geq 0$ for all matroid rank functions r. This is trivial for group G_1 since $r(S) \leq 0$ for all sets S. For group G_2, recall that we have

$$y(1,3) = 1 \quad y(1,4) = -1 \quad y(1,3,4) = -1.$$

Since $r(1) = 0$ and the marginal contributions of matroid rank functions are either 0 or -1, $r(1,3), r(1,4) \in \{-1, 0\}$ and we consider the following three cases:

- If $r(1,3) = -1$ and $r(1,4) = -1$, then by Lemma 5, $r(1,3,4) = -2$. We get $\langle y_{G_2}, r_{G_2} \rangle = 2$.
- If $r(1,3) = -1$ and $r(1,4) = 0$, then by Lemma 5, $r(1,3,4) = -1$ and we get $\langle y_{G_2}, r_{G_2} \rangle = 0$.
- If $r(1,3) = 0$, then $\langle y_{G_2}, r_{G_2} \rangle \geq 0$.

Group G_3 follows similarly as for group G_2. For group G_5, recall that we have

$$y(1,2) = -1 \quad y(1,2,4) = 1 \quad y(1,2,5) = 1 \quad y(1,2,4,5) = -1.$$

Since r is submodular, $r(S) + r(T) \geq r(S \cap T) + r(S \cup T)$ for any sets S, T. Thus, $-r(1,2) + r(1,2,4) + r(1,2,5) - r(1,2,4,5) \geq 0$ and this implies that $\langle y_{G_5}, r_{G_5} \rangle \geq 0$. Group G_6 follows similarly as for group G_5.

It remains to show that $\langle y_{G_4}, r_{G_4} \rangle \geq -1$ and that if $\langle y_{G_4}, r_{G_4} \rangle = -1$, then for at least one other group G, $\langle y_G, r_G \rangle \geq 1$. First, consider the following three possible partitions of G_4.

$$\begin{array}{l} (\{3,5\},\{3,4,5\},\{1,3,5\},\{1,3,4,5\}), (\{2,5\},\{2,4,5\},\{2,3,5\},\{2,3,4,5\}), (\{1,5\},\{1,4,5\}) \\ (\{3,5\},\{3,4,5\},\{2,3,5\},\{2,3,4,5\}), (\{1,5\},\{1,4,5\},\{1,3,5\},\{1,3,4,5\}), (\{2,5\},\{2,4,5\}) \\ (\{2,5\},\{2,4,5\},\{2,3,5\},\{2,3,4,5\}), (\{1,5\},\{1,4,5\},\{1,3,5\},\{1,3,4,5\}), (\{3,5\},\{3,4,5\}) \end{array}$$

For the first two parts G_4' and G_4'' of each possible partition, similarly as for G_5, we obtain that $\langle y_{G_4'}, r_{G_4'} \rangle \geq 0$ and $\langle y_{G_4''}, r_{G_4''} \rangle \geq 0$. For the last part G_5''', it is easy to see that $\langle y_{G_4'''}, r_{G_4'''} \rangle \geq -1$. Thus, we obtain that $\langle y_{G_4}, r_{G_4} \rangle \geq -1$.

Next, consider

$$r(1,4,5) - r(1,5) \quad r(2,4,5) - r(2,5) \quad r(3,4,5) - r(3,5).$$

If one of these three difference is 0, then by considering the corresponding above partition where the two terms in the difference form the last part G_4''' of the partition, we get that $\langle y_{G_4}, r_{G_4} \rangle \geq 0$.

Next, consider the case where these three differences are all equal to -1 and $\langle y_{G_4}, r_{G_4} \rangle = -1$. If $r(4,5) = -1$, then $y(4,5)r(4,5) = 1$ compensates for $\langle y_{G_4}, r_{G_4} \rangle = -1$ and $\langle y, r \rangle \geq 0$. Otherwise, $r(4,5) = 0$. This implies that $r(1,4,5) = r(2,4,5) = r(3,4,5) = -1$ and $r(1,5) = r(2,5) = r(3,5) = 0$ since the above differences are all -1. It must also be the case that $r(2,3,4,5) = -2$ by submodularity since a_2 has marginal contribution -1 to $\{a_4, a_5\}$ and since $r(3,4,5) = -1$. Similarly, $r(1,3,4,5) = -2$.

Next, we focus on G_2 and recall that we have

$$y(1,3) = 1 \quad y(1,4) = -1 \quad y(1,3,4) = -1.$$

Note that $r(1,3,4) \leq -1$ since $r(1,3,4,5) = -2$. Since $r(1,3) \geq r(1,3,4)$, it must be the case that $r(1,4) = 0$, $r(1,3), r(1,3,4) = -1$ for $\langle y_{G_2}, r_{G_2} \rangle = 0$. If that is not the case, then $\langle y_{G_2}, r_{G_2} \rangle \geq 1$ and that compensates for G_4 and we get $\langle y, r \rangle \geq 0$.

Thus, in the remaining case $r(1,3) = -1$. Similarly for G_3 with $r(2,3,4,5) = -2$, the remaining case is if $r(2,4) = 0$ and $r(2,3) = -1$. Next, with $r(2,3) = r(1,3) = -1$, then, by Lemma 5, $r(1,2,3) = -2$, which in turn implies that $r(1,2) = -1$.

Next, we consider G_5. Recall that

$$y(1,2) = -1 \quad y(1,2,4) = 1 \quad y(1,2,5) = 1 \quad y(1,2,4,5) = -1.$$

Since $r(4,5) = 0$, $r(1,4,5) = -1$ and $r(2,4,5) = -1$, by Lemma 5, $r(1,2,4,5) = -2$. If $r(1,2,5), r(1,2,4) = -1$, then with $r(1,2) = -1$, $\langle y_{G_5}, r_{G_5} \rangle \geq 1$ and we are done. Otherwise, $r(1,2,5)$ or $r(1,2,4)$ is equal to -2. But this is impossible since we are in a case where $r(1,5) = 0$ and $r(1,4) = 0$. Thus, if $\langle y_{G_4}, r_{G_4} \rangle = -1$, then for at least one other group G, $\langle y_G, r_G \rangle \geq 1$ and we get $\langle y, r \rangle \geq 0$. □

Combining Corollary 3, Lemma 4 and Lemma 6, we obtain the main result.

Theorem 7. *The gross substitute function v cannot be expressed as a positive linear combination of normalized matroid rank functions. In particular, no affine transformation of v can be expressed as a positive linear combination of matroid rank functions.*

In particular, this implies that the strong quotient sum and tree-concordant-sum operations are not sufficient to construct all gross substitutes from matroid rank functions and that MRS valuations do not exhaust gross substitutes. We note that even though v is non-monotone, the main result holds for monotone gross substitute functions since there exists some affine transformation of v that is monotone. We also extend the negative result from matroid rank functions to weighted matroid rank functions. This follows from the fact that weighted matroid rank functions can be expressed as a positive linear combination of unweighted matroid rank functions, which was proven in [29] and we give a proof in Appendix C for completeness as Lemma 22.

Corollary 8. *No affine transformation of function v can be expressed as a positive linear combination of weighted matroid rank functions.*

5 The Tree-Concordant-Sum Operation

In this section, we define the notion of a tree representation which abstracts the combinatorial structure of *tree-form Hessians* of Hirai and Murota [15]. We first show that this representation has the following nice property: the condition that two functions have such a tree representation that is *compatible* is necessary and sufficient for the summation operation to preserve substitutability. We call this new operation preserving substitutability tree-concordant-sum and show that it also provides a polyhedral characterization of gross substitutes.

We then use this representation to give a positive answer to our main question for $n \leq 4$: a GS over at most 4 elements can be written as a positive linear combination of matroid rank functions. This implies that at least 5 elements are necessary to obtain the negative answer from the previous section.

