HARVARD MATH 101

An Introduction to Algebraic Topology

or: why are we learning this stuff, anyway?

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Abstract

These are notes outlining the basics of Algebraic Topology, written for students in the Fall 2017 iteration of Math 101 at Harvard. As the class is by conception an introduction to proofs, it unfortunately is unable to dive into the interesting details surrounding the objects defined. For instance, we spent nearly three weeks discussing topology, without so much as defining the word "continuous." It is my hope that these notes will pique the interest of some students in the class, or at the very least convince them that this mumbo-jumbo about sets and groups and functions isn't totally dry and worthless.

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1 Introduction

Recall the definition of a topological space, a notion that seems incredibly opaque and complicated:

Definition 1.1. A TOPOLOGICAL SPACE is a pair $(X, \mathscr{T})$ where X is a set and $\mathscr{T}$ is a topology on X. A TOPOLOGY on X is a subset $\mathscr{T} \subseteq \mathscr{P}(X)$ such that

1. the empty set and all of X are in $\mathscr{T}$;
2. if $\{U_i\}_{i \in I}$ is a collection of sets in $\mathscr{T}$, indexed by an arbitrary set I, then
$$\bigcup_{i \in I} U_i \in \mathscr{T};$$
3. if $U_1, \ldots, U_n \in \mathscr{T}$, then the intersection $U_1 \cap \cdots \cap U_n \in \mathscr{T}$ as well. That is to say, $\mathscr{T}$ is closed under finite intersections and arbitrary unions.

A set $U \in \mathscr{T}$ is said to be OPEN in the topological space $(X, \mathscr{T})$. Usually, we abuse notation and refer to X itself as a topological space, when the topology is understood.

One mindset that has hopefully come across during the course is that objects on their own are useless, until we can compare them to other objects by way of suitably defined *functions*. For instance, a group by itself doesn't do much good, but when we can talk about groups being *isomorphic*, then things get interesting. I'm therefore seeking a "good" notion of function between topological spaces; I'll try not to give away the answer too quickly, but rather explain why this definition makes sense.

If the subject of "topology" is supposed to study things like *shapes* or *lines* or *spaces*—all of which are resolutely geometric objects—maps between topological spaces shouldn't "rip" or "tear" the domain in any way. That is to say, we want maps[1] to be *continuous*: points that are near other points—in the sense of mutually being contained in sufficiently small open sets—should be mapped to points close to each other. Recall that a "map" of groups $\varphi : G \to H$ takes products to products; it would thus be a natural guess that a continuous map of topological spaces $f : X \to Y$ would be one that takes open sets to open sets. That is:

Fake Definition 1.2. Let X and Y be topological spaces. A CONTINUOUS MAP $f : X \to Y$ is a set function such that $f(U)$ is open in Y whenever U is open in X.

Now I will explain why this first guess isn't quite the right thing to do, by means of example: consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$. This isn't the most interesting function in the world, but it is certainly "continuous" heuristically. For any open set $U \subseteq X$, we have $f(U) = \{0\}$. But $\{0\}$ is *not* open in $\mathbb{R}$, so we need a different definition.

[1] Whenever I say "map", I really mean "morphism in whatever category makes the most sense given the situation."

Ammended Definition 1.3. Let X and Y be topological spaces. A CONTINUOUS MAP $f : X \to Y$ is a set function such that $f^{-1}(V)$ is open in X whenever V is open in Y.

Perhaps this seems a little strange, but with some contemplation (and maybe some booze) you can convince yourself that it is the right definition to make. In fact, by way of the following exercise, you *will* convince yourself of this (at least in some special cases):

Exercise 1 (Continuity in the metric topology)**.** In this exercise, I'll give a different perspective on the standard topology on $\mathbb{R}$, in a way that may agree more with your intuition[2]. Note that $\mathbb{R}$ comes equipped with a "standard" way of measuring distances: two points $a, b \in \mathbb{R}$ are $|a-b|$ units apart. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d(a,b) = |a-b|$ satisfies the following properties:

1. POSITIVE DEFINITENESS: for all $a, b \in \mathbb{R}$, $d(a,b) \geq 0$. Furthermore, $d(a,b) = 0$ if and only if $a = b$.

