Harvard University Math 231br

Notes on the Adams Spectral Sequence

Reuben Stern

Spring Semester 2017

Abstract

The Adams spectral sequence is a powerful tool for computing homotopy groups of a spectrum, somehow taken with respect to a certain cohomology theory. In particular, it allows one to compute the homotopy groups of certain spaces, given sufficient input data. In this paper, we give a detailed development of the Adams spectral sequence, following Adams 1974 and Ravenel 2015. After a proof of convergence and identification of the E_2-page, we use the Adams spectral sequence to sketch a proof of real Bott periodicity.

Contents

1 Introduction

> With all due respect to anyone who is interested in them, the coefficient groups $\pi_n(\mathbb{S})$ are a mess.
>
> J. Frank Adams 1974

It is a fair bit of an oversimplification to say that algebraic topology is the study of homotopy groups of spheres, but not that far from the truth. Enormous swaths of mathematics have been developed in search of these groups, the computations of which have proven tedious and incomplete. To put things in perspective, we know about 1000 stable homotopy groups of spheres, and even then we only know the 5-primary parts. Perhaps it is not too difficult to believe that computing $\pi_n(\mathbb{S})$ is a gargantuan task: by a theorem of Pontryagin,

$$\pi_n(\mathbb{S}) \cong \Omega_n^{\text{framed}},$$

that is, the n-th stable homotopy group of spheres is isomorphic to the group of framed n-manifolds up to framed cobordism. This group essentially classifies all framed n-manifolds, which is a reasonably hard thing to do.

All is not lost, however: many high-powered tools have been developed and much progress has been made in computing homotopy groups in recent decades, and this paper serves as an exposition of one of the most powerful: the Adams Spectral Sequence. In general, spectral sequences are a powerful bookkeeping tool that help organize (potentially) enormous computations. The Adams Spectral Sequence (unfortunately abbreviated the ASS) in its classical formulation computes (p-completed) homotopy of a space X, $\pi_n(X)^{\wedge_p}$. More generally, it computes E-completed homotopy of a *spectrum* X, where E is a suitable spectrum (and its associated co/homology theory) and E-completion means something specific, to be defined later (Definition 3.3).

I am a novice topologist, so this paper serves ostensibly as a set of "learning notes" for me. As such, the organization may not be ideal, explanations may be verbose and somewhat unnecessary to most readers, and prose may be informal in a way that bothers some. Be warned, dear reader: this paper is not to be used as a reference. For that, I have included an extensive bibliography and cite the literature whenever possible; I'll say upfront that most of this material is covered amply in Doug Ravenel's classic book, *Complex Cobordism and Stable Homotopy Groups of Spheres* (Ravenel 2015). Adams's manuscript *Stable Homotopy and Generalised Homology* (Adams 1974) remains a pertinent and wonderfully-written resource; the author's side remarks alone make it a worthwhile read.

The structure of this article is as follows: I begin with a general discussion of spectra and their associated co/homology theories. In particular, I define the category $\mathsf{Spectra}$ of spectra and their morphisms (Definition 2.4), as well as the homotopy category $\mathsf{hSpectra}$ (Definition 2.8). The homotopy category $\mathsf{hSpectra}$ has a smash product, that makes it into a symmetric monoidal category. With this structure, we can define a **(commutative) ring spectrum** as a (commutative) monoid object in the symmetric monoidal category hSpectra. Finally, I discuss how multiplicative cohomology theories are associated to ring spectra, and give an example or two of both.

Next begins the real heart of the paper: after a brief refresher on how to derive a spectral sequence from an exact couple, I define the notion of an E_*-Adams resolution for a connective spectrum X. Armed with this definition, we are in the perfect place to state Theorem 3.6, the existence and convergence of the Adams Spectral Sequence for E-homology. Its proof will concern the next two sections, as will the identification of the E_2-page as $\mathrm{Ext}^{*,*}_{E_*(E)}(\pi_*(E), E_*(X))$, for a suitable definition of Ext in the abelian category of comodules of a Hopf algebroid (such algebraic nonsense is discussed informally in Appendix A).

The final section takes on a computation of $\pi_*(ko)$, where ko is the spectrum representing real K-theory. The computation is done in three steps: first, we need to compute the mod-2 homology of ko, $H_*(ko; \mathbb{F}_2)$. It turns out that this is the dual algebra of the Hopf algebra quotient $A//A(1)$. We can then apply a certain change-of-rings theorem, which says that

$$\mathrm{Ext}^{*,*}_{A_*}((A//A(1))_*, \mathbb{F}_2) \cong \mathrm{Ext}^{*,*}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2).$$

The Steenrod module $A(1)$ is small enough to get our hands on, and I present a computation of $\mathrm{Ext}^{*,*}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$ that is just too fun to ignore. Proofs of major theorems are often omitted in this section (with reference given to where the interested reader can find them, of course) in order to save space and brainpower. Ultimately, the three parts of this computation give us a "proof" of real Bott periodicity[1]!

A Brief Refresher on Spectral Sequences

I want to conclude this introduction by assembling some basic definitions and facts regarding spectral sequences, so that the reader (and the author!) can have a mostly self-contained experience. Of course, I assume the reader is fairly comfortable with the basics of algebraic topology (that covered in most first courses), because I assume that the reader is me! Anyway, on we go.