Finally, we discuss how the polyhedral understanding of gross substitutes based on this tree representation combined with computational techniques led to the discovery of the counterexample for $n = 5$.

5.1 Substitution Trees

The tree representation of GS is best explained using the discrete derivative property introduced in Section 2. Since it will be convenient to work with non-negative numbers, we introduce the notation:

$$\Delta_{ij}^S(v) = -\partial_{ij} v(S)$$

omitting v when clear from context. In this notation, we can write the GS condition as:

$$\Delta_{ij}^S \geq \min(\Delta_{ik}^S, \Delta_{jk}^S) \geq 0$$

If we permute the identities of i, j, k such that the symbols are sorted, we have the following triangle property: $\Delta_{ij}^S = \Delta_{ik}^S \leq \Delta_{jk}^S$. Hirai and Murota [15] and Bing et al. [5] note that this resembles the definition of *ultra-metrics*, which admit tree-like representations. This enables similar tree-like representations for GS. Below we describe the notion of Hirai and Murota [15] following the presentation in Paes Leme 2017.

Theorem 9 (Hirai and Murota [15]). *A function v satisfies the GS condition iff for every subset S there is a tree $T(v)^S$ having the elements of $[n] \setminus S$ in the leaves and non-negative real number labels in the internal nodes such that:*

- *The label of each internal node is larger than or equal to the label of its parent.*
- *For every $i, j \notin S$, Δ_{ij}^S corresponds to the label of the lowest common ancestor* (lca) *of the leaves corresponding to i and j.*

We observe that the representation of Murota and Hirai has two components: a purely combinatorial structure, which is the collection of trees and a numerical component, which are the values of the labels. If we abstract the numerical component, we obtain what we call a tree structure:

Definition 10 (Tree structure). *A tree structure corresponds to a collection of trees $\{T^S\}_S$ indexed by subsets $S \subseteq [n]$ such that the leaves of tree T^S correspond to the elements of $[n] \setminus S$.*

We say that a valuation function v admits a tree structure $\{T^S\}_S$ if we can represent v in the sense of Theorem 9 using those trees. The tree structure might not be unique. For example, if for a certain v and S, Δ_{ij}^S is given by the matrix in the left of Figure 4, then the two trees in the figure are valid structures for Δ_{ij}^S. There is therefore some flexibility in the choice of the tree structure, which allow us to define the notion of tree-concordant:

Definition 11 (Tree-concordance). *We say that two GS valuations are* tree concordant *if they admit the same tree structure $\{T^S\}_S$.*

A clean way to check when two functions are tree concordant is via the concept of minimal representation. We say that a tree representation for valuation v is *minimal* if no node has the same label as its parent. The tree on the left in Figure 4 is minimal, for example while the one on the right is not. By the definition of minimal, it is clear that each GS valuation has an unique minimal representation. It can be obtained by starting from any representation and collapsing tree edges connecting internal nodes with the same label.

To check when there is one tree structure that two functions simultaneously admit, it is enough to look at the minimal representations. To see that, it is useful to view a tree as a laminar family. Given a tree T^S with elements $[n] \setminus S$ in the leaves, we can represent it by a family of subsets L^S constructed as follows: a set $X \subseteq [n] \setminus S$ is in L^S iff there is an internal node in T^S such that X is the set of leaves below v. Such subset collection is what is called a laminar family:

Definition 12 (Laminar family). *A collection L of subsets is called a laminar family if for every $X, Y \in L$ either: (i) $X \cap Y = \emptyset$; or (ii) $X \subseteq Y$ or (iii) $Y \subseteq X$.*

Now, we can check if two functions u and v are tree-concordant as follows:

$$\begin{bmatrix} * & 2 & 1 & 1 & 1 \\ 2 & * & 1 & 1 & 1 \\ 1 & 1 & * & 3 & 3 \\ 1 & 1 & 3 & * & 3 \\ 1 & 1 & 3 & 3 & * \end{bmatrix}$$

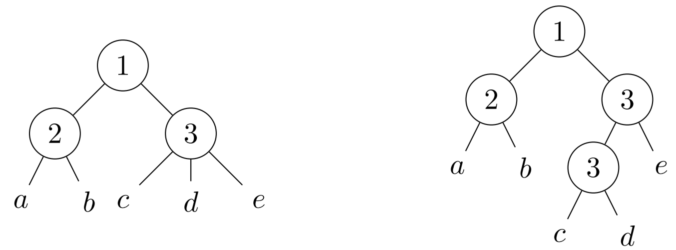

Figure 4: Two valid tree representations for the same matrix $[\Delta^S_{ij}]_{ij}$.

Lemma 13. *If u and v are GS functions and $\{T^S(u)\}$ and $\{T^S(v)\}$ are its minimal tree structures and $\{L^S(u)\}$ and $\{L^S(v)\}$ are its corresponding laminar family representations, then u and v are tree concordant iff for every S, $L^S(u) \cup L^S(v)$ is also a laminar family.*

The proof is straightforward from definitions and the fact that there is natural one-to-one mapping between trees and laminar families. What is interesting about tree concordance is that it provides necessary and sufficient conditions for the sum of two GS functions to be GS:

Theorem 14. *Let u and v be two gross substitute functions. The function $u+v$ is a gross substitute function if and only if u and v are tree-concordant.*

Proof. Proof. We first show that if u and v are tree-concordant, then $u+v$ is a gross-substitute. Consider the tree representation T that has the structure of u and v, which is identical since they are tree-concordant, but with numerical values at each internal node that is the sum of the values at that node for u and v. Since second order derivatives are linear:

$$\Delta^S_{ij}(u+v) = \Delta^S_{ij}(u) + \Delta^S_{ij}(v)$$

and it clearly admits the same tree structure T, so by Theorem 9, $u+v$ is in GS.

Next, we show that if u and v are not tree-concordant, then $u+v$ is not a gross substitute function. Assume u and v are not tree-concordant then by Lemma 13 there is a set S such that for the laminar representations $L^S(u)$ and $L^S(v)$ there are sets with non-trivial intersection, i.e., there are $X \in L^S(u)$ and $Y \in L^S(v)$ such that we can find $i \in X \setminus Y$, $j \in X \cap Y$ and $k \in Y \setminus X$. Therefore:

$$\Delta^S_{ij}(u) > \Delta^S_{ik}(u) = \Delta^S_{kj}(u)$$

$$\Delta^S_{ij}(v) = \Delta^S_{ik}(v) < \Delta^S_{kj}(v)$$

therefore, it must be the case that:

$$\Delta^S_{ik}(u+v) < \min[\Delta^S_{ij}(u+v), \Delta^S_{kj}(u+v)]$$

which is a violation of gross substitutability. □

Since it is a necessary and sufficient condition, this includes the strong-quotient-sum property in [29] as a special case:

Corollary 15. *If a GS function u and a matroid rank function v satisfy the strong-quotient-sum property, then they are tree-concordant.*

5.2 Polyhedral description of $\mathbf{G}^n$

The concept of tree structure and tree concordance provide a good tool for describing the geometry of $\mathbf{G}^n$. Viewing valuations as vectors in $\mathbb{R}^{2^n}$, we can view $\mathbf{G}^n$ as a subset of that space. Lehmann, Lehmann and Nisan [18] observe that the set is a non-convex and has zero measure.

