TOWARD A SYNTHETIC THEORY OF $(\infty, 1)$-CATEGORIES, PART I: PRELIMINARIES

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ABSTRACT. In this first of two papers on ∞-category theory, we present information and resources on the basic prerequisites to parse the Riehl-Verity theory of ∞-cosmoi, the main topic of, e.g, [RV16b]. We provide basic definitions of monoidal categories, enriched categories, model categories, simplicial sets, and quasicategories, leading up to the main definition of [RV17], that of an ∞-*cosmos*. Examples are given throughout, and we ultimately show that the category $\mathsf{QuasiCat}$ of quasicategories is an ∞-cosmos.

CONTENTS

1. INTRODUCTION

The development of category theory is richly intertwined with that of modern algebraic topology, ever since Samuel Eilenberg and Saunders Mac Lane sought to axiomatize homology theory. Their development followed a (soon-to-be) common trope: start with a geometrically meaningful theory, such as singular homology, extract the fundamentally useful material of the theory, find examples of other theories that share the same useful material, and axiomatize. Indeed, this very method of abstraction from geometry initially *begat* category theory, the focus of this paper. More specifically, this paper lays the groundwork for a deeper reading into the theory of ∞-*categories*, a simultaneous generalization of regular categories and topological spaces.

A significant amount of work has gone into elucidating the theory of ∞-categories in the last few decades, since their introduction as *weak Kan complexes* in [Vog73]. Nonetheless, the work has mainly been conducted by choosing a specific *model* of ∞-category, and working hands-on with

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that model. Recent work by Riehl and Verity, summarized concisely in [RV16b] and presented in full in [RV15a]—[RV17], seeks to axiomatize the prevalent models. In particular, they define a suitable "universe of ∞-categories", called an ∞-*cosmos*, which allows one to work axiomatically with ∞-categories in a way familiar to those who work with a particular model: each of the popular models of ∞-categories forms an ∞-cosmos.

In this paper, we outline for the reader the necessary categorical background to build the theory of ∞-cosmoi, pointing the reader to standard references whenever our exposition falls short (which it often does). We begin with a brief outline of enriched category theory, starting with the definition and a few examples of monoidal categories. We define enriched functors and natural transformations, and give an explicit example of a Leibniz construction to form the cotensor of a map $p : E \to B$ in a category $\mathcal{C}$ cotensored over $\mathcal{V}$ by a map $i : U \to V$ in $\mathcal{V}$. Next, we provide the bare minimum familiarity with model categories needed in the rest of the paper: we give the definition of a model category, and show that the class of fibrations (resp. cofibrations) is uniquely determined by the classes of weak equivalences and cofibrations (resp. fibrations).

The next nearly half of the paper focuses on building one model of ∞-categories, commonly called *quasicategories* (i.e., in [Rez17] and [Joy02]) or just ∞-*categories* (i.e., in [Lur09]). We introduce the category of simplicial sets, the total singular complex, geometric realization, and nerve functors, and many examples of simplicial sets. After introducing the notion of *horn*, we define Kan complexes (and show that the total singular complex of a space X is a Kan complex) and, finally, quasicategories. After defining the left adjoint τ_0 to the nerve functor, we introduce functors of quasicategories, explain how quasicategories appear as fibrant objects in the Joyal model structure $\mathsf{sSet}_{\text{Joyal}}$ on the category of simplicial sets, and discuss natural transformations and natural isomorphisms of functors of quasicategories.

Finally, we leave the reader with the central definition to the Riehl-Verity program: that of an ∞-cosmos. After listing the axioms characterizing an ∞-cosmos and its cartesian closed variant, we show that the category $\mathsf{QuasiCat}$ of quasicategories is an ∞-cosmos. We end with a few examples of ∞-cosmoi.

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2. Categorical Background

As the aim for this paper is to be as comprehensive as possible (within reason), we take the time to spell out the "normal" category theory needed to develop the theory of ∞-categories at all. In particular, we outline the basics of monoidal categories, enriched categories, and universal properties in such categories. The canonical reference for this material is G. M. Kelly's book [Kel82]; a brief, yet comprehensive, overview may be found in [SB].

2.1. **Monoidal Categories.** The notion of monoidal category generalizes that of, say, the category Mod_R of modules over a ring R, in that any two objects of a monoidal category can be combined via a "tensor product" written $\otimes$. This product is further required to satisfy some (reasonable) coherence conditions, such as associativity and unitality. Precisely, we have the following definition:

Definition 2.1. A *monoidal structure* on a category $\mathcal{C}$ is the data of a bifunctor $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, called the *tensor product*, and an object $I \in \mathrm{ob}(\mathcal{C})$, called the *unit*, such that for all $X, Y, Z \in \mathrm{ob}(\mathcal{C})$:

(M1) there exist natural isomorphisms $\alpha_{XYZ} : (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$, $l_X : I\otimes X \xrightarrow{\sim} X$, and $r_X : X\otimes I \xrightarrow{\sim} X$, such that the diagrams

(M2)

$$\begin{tikzcd}
 & ((W\otimes X)\otimes Y)\otimes Z \arrow[dl, "a\otimes 1"'] \arrow[dr, "a"] & \\
(W\otimes(X\otimes Y))\otimes Z \arrow[d, "a"'] & & (W\otimes X)\otimes(Y\otimes Z) \arrow[d, "a"] \\
W\otimes((X\otimes Y)\otimes Z) \arrow[rr, "1\otimes a"] & & W\otimes(X\otimes(Y\otimes Z))
\end{tikzcd}$$

and

(M3)

$$\begin{tikzcd}
 & (X\otimes I)\otimes Y \arrow[dl, "a"'] \arrow[dr, "r\otimes 1"] & \\
X\otimes(I\otimes Y) \arrow[rr, "1\otimes l"] & & X\otimes Y
\end{tikzcd}$$

commute.

The data of $(\mathcal{C}, \otimes, I, a, r, l)$ is called a *monoidal category*. In the particular case that the tensor product $-\otimes-$ agrees with the cartesian (i.e., categorical) product on $\mathcal{C}$, we say that $\mathcal{C}$ is a *cartesian monoidal category*.

Examples 2.2. Many monoidal categories are familiar. The category Set is cartesian monoidal, the unit being any specified set with one element. As the notation suggests, the category Mod_R, with the usual tensor product as $\otimes$ and the ring R as unit, is a monoidal category when R is commutative. If R is any ring (not necessarily commutative), then the category biMod_R of R-bimodules is a monoidal category with the same tensor product and unit. Note that if M and N are bimodules over a noncommutative ring R, their tensor product $M\otimes_R N$ is not in general isomorphic to $N\otimes_R M$; this motivates the following definition.