Definition 1.1. A **spectral sequence** $\{E_r, d_r\}$ is an $\mathbb{N}$-indexed collection of $\mathbb{Z}$-bigraded abelian groups $E_r^{*,*}$, equipped with *differential maps* d_r (that is, maps such that $d_r^2 = 0$). We furthermore require, if the bidegree of d_r is (s_r, t_r), that

$$E_{r+1}^{p,q} \cong \frac{\ker d_r : E_r^{p,q} \to E_r^{p+s_r, q+t_r}}{\operatorname{im} d_r : E_r^{p-s_r, q-t_r} \to E_r^{p,q}},$$

the homology at $E_r^{p,q}$.

Most frequently, we will encounter spectral sequences where the d_r differentials have bidegree $(\pm r, \pm(1-r))$. Suppose that in our spectral sequence, for each pair (p, q) there is

[1]The word "proof" is in quotation marks here because the change of rings theorem necessary to this computation assumes real Bott periodicity.

some $r_{p,q}$ such that for all $r \geq r_{p,q}$, we have

$$E_r^{p,q} \cong E_{r_{p,q}}^{p,q}.$$

In this case, we set $E_\infty^{p,q} = E_{r_{p,q}}^{p,q}$, and call the bigraded module $E_\infty^{*,*}$ the **limit term** of the spectral sequence. Often, the E_∞-page of a spectral sequence will be isomorphic (as a bigraded abelian group) to a filtration of some graded group H^* (the reader should be thinking of co/homology, as is the case with the Leray-Serre spectral sequence). In this case, the parlance used is that the spectral sequence $\{E_r, d_r\}$ **converges** or **abuts** to H^*. The general sort of theorem involving spectral sequences goes something like "there exists a spectral sequence with E_2-page given by *[something computable]*, converging to *[something desirable]*". Both the identification of the E_2-page and the object to which the spectral sequence converges are the major parts of such a theorem. For example, the following is a theorem of Serre:

Theorem 1.2 (Serre)**.** *Let $F \hookrightarrow E \to B$ be a fibration, where F has the homology of a sphere. Then there is a spectral sequence with E_2-page given by*

$$E_2^{p,q} = H^p(B, H^q(F)),$$

that converges to $H^*(E)$.

A natural next question is this: how does a spectral sequence arise "in the wild"? Good question, imaginary voice in my head! Alas, we must wait for a satisfactory answer until the beginning of §3, when I will tell you how to get a spectral sequence from an *exact couple* of objects in an abelian category.

Acknowledgments

In the course of writing this paper, stalling for some time, and writing more, I have benefited greatly from the generosity and brilliance of many people. First, I must thank Eric Peterson for teaching the algebraic topology course for which this paper was written[2].

2 Spectra and Cohomology Theories

In this section, we recall the basic notions of spectra, their associated (co)homology theories, and how ring spectra give rise to multiplicative cohomology theories.

Definition 2.1. A **spectrum** E is a sequence of CW-complexes $\{E_n\}_{n\in\mathbb{N}}$, with "suspension maps"

$$\Sigma E_n \to E_{n+1}.$$

[2] More precisely, for which this paper was *started*; it certainly was not turned in during that semester.

Examples 2.2. If X is a CW complex, there is a spectrum $\Sigma^\infty X$ with $(\Sigma^\infty X)_n = \Sigma^n X$, and the linking maps are the identities. These are called **suspension spectra**. The suspension spectrum of the zero sphere S^0 is written $\mathbb{S}^0$ or $\mathbb{S}$; that of the n-sphere is $\mathbb{S}^n$.

Definition 2.3. A **subspectrum** F of a spectrum E is a collection of subcomplexes $F_n \subseteq E_n$ such that the n-th suspension map of E sends ΣF_n into F_{n+1}. In other words, F inherits its linking maps from E. We say that a subspectrum F is **cofinal** if every cell $e^m \subseteq E_n$ eventually appears (after a finite number of suspension map applications) in some $\Sigma^\ell F_n \subseteq \Sigma^\ell E_n$.

Definition 2.4 (The category of spectra)**.** Let E and G be spectra. A **morphism of spectra** is a choice of cofinal subspectrum $F \subseteq E$ and a collection of maps $F_n \to G_n$, commuting with the linking maps of F and G. Two morphisms are considered equivalent if they agree on a mutually cofinal subspectrum:

$$\begin{tikzcd}
F' \arrow[r, hook] \arrow[drr] & E \arrow[dr] & \\
& & G \\
F'' \arrow[uu, hook] \arrow[r, hook] \arrow[urr] & F \arrow[uu, hook] \arrow[ur] &
\end{tikzcd}$$

Definition 2.5 (Homotopies of morphisms of spectra)**.** In the same way that we define a homotopy $\alpha : f \Rightarrow g$ of continuous functions $f, g : X \to Y$ as a function $\widetilde{\alpha} : X \wedge I_+ \to Y$ such that $\widetilde{\alpha}|_0 = f$ and $\widetilde{\alpha}|_1 = g$, we define a **homotopy** $\alpha : \varphi \Rightarrow \psi$ of morphisms $\varphi, \psi : E \to G$ of spectra as a map

$$\widetilde{\alpha} : F'' \wedge I_+ \to G$$

such that $\widetilde{\alpha}|_0 = \varphi$ and $\widetilde{\alpha}|_1 = \psi$. In this situation, F'' is any mutually cofinal subspectrum on which both φ and ψ are defined, I_+ is the interval with disjoint basepoint, and the smash product is taken level-wise ($(F \wedge I_+)_n = F_n \wedge I_+$).