The concepts developed earlier in this section allow us to decompose the space in finitely many convex cones. Since a tree structure is a finite combinatorial object, there are finitely many such structures. Fix a tree structure $\{T^S\}_S$ and consider the subset of $\mathbf{G}^n$ with functions that admit $\{T^S\}_S$. All functions in this subset are tree concordant, so the set is closed under positive linear combination, forming a *convex cone.*

What we would like to do next is to understand how those cones look like. For that we will fix a tree structure and assign labels to internal nodes. The labels must be non-negative and the label of a node cannot be smaller than the label of its parent, but those conditions alone are not sufficient for the existence of a valuation function producing those Δ_{ij}^S. An extra condition that is required is what we call *integrability*[5]:

$$\Delta_{ij}^{S+k} - \Delta_{ij}^S = \Delta_{ik}^{S+j} - \Delta_{ik}^S = \Delta_{jk}^{S+i} - \Delta_{jk}^S \tag{Int}$$

Lemma 16 (Integrability)**.** *Given values Δ_{ij}^S for all i, j, S such that $i, j \notin S$, there is a valuation v such that $\partial_{ij} v(S) = -\Delta_{ij}^S$ if and only if the integrability conditions (Int) hold.*

In Appendix D we provide a discussion on integrability conditions for discrete functions. The lemma above follows directly from Lemma 24 in the appendix.

Since the symbols Δ_{ij}^S have a special tree-form for gross substitutes, it is convenient to re-write the integrability conditions in the following form. We simplify the notation for $S \cup \{i\}$ with $S + i$ for the remainder of the paper.

Lemma 17. *An assignment of labels to the internal nodes of a tree structure corresponds to a representation of GS function if and only if the following condition holds for every $S \subseteq [n]$ and $i, j, k \notin S$:*

$$\begin{aligned} &\text{if} \quad && \Delta_{ik}^S = \Delta_{jk}^S = \Delta_{ij}^S - \alpha, \text{ for some } \alpha \geq 0 \\ &\text{then} \quad && \Delta_{ik}^{S+j} = \Delta_{jk}^{S+i} = \Delta_{ij}^{S+k} - \alpha \end{aligned}$$

The following corollary (which is a rephrasing of Corollary 25 in the appendix) shows how to explicitly reconstruct a function from second order derivatives. We denote by $S_{<i}$ the set of elements smaller than i according to their label, i.e., $S_{<i} = \{j : j < i\}$.

Corollary 18. *Given Δ_{ij}^S satisfying the integrability conditions (Int), the unique normalized function such that $\partial_{ij} v(S) = -\Delta_{ij}^S$ is given by:*

$$v(S) = -\sum_{i,j \in S; i<j} \Delta_{ij}^{S_{<i}}$$

In particular, all the functions such that $\partial_{ij} v(S) = -\Delta_{ij}^S$ are affine transformations of the function defined above.

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[5] We thank Kazuo Murota for his suggestion on a earlier version of this manuscript to phrase this analysis in terms of integrability condition.

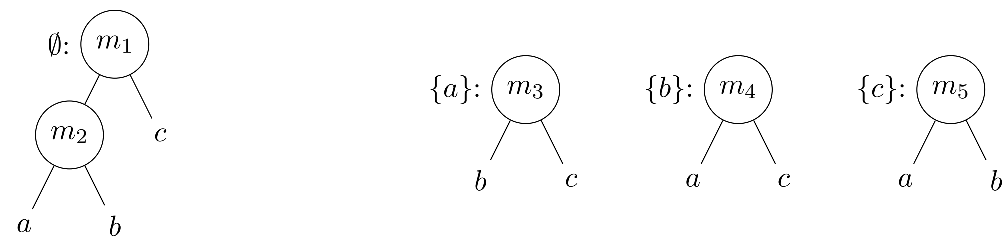


Figure 5: The tree structure for functions in $\mathbf{G}^3$.

5.3 Representations of G

Using the tree representation of gross substitutes, we provide a constructive characterization of gross substitutes $\mathbf{G}^n$ for $n \leq 4$ from matroids. Given a matroid $\mathcal{M}$ we denote its rank by $r[\mathcal{M}]$. We denote the uniform matroid of rank i over j elements by U_j^i.

Given two functions v and $\tilde{v}$ we say that they are equivalent (up to affine transformations) if $v - \tilde{v} \in \mathbf{E}^n$. Given a normalized valuation $v \in \mathbf{G}_0^n$, we will often associate it with the equivalent function $\tilde{v}(S) = v(S) + |S|$. Often, a normalized matroid rank function v is easier to recognize in its $\tilde{v}$ form.

Moreover, we write $X \simeq Y$ to denote a linear isomorphism between two sets, i.e., if there is a linear bijection L such that $L(X) = Y$.

5.3.1 Description of $\mathbf{G}^2$

For $n = 2$, the only constraint is $\Delta_{1,2}^{\emptyset} \geq 0$ so:

$$\mathbf{G}_0^2 = \{(v(\emptyset) = 0, v(1) = 0, v(2) = 0, v(12)); v(12) \leq 0\} \simeq \mathbb{R}_+.$$

Thus, $v = (0, 0, 0, -1)$ is a representative of the class $\mathbf{G}_0^2$ and $\tilde{v} = (0, 1, 1, 1)$, which is the rank function $r[U_2^1]$ of the uniform matroid of rank 1 over 2 elements, is a representative of the class $\mathbf{G}^2$. Since $\mathbf{E}^2 = (1, 1, 1, 1) \cdot \mathbb{R} + (0, 1, 0, 1) \cdot \mathbb{R} + (0, 0, 1, 1) \cdot \mathbb{R} \simeq \mathbb{R}^3$, we have:

$$\mathbf{G}^2 = \mathbf{E}^2 + r[U_2^1] \cdot \mathbb{R}_+ \simeq \mathbb{R}^3 \times \mathbb{R}_+.$$

The set $\mathbf{G}^2$ is not very interesting since the set of gross substitutes on 2 variables is the same as the set of submodular functions on 2 variables, which is known to be a convex set.

5.3.2 Description of $\mathbf{G}^3$

The set of gross substitutes on 3 items is more interesting since it is not convex. We name the items $\{a, b, c\}$. For every $v \in \mathbf{G}_0^3$ and up to the renaming of the elements, there is only one possibility for the substitution tree associated to the empty set, which is depicted in Figure 5. For each singleton set, there is also only one possible substitution tree. Applying Lemma 17 that is required to obtain a well-defined function, with $S = \emptyset$, we obtain the following additional necessary and sufficient constraint for the labels in Figure 5:

$$m_3 = m_4 = m_5 - (m_2 - m_1).$$

We write $m_1 = x$, $m_2 - m_1 = y$ (recall $m_2 \geq m_1$) and $m_3 = z$, so we can parametrize the space of feasible $m = (m_1, m_2, m_3, m_4, m_5)$ by:

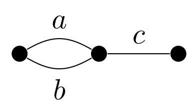
$$(x, y, z) \in \mathbb{R}^3_+ \mapsto m = (x, x + y, z, z, y + z)$$

In other words, the space of feasible values of m is a cone generated by the following vectors: $(1, 1, 0, 0, 0), (0, 0, 1, 1, 1), (0, 1, 0, 0, 1)$. It is particularly interesting to see which valuations they correspond to.

Vector $(1, 1, 0, 0, 0)$ By solving the equation to obtain v from second derivatives Δ (Corollary 18), we obtain the valuation $v \in \mathbf{G}^3_0$ such that $v(ab) = v(bc) = v(ac) = -1$ and $v(abc) = -2$. Thus, $\tilde{v}(S) = v(S) + |S| = 1$ for all $S \neq \emptyset$, which is the rank function $r[U^1_3]$.

Vector $(0, 0, 1, 1, 1)$ Solving the equations for v we obtain $v(ab) = v(bc) = v(ac) = 0$ and $v(abc) = -1$, so $\tilde{v}(S) = \min\{|S|, 2\}$ which is $r[U^2_3]$.