2. SYMMETRY: for all $a, b \in \mathbb{R}$, $d(a,b) = d(b,a)$.

3. TRIANGLE INEQUALITY: for all $a, b, c \in \mathbb{R}$,

$$d(a,b) + d(b,c) \geq d(a,c).$$

You, dear reader, are trusted to verify these properties if you do not believe them immediately. This function d makes $\mathbb{R}$ into a METRIC SPACE. In general, a metric space is a set X equipped with a function $d : X \times X \to \mathbb{R}$ called the METRIC, that satisfies the three properties listed above.

Any metric space gives rise to a topological space, as follows (this applies in general, but feel free to think of $\mathbb{R}$ as the canonical example): say that an *open ball* at x of radius ε in X is the set

$$B_\varepsilon(x) := \{a \in X : d(x,a) < \varepsilon\}$$

of points fewer than ε units away from x. The METRIC TOPOLOGY on X will be the topology $\mathscr{T}^{\text{met}}$ associated to the basis $\mathscr{B}^{\text{met}}$ of open balls[3]:

$$\mathscr{B}^{\text{met}} := \{B_\varepsilon(x) : x \in X, \varepsilon \in \mathbb{R}_{>0}\}.$$

Note that this agrees with our definition of the standard topology on $\mathbb{R}$, as the open balls are precisely the open intervals (in the case $X = \mathbb{R}$, I've really said nothing new here). Now say that a function $f : \mathbb{R} \to \mathbb{R}$ is CONTINUOUS (or *continuous in the metric sense*) at $x \in \mathbb{R}$ if for each $\delta > 0$, there is some $\varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies $d(f(x), f(y)) < \delta$. That is to say, points near x in the domain don't get sent too far from $f(x)$ in the image.

[2]This should feel natural if you've taken a class on real analysis, or if you've ever seen ε-δ proofs.

[3]Check that this is a basis!

Your task is as follows: *show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous in the sense of Definition 1.3 if and only if it is continuous in the metric sense at all points $x \in \mathbb{R}$.*

Exercise 2. As another exercise to familiarize yourself with the notion of continuity, suppose X is a set, equipped with two topologies, $\mathscr{T}_1$ and $\mathscr{T}_2$. Denote by X_1 the topological space $(X, \mathscr{T}_1)$ and X_2 the space $(X, \mathscr{T}_2)$; show that the identity map $\mathbb{1}_X : X_1 \to X_2$ is continuous if and only if $\mathscr{T}_2$ is coarser than $\mathscr{T}_1$. Conclude that if $\mathscr{T}^{\text{ind}}$ is the indiscrete topology on X with corresponding space X^{ind}, the identity function $\mathbb{1}_X : X_1 \to X^{\text{ind}}$ is continuous for any topology $\mathscr{T}_1$. Similarly, if X^{disc} is the set X equipped with the discrete topology, then the identity map $\mathbb{1}_X : X^{\text{disc}} \to X_1$ is continuous.

One can actually prove more about the discrete and indiscrete topologies: show that any set function $X^{\text{disc}} \to Y$ is continuous, for any topological space Y. Dually, show that any set function $Y \to X^{\text{ind}}$ is continuous, for any space Y.

Reuben's Aside 1.4. The previous exercise hints at a deeper fact about the discrete and indiscrete topologies, which I'll phrase in categorical language: the functor $(-)^{\text{disc}} : \mathsf{Set} \to \mathsf{Top}$ taking a set to the corresponding discrete space is left adjoint to the forgetful functor $U_{\mathsf{Set}}^{\mathsf{Top}} : \mathsf{Top} \to \mathsf{Set}$ sending a space to its underlying set. Dually, the functor $(-)^{\text{ind}} : \mathsf{Set} \to \mathsf{Top}$ taking a set to the indiscrete space on that set is *right* adjoint to $U_{\mathsf{Set}}^{\mathsf{Top}}$. This can be summarized in the diagram

$$\begin{tikzcd} \mathsf{Top} \arrow[r] \arrow[r, bend right=60, "\top"'] \arrow[r, bend left=60, "\top"] & \mathsf{Set}. \arrow[l, bend right=60, "(-)^{\text{ind}}"'] \arrow[l, bend left=60, "(-)^{\text{disc}}"] \end{tikzcd}$$

Just as how we think of two groups as being "the same" if there exists an isomorphism between them, we wish to have some notion of sameness for topological spaces, where we'll consider two topological spaces to be identical if they are related via this notion.

Definition 1.5. Let X and Y be topological spaces. A continuous function $f : X \to Y$ is a HOMEOMORPHISM if it is a bijection, and the inverse function $f^{-1} : Y \to X$ is also continuous. If there exists a homeomorphism $f : X \to Y$, then we say that X and Y are HOMEOMORPHIC as spaces.

Note that we really do need to require that f have a continuous inverse to get a valid notion of "sameness": if X^{disc} and X^{ind} denote the set X equipped with the discrete and indiscrete topologies, respectively, then the identity map $\mathbb{1}_X : X^{\text{disc}} \to X^{\text{ind}}$ is a continuous bijection, but the inverse $\mathbb{1}_X : X^{\text{ind}} \to X^{\text{disc}}$ is *not* continuous.

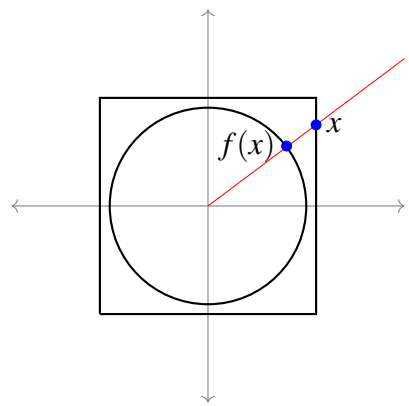
Example 1.6. The square B (of side length 2, centered at the origin) is homeomorphic to the circle S^1 (of radius 1, centered at the origin). To see this, we construct a "normalization" function

$$f : B \to S^1$$

by sending a point $(x,y) \in B$ to the unit vector in the (x,y)-direction:

$$f((x,y)) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

One can visualize the result of this function with the following figure:



You should check that the function f is continuous; it has an inverse function defined by taking a point x on the circle to the unique point on the square intersecting the ray pointing through x from the origin. This function is also continuous (although a little less obviously so), and thus f is a homeomorphism.

This example serves to show that topology cares only about shapes up to squeezing, bending, flattening, or smoothing. You may have heard of the saying that "to a topologist, a coffee cup is the same as a donut"; what one really means by saying that is "the surface of a coffee cup is homeomorphic to the surface of a donut".

1.1 The Goals of Algebraic Topology

That little bit of introductory material was just supposed to get you up to speed with the notion of a continuous function, which we'll use extensively in these notes. I want to conclude these preliminary remarks by giving a very broad description of the goals and techniques of algebraic topology, perhaps as a road map or just a bit of context that may make this document more readable.

Generally speaking, there are a few classes of important problems in mathematics. One large such class is the "classification problem", and takes the following form: *we have defined a type of mathematical object. For this object, we have developed a suitable notion of "equivalence" or "isomorphism". What are all of these objects up to isomorphism?* For the most part, such a question is too broad and generic to answer, so we usually break it up into more manageable pieces. For instance, you may have heard of a result called the "classification of finite simple groups." This is

one of the absolute triumphs of 20th–century mathematics, and occupies *thousands* of pages in the literature. Nonetheless, not every group is finite, and not every finite group is simple[4].

It turns out to be an unreasonable request to classify all topological spaces up to homeomorphism; even in the case of finite topological spaces (where the underlying sets are literally finite), this becomes a (doable, but) computationally impractical task. Furthermore, it is in general quite difficult to determine *a priori* whether or not two topological spaces are homeomorphic from purely geometric arguments (how do you show that *no* homeomorphism exists?) beyond set-theoretic ones. Algebraic topology thus seeks to associate to each space some sort of algebraic *invariant* (to be defined roughly later) that is easier to work with. One such thing might be the number of "holes" in your space, or its dimension, or the number of disjoint pieces, or potentially a more complicated structure like a group.

The process for using algebraic invariants to prove something about topology goes something like this: we want to prove the non-existence of a certain topological construction, say a homeomorphism $f : X \to Y$. Construct an invariant, like a group, in a way that is functorial[5] with respect to continuous maps of topological spaces. Then use facts about algebra to show that the invariant associated to X is different enough from that associated to Y to prevent the existence of a homeomorphism. In this document, I'll describe one such construction (the fundamental group of a space), and use it to show that two common spaces are not homeomorphic.

2 Homotopy and the Fundamental Group

The first important definition in this section will be a notion of *sameness* that is less restrictive than homeomorphism, but still more-or-less reasonable. Our definition will seem more natural after we consider a few examples, so let's do those first.

Example 2.1. For all $a, b \in \mathbb{R}_{>0}$, the intervals $(-a, a)$ and $(-b, b)$ are homeomorphic. If we let $a = 1$ and have b tend to zero, it would make sense to say that $(-1, 1)$ and the single point $\{0\}$ are "similar" topologically; nonetheless, they are not homeomorphic, because $(-1, 1)$ is uncountable, whereas $\{0\}$ is finite (and a homeomorphism is in particular a bijection).

Example 2.2. Consider the circle centered at the origin of radius 1, $S^1 \subseteq \mathbb{R}^2$. The cylinder with base S^1 and height 1 is the space

$$C := \{(a, b, c) \in \mathbb{R}^3 : (a, b) \in S^1, c \in [0, 1]\}.$$

[4]We say that a group is SIMPLE if it has no non-trivial normal subgroups. A subgroup $H \subseteq G$ is called NORMAL if for any $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

[5]We say that an assignment of, e.g., a group $G(X)$ to a space X is FUNCTORIAL with respect to continuous maps if for every continuous map $f : X \to Y$, there is a natural group homomorphism $Gf : G(X) \to G(Y)$; the assignment $f \mapsto Gf$ furthermore respects composition of continuous maps.

Alternatively, we may write $C = S^1 \times [0,1]$ (the cartesian product of two topological spaces has a natural choice of topology making it into a new space).

There is a projection map $\mathrm{pr}_1 : C \to S^1$ given by $(a,b,c) \mapsto (a,b)$, and an inclusion map $\mathbb{1} \times \{0\} : S^1 \to C$ given by $(a,b) \mapsto (a,b,0)$. While these are not homeomorphisms, they seem to show that S^1 and C are suitably alike. In fact, S^1 and C are not homeomorphic (one can see this, for instance, by removing a single point from both spaces; $S^1 - *$ is homeomorphic to $\mathbb{R}$, while $C - \{*\}$ is either homeomorphic to C itself, or to some space with two holes in it), but you can "squash" C down onto S^1 in a continuous manner.

These two examples motivate the following definition:

Definition 2.3. Let $f, g : X \to Y$ be two continuous maps. A HOMOTOPY from f to g is a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Two maps are HOMOTOPIC if there exists a homotopy between them. Two spaces X and Y have the same HOMOTOPY TYPE (or are HOMOTOPY EQUIVALENT) if there is a map $f : X \to Y$ and a "homotopy inverse" map $g : Y \to X$ such that $f \circ g$ is homotopic to $\mathbb{1}_Y$ and $g \circ f$ is homotopic to $\mathbb{1}_X$.

One can assemble the definition of a homotopy into the following commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, "\mathbb{1}\times\{0\}"'] \arrow[drr, bend left, "f"] & & \\
& X \times [0,1] \arrow[r, "H"] & Y \\
X \arrow[ur, "\mathbb{1}\times\{1\}"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

The notion of homotopy is central to the pursuits of algebraic topology, which these days considers two spaces to be "the same" if they are homotopy equivalent[6].

2.1 Homotopy is an equivalence relation

In this section, you will show via a series of exercises that the relation "f is homotopic to g" is an equivalence relation. We'll start off quite easy:

Exercise 3. Show that the relation $f \sim g$, meaning "f is homotopic to g", is reflexive: for every map $f : X \to Y$, $f \sim f$.

Symmetry is not too difficult, either:

Exercise 4. Suppose that two maps $f, g : X \to Y$ are homotopic, via a homotopy $H : X \times [0,1] \to Y$. Construct a homotopy $\widetilde{H}$ to show that g is homotopic to f. (*Hint:* think about flipping the interval.)

[6]This is a gross oversimplification. Many topologists care about spaces up to relations finer than homotopy equivalence, and mathematicians other than topologists care about homotopy.

The (slightly) more difficult property to check is transitivity. In attempting the next exercise, remember that the interval $[0,2]$ is homeomorphic to the interval $[0,1]$ via the map $x \mapsto x/2$, and also that $[0,2]$ is basically just two unit intervals glued together.

Exercise 5. Suppose that $f,g,h : X \to Y$ are continuous maps, where f is homotopic to g via a homotopy H and g is homotopic to h via a homotopy H'. Construct a homotopy H'' to show that f is homotopic to h, thus concluding the proof that homotopy is an equivalence relation.

2.2 The Fundamental Group

Our first algebraic invariant of a topological space will be the *fundamental group* of X relative to a basepoint $x_0 \in X$; this will be denoted $\pi_1(X, x_0)$.

Pseudo-Definition 2.4. An ALGEBRAIC INVARIANT of a space is an algebraic object (e.g., a number, a group, a vector space, etc.) associated to a space in a sufficiently natural (i.e., functorial) way. If two spaces are homotopy equivalent, their associated invariants should be isomorphic.

The fundamental group of a space will be some measure of the "one-dimensional holes in X", in a way that will not be made precise. It tells you to what extent loops in a space can be continuously contracted to a point. Thus, we should probably define what a loop is:

Definition 2.5. Let X be a topological space, and $x_0 \in X$ be a fixed point. A LOOP in X BASED at x_0 is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$.

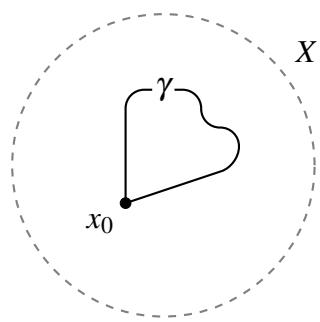


Figure 1: A loop γ in X based at x_0

We can *compose* loops as follows: given two loops $\gamma, \omega : [0,1] \to X$, let $\gamma \cdot \omega : [0,1] \to X$ be the CONCATENATION

$$(\gamma \cdot \omega)(t) = \begin{cases} \gamma(2t) & 0 \le t \le 1/2 \\ \omega(2t-1) & 1/2 \le t \le 1. \end{cases}$$

This is well-defined because $\gamma(1) = \omega(0) = x_0$.

Definition 2.6. Let $\gamma, \omega : [0,1] \to X$ be loops based at x_0. We say that γ is homotopic to ω *relative to* x_0 if there exists a homotopy $H : \gamma \simeq \omega$ where $H(0,t) = H(1,t) = x_0$ for all t in the second factor of $[0,1]$. You should think of such a homotopy as deforming γ into ω while keeping x_0 fixed.

It is a straightforward exercise, similar to those presented above, to show that "homotopy relative to x_0" is an equivalence relation. We will denote this relation by $\sim_{x_0}$.

We are now situated to define the fundamental group:

Definition 2.7. Let X be a space, and fix a basepoint $x_0 \in X$[7]. The LOOP SPACE of X (at x_0) is, as a *set*, the set of loops in X based at x_0:

$$\Omega(X, x_0) := \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = x_0\}.$$

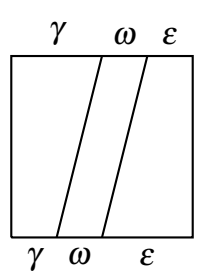
Then the *set* $\pi_1(X, x_0)$ is the set of homotopy equivalence classes of loops, relative to x_0:

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim_{x_0}$$

Proposition/Definition 2.8. *Define a binary operation on $\pi_1(X, x_0)$ by concatenation, as described above. Then $\pi_1(X, x_0)$ forms a group, and is called the* FUNDAMENTAL GROUP *of X based at x_0.*

Proof. There are many things to check here. First, we need to show that the binary operation is well-defined: if $\gamma \sim_{x_0} \gamma'$ and $\omega \sim_{x_0} \omega'$, then $\gamma \cdot \omega \sim_{x_0} \gamma' \cdot \omega'$. To save space and leave you something to do, this will be left as an exercise. Sorry.

The difficulty in proving this proposition is actually showing that concatenation is associative. Without quotienting out by homotopy, concatenation *isn't* associative, as one would end up with two loops being traversed at 1/4 speed, and the other at 1/2 speed; the 1/2-speed loop is different depending on how you parenthesize the concatenation. So, on we go! Let $\gamma, \omega, \varepsilon : [0,1] \to X$ be loops based at x_0. We will construct a homotopy $H : (\gamma \cdot \omega) \cdot \varepsilon \simeq \gamma \cdot (\omega \cdot \varepsilon)$ according to the following diagram (adapted from Peter May):



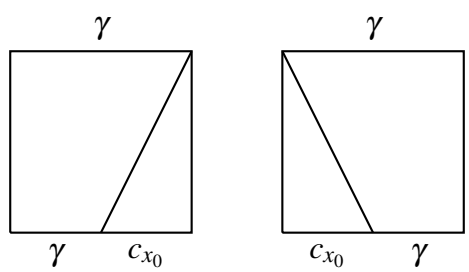
[7] When I say "basepoint", I just mean some chosen point. There's nothing special about it other than that it has been chosen. For what? Who knows...

In this diagram, the vertical axis is supposed to represent the interval $[0,1]$ used in the homotopy H. Thus, we define the homotopy by

$$H(s,t) := \begin{cases} \gamma((4-2t)s) & 0 \leq s \leq 1/4+t/4 \\ \omega(4s+1/4+t/4) & 1/4+t/4 \leq s \leq 1/2+t/4 \\ \varepsilon((2+2t)s) & 1/2+t/4 \leq s \leq 1 \end{cases}$$

While this definition is messy and difficult to parse, it is just reflecting the intuition given in the preceding diagram. To say a few words, the $4-2t$ coefficient of s in the first case of the definition comes from our desire to traverse γ first at 4-times speed, and then at 2-times speed. Feel free to check for yourself that this definition really does what we intend it to.

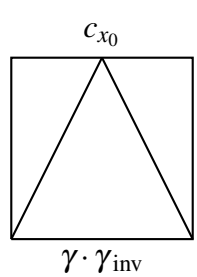
Next, I claim that the *constant loop* at x_0, written c_{x_0} and given by $c_{x_0}(s) = x_0$ for all $s \in [0,1]$, serves as an identity for concatenation in $\pi_1(X, x_0)$. That is, for any loop $\gamma\colon [0,1] \to X$ based at x_0, $\gamma \cdot c_{x_0} \sim_{x_0} \gamma \sim_{x_0} c_{x_0} \cdot \gamma$. The homotopies we will construct are those represented by the following diagrams:



I'll make one of these explicit: the homotopy $H : \gamma \cdot c_{x_0} \simeq \gamma$ is given by

$$H(s,t) := \begin{cases} \gamma((2-t)s) & 0 \leq s \leq 1/2+t/2 \\ x_0 & 1/2+t/2 \leq s \leq 1 \end{cases}$$

Finally, we need to show the existence of inverses with respect to concatenation in $\pi_1(X, x_0)$. There's a natural candidate here: just traverse the loop γ backwards, giving a loop $\gamma_{\text{inv}}(s) = \gamma(1-s)$. Since $1-(1-s) = s$, $(\gamma_{\text{inv}})_{\text{inv}} = \gamma$, and we only need to check that $\gamma \cdot \gamma_{\text{inv}} \sim_{x_0} c_{x_0}$. This homotopy is indicated by the diagram



Explicitly, this is given by

$$H(s,t) := \begin{cases} x_0 & 0 \le s \le t/2 \\ \gamma(2s) & t/2 \le s \le 1/2 \\ \gamma(1-2s) & 1/2 \le s \le 1/2 + t/2 \\ x_0 & 1/2 + t/2 \le s \le 1. \end{cases}$$

The construction of this homotopy completes our proof that $\pi_1(X, x_0)$ is a group under concatenation of loops. $\square$

The next thing we want to see is that a continuous map $f : X \to Y$ of topological spaces induces a group homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y, f(x))$. Certainly, if we follow a loop $\gamma : [0,1] \to X$ by the function f, we get a loop $f \circ \gamma : [0,1] \to Y$ in Y based at $f(x)$, so it remains to check that this map is a homomorphism. That is to say, we must show that $f \circ (\gamma \cdot \omega) \sim_{f(x)} (f \circ \gamma) \cdot (f \circ \omega)$ for any loops γ and ω in X. In this case, we get lucky: $f \circ (\gamma \cdot \omega)$ and $(f \circ \gamma) \cdot (f \circ \omega)$ are literally *equal*, as we can check:

$$\begin{aligned} f \circ (\gamma \cdot \omega)(s) &= \begin{cases} f(\gamma(2s)) & 0 \le s \le 1/2 \\ f(\omega(2s-1)) & 1/2 \le s \le 1 \end{cases} \\ &= \begin{cases} (f \circ \gamma)(2s) & 0 \le s \le 1/2 \\ (f \circ \omega)(2s-1) & 1/2 \le s \le 1 \end{cases} \\ &= (f \circ \gamma) \cdot (f \circ \omega)(s). \end{aligned}$$

Thus the map f_* is a group homomorphism. One can check furthermore that if $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then $g_* \circ f_* = (g \circ f)_*$ as maps from $\pi_1(X,x)$ to $\pi_1(Z, g(f(x)))$, and $\mathbb{1}_* : \pi_1(X,x) \to \pi_1(X,x)$ is the identity homomorphism.

3 The Circle is not the Disk is not the Torus

Of course, none of these definitions would mean anything if we couldn't *use* them to study topological spaces. In this section, I'll describe two comparisons that can be made using the theory of fundamental groups. Details will often be left vague (as I haven't gone at all into the theory of covering spaces), but I'll give ample references to sources where the interested reader may be able to find more information about them.

These start with perhaps the most fundamental (heh) computation in algebraic topology:

Theorem 3.1. $\pi_1(S^1, x) \cong \mathbb{Z}$. *The generator is the class of the "1-fold cover" mapping* $[0,1]$ *onto the circle, injective everywhere except for* 0 *and* 1.

Proof. Details of the proof can be found in every major algebraic topology textbook. Suggested ones include May, Hatcher, Spanier, tom Dieck, Munkres, or Strom. □

Let's say that a space X is SIMPLY CONNECTED if $\pi_1(X, x) \cong e$, the trivial group. A space is CONTRACTIBLE if it is homotopy equivalent to a single point. Since every loop in the space $*$ is equal (and hence homotopic), we see that $\pi_1(*, *) \cong e$. Thus every contractible space is simply connected.