If E and G are spectra, we will denote homotopy classes of maps from E to G as $\mathsf{hSpectra}(E, G)$; when the context is understood, we may also denote this set by $[E, G]$. The n**-th homotopy group** of a spectrum E is the set $\mathsf{hSpectra}(\mathbb{S}^n, E)$.

Remark 2.6. In the same way that homotopy groups of spaces $\pi_n X$ have natural group structures for $n \geq 1$ because $\Sigma(-)$ is a cogroup-valued functor in the category of pointed spaces, $\mathbb{S}^n$ are cogroup objects in the category of spectra. Thus, $\pi_n E$ has a group structure. More generally, $\Sigma(-)$ is cogroup-valued in $\mathsf{Spectra}$, and has an inverse functor $\Sigma^{-1}(-) : \mathsf{Spectra} \to \mathsf{Spectra}$. Because $[\Sigma^2\Sigma^{-2}E, \Sigma^2\Sigma^{-2}G]$ is thus an abelian group, so is $[E, G]$. Interestingly, all homotopy classes of maps of spectra are naturally abelian groups.

Definition 2.7 (Fibrations and cofibrations of spectra)**.** We define **fibrations** and **cofibrations** of spectra analogously to how they are defined in Top. We say that a morphism $\pi : E \to G$ of spectra has the **homotopy lifting property** with respect to a spectrum F

if for any homotopy $\varphi : F \wedge I_+ \to G$, there exists a lift $\widetilde{\varphi} : F \wedge I_+ \to E$ making the relevant diagram commute:

$$\begin{tikzcd} F \arrow[r, "\varphi|_0"] \arrow[d, hook, "\mathrm{id}\wedge\{0\}"'] & E \arrow[d, "\pi"] \\ F \wedge I_+ \arrow[r, "\varphi"] \arrow[ur, dashed, "\widetilde{\varphi}"] & G \end{tikzcd}$$

Definition 2.8 (The homotopy category of spectra)**.** The category $\mathsf{hSpectra}$ is the category whose objects are homotopy types of spectra, and whose morphisms are homotopy classes of maps of spectra.

Aside 2.9 (For those who like model categories)**.** There are many different (that is, not Quillen equivalent) model structures on the category of spectra that give equivalent homotopy categories.

Theorem 2.10 (Adams 1974 §III.4.1)**.** *There is a product bifunctor on the homotopy category of spectra,* $- \wedge - : \mathsf{hSpectra} \times \mathsf{hSpectra} \to \mathsf{hSpectra}$*, that makes* $\mathsf{hSpectra}$ *symmetric monoidal.*

Proof. We give a construction of the smash product of two spectra, following Mandell et al. 2001. Suppose that X and Y are spectra; we define $(X \wedge Y)_n$ for each n, as well as the transition maps $\sigma_n^{X\wedge Y}$.

In even degree, we have

$$(X \wedge Y)_{2n} = X_n \wedge Y_n,$$

and in odd degree,

$$(X \wedge Y)_{2n+1} = S^1 \wedge X_n \wedge Y_n.$$

The structure maps from even degree to odd degree are the identities

$$\sigma_{2n}^{X\wedge Y} : S^1 \wedge X_n \wedge Y_n \to S^1 \wedge X_n \wedge Y_n,$$

and from odd to even degree, they are defined recursively as the composite

$$\sigma_{2n+1}^{X\wedge Y} : S^1 \wedge S^1 \wedge X_n \wedge X_n \simeq S^1 \wedge X_n \wedge S^1 \wedge Y_n \xrightarrow{\sigma_n^X \wedge \sigma_n^Y} X_{n+1} \wedge Y_{n+1} \cong (X \wedge Y)_{2n+2}.$$

Recall that to define a monoidal structure on a category, we must also give a unit object, associativity isomorphisms, and left and right unit multiplication isomorphisms. In $\mathsf{hSpectra}$, the unit for the smash product is the stable sphere $\mathbb{S}$. One can check this easily by noting that $X \wedge \mathbb{S}$ is a cofinal subspectrum of X; the inclusion of a cofinal subspectrum into its parent spectrum is a weak equivalence[3]. As smash products are symmetric even before passing to the homotopy category, we also get the left unit multiplicative identity $\mathbb{S} \wedge X \simeq X$. Checking that the smash product of spectra is associative is slightly more

[3]There is a direct analog of the Whitehead theorem for spectra: a map that induces an isomorphism on all homotopy groups (i.e., a weak equivalence) is a homotopy equivalence.

involved; this is still just a straightforward unwinding of the definitions and is left to the reader as a routine exercise.

The important thing about smash products in the category of spectra is not their explicit construction (of which there are many), but rather that they exist, and satisfy the many axioms for a tensor product. □

The representability theorem of Edwin Brown (first introduced in Brown 1962) tells us that a sufficiently nice (read: cohomological) functor is represented by a CW complex (or, more correctly, a homotopy type of a CW complex). Given a cohomology theory $\widetilde{h}^*$, we can use Brown representability to construct a sequence of spaces $\{\underline{h}_n\}$. This sequence furthermore forms a (homotopy class of a) spectrum, with linking maps induced by the isomorphism

$$\widetilde{h}^{n+1}(\Sigma X) \cong \widetilde{h}^n(X).$$

This construction, furthermore, is functorial, giving a functor

$$\mathsf{CohomThy} \xrightarrow{\text{B.R.}} \mathsf{hSpectra}.$$

Just as easily as we can get a spectrum from a cohomology theory, we can extract a cohomology theory from a spectrum E. We define the **(co)homology theory** associated to E as

$$\begin{aligned} &\text{cohomology: } \widetilde{E}^n(X) = \mathsf{hSpectra}(\Sigma^\infty X, \Sigma^n E) \\ &\text{homology: } \widetilde{E}_n(X) = \pi_n(E \wedge X) = \mathsf{hSpectra}(\mathbb{S}^n, E \wedge X). \end{aligned}$$

We refer the reader to Adams 1974, §III.6.1, for a more in-depth discussion of the following theorem, showing that the above defined (co)homology theories from spectra do, indeed, satisfy the Eilenberg-Steenrod axioms for a generalized (co)homology theory.

Theorem 2.11. *Let E be a spectrum. The homology theory defined above, $\widetilde{E}_*(-)$, is a functor covariant in both E and X, taking values in the category of graded abelian groups. Similarly, the cohomology theory $\widetilde{E}^*(-)$ is a functor contravariant in X and covariant E, taking values in the same category. Both functors, furthermore, satisfy the Eilenberg-Steenrod axioms for generalized cohomology.* □

Aside 2.12. One might expect that the cohomology theory functor $\widetilde{(-)}^*: \mathsf{hSpectra} \to \mathsf{CohomThy}$ form an adjoint equivalence pair; in particular that both functors are fully faithful and essentially surjective. This is, indeed, the case! The categories $\mathsf{hSpectra}$ and $\mathsf{CohomThy}$ "contain the same information." Unfortunately, this fails to be true for the category of homology theories, HomThy. Although the Brown Representability and homology theory functors are essentially surjective, they fail to be fully faithful due to the existence of so-called "phantom maps" in the category of spectra. A **phantom map** is a non-nullhomotopy $\alpha : E \to F$ of spectra such that $\alpha^* : \widetilde{E}^* \to \widetilde{F}^*$ is an isomorphism of homology theories. For more information on phantom maps, see for example Lurie 2010, Lecture 17.

clarify this; there are disagreeing resources online

Definition 2.13 (Ring spectra)**.** A **(commutative) ring spectrum** is a (commutative) monoid object in the symmetric monoidal category $\mathsf{hSpectra}$. That is, it is a spectrum E equipped with a *multiplication* $\mu : E \wedge E \to E$ and a *unit* $\eta : \mathbb{S} \to E$ such that the following diagrams commute up to homotopy:

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mathrm{id}\wedge\mu"] \arrow[d, "\mu\wedge\mathrm{id}"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{S} \wedge E \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & E \wedge E \arrow[d, "\mu"] & E \wedge \mathbb{S} \arrow[l, "\mathrm{id}\wedge\eta"'] \arrow[dl, "\mathrm{id}"] \\ & E & \end{tikzcd}$$

In the latter diagram, it is understood that $S \wedge E \simeq E \wedge S \simeq E$ naturally, via the left and right unitors of the model structure on the homotopy category of spectra. Commutativity comes via a **braiding map** $\tau_{A,B} : A \wedge B \xrightarrow{\simeq} B \wedge A$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau_{E,E}", "\simeq"'] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

Through the correspondence between spectra and (co)homology theories, one might expect that a ring spectrum corresponds to a multiplicative cohomology theory. This is indeed correct: let E be a ring spectrum as defined in Definition 2.13. We define a product in E-cohomology $E^q(X) \otimes E^p(X) \to E^{p+q}(X)$, to give $E^*(X)$ the structure of a graded ring. The natural thing to do is remember that if X is a spectrum, then $E^*(X) = [X, \Sigma^* E]$. If $f \in [X, \Sigma^p E]$ and $g \in [X, \Sigma^q E]$, then

$$f \wedge g \in [X \wedge X, \Sigma^p E \wedge \Sigma^q E].$$

We will first define an *external* product, and then an internal one by composition with the functorial image of the diagonal map. Some details are left out; for a complete treatment, see either Switzer 1975 Chapter 13 or Adams 1974 §III.9.

We have seen in Remark 2.6 that the stable homotopy category $\mathsf{hSpectra}$ is remarkable[4] in that all hom-sets naturally have the structure of abelian groups. Another remarkable feature about this category is that all cofibrations are fibrations:

Theorem 2.14 (Lewis, May, and Steinberger 1986, Theorem III.2.4)**.** *For any morphism*

[4]Pun intended.