Vector $(0, 1, 0, 0, 1)$ More interestingly, by solving for v we obtain: $v(ac) = v(bc) = 0$ and $v(ab) = v(abc) = -1$, so $\tilde{v}$ is such that the singletons have value 1, $\tilde{v}(ab) = 1$ and $\tilde{v}(bc) = \tilde{v}(ac) = \tilde{v}(abc) = 2$. This is the rank function of the graphical matroid associated with the following graph:



We note that when we describe a matroid in terms of a graph in this document we refer to the matroid where the ground set are the edges of the graph and a set is independent if the corresponding edges do not form a cycle. Call the rank of this matroid $r[M_{((ab)c)}]$. Therefore, the set of functions in $\mathbf{G}^3$ associated with the depicted $\emptyset$-tree is given by:

$$\mathbf{E}^3 + r[U^1_3] \cdot \mathbb{R}_+ + r[U^2_3] \cdot \mathbb{R}_+ + r[M_{((ab)c)}] \cdot \mathbb{R}_+$$

Since all the $\emptyset$-trees are symmetric, then all the gross substitute functions over 3 elements are of the form:

$$\mathbf{G}^3 = \mathbf{E}^3 + r[U^1_3] \cdot \mathbb{R}_+ + r[U^2_3] \cdot \mathbb{R}_+ + \left(r[M_{((ab)c)}] \cdot \mathbb{R}_+ \cup r[M_{((ac)b)}] \cdot \mathbb{R}_+ \cup r[M_{((bc)a)}] \cdot \mathbb{R}_+\right)$$

5.3.3 Description of $\mathbf{G}^4$

The description of $\mathbf{G}^4$ follows similarly as for $\mathbf{G}^3$, but with more cases. It is deferred to Appendix E.

5.4 Finding the counterexample

The main idea used to find counterexample v is to exhibit a specific tree structure over 5 elements that is complex enough so that, unlike for $n = 1, 2, 3$, and 4, node labels cannot be decomposed into binary valued vectors which satisfy the integrability condition. We will try to mimic the same proof used for $\mathbf{G}^3$ and $\mathbf{G}^4$ and find a set of trees for which the same proof technique cannot be extended.

Consider the tree defined in Figure 6. We can apply the integrability conditions (Int) to obtain the following relations among the labels:

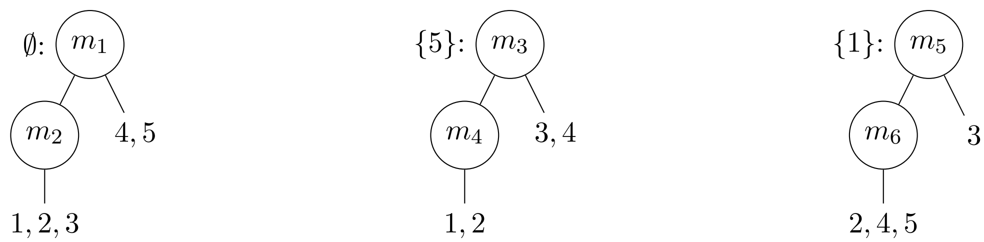


Figure 6: The key part of the tree structure for the counterexample.

- $m_3 - m_5 = m_2 - m_1 \geq 0$ from (Int) with $S = \emptyset$ and $(i, j, k) = (1, 3, 5)$,
- $m_6 - m_3 = m_1 - m_1 = 0$ from (Int) with $S = \emptyset$ and $(i, j, k) = (1, 4, 5)$,
- $m_4 - m_6 = m_2 - m_1 \geq 0$ from (Int) with $S = \emptyset$ and $(i, j, k) = (1, 2, 5)$.

Consider a function that satisfies that tree structure and has $m_2 - m_1 = \Delta > 0$, then $m_4 = m_6 + \Delta = m_3 + \Delta = m_5 + 2\Delta$. Therefore, any parametrization of the space of functions sharing those trees obtained by the same method as used for $\mathbf{G}^3$ and $\mathbf{G}^4$ will have a non-binary coefficient and hence does not decompose the space in combinations of matroid rank functions.

This is yet not a proof, since we do not know yet if there is GS functions with that tree structure and if there is a parametrization obtained by other methods that decomposes the space. Next, we complete this tree description and (computationally) solve a linear program to find a function satisfying that tree structure. Once we get a candidate functions that is the output of this program, we write a second program that tries to write it as a convex combination of matroid rank functions by explicitly enumerating over the set of all matroid rank functions and creating one variable for each in the linear program. Next, we verify that the program is infeasible and obtain a Farkas' type certificate. Finally, since we do not want to rely on the correctness of the enumeration and the computational steps, we give a human readable proof.

6 Conclusion

The class of gross substitutes is a well-studied family of valuation functions that has many different characterizations, but for which we do not know a constructive description. Our main result shows that gross substitutes cannot be constructed via positive linear combinations of matroid rank functions. We also give a new operation, called tree-concordant-sum, which provides a necessary and sufficient condition for the sum operation to preserve substitutability and which is used to find the counterexample for the main result.

In addition to affine transformations, strong quotient sum, and tree-concordant-sum, other operations are known to preserve substitutability. Two important examples are endowment or restriction [13] and convolution [19] or OR [18]. It remains an important open question whether there is a collection of substitutability-preserving operations that allow constructing all gross substitutes from matroid rank functions, or another simple class of functions.

Acknowledgements

We thank Kazuo Murota and Akiyoshi Shioura for their comments on an earlier version of this manuscript and their various suggestions to improve the presentation. We would also like to thank the mantainers of the database of matroids [1] which was in invaluable resource in this project.

Eric is supported by a Google PhD Fellowship.

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Appendix

A Missing Proofs from Section 2

Theorem 19. $\mathbf{M}^n = \{v \in \mathbf{G}^n; v(\emptyset) = 0; \partial_i v(S) \in \{0, 1\}, \forall S \subseteq [n]\}$.

It follows from the following characterization of matroid rank functions that can be found in Section 39.7 of Schrijver [28] (rephrased in the language of discrete derivatives):

Lemma 20. *A valuation function r is in $\mathbf{M}^n$ if and only if $r(\emptyset) = 0$, $\partial_i r(S) \in \{0, 1\}$ and $\partial_{ij} r(S) \leq 0$.*

This means that matroid rank functions are exactly the submodular functions that have $\{0, 1\}$-marginals. Using this lemma, we can now prove Theorem 19:

Proof. Proof of Theorem 19 The inclusion $\supseteq$ follows directly from the previous lemma and the fact that every GS function is submodular.

For the inclusion $\mathbf{M}^n \subseteq \mathbf{G}^n$, we first note that if $r \in \mathbf{M}^n$, then $\partial_{ij} r(S) = \partial_i r(S \cup j) - \partial_i r(S) \in \{0, -1\}$ since the first derivatives are in $\{0, 1\}$ and the second derivatives are non-positive. Moreover, $\partial_{ij} r(S) = -1$ iff $\partial_i r(S \cup j) = \partial_j r(S \cup i) = 0$ and $\partial_i r(S) = \partial_j r(S) = 1$. Therefore the condition:

$$\partial_{ij} v(S) \leq \max[\partial_{ik} v(S), \partial_{kj} v(S)]$$

is violated only when $\partial_{ij} v(S) = 0$ and $\partial_{ik} v(S) = \partial_{kj} v(S) = -1$. This implies that $\partial_i v(S) = \partial_j v(S) = \partial_k v(S) = 1$. But since $\partial_{ij} v(S) = 0$ we must have $v(S \cup ijk) \geq v(S \cup ij) = v(S) + 2$. This implies $\partial_{ij} v(S \cup k) > 0$ which contradicts submodularity. □

B Missing Proofs from Section 3

The main ingredient in Theorem 1 is the following characterization of monotone integer valued submodular functions:

Lemma 21 (Schrijver [28], section 44.6b)**.** *Every monotone integer valued submodular function that evaluates to zero at the empty set can be obtained from a matroid rank function by item grouping. Formally: if $v : 2^{[n]} \to \mathbb{Z}$ is a monotone submodular function (where monotone means that $\partial_i v(S) \geq 0$) with $v(\emptyset) = 0$, then there is a matroid $\mathcal{M}$ defined on a set U and a partition $(X_1, \ldots, X_n)$ of U such that $v(S) = r_{\mathcal{M}}(\cup_{s \in S} X_s)$.*

Therefore we only need to show that all submodular functions can be constructed from monotone integer valued submodular functions using positive linear combinations and affine transformations. To show this we start by viewing each submodular function as a vector in $\mathbb{R}^{2^n}$ indexed by the subsets of $[n]$. From this perspective, the set of submodular functions $\mathbf{S}^n$ correspond to the set of $\mathbb{R}^{2^n}$-points satisfying the linear inequalities given by $\partial_{ij} v(S) \leq 0$, which is a system of homogeneous linear inequalities. The set of solutions of such system is usually called a *polyhedral cone.* A classic result in convex analysis (see [26] for example) says that every polyhedral cone is finitely generated, i.e., every point can be written as a positive combination of a finite set of points. In other words, there is a finite set $v_1, \ldots, v_k \in \mathbf{S}^n$ such that:

$$\mathbf{S}^n = \left\{ \sum_{i=1}^{k} \alpha_i v_i; \alpha_i \geq 0 \right\}$$

When this set is minimal, those are called extremal rays of the cone. Also from convex analysis, if the constraints have rational coefficients then there is a set of extremal rays with integer coefficients.

Finally, observe that we can construct general integer value submodular functions from monotone ones by applying an affine transformation. For each v_i, let $M = \min_{S \subseteq [n], j \notin S} \partial_j v_i(S)$ and then define a normalized version of v_i as:

$$\bar{v}_i(S) = v_i(S) - M\,|S| - v_i(\emptyset)$$

It is simple to see that $\bar{v}_i(S)$ is monotone and evaluates to zero at the empty set and that v_i can be constructed from $\bar{v}_i$ using an affine transformation.

Extremal submodular functions One can ask whether the *item grouping* operation is necessary. Equivalently, are all the extremal rays matroid rank functions? For $\mathbf{S}^2$ and $\mathbf{S}^3$ all submodular functions are convex combinations of matroid rank functions (modulo affine transformations). For $n = 4$, however, the following submodular function is extremal and is not a matroid rank function: define $f(S)$ over $\{a, b, c, d\}$ such that $f(S) = 0$ for $|S| \leq 1$, $f(ab) = f(bd) = f(bc) = f(acd) = -1$, $f(ac) = f(ad) = f(cd) = 0$, $f(abc) = f(abd) = f(bcd) = -2$ and $f(abcd) = -3$.

In general the set of extremal submodular functions can be obtained using the standard technique of converting between the H-representation and V-representation of a cone. See Ziegler [31] for a complete discussion and the LRS package [2] for an implementation of such algorithms.

C Missing Proofs from Section 4

C.1 Checking $\mathbf{G}_0^5$-conditions for candidate

Below we check conditions $\partial_{ij} v(S) \leq \max[\partial_{ik} v(S), \partial_{kj} v(S)] \leq 0$ for the candidate function in Table 1. There are 40 inequalities to be checked, which we do below.

$$-1 = \partial_{3,2} v(\emptyset) \leq \max(\partial_{3,1} v(\emptyset), \partial_{1,2} v(\emptyset)) = \max(-1,-1) = -1 \leq 0$$
$$0 = \partial_{4,2} v(\emptyset) \leq \max(\partial_{4,1} v(\emptyset), \partial_{1,2} v(\emptyset)) = \max(0,-1) = 0 \leq 0$$
$$0 = \partial_{4,3} v(\emptyset) \leq \max(\partial_{4,1} v(\emptyset), \partial_{1,3} v(\emptyset)) = \max(0,-1) = 0 \leq 0$$
$$0 = \partial_{4,3} v(\emptyset) \leq \max(\partial_{4,2} v(\emptyset), \partial_{2,3} v(\emptyset)) = \max(0,-1) = 0 \leq 0$$
$$0 = \partial_{5,2} v(\emptyset) \leq \max(\partial_{5,1} v(\emptyset), \partial_{1,2} v(\emptyset)) = \max(0,-1) = 0 \leq 0$$
$$0 = \partial_{5,3} v(\emptyset) \leq \max(\partial_{5,1} v(\emptyset), \partial_{1,3} v(\emptyset)) = \max(0,-1) = 0 \leq 0$$
$$0 = \partial_{5,3} v(\emptyset) \leq \max(\partial_{5,2} v(\emptyset), \partial_{2,3} v(\emptyset)) = \max(0,-1) = 0 \leq 0$$
$$0 = \partial_{5,4} v(\emptyset) \leq \max(\partial_{5,1} v(\emptyset), \partial_{1,4} v(\emptyset)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{5,4} v(\emptyset) \leq \max(\partial_{5,2} v(\emptyset), \partial_{2,4} v(\emptyset)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{5,4} v(\emptyset) \leq \max(\partial_{5,3} v(\emptyset), \partial_{3,4} v(\emptyset)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{4,3} v(1) \leq \max(\partial_{4,2} v(1), \partial_{2,3} v(1)) = \max(-1,0) = 0 \leq 0$$
$$0 = \partial_{5,3} v(1) \leq \max(\partial_{5,2} v(1), \partial_{2,3} v(1)) = \max(-1,0) = 0 \leq 0$$
$$-1 = \partial_{5,4} v(1) \leq \max(\partial_{5,2} v(1), \partial_{2,4} v(1)) = \max(-1,-1) = -1 \leq 0$$
$$-1 = \partial_{5,4} v(1) \leq \max(\partial_{5,3} v(1), \partial_{3,4} v(1)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{4,3} v(2) \leq \max(\partial_{4,1} v(2), \partial_{1,3} v(2)) = \max(-1,0) = 0 \leq 0$$
$$0 = \partial_{5,3} v(2) \leq \max(\partial_{5,1} v(2), \partial_{1,3} v(2)) = \max(-1,0) = 0 \leq 0$$
$$-1 = \partial_{5,4} v(2) \leq \max(\partial_{5,1} v(2), \partial_{1,4} v(2)) = \max(-1,-1) = -1 \leq 0$$
$$-1 = \partial_{5,4} v(2) \leq \max(\partial_{5,3} v(2), \partial_{3,4} v(2)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{5,4} v(1,2) \leq \max(\partial_{5,3} v(1,2), \partial_{3,4} v(1,2)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{4,2} v(3) \leq \max(\partial_{4,1} v(3), \partial_{1,2} v(3)) = \max(0,0) = 0 \leq 0$$