The unit disk in $\mathbb{R}^2$ is the space $D^2 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. This is contractible, as you can easily verify. Knowing this helps us answer the following question:

Question 3.2. Is the circle S^1 a retract of D^2? We say that a space X is a RETRACT of a space Y if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathbb{1}_X$.

Answer. Suppose that such maps $f : S^1 \to D^2$ and $g : D^2 \to S^1$ existed. Then upon passage to fundamental groups, we'd see that $(g \circ f)_* = g_* \circ f_* = \mathbb{1}_{\mathbb{Z}}$. In particular, f_* would have to be an injective group homomorphism. But no map $\mathbb{Z} \to e$ is injective, so S^1 cannot be a retract of D^2. □

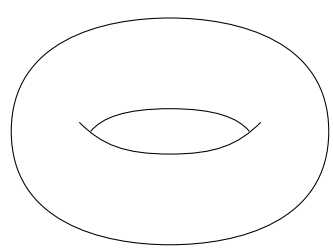
The next question we'll answer relies on the following theorem:

Theorem 3.3. *Let X and Y be topological spaces, and $x \in X$, $y \in Y$ basepoints. Then*

$$\pi_1(X \times Y, (x,y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

Proof sketch. To give a loop in $X \times Y$ based at (x,y) is equivalent to giving a loop in X based at x and a loop in Y based at y. The result follows more precisely from categorical nonsense. □

Definition 3.4. The TORUS $\mathbb{T}^2$ is the product of two circles: $\mathbb{T}^2 = S^1 \times S^1$. It looks like the surface of a donut:



By the theorem, we see that $\pi_1(\mathbb{T}^2, x) \cong \pi_1(S^1, x)^2 \cong \mathbb{Z}^2$. This immediately answers the following question in the negative:

Question 3.5. Is the circle S^1 homotopy equivalent to the torus $\mathbb{T}^2$?

4 Epilogue: what else is out there?

While the fundamental group is a powerful enough invariant to distinguish some simple spaces (particularly in low dimensions), it is not strong enough even to distinguish the 2-sphere S^2 from the 3-sphere S^3, both of which are simply connected. There are a few natural generalizations of the fundamental group, which I'll name but not discuss in detail:

1. The notation $\pi_1(X,x)$ suggests that there may be "higher fundamental groups". These do exist, and are called the "higher homotopy groups" of a space, written $\pi_n(X,x)$. It is thought that these groups capture all the information about a space, but they are in general *very* hard to compute.

2. Whereas the fundamental group was constructed by probing our space with loops, there are many other things we could use to probe spaces. If we probe with manifolds (spaces that look locally like $\mathbb{R}^n$), we end up with a construction called BORDISM HOMOLOGY. While bordism homology is much less fine an invariant than homotopy groups, it has the advantage of being more computable.

3. If we probe with triangles and their higher-dimensional analogues, we end up with something called SINGULAR HOMOLOGY. This comes with a dual theory known as SINGULAR COHOMOLOGY. These are both more difficult to define than homotopy groups, but they end up being much easier to compute and still capture a lot about a space.

All of this algebro-topological nonsense has been used in some interesting ways, some of which I'll list here:

1. The Brouwer Fixed Point Theorem has the interesting consequence that if you pick a piece of paper up off a table, crumple it up, and place it back down within the boundaries of where it was before, at least one point of the paper will be directly above its previous location. (In fancier terms, every continuous map $D^n \to D^n$ has a fixed point.)

2. There is an established theory of *topological data analysis*, wherein researchers analyze large data sets by associating to them certain topological spaces for which one can compute singular homology.

3. Algebraic topology shows up everywhere in physics. For instance, Topological Quantum Field Theories (TQFTs) are of active research in both math and physics circles.

There is much more out there—algebraic topology is an enormous field of study! I hope that with this document (if you're not too confused or turned off from it) you may look into more topology later on. Do feel free to approach me with any questions, or just to chat!

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