$f : X \to Y$ in $\mathsf{hSpectra}$*, there is an isomorphism*

$$\phi : \mathrm{hofib}(f) \xrightarrow{\simeq} \Omega \mathrm{hocof}(f)$$

between the homotopy fiber of f and the looping of the homotopy cofiber of f that fits into the following commutative diagram:

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d, equal] & \mathrm{hofib}(f) \arrow[r] \arrow[d, "\phi"] & X \arrow[d, "\simeq"] \\
\Omega Y \arrow[r] & \Omega \mathrm{hocof}(f) \arrow[r] & \Omega \Sigma X.
\end{tikzcd}$$

In particular, a sequence is a cofibration in $\mathsf{hSpectra}$ *if and only if it is a fibration.*

Proof. This proof is given in two parts. The first, following Hovey, Shipley, and Smith 2000, shows that the stable homotopy category $\mathsf{hSpectra}$ really is stable, in a sense to be made precise. The second, after Hovey 1998, proves that in any homotopy category, homotopy pushout squares coincide with homotopy pullback squares: the square

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

is a homotopy pullback square if and only if it is a homotopy pushout square.

Recall that in any pointed model category, the **loop space functor** $\Omega(-) : \mathrm{HO}(\mathscr{C}) \to \mathrm{HO}(\mathscr{C})$ is the functor given on objects by the homotopy pullback of the diagram $* \to X \leftarrow *$:

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X.
\end{tikzcd}$$

Dually, the **suspension functor** $\Sigma(-) : \mathrm{HO}(\mathscr{C}) \to \mathrm{HO}(\mathscr{C})$ is given by objects as the homotopy pushout of the diagram $* \leftarrow X \to *$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \Sigma X.
\end{tikzcd}$$

We say that the homotopy category of a pointed model category is **stable** if the functors Σ

and Ω form an inverse equivalence pair:

$$\begin{tikzcd} \Omega : \mathrm{HO}(\mathscr{C}) \arrow[r, bend right, "\simeq"'] & \mathrm{HO}(\mathscr{C}) : \Sigma \arrow[l, bend right, "\simeq"'] \end{tikzcd}$$

The important thing to note is that the category $\mathsf{hSpectra}$ is, indeed, a stable homotopy category:

Lemma 2.15. *The category* $\mathsf{hSpectra}$ *is a stable homotopy category: the suspension functor* $\Sigma : \mathsf{hSpectra} \to \mathsf{hSpectra}$ *is an equivalence, with inverse* Ω*.*

Proof. Omitted for now. □

Now, we have the following lemma which categorizes homotopy pushouts and pullbacks in any stable homotopy category. □

3 Construction of the Adams Spectral Sequence

Before embarking on the main goal of this article—the construction of the Adams spectral sequence—we take a few moments to remind the reader how to extract a spectral sequence from an exact couple. This exposition may be unfortunately terse; for a more leisurely (and complete) stroll through the basics of spectral sequences in general, we recommend taking a look at McCleary 2001, in particular Part I, Chapters 2 and 3.

The theory of spectral sequences can be developed in any abelian category; for simplicity, we invite the reader to imagine the category Ab of abelian groups.

Definition 3.1. An **exact couple**[5] of abelian groups (or objects in an abelian category $\mathscr{A}$) is a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\alpha"] & & A \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

that is exact at every point:

$$\operatorname{im}\alpha = \ker\beta, \operatorname{im}\beta = \ker\gamma, \text{ and } \operatorname{im}\gamma = \ker\alpha.$$

Often we will consider exact triangles in a category of *graded* objects, and the map α will be taken to have a fixed degree (usually -1).

Let $d : E \to E$ be the composite map $d = \gamma \circ \beta$ (note that both A's are the same object!). By exactness, we have $d \circ d = 0$, and so it makes sense to take the homology of

[5] This may also be called an **exact triangle**, because while there are two objects in sight, there are three points on the triangle. I may use the terminology interchangeably.

the exact triangle: set
$$E' = \frac{\ker d}{\operatorname{im} d} \quad \text{and} \quad A' = \operatorname{im} \alpha.$$
We can define a new couple
$$\begin{tikzcd} A' \arrow[rr, "\alpha'"] & & A' \arrow[dl, "\beta'"] \\ & E' \arrow[ul, "\gamma'"] & \end{tikzcd}$$
where

- $\alpha' : A' \to A'$ is the restriction of α to A',
- $\beta' : A' \to E'$ is given by $\beta'(\alpha(a)) = [\beta(a)]$, and
- $\gamma' : E' \to A'$ is defined by $\gamma'([e]) = \gamma(e)$.

It is a routine exercise left to the reader to show that the data $(E', A', \alpha', \beta', \gamma')$.