$$0 = \partial_{5,2} v(3) \leq \max(\partial_{5,1} v(3), \partial_{1,2} v(3)) = \max(0,0) = 0 \leq 0$$
$$-1 = \partial_{5,4} v(3) \leq \max(\partial_{5,1} v(3), \partial_{1,4} v(3)) = \max(0,0) = 0 \leq 0$$
$$-1 = \partial_{5,4} v(3) \leq \max(\partial_{5,2} v(3), \partial_{2,4} v(3)) = \max(0,0) = 0 \leq 0$$
$$-1 = \partial_{5,4} v(1,3) \leq \max(\partial_{5,2} v(1,3), \partial_{2,4} v(1,3)) = \max(-1,-1) = -1 \leq 0$$
$$-1 = \partial_{5,4} v(2,3) \leq \max(\partial_{5,1} v(2,3), \partial_{1,4} v(2,3)) = \max(-1,-1) = -1 \leq 0$$
$$-1 = \partial_{3,2} v(4) \leq \max(\partial_{3,1} v(4), \partial_{1,2} v(4)) = \max(-1,-2) = -1 \leq 0$$
$$-1 = \partial_{5,2} v(4) \leq \max(\partial_{5,1} v(4), \partial_{1,2} v(4)) = \max(-1,-2) = -1 \leq 0$$
$$-1 = \partial_{5,3} v(4) \leq \max(\partial_{5,1} v(4), \partial_{1,3} v(4)) = \max(-1,-1) = -1 \leq 0$$
$$-1 = \partial_{5,3} v(4) \leq \max(\partial_{5,2} v(4), \partial_{2,3} v(4)) = \max(-1,-1) = -1 \leq 0$$
$$0 = \partial_{5,3} v(1,4) \leq \max(\partial_{5,2} v(1,4), \partial_{2,3} v(1,4)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{5,3} v(2,4) \leq \max(\partial_{5,1} v(2,4), \partial_{1,3} v(2,4)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{5,2} v(3,4) \leq \max(\partial_{5,1} v(3,4), \partial_{1,2} v(3,4)) = \max(0,-1) = 0 \leq 0$$
$$-1 = \partial_{3,2} v(5) \leq \max(\partial_{3,1} v(5), \partial_{1,2} v(5)) = \max(-1,-2) = -1 \leq 0$$
$$-1 = \partial_{4,2} v(5) \leq \max(\partial_{4,1} v(5), \partial_{1,2} v(5)) = \max(-1,-2) = -1 \leq 0$$
$$-1 = \partial_{4,3} v(5) \leq \max(\partial_{4,1} v(5), \partial_{1,3} v(5)) = \max(-1,-1) = -1 \leq 0$$
$$-1 = \partial_{4,3} v(5) \leq \max(\partial_{4,2} v(5), \partial_{2,3} v(5)) = \max(-1,-1) = -1 \leq 0$$
$$0 = \partial_{4,3} v(1,5) \leq \max(\partial_{4,2} v(1,5), \partial_{2,3} v(1,5)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{4,3} v(2,5) \leq \max(\partial_{4,1} v(2,5), \partial_{1,3} v(2,5)) = \max(0,0) = 0 \leq 0$$
$$0 = \partial_{4,2} v(3,5) \leq \max(\partial_{4,1} v(3,5), \partial_{1,2} v(3,5)) = \max(0,-1) = 0 \leq 0$$
$$0 = \partial_{3,2} v(4,5) \leq \max(\partial_{3,1} v(4,5), \partial_{1,2} v(4,5)) = \max(0,-1) = 0 \leq 0$$

C.2 Weighted matroids

Lemma 22 (Shioura [29]). *Any weighted matroid rank functions can be written as a positive linear combination of unweighted matroid rank functions.*

Proof. Proof. Let $w_1 \geq \ldots \geq w_n \geq 0$ be the weights of the n elements $[n] := \{1, \ldots, n\}$ of a weighted matroid rank function v associated to matroid $\mathcal{M}$. Let $\mathcal{M}_i$ be the matroid $\mathcal{M}$ restricted to elements $[i]$ and r_i be the unweighted matroid rank function over elements $[i]$ associated with matroid $\mathcal{M}_i$. Since the greedy algorithm finds a maximum weight base of a weighted matroid, we have

$$\begin{aligned} v(S) &= \sum_{i \in [n]} w_i \left(r(S \cap [i]) - r(S \cap [i-1]) \right) \\ &= \sum_{i \in [n]} \sum_{j=i}^{n} (w_j - w_{j+1}) \left(r(S \cap [i]) - r(S \cap [i-1]) \right) \\ &= \sum_{j \in [n]} (w_j - w_{j+1}) \cdot r(S \cap \{j\}) \\ &= \sum_{j \in [n]} (w_j - w_{j+1}) \cdot r_j(S). \end{aligned}$$

□

D Integrability Conditions

Given differentiable functions $b_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n$, it is well known that there is a function f such that $b_i(x) = \partial f(x)/\partial x_i$ iff the functions b_i satisfy the conditions $\partial b_i/\partial x_j = \partial b_j/\partial x_i$. In physics, those correspond to the necessary and sufficient conditions for a field to be a conservative field. We refer to Section 10.16 of [**?**] for a complete discussion. Those conditions can also be derived as a special case of Stoke's Theorem. The exact same condition provides integrability over the hypercube:

Lemma 23. *Given functions $\beta_i : 2^{[n]\setminus i} \to \mathbb{R}$ for $i \in [n]$, then there exists a function v such that $\beta_i = \partial_i v$ for all i iff $\partial_i \beta_j = \partial_j \beta_i$ for all $i \neq j$.*

Proof. Proof. Given β_i satisfying $\partial_i \beta_j = \partial_j \beta_i$, fix any order among the elements in $[n]$ and let $S_{<i} = \{j \in S; j < i\}$. Now, define v as follows:

$$v(S) = \sum_{i \in S} \beta_i(S_{<i})$$

First we argue that the definition is order independent, i.e., for any ordering of the elements, we construct the same function v. To see that, start for an arbitrary order and swap a pair of adjacent elements $i < j$. Then if only one is in S, this doesn't change $v(S)$. If both are in S, we change the definition of $v(S)$ from:

$$\ldots + \beta_i(T) + \beta_j(T \cup i) + \ldots$$

to the following (where the terms in ... are left unchanged):

$$\ldots + \beta_j(T) + \beta_i(T \cup j) + \ldots$$

Since $\partial_j \beta_i(T) = \partial_i \beta_j(T)$ we have $\beta_i(T) + \beta_j(T \cup i) = \beta_j(T) + \beta_i(T \cup j)$. This is equivalent to the notion of *path-independence* for continuous functions (see Section 10.17 of [**?**]).

Now, fixed j, we can assume without loss of generality that j is placed in the end of the ordering. Therefore by the definition of v we have $v(S \cup j) = v(S) + \beta_j(S)$ so $\beta_j(S) = \partial_j v(S)$. □

We can now obtain second order integrability conditions from the first order ones easily:

Lemma 24. *Given functions $\alpha_{ij} : 2^{[n] \setminus ij} \to \mathbb{R}$ for $i \neq j$, then there exists a function v such that $\alpha_{ij} = \partial_{ij} v$ for all i, j iff $\alpha_{ij} = \alpha_{ji}$ and*

$$\partial_i \alpha_{jk} = \partial_j \alpha_{ik} = \partial_k \alpha_{ij} \quad \forall \text{ distinct } i, j, k.$$

Proof. Proof. First observe that by first order integrability conditions (Lemma 23) we can find for each i, a function β_i such that $\alpha_{ij} = \partial_j \beta_i$ since $\partial_k \alpha_{ij} = \partial_j \alpha_{ik}$. Now observe that the functions $\beta_1, \ldots, \beta_n$ constructed satisfy first order integrability conditions, since $\partial_j \beta_i = \alpha_{ij} = \partial_i \beta_j$ so there is v such that $\beta_i = \partial_i v$. □