Definition 2.3. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, r, l)$ be a monoidal category. A *symmetry* c for $\mathcal{C}$ is a natural isomorphism $c_{XY} : X\otimes Y \xrightarrow{\sim} Y\otimes X$ for each pair $X, Y \in \mathrm{ob}(\mathcal{C})$, such that the following diagrams commute:

$$\begin{tikzcd}
X\otimes Y \arrow[r, "c_{XY}"] \arrow[dr, "1"'] & Y\otimes X \arrow[d, "c_{YX}"] \\
 & X\otimes Y,
\end{tikzcd} \tag{2.1}$$

$$
\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "a_{XYZ}"] \arrow[dl, "c_{XY} \otimes 1"'] & X \otimes (Y \otimes Z) \arrow[dr, "c_{X(Y \otimes Z)}"] & \\
(Y \otimes X) \otimes Z \arrow[dr, "a_{YXZ}"'] & & & (Y \otimes Z) \otimes X \arrow[dl, "a_{YZX}"] \\
 & Y \otimes (X \otimes Z) \arrow[r, "1 \otimes c_{XZ}"] & Y \otimes (Z \otimes X), &
\end{tikzcd}
\tag{2.2}
$$

$$
\begin{tikzcd}
I \otimes X \arrow[r, "c_{IX}"] \arrow[dr, "l"'] & X \otimes I \arrow[d, "r"] \\
 & X.
\end{tikzcd}
\tag{2.3}
$$

A monoidal category $\mathcal{C}$ with a symmetry c is said to be a *symmetric monoidal category.*

Examples 2.4. The categories Set and Mod_R where R is a commutative ring, with their usual tensor products (i.e., $\times$ and $\otimes_R$, respectively), are symmetric monoidal. As noted above, if R is a noncommutative ring, the category biMod_R of R-bimodules (with the usual tensor product $\otimes_R$) is *not* symmetric monoidal.

Definition 2.5. Let $(\mathcal{C}, \otimes)$ be a monoidal category. The *internal hom functor* $\mathcal{C}(X, -)$ is the right adjoint to the tensor product functor $- \otimes X$. If this adjoint exists for all X, we say that $(\mathcal{C}, \otimes)$ is a *closed* monoidal category. More precisely, an *internal hom* is a bifunctor

$$[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that for every object $X \in \mathcal{C}$, we have an adjunction

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "- \otimes X"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "{[X,-]}"]
\end{tikzcd}
$$

Examples 2.6. The category Set is *cartesian* closed: the internal hom functor $\mathsf{Set}(A, -)$ is just the covariant representable functor with representing object A. Write $\mathsf{Ch}_\bullet(R)$ for the category of chain complexes of R-modules, where R is a commutative ring. This category has a closed monoidal structure, where the tensor product is given as follows: for complexes $X, Y \in \mathsf{Ch}_\bullet(R)$, the *tensor product* is the complex $(X \otimes Y)\bullet$ whose degree-n component is given by

$$(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes_R Y_j,$$

and whose differential is given on homogeneous-degree elements (x, y) by

$$\partial^{X \otimes Y}(x, y) = (\partial^X x, y) + (-1)^{\deg(x)}(x, \partial^Y y).$$

For objects $X, Y \in \mathsf{Ch}_\bullet(R)$, we define the *internal hom chain complex* $[X, Y] \in \mathsf{Ch}_\bullet(R)$ to have degree-n component

$$[X, Y]_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Mod}_R}(X_i, Y_{i+n}),$$

and differential defined on homogeneously-graded $f \in [X, Y]_n$ by

$$df = d_Y \circ f - (-1)^n f \circ d_X.$$

It is straightforward to check that the category $\mathsf{Ch}_\bullet(R)$ is closed with the given tensor product and internal hom functor.

2.2. **Enriched Categories.** Central to higher category theory is the theory of enriched categories, which intuitively are categories whose hom-sets have additional structure (i.e., are objects of some category). For instance, any locally small category is enriched over Set, and the category Mod_R of modules over a ring is enriched in Ab, the category of abelian groups. In a very weak sense, a metric space is a category enriched over the poset $([0,\infty],\le)$ of the extended positive real numbers; the curious reader may learn more about these *Lawvere metric spaces* in [Law02]. Unfortunately, the mere fact that hom-sets are objects of some category is not enough to guarantee some basic and reasonable facts about enriched categories; this motivates the following definitions.

Definition 2.7. Let $\mathcal{V}$ be a monoidal category. A *$\mathcal{V}$-category* $\mathcal{A}$ is the data of a set ob $\mathcal{A}$ of *objects*, a *hom-object* $\mathcal{A}(A,B)\in\mathcal{V}$ for each pair A,B of objects of $\mathcal{A}$, a *composition law* $\mu=\mu_{A,B,C}:\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\to\mathcal{A}(A,C)$ for each triple of objects, and for each object, an *identity element* $j_A:I\to\mathcal{A}(A,A)$. We furthermore require that these data satisfy certain associativity and unit conditions, i.e., that the diagrams

$$\begin{tikzcd}
(\mathcal{A}(C,D)\otimes\mathcal{A}(B,C))\otimes\mathcal{A}(A,B) \arrow[rr, "a"] \arrow[d, "\mu\otimes 1"'] & & \mathcal{A}(C,D)\otimes(\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)) \arrow[d, "1\otimes\mu"] \\
\mathcal{A}(B,D)\otimes\mathcal{A}(A,B) \arrow[dr, "\mu"'] & & \mathcal{A}(C,D)\otimes\mathcal{A}(A,C) \arrow[dl, "\mu"] \\
& \mathcal{A}(A,D) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
I\otimes\mathcal{A}(A,B) \arrow[r, "l"] \arrow[d, "j\otimes 1"'] & \mathcal{A}(A,B) & \mathcal{A}(A,B)\otimes I \arrow[l, "r"'] \arrow[d, "1\otimes j"] \\
\mathcal{A}(B,B)\otimes\mathcal{A}(A,B) \arrow[ur, "\mu"'] & & \mathcal{A}(A,B)\otimes\mathcal{A}(B,B) \arrow[ul, "\mu"]
\end{tikzcd}$$

commute. We may also call a $\mathcal{V}$-category a *category enriched over* $\mathcal{V}$ (or enriched *in* $\mathcal{V}$).

Exercise 2.8. The examples given in the preceeding paragraph are indeed enriched over the specified categories, with their usual monoidal structure (the tensor product on $[0,\infty]$ is addition).

Important Example 2.9. The category Cat of locally small categories is cartesian monoidal (recall that the product of two categories $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C}\times\mathcal{D}$ with objects ordered pairs (C,D) with $C\in\mathcal{C}$ and $D\in\mathcal{D}$, and morphisms ordered pairs of morphisms (f,g) where $f:C\to C'\in\mathcal{C}$ and $g:D\to D'\in\mathcal{D}$). A category enriched in Cat is called a *(strict) 2-category*, the collection of which forms a category called 2-Cat. Once we define the notions of $\mathcal{V}$-functor and $\mathcal{V}$-natural transformation, the reader may derive immediately the definition of *2-functor*, a functor of 2-categories. These (and other higher categories) play a very important role in modern homotopy theory and category theory.

Definition 2.10. We now define the notion of a *$\mathcal{V}$-functor*, to make the collection of all $\mathcal{V}$-categories into a category $\mathcal{V}$-Cat. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-functor* $F:\mathcal{A}\to\mathcal{B}$ is the data of

- for each object $A\in\mathcal{A}$, an object $F(A)\in\mathcal{B}$ and
- for every pair of objects $A,A'\in\mathcal{A}$, a $\mathcal{V}$-morphism

$$F_{A,A'}:\mathcal{A}(A,A')\longrightarrow\mathcal{B}(F(A),F(A'))$$

such that the diagram

$$\begin{tikzcd}
\mathcal{A}(A',A'')\otimes\mathcal{A}(A,A') \arrow[r, "F\otimes F"] \arrow[d, "\mu_{A,A',A''}"] & \mathcal{B}\left(F(A'),F(A'')\right)\otimes\mathcal{B}\left(F(A),F(A')\right) \arrow[d, "\mu_{F(A),F(A'),F(A'')}"] \\
\mathcal{A}(A,A'') \arrow[r, "F"] & \mathcal{B}\left(F(A),F(A'')\right)
\end{tikzcd}$$

commutes, and one has the equality $F_{A,A}\circ j_A = j_{F(A)}$. These two conditions encapsulate the "functoriality" of a $\mathcal{V}$-functor.