This new couple is known as the **derived couple** of the couple $(E, A, \alpha, \beta, \gamma)$; we can thus keep "turning the crank" of taking derived couples to get an entire sequence of them. If the initial data consist of objects A_1 and E_1, we will denote by A_2 and E_2 the objects A_1' and E_1'; inductively, let $E_i = E_{i-1}'$. With a similar notation for the derived maps in the couple, we let $d_i = \beta_{i-1}' \circ \gamma_{i-1}'$. We are thus left with a sequence of objects $\{E_i\}$ equipped with differentials d_i; the data of the sequence $\{(E_i, d_i)\}$ are known as the **spectral sequence** associated to the exact couple $(E, A, \alpha, \beta, \gamma)$.

Remark 3.2. It is perhaps unsatisfying to the reader to see that a spectral sequence associated to an exact couple does not consist of *bigraded* abelian groups. Frequently, the exact couple we work with *comes* with a bigrading already; for instance, we will see that filtering a spectrum by a Postnikov-like tower (called an E_*-Adams resolution) gives one grading, and taking E-homology gives another. These two gradings lead to the double complex of the Adams Spectral Sequence.

Definition 3.3. Let E be a spectrum. An **E-completion** $\widehat{X}$ of a spectrum X is an E-pseudo-equivalent spectrum that admits an E_*-Adams $\{\widehat{X}_s\}$ with $\varinjlim \widehat{X}_s = \mathrm{pt}$.

Definition 3.4. Let G be a group. The **Moore spectrum** associated to G is the spectrum SG such that

1. $\pi_{<0}(SG) = 0$,
2. $\pi_0(SG) = G$, and
3. $\pi_{>0}(SG \wedge H\mathbb{Z}) = 0$, where $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum of the integers.

A model of the Moore spectrum for an abelian group G is constructed in Adams 1974, §III.6. For completeness, we restate the construction as Construction B.3.

Theorem 3.5 (Adams 1974, Theorem III.14.6)**.** *Let E be a ring spectrum satisfying the conditions $(*)$. (i) If $\pi_0(E) = R$ is a subring of the rationals, then*

$$\widehat{X}^E = XR$$

is the spectrum $X \wedge SR$.

(ii) If $\pi_0(E) = \mathbb{Z}/m$ and $\pi_r(E)$ is finitely generated for all $r > 0$, then

$$\widehat{X}^E = X_p := X\mathbb{Z}_p$$

is the spectrum $X \wedge S\mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. □

The main theorem of this section (and, indeed, of this paper) is the following, stating the existence and convergence of the Adams spectral sequence, as well as an identification of the E_2-page.

Theorem 3.6. *Let E be a spectrum satisfying the conditions $(*)$, and let X be any spectrum. An E_*-Adams resolution for X gives rise to a natural spectral sequence $E_*^{*,*}(X)$ with $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that*

(a) $E_2^{*,*} = \mathrm{Ext}^{*,*}_{E_*(E)}(E_*(\mathbb{S}), E_*(X))$ and

(b) the spectral sequence converges to $\pi_*(\widehat{X}^E)$; i.e., there is a filtration of $\pi_*(\widehat{X}^E)$ giving a double complex that is isomorphic (as a doubly-graded abelian group) to $E_\infty^{*,*}$. This filtration will furthermore be identified explicitly in Theorem 3.7.

Theorem 3.7. *We can identify the filtration of $\pi_*(\widehat{X}^E)$ mentioned in Theorem 3.6 as follows: a map $f : \mathbb{S}^n \to X$ (or rather a homotopy class thereof) is considered to be in filtration degree s if s is the least integer such that f can be factored into s maps, each of which becomes trivial after smashing the target with E. That is, we can write*

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "g_s"] \arrow[rrrr, bend right, "f"'] & Y_{s-1} \arrow[r, "g_{s-1}"] & Y_{s-2} \arrow[r, "g_{s-2}"] & \cdots \arrow[r, "g_0"] & X, \end{tikzcd}$$

where every composite

$$\begin{tikzcd} Y_i \arrow[r, "g_i"'] \arrow[rr, bend left, "\widetilde{g}_i"] & Y_{i-1} \arrow[r] & Y_{i-1} \wedge E \end{tikzcd}$$

is nullhomotopic. The map $Y_{i-1} \to Y_{i-1} \wedge E$ is the composite $Y_{i-1} \simeq Y_{i-1} \wedge \mathbb{S} \xrightarrow{\mathrm{id}_{Y_{i-1}} \wedge \eta} Y_{i-1} \wedge E$.

4 A Proof of Convergence

5 Identification of the E_2-page

6 Application to Real Bott Periodicity

A Algebraic Hogwash

In this section, we discuss some of the algebraic background necessary to complete the proofs of, for example, [insert theorems here]. It is the opinion of many that serious work with Hopf algebroids is largely unnecessary to working with the Adams SS; I happen to agree. However, as this paper is ostensibly a set of "learning notes," I would think it a shame to ignore this part of the theory altogether. We begin with the definition of a Hopf algebroid, which generalizes that of a Hopf algebra (a cogroup object in the category of K-algebras for some ring K). The true purpose of this appendix is to define the functor Ext in the category of comodules over a Hopf algebroid.

B Other Miscellany

In this appendix, I collect various results and definitions that I find interesting, but nonetheless unimportant to the exposition above.

Definition B.1 (Triangulated categories; ignore if category theory isn't really your thing)**.** Throughout this definition, you should be thinking of the category $\mathsf{hSpectra}$. A **triangulated category** is the following data:

1. an additive category $\mathscr{H}$ (i.e., an Ab-enriched category admitting finite coproducts);
2. an auto-equivalence of categories called the **suspension functor**
$$\Sigma : \mathscr{H} \xrightarrow{\simeq} \mathscr{H};$$
3. and a subclass Cof of triples of composable morphisms, called the class of **distinguished triangles**, where each triple starts with E and ends with ΣE; we write these as
$$E \longrightarrow B \longrightarrow B/E \longrightarrow \Sigma E.$$
Alternatively, these may be written
$$\begin{tikzcd} E \arrow[rr] & & B \arrow[dl] \\ & B/E, \arrow[ul] & \end{tikzcd}$$
with the understanding that the map $B/E \to E$ is taken to be $B/E \to \Sigma E$.

These data are subject to the following axioms:

TC0 For every morphism $f : A \to B$, there exists a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow B/A \longrightarrow \Sigma A.$$

If furthermore the top row of the following commutative diagram is a distinguished triangle and all the vertical arrows are isomorphisms, then so is the bottom row distinguished:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\simeq"] & B \arrow[r, "g"] \arrow[d, "\simeq"] & B/A \arrow[r, "h"] \arrow[d, "\simeq"] & \Sigma A \arrow[d, "\simeq"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & B'/A' \arrow[r, "h'"] & \Sigma A' \end{tikzcd}$$

TC1 For every $E \in \mathscr{H}$,

$$0 \longrightarrow E \xrightarrow{\mathrm{id}_E} E \longrightarrow 0$$

is a distinguished triangle.

TC2 If the triple (f, g, h) is a distinguished triangle, then so is the triple $(g, h, -\Sigma f)$ (where $-\Sigma$ is the inverse equivalence to Σ):

$$A \xrightarrow{f} B \xrightarrow{g} B/A \xrightarrow{h} \Sigma A \quad \text{distinguished implies}$$

$$B \xrightarrow{g} B/A \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B \quad \text{is also distinguished.}$$

TC3 Given a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_A"] & B \arrow[r, "g"] \arrow[d, "\phi_B"] & B/A \arrow[r, "h"] & \Sigma A \arrow[d, "\Sigma\phi_A"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & B'/A' \arrow[r, "h'"] & \Sigma A', \end{tikzcd}$$

where the top and bottom rows are distinguished triangles, there exists a map $\psi : B/A \to B'/A'$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_A"] & B \arrow[r, "g"] \arrow[d, "\phi_B"] & B/A \arrow[r, "h"] \arrow[d, dashed, "\exists\,\psi"] & \Sigma A \arrow[d, "\Sigma\phi_A"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & B'/A' \arrow[r, "h'"] & \Sigma A' \end{tikzcd}$$

commutes.

TC4 Suppose given two composable morphisms $f : A \to B$ and $f' : B \to C$. Then the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f'"] & B/A \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f'\circ f"] & C \arrow[r, "g''"] \arrow[d, "g'"] & C/A \arrow[r, "h''"] \arrow[d, "y"] & \Sigma A \\
& C/B \arrow[r, "\simeq"] \arrow[d, "h'"] & C/B \arrow[d, "(\Sigma g)\circ h'"] & \\
& \Sigma B \arrow[r, "\Sigma g"] & \Sigma(B/A) &
\end{tikzcd}$$

Furthermore, both long rows and both long columns are distinguished triangles.

Definition B.2 (Tensor triangulated categories; category-haters keep ignoring)**.** A **tensor triangulated category** is a category $\mathscr{C}$ with the structure of a symmetric monoidal category[6] $(\mathscr{C}, \otimes, \mathbb{1}, \tau)$, the structure of a triangulated category $(\mathscr{C}, \Sigma, \mathsf{Cof})$, and for all objects $X, Y \in \mathscr{C}$, natural isomorphisms

$$e_{X,Y} : (\Sigma X) \otimes Y \xrightarrow{\simeq} \Sigma(X \otimes Y).$$

These data are required to satisfy the following axioms:

TT1 For all $X \in \mathscr{C}$, the functors $X \otimes (-) \simeq (-) \otimes X$ are additive (preserve finite coproducts);

TT2 For all $X \in \mathscr{C}$, the functors $X \otimes (-) \simeq (-) \otimes X$ preserve distinguished triangles: if

$$X \xrightarrow{f} Y \xrightarrow{g} Y/X \xrightarrow{h} \Sigma X$$

is in Cof, then so is

$$V \otimes X \xrightarrow{1_V \otimes f} V \otimes Y \xrightarrow{1_V \otimes g} V \otimes Y/X \xrightarrow{1_V \otimes h} V \otimes (\Sigma X) \simeq \Sigma(V \otimes X),$$

where the final equivalence is the map $e_{X,V} \circ \tau_{V,\Sigma X}$.

The triangulated structure and the symmetric monoidal structure are **compatible** if they satisfy the following coherence axioms:

[6] Recall that the data of a symmetric monoidal category consist of a *tensor product* $\otimes$, a *unit* $\mathbb{1}$, and a *braiding map* τ

TT3 For all $X, Y, Z \in \mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
 & \Sigma(X \otimes Y) \otimes Z \arrow[dr, "e_{X\otimes Y,Z}"] & \\
(\Sigma X \otimes Y) \otimes Z \arrow[ur, "e_{X,Y}\otimes 1_Z"] \arrow[d, "a_{\Sigma X,Y,Z}"] & & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma a_{X,Y,Z}"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X,Y\otimes Z}"] & & \Sigma(X \otimes (Y \otimes Z)).
\end{tikzcd}$$

TT4 For all integers n and $m \in \mathbb{Z}$, the following diagram commutes:

$$\begin{tikzcd}
(\Sigma^n 1) \otimes (\Sigma^m 1) \arrow[r, "\simeq"] \arrow[d, "\tau_{\Sigma^n 1, \Sigma^m 1}"'] & \Sigma^{n+m} 1 \arrow[d, "(-1)^{n\cdot m}"] \\
(\Sigma^m 1) \otimes (\Sigma^n 1) \arrow[r, "\simeq"] & \Sigma^{n+m} 1
\end{tikzcd}$$

Construction B.3 (A model of Moore spectra)**.** In this section, we do what was promised in Definition 3.4 and restate the construction of a model of the Moore spectrum for an abelian group G given in Adams 1974, §III.6.

Take a free $\mathbb{Z}$-resolution of G: $0 \longrightarrow R \xrightarrow{i} F \longrightarrow G \longrightarrow 0$. By, say, Van Kampen's theorem, we can construct bouquets of stable spheres $\bigvee_{\alpha\in A} \mathbb{S}$ and $\bigvee_{\beta\in B} \mathbb{S}$ such that

$$\pi_0 \left(\bigvee_{\alpha\in A} \mathbb{S} \right) = R \quad \text{and} \quad \pi_0 \left(\bigvee_{\beta\in B} \mathbb{S} \right) = F.$$

It is possible as well to take a map $f : \bigvee_{\alpha\in A} \mathbb{S} \to \bigvee_{\beta\in B} \mathbb{S}$ that induces i in π_0. We then let the spectrum SG be the homotopy cofiber[7] of this map f:

$$SG = \left(\bigvee_{\beta\in B} \mathbb{S} \right) \cup_f C \left(\bigvee_{\alpha\in A} \mathbb{S} \right).$$

It's fairly clear that this construction does exactly what we need it to: because wedges of sphere spectra are connective, so is the homotopy cofiber of a map between such spectra; this knocks part 1 of Definition 3.4 off the list. Part 2 follows by the long exact sequence in homotopy associated to a cofibration, and part 3 comes from the long exact sequence in singular homology (i.e., that in homotopy associated to the cofibration that comes from application of the functor $- \wedge H\mathbb{Z}$).

[7] The cone CX on a spectrum X is formed analogously to that on a *space* X: $CX = X \wedge I$, where I is the interval with basepoint 1.

References

[1] J. Frank Adams. *Stable Homotopy and Generalised Homology*. Chicago Lectures in Mathematics. The University of Chicago Press, 1974.

[2] Edgar H. Brown. "Cohomology Theories". In: *Annals of Mathematics* 75.3 (1962), pp. 467–484. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970209.

[3] Anatoly Fomenko and Dmitry Fuchs. *Homotopical Topology*. 3rd ed. Graduate Texts in Mathematics 273. Springer International Publishing, 2016.

[4] Mark Hovey. *Model Categories*. Mathematical Surveys and Monographs 63. American Mathematical Society, 1998.

[5] Mark Hovey, Brooke Shipley, and Jeff Smith. "Symmetric Spectra". In: *Journal of the American Mathematical Society* 13.1 (2000), pp. 149–208. URL: http://www.ams.org/journals/jams/2000-13-01/S0894-0347-99-00320-3/S0894-0347-99-00320-3.pdf.

[6] L.G. Lewis, J.P. May, and M. Steinberger. *Equivariant Stable Homotopy Theory*. Lecture Notes in Mathematics. Springer-Verlag, 1986. URL: http://www.math.uchicago.edu/~may/BOOKS/equi.pdf.

[7] Jacob Lurie. *Chromatic Homotopy Theory*. Lecture Notes Collection, 2010. URL: http://www.math.harvard.edu/~lurie/252x.html/.

[8] M. A. Mandell et al. "Model Categories of Diagram Spectra". In: *Proceedings of the London Mathematical Society* 82.2 (2001), p. 441. DOI: 10.1112/S0024611501012692. URL: http://dx.doi.org/10.1112/S0024611501012692.

[9] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001. ISBN: 9780521567596. URL: https://books.google.com/books?id=NijkPwesh-EC.

[10] Doug C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. 2015.

[11] Urs Schreiber. *Introduction to Stable Homotopy Theory*. 2016. URL: https://ncatlab.org/nlab/show/Introduction+to+Stable+homotopy+theory+--+1-1.

[12] Robert M. Switzer. *Algebraic Topology – Homotopy and Homology*. Grundlehren der mathematischen Wissenschaften 212. Berlin: Springer-Verlag, 1975.

Index