Corollary 25. *Given functions $\alpha_{ij} : 2^{[n] \setminus ij} \to \mathbb{R}$ satisfying integrability conditions in the previous lemma, all $v : 2^{[n]} \to \mathbb{R}$ are affine transformations of the function:*

$$v(S) = \sum_{i<j; i,j \in S} \alpha_{ij}(S_{<i})$$

Proof. Proof. Using Lemma 23 we can reconstruct the functions β_i as:

$$\beta_i(S) = \beta_i(\emptyset) + \sum_{j \in S} \alpha_{ij}(S_{<j})$$

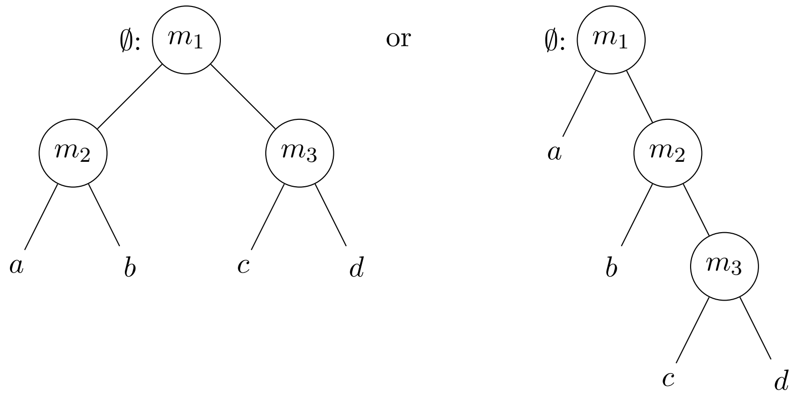
Applying the same process for reconstructing v from β_i, we get:

$$\begin{aligned} v(S) &= v(\emptyset) + \sum_{i \in S} \beta_i(S_{<i}) = v(\emptyset) + \sum_{i \in S} \left[\beta_i(\emptyset) + \sum_{j \in S_{<i}} \alpha_{ij}(S_{<j}) \right] \\ &= v(\emptyset) + \sum_{i \in S} \beta_i(\emptyset) + \sum_{j<i; i,j \in S} \alpha_{ij}(S_{<j}) \end{aligned}$$

□

E Description of $\mathbf{G}^4$

For the case of $\mathbf{G}^4$, we start by observing that up to renaming the items, the $\emptyset$-tree must have one of two forms, which we will refer as the shallow tree and the deep tree respectively.



Shallow Tree Case Assume we have a valuation function $v \in \mathbf{G}_0^4$ whose $\emptyset$-tree is shallow (i.e., is like the left diagram above). We know $m_1 \leq m_2$ and $m_1 \leq m_3$. For convenience of notation we will refer to $m_1 = x$, $m_2 = x + y$ and $m_3 = x + z$ for $x, y, z \geq 0$. Applying Lemma 17 with $S = \emptyset$ and for every triple of elements, we get:

$$\begin{aligned} w_1 &=: \Delta^d_{ac} = \Delta^c_{ad} = \Delta^a_{cd} - z \\ w_2 &=: \Delta^d_{bc} = \Delta^c_{bd} = \Delta^b_{cd} - z \\ w_3 &=: \Delta^b_{ac} = \Delta^a_{bc} = \Delta^c_{ab} - y \\ w_4 &=: \Delta^b_{ad} = \Delta^a_{bd} = \Delta^d_{ab} - y \end{aligned}$$

If $w_3 \neq w_4$ we can wlog (up to permuting the identity of the items) assume that $w_3 < w_4$. In such case, observe that: $\Delta^b_{ac} = w_3$, $\Delta^b_{ad} = w_4$, and $\Delta^b_{cd} = w_2 + z$. Since the minimum value is repeated among Δ^b_{ac}, Δ^b_{ad} and Δ^b_{cd} we must have $w_3 = w_2 + z$. Now, looking at the substitution symbols for the $\{a\}$-tree, we get: $\Delta^a_{bc} = w_3$, $\Delta^a_{bd} = w_4$, and $\Delta^a_{cd} = w_1 + z$; so we must have by the same argument: $w_3 = w_1 + z$. In particular we will get: $w_1 = w_2 = w_3 - z = w_4 - z - u$ for some $u \geq 0$. This means that we can write:

$$w_1 = w \qquad w_2 = w \qquad w_3 = w + z \qquad w_4 = w + z + u. \tag{S}$$

If $w_3 = w_4$, then there are two cases. If $w_1 \neq w_2$, then up to permuting the identity of items this case is identical to $w_3 \neq w_4$ and we can obtain (S) with indices permuted. The other case is if $w_1 = w_2$ and $w_3 = w_4$. Looking at the substitution symbols for the $\{b\}$-tree, the minimum value of $\Delta^b_{ac} = w_3, \Delta^b_{ad} = w_4$, and $\Delta^b_{cd} = w_2 + z$ is repeated and we get $w_3 = w_4 \leq w_2 + z$. Similarly, by observing the substitution symbols for the $\{c\}$-tree, we get $w_1 = w_2 \leq w_3 + y$. Thus, $w_1 = w_2 = w_3 = w_4$, which is a special case of (S) with $y = z = u = 0$. Therefore it is w.l.o.g. to focus on the setup in (S).

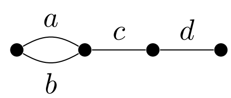
The values of w_1, w_2, w_3 and w_4 define the $\{i\}$-trees for all $i = a, b, c, d$. It is possible now to reconstruct $v(S)$ for $S \neq \emptyset$ with Corollary 18:

$$\begin{aligned} v(abc) &= -\Delta^\emptyset_{ab} - \Delta^\emptyset_{ac} - \Delta^a_{bc} = -2x - y - w - z \\ v(abd) &= -\Delta^\emptyset_{ab} - \Delta^\emptyset_{ad} - \Delta^a_{bd} = -2x - y - w - z - u \\ v(acd) &= -\Delta^\emptyset_{ac} - \Delta^\emptyset_{ad} - \Delta^a_{cd} = -2x - w - z \\ v(bcd) &= -\Delta^\emptyset_{bc} - \Delta^\emptyset_{bd} - \Delta^b_{cd} = -2x - z - w \\ v(abcd) &= -3x - y - 2z - 2w - u - t \end{aligned}$$

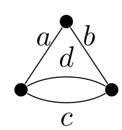
for some $t \geq 0$. This gives us a valuation v in gross substitutes parametrized by $(x, y, z, w, u, t) \in \mathbb{R}^6_+$. Therefore, every valuation v in $\mathbf{G}^4_0$ whose $\emptyset$-tree is shallow can be written as a non-negative combination of 6 *extremal* valuation functions. The extremal valuations are obtained when we set one of the coefficients to one and all coefficients to zero. It is instructive to see which valuations are those.

- $x = 1$ and all other coefficients are zero. We obtain a valuation function such that $v(S) = 0$ for $|S| \leq 1$, $v(S) = -1$ for $|S| = 2$, $v(S) = -2$ for $|S| = 3$ and $v(S) = -3$ for $|S| = 4$. We get $\tilde{v}(S) = 0$ for $S = \emptyset$ and $\tilde{v}(S) = 1$ for $S \neq \emptyset$, which is $r[U^1_4]$.

- $y = 1$ and all other coefficients are zero. Then $v(S) = -1$ if $\{a, b\} \subseteq S$ and $v(S) = 0$ otherwise, and $\tilde{v}$ is the rank function of the following matroid:

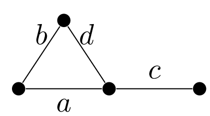


- $z = 1$ and all other coefficients are zero. Then $v(S) = 0$ for $|S| \leq 1$, $v(cd) = -1$, $v(S) = 0$ for all other S with $|S| = 2$, $v(S) = -1$ for all S with $|S| = 3$, and finally $v(abcd) = -2$. So $\tilde{v}$ is the rank function of the matroid:



- $w = 1$ and all other coefficients are zero, then $v(S) = 0$ for $|S| \leq 2$, $v(S) = -1$ for $|S| = 3$ and $v(S) = -2$ for $|S| = 4$, and $\tilde{v}$ is $r[U^2_4]$.

- $u = 1$ and all other coefficients are zero, then $v(S) = -1$ if $\{a, b, d\} \subseteq S$ and $v(S) = 0$ otherwise. So, $\tilde{v}(S) = v(S) + |S|$ is the rank function of the matroid:



- if $t = 1$ and all other coefficients are zero, then $v(S) = 0$ if $|S| \leq 3$ and $v(S) = -1$ for $|S| = 4$, therefore, $\tilde{v}(S)$ is $r[U^3_4]$.

