Mathematical Card Tricks

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Introduction

For about as long as playing cards have existed, humans have experimented with methods and techniques to manipulate them. Historically known as a job for magicians and tricksters, most people see card tricks as simply a manipulation of the deck or a slight of hand. However, the methods behind card tricks actually find their roots within mathematics and since the 1900s, there have been a large amount of publications and writings on these mathematical techniques.

In their essence, playing cards are information carriers. However, cards in particular carry more information than almost any other physical object. They are able to be used simply as a counting methodology, just like coins, rocks, or sticks may be used, but with the additional properties of being numbered, colored, and suited. This bounty of information allows for a wide variety of tricks to be played with them.

In this paper, I will analyze two different card tricks and the mathematics behind them. Each of these card tricks deals with a different method of utilizing the information that is held within playing cards. First I will look at The Color Scheme, which uses the color component of cards and deals with the transposition of information. I will also analyze the 27 Card Trick, which is perhaps the most well covered card trick in mathematics. This trick provides insight into the process of sorting information. Upon further generalization, the 27 Card Trick also reveals a fascinating world in which massive quantities of information can be sorted through a few simple techniques. These card tricks will show some of the various ways that complicated amounts of information can be manipulated using mathematics.

The Color Scheme

The first card trick I will show uses the concept of information transposition. This card trick, invented by Oscar Weigle in 1949, is known as The Color Scheme. This is how it works:

The Trick

1. Organize a deck of 20 cards such that the colors alternate.
2. Have the participant take the deck and hold it under the table.
3. The participant will now shuffle the deck in this specific manner: i) Flip the top two cards over, holding them together as if they were one, ii) cut the deck. The participant may shuffle in this manner for as long as desired.

4. The participant will now take the top card from the deck and move it to the bottom. The next card will be flipped over and placed onto the table.
5. Repeat step 4 until there are ten cards on the table. As you will observe, all of the face-up cards will be of the same color and all of the face-down cards will be of the same color. If you continue this process for the other ten cards, you will find that the same effect takes place but with the colors reversed.

This trick relies upon the fact that even though you are creating a deck that is seemingly random, you are in fact maintaining the color properties of the cards. While Martin Gardner writes about this trick in his book Wheels, Life and Other Mathematical Amusements, he does not go into the specifics of how the trick works. However, I deduced the solution by looking at isolated trivial cases and putting the pieces together.

If we look at the original organization of the deck, it is easy to see that the final step will yield the desired result because the cards are alternating colors. Without flipping any cards, it is also trivial to see that cutting the deck will not change the fact that the cards alternate. Additionally, any cards that are flipped will be alternating with respect to each other and they maintain this property even after the deck is cut.

Now let us examine the situation where the cards are flipped and placed at a random point in the deck. This process now places a red card next to another red and a black card next to another black. While this may seem like it would cause a problem in the final step, the solution comes in the fact that the cards are facing opposite directions. The act of flipping it originally causes the card to undergo a half-cycle while the fact that the red card is now where a black card should be causes that half-cycle to essentially reverse. Thus, when the cards are placed on the table, every card will face the same direction as all other cards of its color.

Put in other words, we can look at the fact that face-up red cards and face-down black cards can essentially be treated as equivalent, while face-down reds and face-up blacks behave the same way as well. If we look at the cards in this way, then we can see that no matter how many times we shuffle the deck, the cards remain alternating.

Of course, the number of cards that are face-up when the ten cards are placed on the table is not always five. This is a chance based activity that depends on the size of the cut that is made. If there is an imbalance in the number of flipped cards that end up on the bottom half of the deck, then there will be an unbalanced number of face-up cards.

The color that is face up is also based on chance. This property depends on where the cuts were made. If the final cut places either a face-down black or a face-up red on the top, then the first ten cards will only reveal red cards. If the final cut places either a face-down red or a face-up black on top, then the first ten card will only reveal black cards.

Another interesting fact is that the last ten cards will reverse the color that is shown. This is a result of the fact that these ten cards were the cards that were placed on the bottom of the deck during the original sieving. Thus, these cards will have the opposite properties of the first ten cards.

Summarization

This card trick does not involve very much math beyond simple logic and reasoning. This trick does show, however, that seemingly randomized transposition of information could actually be nothing more than a simple reordering of the cards themselves, with no changes to the color properties. Following step 3 in the process, the deck has become one that is randomized in terms of which cards are face-up vs face-down as well as the order of the card color. However, combining the two pieces of information of direction faced and color, it becomes clear that there is a simple, alternating pattern of red-up/black-down and red-down/black-up.

The 27 Card Trick

The other aspect of card tricks that I will cover in this paper is the concept of information sorting. To do this, lets examine the case of the 27 Card Trick:

The Trick

1. The participant chooses a random card from a deck of 27, memorizes it, replaces it, and shuffles the deck as much as desired. The participant also states a random number between 1 and 27 (call this number n).
2. The dealer creates three face-up piles of nine cards, dealing the top card onto the left, middle, then right deck and repeating until all the cards are gone.
3. The participant then points to which pile contains the chosen card.
4. The dealer then re-stacks the piles into a new deck. The dealer must maintain the order of the specific piles themselves however.
5. Steps 2-4 are repeated twice.
6. The dealer then flips over n-1 cards from the top and reveals the nth card to be the original chosen card.

The secret to this trick relies on the order in which the dealer restacks the piles in step 4. For an easy example, let us assume that the dealer places the chosen pile on the top of the new deck every time. After the first restack, we know that the chosen card must be one of the first 9 cards in the new deck. When the cards are dealt again, we know the chosen card is one of the bottom three in each of the new piles. Therefore, after the second restack, we know the chosen card is among the first three on the new deck. These three cards are then going to be placed at the very bottom of the newly dealt piles and after the final selection, the dealer knows what the chosen card is. Stacking these one final time, the chosen card will end up on the top of the deck.

This same method can be used to place the chosen card in any position desired (i.e. the number that the participant chooses at the beginning). The final restack determines if the card will be between 1-9, 10-18, or 19-27. The second restack determines which set of three within that nine the card will end up in. The initial restack determines if the card will be the first, second, or third card within that subset of three.

A more mathematical way of explaining this process is through the use of a ternary number system. Martin Gardner, in his book Mathematics, Magic, and Mystery, covers how the use of ternary numbers can be applied to this problem. If we want a card to be in the nth position, then we take the ternary form of n-1. If we associate 0 with top, 1 with middle, and 2 with bottom position, then we can easily determine how to stack the cards by reading the ternary digits from right to left. For example, if we want the card to end up in the 8th position, then we write out the ternary form of 7, which is 021. By first reversing the digits to 120 and then associating them with the respective deck positions, we get that we need to stack the decks in the order middle, bottom, top.

In a more elementary form, we can determine exactly what order to restack the decks by satisfying the following equation:

$$n - 1 = a * 3^0 + b * 3^1 + c * 3^2$$

Where n is the chosen card position, a is the value of the position of the first restack, b is the second, and c is the third. a, b, and c are all less than 3. To reuse the example from above, if we wanted to place the card in the eighth position, we would satisfy the above equation with a=1, b=2, and c=0. Therefore, we would want to place the chosen deck at middle, bottom, then top.

Generalizations

This trick works easily with 27 cards, however it is also possible with different numbers of cards. One of these situations, proven by French mathematician Joseph Diaz Gergonne, is when the number of cards in the deck can be written as m^m and they are arranged in m piles with m^{m-1} cards in each deck. If we want to place the chosen card in the nth position after m number of deals, we must satisfy the following equation if m is even:

$$n = km^{m-1} - jm^{m-2} + ... + bm - a + 1$$

And the following if m is odd:

$$n = km^{m-1} - jm^{m-2} + ... - bm + a$$

Where during the first restack, the pile with the chosen card is taken up ath, during the second restack, the pile with the chosen card is taken up bth, and so on. The proof for this is summarized in Mathematical Recreations and Essays, by W.W. Rouse Ball and H.S.M Coxeter and is formalized in Gergonnes Annales de Mathematiques. The general idea behind this is that given n, we can determine a, b, c, etc. by dividing n by m and letting each letter equal to the remainder in each step. We must also alternate the sign of each value. The restriction on this is that a, b, c, etc. may not be greater than m or less than one.

This can be further generalized to a pack of lm number of cards. C.T. Hudson and L.E. Dickson proved the general case where the pack is dealt n times into l piles of m cards. They have also shown how the piles should be organized such that the chosen card appears at the rth position in the deck. While this proof is rather complicated, one of the obvious corollaries that we can draw from it is the case where the number of cards in the deck is a power.

If the number of cards that are in the deck can be represented by pk, where p and k are both non-negative integers, then we know that we can put any card in any given position using the same technique used in the 27 card trick. If there are pk number of cards, then we must deal p number of decks and do the restack process k number of times. For example, we used three decks and three restacks in the case of the 27 (3^3) card trick. If there were 8 (2^3) cards, then we would only need two decks and if there were 81 (3^4) cards, then we would have needed four restacks.

The order in which we would restack the decks would be determined with the same methods we used for 27 cards. By writing out the value of n-1 in the pth arity and reversing the order of the digits, we can determine the position in which to place the deck after each dealing.

Summarization

Although the 27 Card Trick is not an overly difficult puzzle crack in itself, it provides a very interesting method in which information can be sorted. The further generalizations that have been proven show the many ways in which this method can be used to organize sets of information, not necessarily just playing cards. By organizing the information by powers, we can reposition any piece of information at incredibly fast speeds. A vivid example of this is the Gargantua deck, which is composed of 1 billion unique cards. Using the same methodology outlined above, we can reposition any card to any other position after only ten restacks if we use ten piles with a billion cards in each deck.

Conclusion

These card tricks have existed for a long time and can be performed by any regular person. However, by examining the mathematics behind the tricks, an intricate world of logic and information is revealed. We saw in The Color Scheme that two independent pieces of information, direction faced and card color, tell us nothing but that combining the two reveals a very obvious pattern. The 27 Card Trick showed that large quantities of information can be organized in a very small number of steps. These tricks provide more than just amusement, they reveal incredibly interesting properties that playing cards hold and ways in which these properties can be manipulated.

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