Exercise 2.11. Convince yourself that a Set-functor between categories enriched in Set is the same thing as a classical functor.

Definition 2.12. Let $F,G:\mathcal{A}\to\mathcal{B}$ be $\mathcal{V}$-functors. A $\mathcal{V}$-*natural transformation* $\alpha:F\implies G$ is the data of a *component* $\alpha_A : I\to\mathcal{B}(FA,GA)$ for each $A\in\mathrm{ob}(\mathcal{A})$, satisfying the $\mathcal{V}$-naturality condition encoded in the commutativity of

$$\begin{tikzcd}
& I\otimes\mathcal{A}(A,B) \arrow[r, "\alpha_B\otimes G"] & \mathcal{B}(FB,GB)\otimes\mathcal{B}(FA,FB) \arrow[dr, "\mu"] & \\
\mathcal{A}(A,B) \arrow[ur, "l^{-1}"] \arrow[dr, "r^{-1}"'] & & & \mathcal{B}(FA,GB). \\
& \mathcal{A}(A,B)\otimes I \arrow[r, "F\otimes\alpha_B"] & \mathcal{B}(GA,GB)\otimes\mathcal{B}(FA,GA) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Definition 2.13. Let $\mathcal{V}$ be a closed monoidal category, not necessarily symmetric, and $\mathcal{C}$ a category enriched over $\mathcal{V}$. The *power* or *cotensor* of an object $A\in\mathcal{C}$ by an object $U\in\mathcal{V}$ is an object A^U (also written $\pitchfork(A,U)$, in particular in [RV15a]) in $\mathcal{C}$, defined by the natural isomorphism

$$\mathcal{C}(B,A^U)\cong\mathcal{V}(U,\mathcal{C}(B,A)),$$

natural in all objects. If the object A^U exists for all $A\in\mathcal{C}$ and $U\in\mathcal{V}$, we say that $\mathcal{C}$ is *cotensored* over $\mathcal{V}$.

Example 2.14. Suppose $(\mathcal{V},\otimes)$ is a closed monoidal category, and $\mathcal{C}$ is cotensored over $\mathcal{V}$. Let $p:E\to B$ be a map in $\mathcal{C}$, and $i:U\to V$ a map in $\mathcal{V}$. We can then construct a map

$$i\,\widehat{\pitchfork}\,p : E^V \longrightarrow E^U\times_{B^U} B^V$$

as follows: from the canonical map $j_{E^V}: I\to\mathcal{C}(E^V,E^V)$, there is a natural morphism $\theta_{E,V}: I\to\mathcal{V}(V,\mathcal{C}(E^V,E))$. We thus have a map $E^i : E^V\to E^U$ given as the image of I under the composite

$$I\xrightarrow{\theta_{E,V}}\mathcal{V}(V,\mathcal{C}(E^V,E))\xrightarrow{-\circ i}\mathcal{V}(U,\mathcal{C}(E^V,E))\cong\mathcal{C}(E^V,E^U).$$

Maps $B^i: B^V\to B^U$, $p^U: E^U\to B^U$, and $p^V: E^V\to B^V$ may be constructed analogously. This gives us commutative squares

$$\begin{tikzcd}
E^V \arrow[r] \arrow[d] & B^V \arrow[d] \\
E^U \arrow[r] & B^U
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
E^U\times_{B^U}B^V \arrow[r] \arrow[d] & B^V \arrow[d] \\
E^U \arrow[r] & B^U
\end{tikzcd}$$

By the universal property of pullbacks, the map $i \mathbin{\widehat{\pitchfork}} p$ is the induced (dashed) arrow in the diagram

$$\begin{tikzcd}
E^V \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& E^U \times_{B^U} B^V \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B^V \arrow[d, "i^*"] \\
& E^U \arrow[r, "p_*"] & B^U
\end{tikzcd}$$

We call the map $i \mathbin{\widehat{\pitchfork}} p$ a *Leibniz cotensor* or the *pullback product* of i and p.

3. A Brief Introduction to Model Categories

Quillen's theory of model categories is extremely important in algebraic topology, and forms in some sense the foundation of homotopy theory. Often in mathematics, there is an ambient category that we are working in, with a specified class of morphisms that one would like to formally invert. The prototypical example to keep in mind is that of the category Top with the class of weak equivalences (i.e., maps that induce isomorphisms on all homotopy groups). With Voevodsky's proof of the Milnor conjecture in [cite here], the theory of model categories became crucial to algebraic geometry. Indeed, the notion of derived functor is ubiquitous in algebraic geometry, a notion central to model category theory. We give here the definition of a model structure on a category $\mathcal{C}$ without much context. The interested reader (which realistically should be every reader who decided to leaf through a paper on ∞-categories) may consult, for instance, [Hov98]. Our motivation for the definitions of a model category will thus be Top; the reader is invited to verify that the model category axioms are satisfied for Top, with the model structure given below.

Definition 3.1. We first recall two preliminary definitions. Let $\mathcal{C}$ be any category. A map $f : A \to B$ is a *retract* of a map $g : C \to D$ if there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & C \arrow[r] & B.
\end{tikzcd}$$

A *functorial factorization* is an ordered pair (α, β) of functors $\mathcal{C} \to \mathcal{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all morphisms f in $\mathcal{C}$. Note that the domain of $\alpha(f)$ is the domain of f; the codomain of $\beta(f)$ is the codomain of f. One may picture this scenario as

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\alpha(f)"'] & & B \\
& C_f \arrow[ur, "\beta(f)"'] &
\end{tikzcd}$$

Suppose we are given the following diagram, where the outer square commutes:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

We say that i has the *left lifting property* with respect to p, or equivalently p has the *right lifting property* with respect to i, if the dashed arrow exists, making the diagram commute. If $\mathcal{X}$ is some set of morphisms, we write $i \in \mathsf{llp}(\mathcal{X})$ (respectively, $i \in \mathsf{rlp}(\mathcal{X})$) if i has the left (resp. right) lifting property with respect to all morphisms in $\mathcal{X}$.

Definition 3.2. Let $\mathcal{C}$ be category. A *model structure* on $\mathcal{C}$ is a collection of three wide subcategories[1]:

- $\mathcal{W}$, called *weak equivalences*, written $\xrightarrow{\sim}$,
- $\underline{\text{fib}}$, called *fibrations*, written $\twoheadrightarrow$, and
- $\underline{\text{cof}}$, called *cofibrations*, written $\rightarrowtail$,

These three classes are required to satisfy the following axioms:

(MC2) $\mathcal{W}$ has the "2-out-of-3" property: if f and g are composable arrows, and two of $\{f, g, gf\}$ are weak equivalences, then so is the third.

(MC3) All three of $\mathcal{W}$, $\underline{\text{fib}}$, and $\underline{\text{cof}}$ are closed under retracts. That is, if f is a retract of g and g is a weak equivalence (resp. co/fibration), then f is also a weak equivalence (resp. co/fibration).

(MC4) All cofibrations have the left lifting property with respect to fibrations that are weak equivalences; all fibrations have the right lifting property with respect to cofibrations that are weak equivalences[2]. We may write this as

$$\underline{\text{cof}} \subseteq \mathsf{llp}(\mathcal{W} \cap \underline{\text{fib}}) \quad \text{and} \quad \underline{\text{fib}} \subseteq \mathsf{rlp}(\mathcal{W} \cap \underline{\text{cof}}).$$

(MC5) There exist functorial factorizations (α, β) and (γ, δ), such that for any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is an acyclic fibration, $\gamma(f)$ is an acyclic cofibration, and $\delta(f)$ is a fibration:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, tail] & & B \\ & C_f \arrow[ur, two heads, "\sim"] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[rr] \arrow[dr, tail, "\sim"'] & & B \\ & D_f \arrow[ur, two heads] & \end{tikzcd}$$

Definition 3.3. A category $\mathcal{C}$ equipped with a model structure is said to be a *model category* if it satisfies the additional axiom

(MC1) $\mathcal{C}$ has all small limits and colimits.

Examples 3.4. Let $\mathcal{C}$ be any category with all small limits and colimits. We can put three different model structures on $\mathcal{C}$ by taking one of the classes of morphisms to be the isomorphisms, and the other two to be all morphisms.

As the theory of model categories was motivated by homotopy theory, one may rightly expect the category Top of topological spaces and continuous maps to have a model structure. There is a particularly nice model structure on Top with $\mathcal{W} = \{\text{weak homotopy equivalences}\}$ (i.e., maps that induce isomorphisms on homotopy groups), $\underline{\text{fib}} = \{\text{Serre fibrations}\}$ (maps with the homotopy lifting property with respect to all CW-complexes), and $\underline{\text{cof}} = \{\text{retracts of cell complexes}\}$.

If $\mathcal{C}$ and $\mathcal{D}$ are model categories, we can put a model structure on $\mathcal{C} \times \mathcal{D}$, called the *product model structure*: a map (f, g) is a weak equivalence (fibration, weak equivalence) if and only if both f and g are weak equivalences (fibrations, cofibrations).

[1]A *wide subcategory* is a subcategory with all the objects of the original category. The point is that we're specifying just the maps

[2]The classes $\mathcal{W} \cap \underline{\text{fib}}$ and $\mathcal{W} \cap \underline{\text{cof}}$ are called *acyclic fibrations* and *acyclic cofibrations*, respectively.

Definition 3.5. Let $\mathcal{C}$ be a model category. Because $\mathcal{C}$ has all small limits and colimits, it has an initial object 0 and a terminal object $*$. We call an object $A \in \mathcal{C}$ *fibrant* if the natural map $0 \to A$ is a fibration, and *cofibrant* if the map $A \to *$ is a cofibration.

Lemma 3.6. *Let $\mathcal{C}$ be a model category. A morphism $f \in \mathcal{C}$ is a cofibration (resp. acyclic cofibration) if and only if $f \in \mathsf{llp}(\mathcal{W} \cap \underline{\mathrm{fib}})$ (resp. $f \in \mathsf{llp}(\underline{\mathrm{fib}})$). Dually, f is a fibration (resp. acyclic fibration) if and only if $f \in \mathsf{rlp}(\mathcal{W} \cap \underline{\mathrm{cof}})$ (resp. $f \in \mathsf{rlp}(\underline{\mathrm{cof}})$).*

Proof. We'll demonstrate $\underline{\mathrm{cof}} = \mathsf{llp}(\mathcal{W} \cap \underline{\mathrm{fib}})$ an example to show how these arguments work; the other four cases are essentially the same, and left as an exercise to the reader.

Note that we only need to prove the "$\supseteq$" direction. Suppose $f \in \mathsf{llp}(\mathcal{W} \cap \underline{\mathrm{fib}})$. Factor f as

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d, two heads, "\wr\, p"] \\ X \arrow[r, equal] \arrow[ur, dashed, "r"] & X; \end{tikzcd}$$

so the lift r exists. We can redraw this diagram as

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, tail, "i"] & A \arrow[d, "f"] \\ X \arrow[r, "r"] & B \arrow[r, two heads, "p", "\sim"'] & X, \end{tikzcd}$$

which is a retract diagram. As i is a cofibration, (MC3) gives that f is also a cofibration. $\square$

4. Simplicial Sets

The simplices of simplicial homology theory have a categorical analogue, which allows for a combinatorial approach to homotopy theory. This combinatorial data is encoded in an object called a *simplicial set* (or more generally, a *simplicial object*), which is a contravariant functor off a certain *simplex category* Δ: a simplicial set is a functor $X : \Delta^{\mathrm{op}} \to \mathsf{Sets}$ (and a simplicial object a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$). There is a category of simplicial sets sSet, and a fully faithful functor $N : \mathsf{Cat} \to \mathsf{sSet}$, so we can often view simplicial sets as generalizations of categories; this leads to the notion of *quasicategory*, discussed in the next section as a model of ∞-category.

Definition 4.1. The *simplex category* Δ is the category with object the finite, nonzero ordinals, and maps are order preserving maps. The symbol $[n]$ will stand for the set $\{0, 1, \ldots, n\}$, i.e., the $(n+10)$-th ordinal. Equivalently, we may view Δ as the full subcategory of Cat whose objects are the posets defining finite, non-empty ordinals, regarded as categories.

There are two special classes of morphisms in Δ, which in fact generate all other morphisms. For every $n \geq 0$, there are morphisms $d^i : [n-1] \to [n]$ called *coface maps* and $s^i : [n+1] \to [n]$ called *codegeneracy maps* defined as follows:

$$d^i(k) = \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases} \quad \text{and}$$

$$s^i(k) = \begin{cases} k, & k \leq i \\ k-1, & k > i \end{cases},$$

where $0 \leq i \leq n$. These morphisms satisfy the *cosimplicial relations*

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, \quad i < j \\ s^j s^i &= s^i s^{j+1}, \quad i \leq j \\ s^j d^i &= \begin{cases} 1, & i = j, j+1 \\ d^i s^{j-1}, & i < j \\ d^{i-1} s^j, & i > j+1 \end{cases} . \end{aligned}$$

Proposition 4.2. *Every morphism in Δ can be written as a composite of the coface and codegeneracy maps.* □

Definition 4.3. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. In the special case $\mathcal{C} = \mathsf{Set}$, we call $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ a *simplicial set*. A *morphism* of simplicial sets $f : X \to Y$ is a natural transformation of their representing functors $f : X \Rightarrow Y : \Delta^{\mathrm{op}} \to \mathsf{Set}$. In other words, we can think of the category sSet of all simplicial sets as the functor category $\mathsf{Set}^{\Delta^{\mathrm{op}}}$. More generally, the category $\mathsf{s}\mathcal{C}$ of simplicial objects in $\mathcal{C}$ is the functor category $\mathcal{C}^{\Delta^{\mathrm{op}}}$.

If X is a simplicial set, we refer to the set $X[n]$ as the set of *n-simplices* in X, and write this as X_n. Geometrically, you should imagine an n-simplex as a (potentially singular) n-dimensional topological simplex, with a labeled set of vertices.

Given a simplicial set X, we write the images of the coface and codegeneracy maps d^i and s^i as

$$\begin{aligned} d_i &= X d^i : X_n \to X_{n-1} & & 0 \leq i \leq n \\ s_i &= X s^i : X_n \to X_{n+1} & & 0 \leq i \leq n, \end{aligned}$$

and call these the *face* and *degeneracy* maps, respectively. Your inner geometric thinking should kick in now, and imagine that the i-th face map d_i, applied to every X_n, collapses a simplex to its i-th face. Similarly, the degeneracy maps embed an n-dimensional simplex into an $n+1$-dimensional simplex, as a degenerate simplex. This geometric thinking will be made rigorous once we introduce the geometric realization functor $|-| : \mathsf{sSet} \to \mathsf{Top}$. Until then, you should keep the geometry in the back (or front!) of your mind. After all, this wave of generality is all motivated by topology!

The face and degeneracy maps satisfy relations dual to the cosimplicial relations, aptly named *simplicial* relations. Indeed, one may define a simplicial set in a more "hands-on" way, i.e., without mentioning functors:

Alternative Definition 4.4. A *simplicial set* X is a collection of sets X_n indexed by $\mathbb{N}$, equipped with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for every $0 \leq i \leq n$ and each n, which satisfy the simplicial relations.

Recall that the Yoneda lemma states (in part) that there is a fully faithful embedding

$$\mathcal{C} \hookrightarrow \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

for all categories $\mathcal{C}$, called the *Yoneda embedding*. For each $[n] \in \Delta$, we let $\Delta^n = \Delta(-, [n])$ be the image of $[n]$ under the Yoneda embedding

$$y : \Delta \hookrightarrow \mathsf{Set}^{\Delta^{\mathrm{op}}} = \mathsf{sSet}.$$

We call this simplicial set the *standard n-simplex*. One sees directly that the set of k-simplices Δ^n_k in Δ^n is the hom-set $\Delta([k], [n])$, i.e., maps $[k] \to [n]$ in Δ. Geometrically, one may think of the standard n-simplex as the standard topological n-simplex; while this is not precisely correct, it is justified.

Example 4.5 (the total singular complex of a topological space)**.** A particularly geometric (and familiar) example of simplicial sets comes from the category of topological spaces. First, recall that the *standard topological n-simplex* is the space

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1, x_i \geq 0\}.$$

There is an evident *covariant* functor $\delta : \Delta \to \mathsf{Top}$ that associates to each $[n] \in \Delta$ the standard topological n-simplex Δ_n. The images of the coface and codegeneracy maps (which we will also denote by d^i and s^i) are described by $d^i : \Delta_{n-1} \to \Delta_n$ adding a zero in the i-th coordinate, while $s^i : \Delta_{n+1} \to \Delta_n$ adding the i-th and $(i+1)$-st coordinates.

We will now define a functor $\mathrm{Sing}(-) : \mathsf{Top} \to \mathsf{sSet}$ which assigns to each topological space Y its *total singular complex*. Let Y be a topological space; the n-simplices of $\mathrm{Sing}(Y)$ will be the set of maps $\Delta_n \to Y$ into Y from the standard topological n-simplex, i.e., $\mathsf{Top}(\Delta_n, Y)$. The face maps are given by pre-composition with d^i:

$$d_i = - \circ d^i : \mathsf{Top}(\Delta_n, Y) \to \mathsf{Top}(\Delta_{n-1}, Y),$$

and the degeneracy maps by pre-composition with s^i,

$$s_i = - \circ s^i : \mathsf{Top}(\Delta_n, Y) \to \mathsf{Top}(\Delta_{n+1}, Y).$$

These maps satisfy the simplicial identities because the dual maps d^i and s^i satisfy the *co*simplicial identities; thus $\mathrm{Sing}(Y)$ is a simplicial set.

We define the action of $\mathrm{Sing}(-)$ on continuous maps $f : X \to Y$ of topological spaces in the obvious way, i.e., on simplices by post-composition with f.

Exercise 4.6. Check that $\mathrm{Sing}(-) : \mathsf{Top} \to \mathsf{sSet}$ really is a (covariant) functor, by showing that it respects morphisms of topological spaces properly.

Example 4.7 (geometric realization of a simplicial set)**.** Adjoint to the singular complex functor is the *geometric realization* functor $|-| : \mathsf{sSet} \to \mathsf{Top}$. As before, let Δ_n be the standard topological n-simplex. If X is a simplicial set, we give X_n the discrete topology. The idea of geometric realization is to take a topological n-simplex for each element in X_n, and glue all of them together with the data included in the face and degeneracy maps. Precisely, we let $|X|$ be the quotient space

$$|X| = \left(\coprod_n X_n \times \Delta_n \right) / \sim,$$

where $(x, p) \sim (y, q)$ if either $d_i x = y$ and $d^i p = q$, or $s_j x = y$ and $s^j p = q$. Here, the d^i and s^i are the coface and codegeneracy maps *induced* by the functor $\delta : \Delta \to \mathsf{Top}$ defined above.

We now give perhaps the most important example of simplicial sets. Let $\mathcal{C}$ be any category. The *nerve* of $\mathcal{C}$, written $N\mathcal{C}$, is the simplicial set defined by

$$\begin{aligned} N\mathcal{C}_0 &= \mathrm{ob}\ \mathcal{C} \\ N\mathcal{C}_1 &= \mathrm{mor}\ \mathcal{C} \\ N\mathcal{C}_2 &= \{\text{pairs of composable maps } \bullet \to \bullet \to \bullet \text{ in } \mathcal{C}\} \\ &\vdots \\ N\mathcal{C}_n &= \{\text{strings of } n \text{ composable arrows } \bullet \to \bullet \to \cdots \to \bullet \text{ in } \mathcal{C}\}. \end{aligned}$$

The face maps $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ compose the i-th and $(i+1)$-st arrows when $0 < i < n$, and leaves out the first or last arrow if $i = 0$ or n, respectively. The degeneracy maps $s_i : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ take a string

$$x_0 \to x_1 \to \cdots \to x_i \to \cdots \to x_n$$

of n composable arrows and inserts the identity id_{x_i} at the i-th spot. It is straightforward to check that these d_i and s_i satisfy the simplicial relations; the eager reader may willingly check what we are unwilling to write out.

Exercise 4.8. Some authors (notably [Rez17]) choose to define $N\mathcal{C}_n$ as the functor set $\mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C})$ from the poset category n into $\mathcal{C}$. Show that this agrees with our definition.

The important thing about nerves is that $N\mathcal{C}$ "knows everything" about $\mathcal{C}$. In particular, one may recover $\mathcal{C}$ up to isomorphism (not just equivalence!) from $N\mathcal{C}$. The really important result is that $N : \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful:

Proposition 4.9. *The functor $N : \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful, i.e., if $\mathcal{C}$ and $\mathcal{D}$ are categories, every map $g : N\mathcal{C} \to N\mathcal{D}$ of simplicial sets is uniquely given as $g = Nf$ for a functor $f : \mathcal{C} \to \mathcal{D}$.*

Proof sketch. (Following [Rez17]) The point is to show that $\mathrm{fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{sSet}}(N\mathcal{C}, N\mathcal{D})$ is a bijection for all $\mathcal{C}$ and $\mathcal{D}$. It is clearly injective, for a functor is defined by its action on objects and morphisms (0-simplices and 1-simplices). Surjectivity is a little trickier; in particular, we must make use of a bijection $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(I^n, X)$, where I^n is a simplicial set called the *spine* of the simplex Δ^n. □

Example 4.10 ([Rez17] section 7.8)**.** There is another interesting construction using the nerve of a category. Let A be an abelian group and $d \geq 0$ be an integer. We can construct a simplicial set $K(A, d)$ by specifying the n-simplices for each n and the action of all simplicial operators $f : [m] \to [n]$ on the $K(A, d)_n$. The set $K(A, d)_n$ will be the set of data $a = (a_{i_0 \cdots i_d})$ which consist of

- for each $0 \leq i_0 \leq \cdots \leq i_d \leq n$, a choice of element $a_{i_0 \cdots i_d} \in A$ such that
- $a_{i_0 \cdots i_d} = 0$ if $i_{u-1} = i_u$ for any u, and
- for each $0 \leq j_0 \leq j_{d+1} \leq n$,

$$\sum_u (-1)^u a_{j_0 \cdots \widehat{j_u} \cdots j_{d+1}} = 0.$$

A simplicial operator $f : [m] \to [n]$ acts on $a_{i_0 \cdots i_d}$ via

$$(af)_{i_0 \cdots i_d} = a_{f(i_0) \cdots f(i_d)}.$$

This gives a simplicial set. In the case $d = 1$, it is isomorphic (as a simplicial set) to the nerve of A, where we consider A to be a groupoid with one element. The geometric realization of $K(A, d)$ gives the usual notion of *Eilenberg-MacLane space*, and so we call the simplicial set $K(A, d)$ an *Eilenberg-MacLane object*.

In general, one can define an Eilenberg MacLane object in any model category, as an object X that is both n-connected and n-truncated (i.e., $\pi_{<n}X = 0$ and $\pi_{>n}X = 0$). The Eilenberg-MacLane object is a simplicial object in the model category[3] sSet.

We are almost done with our whirlwind tour of simplicial sets. What is left is to define a few important *subcomplexes* of simplicial sets.

Definition 4.11. Let X be a simplicial set. We say that Y is a *subcomplex* or *subset* of X if $Y_n \subseteq X_n$ for all n, and if the restriction of Xf to Y_n $Xf|_{Y_n}$ is precisely Yf for all $f : [m] \to [n]$, i.e., the "inclusion" $Y \subseteq X$ preserves the action of simplex operators. Succinctly, we say that Y is a subcomplex if there is a monomorphism of simplicial sets $Y \overset{f}{\hookrightarrow} X$. If Y is a subcomplex of X, we write $Y \subseteq X$.

[3]Admittedly, the author is unsure as to *which* model structure.

We can specify subcomplexes of X by giving *generators*, which are usually found as a subset $S \subset X_n$. We then say that the simplicial set *generated* by S is the smallest subcomplex of X containing S. In particular, the k-simplices of this subcomplex are the k-simplices of X that arise as the image of S when acted on the right by some $f : [k] \to [n] \in \Delta$.

Examples 4.12. A few important examples of subcomplexes come from analogy with geometry. The *i-th face* $\partial_i \Delta^n$ of the standard n-simplex is the subcomplex generated by the coface maps $d^i \in \Delta^n_{n-1}$. That is to say, $\partial_i \Delta^n$ is isomorphic to the subset Δ^{n-1} that includes into Δ^n via d^i. The *simplicial horn* Λ^n_k is the union of all the faces of Δ^n *except* for the k-th face $\partial_k \Delta^n$, i.e., the subcomplex generated by the set

$$S = \{d^0, \dots, d^{k-1}, d^{k+1}, \dots, d^n\} \subset \Delta^n_{n-1}.$$

Equivalently, Λ^n_k is the colimit in sSet of the diagram

$$\begin{tikzcd}
\partial_0 \Delta^n \arrow[rrrd, bend left, "d^0"] & & & \\
\partial_1 \Delta^n \arrow[rrr, "d^1"] & & & \Delta^n \\
\vdots & & & \\
\partial_n \Delta^n \arrow[rrruu, bend right, "d^n"'] & & &
\end{tikzcd}$$

where $\partial_k \Delta^n$ is left out. Geometrically, we think of Λ^n_k as the boundary of the topological n-simplex, without the k-th face. Specifically, the geometric realization of these objects corresponds with their names: $|\partial_i \Delta^n|$ is the i-th face of the topological n-simplex, and Λ^n_k is all *but* the k-th face of that simplex.

Definition 4.13 (The monoidal structure on sSet)**.** The *product* of two simplicial sets $X, Y : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is their levelwise product (i.e., the product as functors). The category sSet is cartesian monoidal, with the cartesian tensor product $X \otimes Y = X \times Y$:

$$(X \otimes Y)_n = X_n \times Y_n.$$

The unit object is the *terminal simplex* Δ^0, the maps α are equalities, and the maps r and l are the natural isomorphisms. You may eagerly and easily check that $(\mathsf{sSet}, \times, \Delta^0, \alpha, r, l)$ is monoidal.

Furthermore, sSet is cartesian *closed*, i.e., the functor $- \times Y$ has a right adjoint $[Y, -] : \mathsf{sSet} \to \mathsf{sSet}$ for all simplicial sets Y. This is given on objects $X \in \mathsf{sSet}$ by

$$[Y, X]_n = \mathrm{Hom}_{\mathsf{sSet}}(Y \times \Delta^n, X)$$

We will often write the internal hom $[Y, X]$ as X^Y.

5. One Model of ∞-categories

The Riehl-Verity theory of ∞-cosmoi seeks to provide a synthetic (i.e., *axiomatic*) theory of $(\infty, 1)$-categories. An ∞-category should be akin to a category with extra (higher) morphisms, and thus have some notion of *objects*, *1-morphisms* between objects, corresponding to regular morphisms in a category, *2-morphisms* between 1-morphisms, and so on. An $(\infty, 1)$-category is then an ∞-category where the n-morphisms for $n \geq 2$ are invertible. Moreover, these morphisms should interact weakly in a way not dissimilar from how natural transformations (2-morphisms in

Cat) involve functors (1-morphisms). Specifically, the picture to have in mind is the commutative square

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \big\downarrow{\scriptstyle Ff} & & \big\downarrow{\scriptstyle Gf} \\ F(A') & \xrightarrow{\alpha'_A} & G(A'). \end{array}$$

I intentionally leave things vague.

Example 5.1. Topological spaces provide a nice motivation for discussing ∞-categories. For a space X, we can construct an ∞-*groupoid* (i.e., ∞-category in which all morphisms of all levels are invertible) $\Pi_\infty X$ as the notation suggests: the objects of $\Pi_\infty X$ are points of X, 1-morphisms are paths $\gamma : [0,1] \to X$, 2-morphisms are homotopies of paths, 3-morphisms are homotopies of homotopies, etc. We also consider two n-morphisms to be the same if they are homotopic. More often than not, the subscript on Π_∞ is left out, and the ∞-groupoid ΠX is called the *fundamental* ∞-*groupoid* of X.

Much of the existing literature on ∞-categories chooses a specific model of ∞-category and works internally to that model. The Riehl-Verity theory of ∞-cosmoi, for which this paper seeks to provide the background, develops a synthetic (i.e., axiomatic) framework in which to deal with ∞-categories, independent of model. Notably, the several prevalent models of ∞-categories are all equivalent, and they each form an ∞-cosmos. One may then work purely with the axioms characterizing the ∞-cosmos, and then transfer those results back to the specific model. The results will be the same regardless of choice of model.

We feel that it would be silly to give an introduction to the theory of ∞-cosmoi without first giving an unfortunately brief description of one of the most prevalent models, that of a *quasicategory*. Much of the work of Lurie (in [Lur09]) and Joyal (in [Joy02] and [Joy08]) centers specifically around quasicategories (although Lurie refers to them simply as ∞-categories). It would thus be wise (pretending for a moment that the author of this survey is wise) to treat you to a little morsel of quasicategory theory.

Definition 5.2. The horn $\Lambda^n_k \to X$ in a simplicial set X is said to have a *filler* if there is an extension (not necessarily unique!) along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ such that the triangle shown commutes:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

A *Kan complex* is a simplicial set X for which *every* horn inclusion $\Lambda^n_k \to X$ has a filler. We already know of a very large class of Kan complexes:

Lemma 5.3. *If X is a topological space, the total singular complex of X $\mathrm{Sing}(X)$ is a Kan complex.*

Proof. We make use of the adjunction between $\mathrm{Sing}(-)$ and the geometric realization functor $|-|$. Indeed, the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

in sSet corresponds precisely to the diagram

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

in Top. The topological (n, k)-horn $|\Lambda^n_k|$ is a deformation retract of the topological n-simplex $|\Delta^n|$, so the lift $|\Delta^n| \to X$ exists in Top. Pulling this back across the adjunction, we get a lift $\Delta^n \to \mathrm{Sing}(X)$ in sSet. □

Definition 5.4. In many cases, asking for *all* horns to have fillers is too strong a condition, for that is the same as asking that all morphisms be invertible. A horn Λ^n_k is said to be *inner* if $0 < k < n$, and *outer* otherwise. A *quasicategory* is then a simplicial set X such that all *inner* horns have extensions. The close relation to Kan complexes why quasicategories were initially callled *weak Kan complexes* in [Vog73].

Example 5.5. We also know a large and important class of quasicategories already. If $\mathcal{C}$ is a category, then $N\mathcal{C}$ is a quasicategory. In particular, the following stronger condition holds:

Proposition 5.6 ([Rez17] prop. 4.7)**.** *If $\mathcal{C}$ is a category, then every inner horn $\Lambda^n_k \subset \Delta^n$ has a* unique *extension. In other words, the restriction map*

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, N\mathcal{C}) \to \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^n_k, N\mathcal{C})$$

is a bijection. □

Definition 5.7. Let X and Y be quasicategories. One says that a map $f : X \to Y$ of simplicial sets is a *quasi-functor* (or just *functor*) of quasicategories. That is, if $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ and $Y : \Delta^{\mathrm{op}} \to \mathsf{Set}$ are quasicategories, then a quasi-functor $f : X \to Y$ is a natural transformation of functors $f : X \Rightarrow Y$. It is easy to check that quasicategories and functors form a category QCat, which is obviously a full subcategory of sSet.

Definition 5.8. The nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$ participates in an adjunction

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, bend left, "N"] \arrow[r, phantom, "\top"] & \mathsf{sSet} \arrow[l, bend left, "\tau_0"] \end{tikzcd}$$

The left adjoint $\tau_0 : \mathsf{sSet} \to \mathsf{Cat}$ is called the *fundamental category*, in homage to the fundamental groupoid of a topological space. Explicitly, given a simplicial set X, $\tau_0(X)$ is the category whose objects are the 0-simplices of X, and morphisms in $\tau_0(X)$ between 0-simplices a and b are finite sequences of 1-simplices $(f_n, f_{n-1}, \ldots, f_1, f_0)$ of X such that $d_0 f_0 = a$, $d_1 f_n = b$, and $d_1 f_i = d_0 f_{i+1}$ for all $0 \leq i < n$, modulo the relations

$$s_0(x) \sim \mathrm{id}_x \text{ for } x \in X_0, \quad \text{and} \quad d_1(x) \sim d_0(x) \circ d_2(x) \text{ for } x \in X_2.$$

In the special case where X is a quasicategory, we call $\tau_0(X)$ the *homotopy 2-category* (or just *homotopy category*) $\mathrm{Ho}(X)$ of X. We thus have a diagram of functors

$$\begin{tikzcd} \mathsf{QuasiCat} \arrow[rr, hook, "N"] \arrow[dr, "\mathrm{Ho}"'] & & \mathsf{sSet} \arrow[dl, "\tau_0"] \\ & \mathsf{Cat} & \end{tikzcd}$$

Definition 5.9. There is a model structure on sSet, which we denote by $\mathsf{sSet}_{\mathrm{Joyal}}$, due to Andre Joyal (see, e.g., [Joy02]). In this model structure, we have the following distinguished classes of morphisms:

- The cofibrations Cof are the monomorphisms of simplicial sets
- The weak equivalences $\mathcal{W}$ known as *weak categorical equivalences*. These are morphisms $u : A \to B$ such that the induced map $\tau_0(u^*) : \tau_0(X^B) \to \tau_0(X^A)$ is an isomorphism for all quasicategories X.
- The fibrations Fib are determined by Cof and $\mathcal{W}$.

All objects in $\mathsf{sSet}_{\mathrm{Joyal}}$ are cofibrant; the fibrant objects are precisely the quasicategories. We will call a fibration between fibrant objects an *isofibration*.

Using the words "quasicategory" and "functor" makes it seem like we are going to introduce a notion of natural transformation for functors between quasicategories, to make $\mathsf{QuasiCat}$ into some form of 2-category. Indeed, this is what we're going to do; we first recall the definition of a natural transformation of functors in Cat. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \Longrightarrow G$ is the data of maps $\alpha(c) : F(c) \to G(c)$ such that for any map $f : c \to c'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(c) \arrow[r, "\alpha(c)"] \arrow[d, "Ff"'] & G(c) \arrow[d, "Gf"] \\ F(c') \arrow[r, "\alpha(c')"'] & G(c') \end{tikzcd}$$

commutes. Equivalently, we may say that a natural transformation is a functor $H : \mathcal{C} \times [1] \to \mathcal{D}$ such that $H|_{\mathcal{C}\times\{0\}} = F$, $H|_{\mathcal{C}\times\{1\}} = G$, and $H|_{\{c\}\times[1]} = \alpha(c)$ for each $c \in \mathrm{ob}(\mathcal{C})$. One may define natural transformations of functors of quasicategories analogously:

Definition 5.10. A *natural transformation* $\alpha : F \Rightarrow G$ of functors $F, G : C \to D$ between quasicategories C and D is a map

$$\alpha : C \times N[1] = C \times \Delta^1 \to D$$

of simplicial sets such that $\alpha|_{C\times\{0\}} = F$ and $\alpha|_{C\times\{1\}} = G$. If C and D are nerves of categories $\mathcal{C}$ and $\mathcal{D}$, and F and G are functors of quasicategories given by functors $\widetilde{F}, \widetilde{G} : \mathcal{C} \to \mathcal{D}$, then the data of a natural transformation $\alpha : F \Rightarrow G$ agrees with that of a natural transformation $\widetilde{\alpha} : \widetilde{F} \Rightarrow \widetilde{G}$.

Note that Δ^1 is the nerve of the *walking arrow*, the category represented schematically by $\bullet \to \bullet$. The *walking isomorphism* is the category $\bullet \simeq \bullet$. Let us denote the nerve of the walking isomorphism by $\blacktriangle$. A natural transformation $\alpha : F \Rightarrow G$ of quasicategories is a *natural isomorphism* if there is an extension of $\alpha : C \times \Delta^1 \to D$ to $\widetilde{\alpha} : C \times \blacktriangle \to D$:

$$\begin{tikzcd} C \times \Delta^1 \arrow[r, "\alpha"] \arrow[d, hook] & D \\ C \times \blacktriangle \arrow[ur, dashed, "\widetilde{\alpha}"'] & \end{tikzcd}$$

Equivalently, a natural isomorphism is an invertible 2-morphism in the 2-category $\mathsf{QuasiCat}$ (see, e.g., [RV15a]).

6. A Stray Definition, and an Incomplete Proof

The key definition in the Riehl-Verity program—which we spend much more time on in the second paper in this series—is that of an ∞-cosmos. The notion of an ∞-cosmos encapsulates nearly everything about quasicategories and other models of ∞-categories that make them meaningful for topologists. In many ways, the current paper has been a build-up to this very definition, and it will be deeply unsatisfying for the reader if we end without giving it. Thus, we give it here.

Definition 6.1 (∞-cosmos, from [RV17])**.** An ∞*-cosmos* is a certain type of simplicially enriched category $\mathcal{K} \in \mathsf{sSet}\text{-}\mathsf{Cat}$, consisting of

- objects called ∞*-categories* with morphisms $A \to B \in \mathcal{K}$ called functors,
- where hom simplicial sets (written $\mathrm{fun}(A, B)$) are all quasicategories,

and equipped with a specified wide subcategory of maps called *isofibrations*, written "$\twoheadrightarrow$", such that $\mathcal{K}$ satisfies the following axioms:

(IC1) *(completeness)* As a simplicially-enriched category, $\mathcal{K}$ has a terminal object 1 and pullbacks of isofibrations along any functor, and $\mathcal{K}$ is cotensored over sSet.

(IC2) *(isofibrations)* the class of isofibrations contains the isomorphisms of $\mathcal{K}$ and all functors $A \twoheadrightarrow 1$ and is stable under pullback along all functors and under formation of *Leibniz cotensors*: if $p : E \twoheadrightarrow B$ is an isofibration and $i : U \hookrightarrow V$ is a monomorphism of simplicial sets, the Leibniz cotensor $i \mathbin{\widehat{\pitchfork}} p : E^V \twoheadrightarrow E^U \times_{B^U} B^V$ (defined in Example 2.14) is an isofibration. Furthermore, if $X \in \mathrm{ob}(\mathcal{K})$ and $p : E \twoheadrightarrow B$ is an isofibration, then $\mathrm{fun}(X, p) : \mathrm{fun}(X, E) \twoheadrightarrow \mathrm{fun}(X, B)$ is an isofibration of quasicategories, i.e., a fibration between fibrant objects in the Joyal model structure.

Recall that a collection of morphisms $\mathcal{W} \subseteq \mathrm{Mor}(\mathcal{C})$ in a category $\mathcal{C}$ is said to possess the *2-of-6 property* if for any sequence of composable maps

$$A \xrightarrow{v} B \xrightarrow{u} C \xrightarrow{w} D,$$

if wv and vu are in $\mathcal{W}$, then so are u, v, w, and wvu. The category underlying the enriched ∞-cosmos $\mathcal{K}$ has a subcategory of *equivalences*, which satisfy the 2-of-6 property. We say that an arrow $f : A \to B$ is an *equivalence* in $\mathcal{K}$ exactly when the induced quasi-functor $\mathrm{fun}(X, f) : \mathrm{fun}(X, A) \to \mathrm{fun}(X, B)$ is an equivalence of quasicategories for all $X \in \mathrm{ob}(\mathcal{K})$. We say than an isofibration in $\mathcal{K}$ is a *trivial fibration*, written " $\overset{\sim}{\twoheadrightarrow}$ ", if it is also an equivalence.

(IC3) *(cofibrancy)* All objects have the left lifting property with respect to all trivial cofibrations in $\mathcal{K}$:

$$\begin{tikzcd} & E \arrow[d, two heads, "\wr"] \\ A \arrow[ur, dashed, "\exists"] \arrow[r] & B \end{tikzcd}$$

We call such an object *cofibrant*.

Axioms IC1–IC3 imply

(IC4) *(trivial fibrations)* The Leibniz cotensor $i \mathbin{\widehat{\pitchfork}} p : E^V \overset{\sim}{\twoheadrightarrow} E^U \times_{B^U} B^V$ of an isofibration $p : E \twoheadrightarrow B$ and a monomorphism $i : U \hookrightarrow V$ of simplicial sets is a trivial fibration when either p is a trivial fibration in $\mathcal{K}$ or i is a trivial fibration in $\mathsf{sSet}_{\mathrm{Joyal}}$. The subcategory of trivial fibrations contain the isomorphisms and are stable under formation of pullback along all functors.

(IC5) *(factorization)* Any functor $f : A \to B$ factors as $f = pj$, where $p : N_f \twoheadrightarrow B$ is an isofibration and $j : A \overset{\sim}{\hookrightarrow} N_f$ is right inverse to a trivial fibration $q : N_f \overset{\sim}{\twoheadrightarrow} A$:

$$\begin{tikzcd} & N_f \arrow[dl, two heads, bend right, "q"', "\sim"] \arrow[dr, two heads, "p"] & \\ A \arrow[ur, "j"', "\sim"] \arrow[rr, "f"] & & B. \end{tikzcd}$$

An ∞-cosmos is said to be *cartesian closed* if in addition it satisfies the axiom

(IC6) *(cartesian closure)* The product bifunctor $- \times - : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ participates in a simplicially-enriched two-variable adjunction

$$\mathrm{fun}(A \times B, C) \cong \mathrm{fun}(A, C^B) \cong \mathrm{fun}(B, C^A).$$

Note that the notation, e.g., C^B does *not* mean the same thing as the cotensor E^U of an object $E \in \mathcal{K}$ by a simplicial set U.

Theorem 6.2. *The category* $\mathsf{QuasiCat}$ *of quasicategories and quasi-functors is an* ∞*-cosmos.*

Proof idea. We make $\mathsf{QuasiCat}$ into a simplicially enriched category by taking $\mathrm{map}(A, B)$ to be the simplicial inner hom $[A, B]$ (also written B^A), and take the isofibrations to be isofibrations in the Joyal model structure. Most of the proof is just looking at the definition of the Joyal model structure; we thus comment briefly on each of the axioms:

(IC1) The simplicial set Δ^0 is terminal in $\mathsf{QuasiCat}$. Pullbacks of isofibrations exist because a model category is (finitely) bicomplete; cotensors of cofibrant objects in $\mathsf{sSet}_{\mathrm{Joyal}}$ are again cofibrant.

(IC2) That any map $A \twoheadrightarrow 1$ is an isofibration is immediate from the definitions. Stability along pullback follows from the model category axioms; stability under formation of Leibniz cotensors comes from the fact that $i \,\widehat{\pitchfork}\, p$ is a fibration between fibrant objects.

(IC3) Follows from the model category axioms and the definition of the Joyal model structure.

(IC6) This is obvious because sSet is cartesian closed.

□

Examples 6.3 (Other examples of ∞-cosmoi)**.** The following are ∞-cosmoi:

- Cat, with objects ordinary categories, and isofibrations and equivalences the usual categorical ones;
- CSS, with objects complete Segal spaces ([RV17], §2.2.5);
- θ_n-Sp, with objects called θ_n*-spaces*, a model of (∞, n)-categories ([RV17], §2.2.10).

Many of these examples are discussed in greater detail in the second paper in this series. We apologize deeply to the reader who expected to be somewhat satisfied by this overview of the theory of ∞-cosmoi, as promised in the introduction to this section. Perhaps reading the next paper will satiate you; perhaps not. *C'est la vie.*

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