Therefore we identified the following cone which is a subset of $\mathbf{G}^4$:

$$\mathbf{E}^4 + \sum_{j=1}^{6} r_j \cdot \mathbb{R}_+$$

where $r_1, \ldots, r_6$ are the rank functions of the matroids identified in the previous items. Also, since $\mathbf{G}^4$ is symmetric with respect to permutations of the identities of the items, we can obtain 11 other cones by permuting the identities of the items.

Deep Tree Case. Assume now that we have a valuation $v \in \mathbf{G}_0^4$ whose $\emptyset$-tree is deep. We know $m_1 \leq m_2 \leq m_3$, so we will refer for convenience to $m_1 = x$, $m_2 = x + y$ and $m_3 = x + y + z$ for $x, y, z \geq 0$. By applying again Lemma 17 we obtain:

$$\begin{aligned} w_1 &=: \Delta^c_{ab} = \Delta^b_{ac} = \Delta^a_{bc} - y \\ w_2 &=: \Delta^d_{ab} = \Delta^b_{ad} = \Delta^a_{bd} - y \\ w_3 &=: \Delta^d_{ac} = \Delta^c_{ad} = \Delta^a_{cd} - y - z \\ w_4 &=: \Delta^d_{bc} = \Delta^c_{bd} = \Delta^b_{cd} - z \end{aligned}$$

It is instructive to re-write the substitution symbols to write together the symbols for the same tree:

a :	b :	c :	d :
$\Delta^a_{bc} = w_1 + y$	$\Delta^b_{ac} = w_1$	$\Delta^c_{ab} = w_1$	$\Delta^d_{ab} = w_2$
$\Delta^a_{bd} = w_2 + y$	$\Delta^b_{ad} = w_2$	$\Delta^c_{ad} = w_3$	$\Delta^d_{ac} = w_3$
$\Delta^a_{cd} = w_3 + y + z$	$\Delta^b_{cd} = w_4 + z$	$\Delta^c_{bd} = w_4$	$\Delta^d_{bc} = w_4$

In each column, at least two values are equal and the third value is at least as large as the two equal values. Let's look at the symbols for the $\{c\}$-tree. There are four possibilities:

1. $w_1 = w_3 < w_4$. In such case, by looking at the $\{d\}$-tree, since $w_3 < w_4$, it must be the case that $w_2 = w_3$ since the minimum substitution symbol in the $\{d\}$-tree must be repeated. Therefore we must have $w_1 = w_2 = w_3 = w$ and $w_4 = w + u$. It is simple to see that if this condition is true, for every $\{i\}$-tree, the minimum value is repeated.

2. $w_1 = w_4 < w_3$. In such case, by looking at the $\{d\}$-tree, since $w_4 < w_3$, it must be the case that $w_2 = w_4$ since the minimum substitution symbol in the $\{d\}$-tree must be repeated. Therefore we must have $w_1 = w_2 = w_4 = w$ and $w_3 = w + u$.

3. $w_3 = w_4 < w_1$. Now, we consider other three possibilities for the $\{b\}$-tree:
   (a) $w_1 = w_4 + z \leq w_2$, so we have $w_1 = w + z$, $w_2 = w + z + u$ and $w_3 = w_4 = w$,
   (b) $w_2 = w_4 + z < w_1$, so we have: $w_1 = w + z + u$, $w_2 = w + z$, and $w_3 = w_4 = w$.
   (c) $w_1 = w_2 < w_4 + z$, so we have $w_1 = w + u$, $w_2 = w + u$, $w_3 = w_4 = w$, and $z = u + v$.

4. $w_1 = w_3 = w_4$. By inspecting the $\{b\}$-tree, we must have $w_2 = w_1$. Since this is a special case of the previous cases, we ignore this case from now on.

We note that in either case, we have:

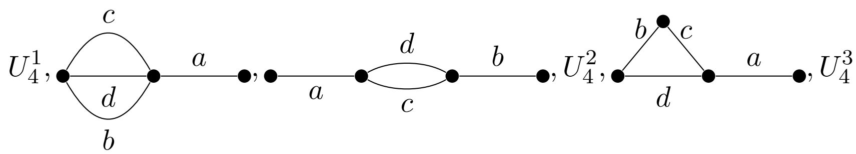
$$\begin{aligned} v(abc) &= -\Delta^\emptyset_{ab} - \Delta^\emptyset_{ac} - \Delta^a_{bc} = -2x - y - w_1 \\ v(abd) &= -\Delta^\emptyset_{ab} - \Delta^\emptyset_{ad} - \Delta^a_{bd} = -2x - y - w_2 \\ v(acd) &= -\Delta^\emptyset_{ac} - \Delta^\emptyset_{ad} - \Delta^a_{cd} = -2x - y - z - w_3 \\ v(bcd) &= -\Delta^\emptyset_{bc} - \Delta^\emptyset_{bd} - \Delta^b_{cd} = -2x - 2y - z - w_4 \end{aligned}$$

By Corollary 18, $v(abcd) = -\Delta_{ab}^{\emptyset} - \Delta_{ac}^{\emptyset} - \Delta_{ad}^{\emptyset} - \Delta_{bc}^{a} - \Delta_{bd}^{a} - \Delta_{cd}^{ab}$. By Lemma 17 with $S = \{c\}$ and triplet abd, and also with $S = \{d\}$ and triplet abc, $\Delta_{cd}^{ab} = t + z + u$ for some $t \geq 0$. In addition, we also have $\Delta_{ab}^{\emptyset} = \Delta_{ac}^{\emptyset} = \Delta_{ad}^{\emptyset} = x$, $\Delta_{bc}^{a} = w_1 + y$, and $\Delta_{bd}^{a} = w_2 + y$. Thus,

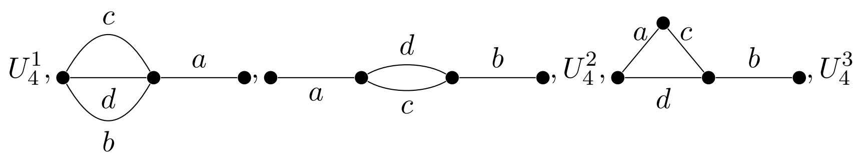
$$v(abcd) = -3x - 2y - w_1 - w_2 - z - u - t.$$

Now, following the same procedure used in the previous section, we have that in each case we have:

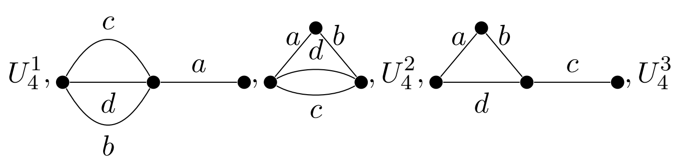
- Case 1: $v \in \mathbf{E}^4 + \sum_{j=1}^{6} r_j \cdot \mathbb{R}_+$ where $r_1, \ldots, r_6$ are the ranks of the following matroids:



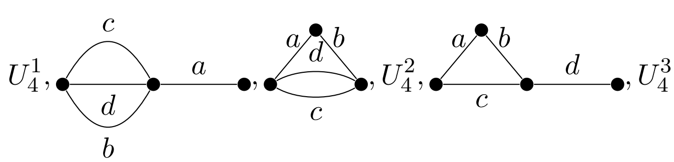
- Case 2: Same as before but with the rank functions of the following matroids:



- Case 3a: Same as before but with the rank functions of the following matroids:



- Case 3b: Same as before but with the rank functions of the following matroids:



- Case 3c: Same as before but with the rank functions of the following matroids: