THE LERAY-SERRE SPECTRAL SEQUENCE

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Abstract. Spectral sequences are a powerful bookkeeping tool, used to handle large amounts of information. As such, they have become nearly ubiquitous in algebraic topology and algebraic geometry. In this paper, we take a few results on faith (i.e., without proof, pointing to books in which proof may be found) in order to streamline and simplify the exposition. From the exact couple formulation of spectral sequences, we introduce a special case of the Leray-Serre spectral sequence and use it to compute $H^*(\mathbb{CP}^n;\mathbb{Z})$.

Contents

1. Spectral Sequences

The *modus operandi* of algebraic topology is that "algebra is easy; topology is hard." By associating to a space X an algebraic invariant (the (co)homology groups $H_n(X)$ or $H^n(X)$, and the homotopy groups $\pi_n(X)$), with which it is more straightforward to prove theorems and explore structure. For certain computations, often involving (co)homology, it is perhaps difficult to determine an invariant directly; one may side-step this computation by approximating it to increasing degrees of accuracy. This approximation is bundled into an object (or a series of objects) known as a *spectral sequence*. Although spectral sequences often appear formidable to the uninitiated, they provide an invaluable tool to the working topologist, and show their faces throughout algebraic geometry and beyond.

Loosely speaking, a spectral sequence $\{E_r^{*,*}, d_r\}$ is a collection of bigraded modules or vector spaces $E_r^{*,*}$, equipped with a differential map d_r (i.e., $d_r \circ d_r = 0$), such that

$$E_{r+1}^{*,*} = H(E_r^{*,*}, d_r).$$

That is, a bigraded module in the sequence comes from the previous one by taking homology. Allow us to elucidate some terminology: a *bigraded R-module* E is a family of R-modules $E^{p,q}$ indexed by $p, q \in \mathbb{Z}$. We will also write $\{E^{p,q}\}$ to refer to the same object. There are two natural ways to make a bigraded R-module into an R-module: the first is by taking the direct product

$$E = \prod_{p,q\in\mathbb{Z}} E^{p,q}$$

and the second by taking the direct sum

$$E = \bigoplus_{p,q\in\mathbb{Z}} E^{p,q}.$$

An R-linear map $d : E_1^{*,*} \to E_2^{*,*}$ is a map of bigraded modules if $d : E^{p,q} \to E^{p',q'}$ is a map of R-modules for all p, q, p', q'.

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Definition 1.1. Let $E^{*,*}$ be a bigraded R-module, and let $d : E^{*,*} \to E^{*,*}$ be a differential R-linear map (i.e., a map with $d^2 = 0$). The *bidegree* of d is the ordered pair (r, s) such that $d : E^{p,q} \longrightarrow E^{p+r,q+s}$ for all $p, q \in \mathbb{Z}$. A bigraded R-module is said to be *differential* if it is equipped with a differential map $d : E^{*,*} \to E^{*,*}$, with bidegree $(s, 1-s)$ or $(-s, s-1)$.

Because the map d associated to a differential bigraded module $\{E^{p,q}\}$ has $d^2 = 0$, we can take the *homology* of the bigraded module at the module $E^{p,q}$:

$$H^{p,q}(E^{*,*}, d) := {}^{\ker d: E^{p,q} \to E^{p+s,q-s+1}} \big/ {}_{\operatorname{im} d: E^{p-s,q+s-1} \to E^{p,q}}.$$

Finally, we are ready to define precisely the notion of a spectral sequence:

Definition 1.2. A *spectral sequence* is a series of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$ for $r = 1, 2, \ldots$, where the differentials are either all of bidegree $(-r, r-1)$ or of bidegree $(r, 1-r)$. In the first case, we say the spectral sequence is *homological* or *of homological type*, while in the second, we say it is *cohomological* or *of cohomological type*. Finally, for all p, q, r, we require that

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r),$$

with homology as defined above.

The differential bigraded module $\{E_r^{*,*}, d_r\}$ is often called the E_r-*term* or E_r-*page* of the spectral sequence, reminiscent of pages in a book. The analogy is that a bigraded module can be represented pictorially as an integer lattice in the plane (like a sheet of paper), and by taking homology, one "turns the page" of the book, getting to the E_{r+1} page from the E_r one. Note that while $E_r^{*,*}$ and d_r precisely determine $E_{r+1}^{*,*}$, they do not determine the differential d_{r+1}. It is often the case that one requires geometric knowledge of what one is trying to approximate in order to move forward with the next page.

Suppose for the remainder of this paper that the spectral sequence $\{E_r^{*,*}, d_r\}$ is concentrated in the first quadrant; that is, $E_2^{p,q} = 0$ whenever p or q are negative. We can view this sequence pictorially as

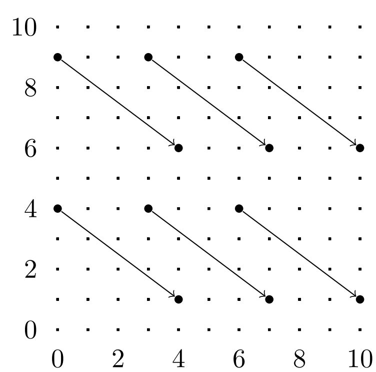

FIGURE 1. Differentials on the E_3-page of a cohomological spectral sequence

where each dot represents a summand $E_3^{p,q}$.

Consider the differential $d_{q+2} : E_{q+2}^{p+1,q} \to E_{q+2}^{p+q+3,-1}$; this must be the zero map, because $E_{q+2}^{*,-1} \cong \{0\}$ for all q. Furthermore, the map $d_{p+1} : E_{p+1}^{-1,q-p+2} \to E_{p+1}^{p,q}$ is the zero map as well. Thus, for $s = \max(p+1, q+2)$, we must have

$$E_s^{p,q} \cong E_{s+1}^{p,q} \cong \cdots,$$

i.e., the spectral sequence stabilizes at every point in finite time. Let us denote this common module by $E_\infty^{p,q}$; we then say that the spectral sequence *converges* to the differential bigraded R-module $E_\infty^{*,*}$, or that it *abuts* to this module. To be more precise, we say that a spectral sequence converges to a module $H^{*,*}$ if there are isomorphisms

$$E_\infty^{p,q} \xrightarrow{\cong} H^{p,q}.$$

The general case of using spectral sequences is when one has "there exists a spectral sequence with E_2-page given by *[something computable]*, converging to *[something desirable]*."

The natural next question then becomes, "how does a spectral sequence arise in the wild?" There are two prominant situations one may find oneself in, which bare as fruit a spectral sequence: first, having a *filtration* on an R-module A (i.e., a family of submodules F^pA for $p \in \mathbb{Z}$ such that

$$\cdots \subset F^{p+1}A \subset F^pA \subset F^{p-1}A \subset \cdots \subset A;$$

specifically this is a *descending filtration*; *increasing* filtrations may be defined dually) allows one to create a graded module $E_0^*(A)$ by setting

$$E_0^p(A) = \begin{cases} F^pA/F^{p+1}A & \text{for a decreasing filtration,} \\ F^pA/F^{p-1}A & \text{for an increasing filtration.} \end{cases}$$

If further there is a filtration F on a graded R-module H^*, we may use the filtration to assign a bigrading to H^*, and thus get a spectral sequence.

Alternatively, and most immediately for us, one may arrive at a spectral sequence through the theory of *exact couples.*

Definition 1.3. Suppose that M and N are R-modules (potentially bigraded), and let $i : M \to M$, $j : M \to N$, and $k : N \to M$ be R-module homomorphisms. The data $\{M, N, i, j, k\}$ are called an *exact couple* if the associated diagram

$$\begin{tikzcd} M \arrow[rr, "i"] & & M \arrow[dl, "j"] \\ & N \arrow[ul, "k"] & \end{tikzcd}$$

is exact at each point. We note that with the map $d : N \to N$ given by $d = j \circ k$, N becomes a differential R-module: $d \circ d = (j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = 0$.

Given an exact couple $\mathcal{C} = \{M, N, i, j, k\}$, we can perform an operation that smells exceedingly like "turning the page" of a spectral sequence: set

$$N' = H(N, d) = \ker d / \operatorname{im} d = \ker(j \circ k)/\operatorname{im}(j \circ k),$$

and put

$$M' = i(M) = \ker j.$$

Furthermore, define $i' = i|_{i(M)} : M' \to M'$ and $j' : M' \to N'$ by letting $j'(i(x)) = j(x) + dN \in N'$. We also set $k' : N' \to M'$ to be the map defined by $k'(e + dN) = k(e)$ (the reader may wish to check that both j' and k' are indeed well-defined). We call the new data $\mathcal{C}' = \{M', N', i', j', k'\}$ the *derived couple* of $\mathcal{C}$. The important point about $\mathcal{C}$ is that it too is exact:

Proposition 1.4. *The derived couple $\mathcal{C}' = \{M', N', i', j', k'\}$ is exact.*

Proof. We show exactness at the three points of the triangle:

(a) at $N' \xrightarrow{k'} M' \xrightarrow{i'} M'$: we have

$$\begin{aligned} \ker i' &= \operatorname{im} i \cap \ker i = \ker j \cap \operatorname{im} k \\ &= k(k^{-1}(\ker j)) = k(\ker d) = k'(\ker d/\operatorname{im} d) \\ &= \operatorname{im} k'. \end{aligned}$$

(b) at $M' \xrightarrow{i'} M' \xrightarrow{j'} N'$: noting that there is a isomorphism $M' = iM \cong M/\ker i$, we can write

$$\begin{aligned} \ker j' &= j^{-1}(\operatorname{im} d)/\ker i = j^{-1}(j(\operatorname{im} k)) \\ &= (\operatorname{im} k + \ker j)/\ker i = (\ker i + \ker j)/\ker i \\ &= i(\ker j) = i(\operatorname{im} i) = \operatorname{im} i'. \end{aligned}$$

(c) at $M' \xrightarrow{j'} N' \xrightarrow{k'} M'$: we have $\ker k' = \ker k / \operatorname{im} d = \operatorname{im} j / \operatorname{im} d = jM / \operatorname{im} d$, which, as $j \circ i = 0$, is precisely $\operatorname{im} j'$. □

Thus, we can repeatedly "turn the page" on an exact couple to produce a sequence of derived couples. In particular, if we have an exact couple of bigraded R-modules, we can get a spectral sequence:

Theorem 1.5 (McCleary, Thm 2.8)**.** *Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are two bigraded R-modules, with R-module homomorphisms i, j, and k, of bidegrees $(-1,1)$, $(0,0)$, and $(1,0)$, respectively, forming an exact couple:*

$$\begin{tikzcd} D^{*,*} \arrow[rr, "i"] & & D^{*,*} \arrow[dl, "j"] \\ & E^{*,*} \arrow[ul, "k"] & \end{tikzcd}$$

Then these data give rise to a cohomological spectral sequence $\{E_r, d_r\}$ for $r = 1, 2, \ldots$, where $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$-st derived module (i.e., corresponding module in the derived couple) of $E^{*,*}$, and $d_r = j^{(r)} \circ k^{(r)}$.

In particular, the spectral sequence from a derived couple tells its user the differential of every page.

Proof. It suffices to check that the derived differentials d_r have the correct bidegree, $(r, 1-r)$, which we show by induction. For the base case, let $E_1 = E^{*,*}$ and $d_1 = j \circ k$, so d_1 has bidegree $(1,0)$ (bidegree is additive). Now suppose by induction that $j^{(r-1)}$ has bidegree $(r-2, 2-r)$ and $k^{(r-1)}$ has bidegree $(1,0)$. By definition,

$$j^{(r)}(i^{(r-1)}(x)) = j^{(r-1)}(x) + d^{(r-1)}E^{(r-1)},$$

so the image of $j^{(r)}$ in $(E^{p,q})^{(r)}$ must come from

$$i^{(r-1)}(D^{p-r+2,q+r-2})^{(r-1)} = (D^{p-r+1,q+r-1})^{(r)},$$

so $j^{(r)}$ has bidegree $(r-1, 1-r)$. Furthermore, as

$$k^{(r)}(e + d^{(r-1)}E^{(r-1)}) = k^{(r-1)}(e)$$

and $k^{(r-1)}$ has bidegree $(1,0)$, so does $k^{(r)}$. Thus by induction, we find that $d^{(r)}$ has bidegree $(r, 1-r)$, and so $\{E_r, d_r\}$ is a cohomological spectral sequence. □

2. Fibrations and the Leray-Serre Spectral Sequence

Definition 2.1. We say that a map $\pi : E \to B$ is a *fibration* if it satisfies the *homotopy lifting property* with respect to all spaces Y. Let Y be a space, $G : Y \times I \to B$ a homotopy, and $g : Y \times \{0\} \to E$ a map such that $\pi \circ g(y,0) = G(y,0)$. We say that the map π *has the homotopy lifting property* with respect to the space Y if there exists a homotopy $\widetilde{G} : Y \times I \to E$ such that $\widetilde{G}(y,0) = g(y,0)$, and $\pi \circ \widetilde{G} = G$. Pictorially, the following diagram commutes:

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d] & E \arrow[d, "\pi"] \\ Y \times I \arrow[r, "G"] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$

Some authors refer to this construction as a *Hurewicz fibration*, and they refer to a map with the HLP for all n-cells a *Serre fibration*. In particular, all fiber bundles are Serre fibrations[1]. For $\pi : E \to B$ a fibration, we call $F_b = \pi^{-1}(b)$ the *fiber of π over b*. One may think then of fibrations as fiber bundles where the fibers are allowed to be non-homeomorphic. We have, however, the following restriction:

Proposition 2.2. *Let $\pi : E \to B$ be a fibration, and suppose that the base space B is path-connected. Then for all $b, b' \in B$, F_b is homotopy-equivalent to $F_{b'}$.*

[1]In fact, something much stronger is true: a fiber *bundle* $\pi : E \to B$ has the homotopy lifting property with respect to all CW pairs (X, A). See, for example, [4], Proposition 4.48.

Proof. See [1], Proposition 4.26. ☐

Remark 2.3. By the proposition, we may appeal to the usual abuse of language, referring to "the fiber" of $\pi : E \to B$ whenever B is path-connected. Indeed, for our purposes, we will always have B path connected, and we never care to distinguish F more precisely than up to homotopy. In this case, we will write $\pi : E \to B$ as $F \hookrightarrow E \xrightarrow{\pi} B$.

Proposition 2.4. *Given a fibration $F \hookrightarrow E \xrightarrow{p} B$, with B path-connected, there is a long exact sequence*

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(B) \longrightarrow \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B).$$

Proof. This follows from the long exact sequence of a pair, together with an isomorphism

$$p_* : \pi_n(E, F) \xrightarrow{\cong} \pi_n(B).$$

For a complete proof, see [2], page 66. ☐

We are now set to introduce the Leray-Serre spectral sequence in both its homological and cohomological form[2]:

Theorem 2.5 (The homological Leray-Serre spectral sequence)**.** *Let A be a commutative ring with unity, and suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where F and B are path connected, and $\pi_1(B) = 0$. Then there is a first-quadrant spectral sequence $\{E^r_{*,*}, d^r\}$, converging to $H_*(E; G)$, with*

$$E^2_{p,q} \cong H_p(B; H_q(F; A)),$$

natural with respect to fiber-preserving maps of fibrations.

Theorem 2.6 (The cohomological Leray-Serre spectral sequence)**.** *Let R be a commutative ring with unity. Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where F and B are path connected, and $\pi_1(B) = 0$. Then there is a first-quadrant spectral sequence* of algebras, *$\{E^{*,*}_r, d_r\}$, converging to $H^*(E; R)$* as an algebra, *with*

$$E^{p,q}_2 \cong H^p(B; H^q(F; R)),$$

natural with respect to fiber-preserving maps of fibrations. Moreover, the cup product $\smile$ on cohomology and the product $\cdot_2$ on $E^{*,*}_2$ are related by $u \cdot_2 v = (-1)^{p'q} u \smile v$, when $u \in E^{p,q}_2$ and $v \in E^{p',q'}_2$.

While we will not present the proofs of either theorems, we will take the time to explain how one constructs an exact couple, which then gives rise to the homological Leray-Serre spectral sequence. For a complete proof of both theorems, we direct the reader to J.P. Serre's seminal work, *Homologie singuliére des espaces fibrés*, [5].

Suppose for simplicity that in our fibration $F \hookrightarrow E \xrightarrow{\pi} B$, the base space B has the homotopy type of a CW complex. Thus, we can take B to be a homotopy-equivalent CW complex[3]. Note that every CW complex comes with a filtration, from looking at the n-skeleta: thus we can lift the filtration on B to one on E by $X^s = \pi^{-1}(B^{(s)}) \subset E$. This gives a filtration on E:

$$E \supset \cdots \supset X^s \subset X^{s-1} \supset \cdots \supset X^0 \supset \varnothing.$$

[2]The expert will notice that the theorems we provide are not the Leray-Serre spectral sequence in its fullest generality. Indeed, we have judiciously added in the assumption that $\pi_1(B) = 0$ in order to avoid working with local systems of coefficients, a topic we do not have the space to explain in this paper. For a full explanation of the use of local coefficients in the Leray-Serre spectral sequence, see [1], §5.3.

[3]As our main examples take B to *be* a CW complex, this is not a particularly restrictive condition. Most nice spaces are homotopy-equivalent to CW complexes.

From this filtration, the long exact sequence of the pairs (X^s, X^{s-1}) for each s gives an exact couple

$$\begin{tikzcd}
 & H_r(J^{s-1};G) \arrow[rr, "i_*"] & & H_r(J^s;G) \arrow[dl] \\
H_{r-1}(J^{s-1};G) & & H_r(J^s, J^{s-1};G) \arrow[ll, "\partial"] &
\end{tikzcd}$$

which determines a spectral sequence with $E^1_{p,q} = H_{p+q}(J^p, J^{p-1};G)$, and d^1 given by the composition

$$j_* \circ \partial : H_r(J^s, J^{s-1};G) \xrightarrow{\partial} H_{r-1}(J^{s-1};G) \xrightarrow{j_*} H_{r-1}(J^{s-1}, J^{s-2};G).$$

3. The Gysin Sequence and the Cohomology Ring $H^*(\mathbb{CP}^n;R)$

We begin to explore the power of the Serre spectral sequence by showing

$$H^*(\mathbb{CP}^n;R) \cong R[x]/(x^{n+1}).$$

Certainly, this computation can be performed without the use of spectral sequences, invoking the geometric structure of $\mathbb{CP}^n$ and the naturality of the cup product (see [4], Theorem 3.12, for a proof of this form). Proving this with spectral sequences, however, will give the reader a good taste of how useful spectral sequences may be, and how to compute with them.

Definition 3.1. A space X is a *homology n-sphere* if there is some $n \geq 1$ such that $H_*(X) \cong H_*(S^n)$, i.e. $H_i(X) \cong H_i(S^n)$ for all $i \geq 0$.

We will exhibit the ring structure of $H^*(\mathbb{CP}^n;R)$ as a corollary of the following main theorem:

Theorem 3.2 (McCleary, Ex. 5C)**.** *Suppose that $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, B is path-connected, and the system of local coefficients on B induced by the fiber F is simple. If F is a homology n-sphere for some $n \geq 1$, then there is an exact sequence*

$$\longrightarrow H^k(B;R) \xrightarrow{\gamma} H^{n+k+1}(B;R) \xrightarrow{\pi^*} H^{n+k+1}(E;R) \xrightarrow{Q} H^{k+1}(B;R) \longrightarrow$$

where $\gamma(u) = z \smile u$ for some $z \in H^{n+1}(B;R)$ and, if n is even and R has characteristic $\neq 2$, then $2z = 0$.

The technical heart of the theorem is mainly in the following proposition, which associates to a specific, "nicely-behaved" spectral sequence a convenient long exact sequence.

Proposition 3.3. *Suppose $\{E_r^{*,*}, d_r\}$ is a spectral sequence with $E_2^{p,q} = \{0\}$ unless either $q = 0$ or $q = n$ for some fixed $n \geq 2$. Suppose further that $\{E_r^{*,*}, d_r\}$ converges to a graded R-module H^*. Then there is a long exact sequence*

$$\begin{aligned}
\cdots \longrightarrow H^{p+n} \longrightarrow E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} &\longrightarrow \\
H^{p+n+1} \longrightarrow E_2^{p+1,n} \xrightarrow{d_{n+1}} E_2^{p+n+2,0} &\longrightarrow \cdots
\end{aligned}$$

Proof of Proposition. By the arrangement of zero modules $E_2^{p,q}$, the only maps that could possibly be *non-zero* are the $d_{n+1} : E_2^{p,n} \to E_2^{p+n+1,0}$. Thus the spectral sequence abuts at the E_3-page, so we have $E_\infty^{*,0} \cong E_3^{*,0} = E_2^{*,0}/(\operatorname{im} d_{n+1} : E_2^{*,n} \to E_2^{*,0})$ and $E_\infty^{*,n} \cong E_3^{*,n} = \ker d_{n+1} : E_2^{*,n} \to E_2^{*,0}$. From this, it follows that the sequence

$$0 \longrightarrow E_\infty^{p,n} \longrightarrow E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \longrightarrow E_\infty^{p+n+1,0} \longrightarrow 0 \tag{3.4}$$

is exact. Furthermore, [1] p. 9 gives the short exact sequence for each p,

$$0 \longrightarrow E_\infty^{p+n+1,0} \longrightarrow H^{p+n} \longrightarrow E_\infty^{p+n} \longrightarrow 0.$$

Using these short exact sequences, we can splice the sequences (3.4) into

$$\begin{tikzcd}
{} \arrow[dr, dashed] & {} \arrow[d] & & & & \\
& H^{p+n-1} \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E_\infty^{p,n} \arrow[r] \arrow[d] & E_2^{p,n} \arrow[r, "d_{n+1}"] & E_2^{p+n+1,0} \arrow[r] \arrow[dr, dashed] & E_\infty^{p+n+1,0} \arrow[r] \arrow[d] & 0 \\
& 0 & & & H^{p+n} \arrow[d] \arrow[dr, dashed] & \\
& & & 0 \arrow[r] & E_\infty^{p+1,n} \arrow[r] \arrow[d] & E_2^{p+1,n} \\
& & & & 0 &
\end{tikzcd}$$

Following the dashed arrows (each of which is just the composite of the two "legs" of the right triangle) gives the desired long exact sequence. This is known as the *Gysin sequence*. ☐

We can now proceed to the proof of Theorem 3.2:

Proof of Theorem. Because F has the homology of an n-sphere, $H^m(F;R) \cong R$ for $m = 0$ or $m = n$, and $H^m(F;R)$ is trivial otherwise. When B is path-connected, the E_2-page of the Leray-Serre spectral sequence is given by

$$E_2^{p,q} \cong \begin{cases} H^p(B;R) & q = 0 \text{ or } q = n \\ 0 & \text{otherwise} \end{cases}$$

By Proposition 3.3, we have a long exact sequence

$$\longrightarrow E_2^{k,n} \xrightarrow{d_{n+1}} E_2^{k+n+1,0} \longrightarrow H^{k+n+1}(E;R) \longrightarrow E_2^{k+1,n} \xrightarrow{d_{n+1}} E_2^{k+n+2,0} \longrightarrow$$

Let h be a generator for $H^n(F;R)$. As the system of local coefficients is simple, [1, p. 139] gives an isomorphism

$$E_{n+1}^{*,*} \cong (H^*(B;R) \otimes 1) \oplus (H^*(B;R) \otimes h).$$

Now if we let z be the unique element in $H^{n+1}(B;R)$ such that $d_{n+1}(1 \otimes h) = z \otimes 1$, [1, p. 144] gives a multiplicative structure on the spectral sequence corresponding to the cup product, and so

$$\begin{aligned}
(-1)^{n \deg x} d_{n+1}(x \otimes h) &= d_{n+1}((1 \otimes h) \smile (x \otimes 1)) \\
&= d_{n+1}(1 \otimes h) \smile (x \otimes 1) + (-1)^n (1 \otimes h) \smile d_{n+1}(x \otimes 1) \\
&= (z \otimes 1) \smile (x \otimes 1) = (z \smile x) \otimes 1.
\end{aligned}$$

By the identification $E_2^{k,n} = H^k(B;R) \otimes h$ and $E_2^{n+k+1,0} = H^{n+k+1}(B;R) \otimes 1$, we can define $\gamma(x) = d_{n+1}(x \otimes h)$, which gives the exact sequence of the proposition.

Now if n is even, we can use the interaction of the boundary map with the cup product along with $h \smile h = 0$ (because F has zero cohomology in dimensions greater than n) to find

$$\begin{aligned}
0 = d_{n+1}(1 \otimes (h \smile h)) &= d_{n+1}((1 \otimes h) \smile (1 \otimes h)) \\
&= (z \otimes 1) \smile (1 \otimes h) + (-1)^n (1 \otimes h) \smile (z \otimes 1) \\
&= 2z \otimes h.
\end{aligned}$$

As $2h$ is nonzero, we must have $2z = 0$. This proves the proposition. ☐

As a corollary to this proposition, we will compute the polynomial algebra of $H^*(\mathbb{CP}^n; R)$. We note first that there is a fibration $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n$. Since $\mathbb{CP}^n$ is simply connected for all n, the system of local coefficients induced by the fibration is always simple. Thus, we will be able to apply Theorem 3.2 in the following:

Corollary 3.5. *Let R be an arbitrary commutative ring with unit, and $\mathbb{CP}^n$ denote complex projective n-space. The cohomology ring of $\mathbb{CP}^n$ is given by*

$$H^*(\mathbb{CP}^n; R) \cong R[x_2]/(x_2^{n+1}),$$

where x_2 is the generator, of degree 2 (that is, living in $H^2(\mathbb{CP}^n; R)$).

Proof. The beginning of the Gysin sequence for the fibration $S^{2n+1} \to \mathbb{CP}^n$ is

$$H^0(\mathbb{CP}^n; R) \xrightarrow{\gamma} H^2(\mathbb{CP}^n; R) \longrightarrow H^2(S^{2n+1}; R) \longrightarrow \cdots$$

With γ defined as above, it is clear that $\gamma(1)$ generates $H^2(\mathbb{CP}^n; R)$, so we can take $x_2 = \gamma(1)$. By the CW structure on $\mathbb{CP}^n$, we know that $\mathbb{CP}^n$ has trivial cohomology in all odd dimensions. Suppose inductively that $(x_2)^k$ generates $H^{2k}(\mathbb{CP}^n; R)$, and consider the part of the Gysin sequence

$$\longrightarrow H^{2k}(\mathbb{CP}^n; R) \xrightarrow{\gamma} H^{2k+2}(\mathbb{CP}^n; R) \longrightarrow H^{2k+2}(S^{2n+1}; R) \longrightarrow .$$

Whenever $k < n$, γ is an isomorphism, so it follows that $x_2 \smile (x_2)^k = (x_2)^{k+1}$ generates $H^{2k+2}(\mathbb{CP}^n; R)$. When $k = n$, the Gysin sequence gives the following (short) exact sequence:

$$0 \longrightarrow H^{2n+1}(\mathbb{CP}^n; R) \longrightarrow H^{2n+1}(S^{2n+1}; R) \xrightarrow{Q} H^{2n}(\mathbb{CP}^n; R) \xrightarrow{\gamma} H^{2n+2}(\mathbb{CP}^n; R) \longrightarrow 0.$$

because $\mathbb{CP}^n$ is a $2n$-dimensional manifold, $H^{2n+2}(\mathbb{CP}^n; R)$ is trivial, so γ is the zero map, and the map Q induced by the quotient is an isomorphism. As $\gamma((x_2)^n) = 0$ but also $\gamma((x_2)^n) = (x_2)^{n+1}$, this gives $(x_2)^{n+1} = 0$. $\square$

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