From the Desk of Reuben A. Stern

What are Spectra?

A Poor Man's Attempt to Learn Stable Homotopy Theory

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1 Introduction

Herein lie my struggles, my time spent wrestling with one simple question: *what the hell are spectra?* This seemingly-innocent question has led to many internet rabbit-whole chases, which only lead to more questions. So I thought to myself, why not document my progress and the things I've discovered? Perhaps by laying bare my thoughts on learning the subject of stable homotopy theory, others might benefit themselves.

1.1 Acknowledgments

1.2 Notation

As I sometimes get annoyed when authors don't lay out their notational preferences in advance, I will lay mine out in way more detail than anyone would ever reasonably include in a paper or book. Named categories will be typeset in $\mathsf{SansSerif}$ font; unnamed categories will usually be given script letters ($\mathscr{C}, \mathscr{D}, \mathscr{A}, \ldots$). Functors between categories are usually given capital Roman letters ($F, G, H, L, \ldots$), and natural transformations between functors will be written in lowercase Greek letters ($\alpha, \beta, \gamma, \ldots$) and denoted by a doubled arrow ($\alpha : F \Rightarrow G$). If $\mathscr{M}$ is a model category, its homotopy category will be denoted $\mathsf{h}\mathscr{M}$.

Standard uses for blackboard bold letters are in place: $\mathbb{N}$ are the natural numbers (≥ 0), $\mathbb{Z}$ are the integers, $\mathbb{Q}$ are the rationals, $\mathbb{R}$ are the reals, and $\mathbb{C}$ are the complex numbers. Projective spaces will be written according to the topologists' convention: $\mathbb{CP}^n$ denotes n-dimensional complex projective space, for instance. The letter $\mathbb{S}$ denotes the sphere spectrum, $\Sigma^\infty S^0$, and $\mathbb{S}^n$ denotes its n-fold suspension. Other blackboard bold letters are up for grabs, as I see fit throughout the text.

The topological n-sphere will be written S^n; the disk that it bounds will be D^n. All spaces are taken to be pointed unless explicitly mentioned otherwise; the suspension $\Sigma(-)$ and cone $C(-)$ functors are understood to be the reduced versions. Eilenberg–MacLane spaces will be written $K(G, n)$; I will try not to use π as a stand-in for a general group, as it annoys me.

2 Cohomology theories beget spectra

I guess the notion of a spectrum in algebraic topology is due to Lima in his Ph.D thesis ([Lim58]); this reference is in turn due to Adams ([Ada74]). Unfortunately, I can't verify this, as I have been unable to procure a copy of Lima's thesis. A lot of this section is expanding upon things I first learned while taking Eric Peterson's Math 231br course, taught at Harvard in the Spring of 2017 ([Pet17]). That course was a very positive experience for me; I will likely mention it more. The other main reference for this section is Part III of Adams's notes [Ada74].

We'll start by recalling the axioms for a generalized (co)homology theory, due to Eilenberg and Steenrod [ES45].

Axioms 2.1. A GENERALIZED HOMOLOGY THEORY is a covariant functor $\widetilde{h}_* : (\mathsf{Top}, \mathsf{Top})^{\mathrm{op}}_{*/} \to \mathsf{Ab}^{\mathbb{Z}}$ off the category of pairs of pointed topological spaces, valued in graded abelian groups. We will write $\widetilde{h}_*(X)$ for $h_*(X, *)$. We ask that $\widetilde{h}_*$ satisfy:

1. If $f : (X, A) \to (Y, B)$ is homotopic to $g : (X, A) \to (Y, B)$, then $f_* = g_*$.
2. Given a pair $(X, A) \in (\mathsf{Top}, \mathsf{Top})_{/*}$, there is an exact sequence

$$\begin{tikzcd} \widetilde{h}_*(A, *) \arrow[rr] & & \widetilde{h}_*(X, *) \arrow[dl] \\ & \widetilde{h}_*(X, A) \arrow[ul, "(-1)"] & \end{tikzcd}$$

3. The functor h_* takes coproducts to coproducts[1]:

$$\widetilde{h}_*\left(\bigvee_\alpha X_\alpha\right) \cong \bigoplus_\alpha \widetilde{h}_*(X_\alpha).$$

4. If (X, A) is a pair and $U \subset X$ is a subset such that $\overline{U} \subset \mathrm{int}(A)$, then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$\widetilde{h}_*(X \setminus U, A \setminus U) \cong \widetilde{h}_*(X, A).$$

A GENERALIZED COHOMOLOGY THEORY is the same thing but contravariant, written $\widetilde{h}^*$. We write $\widetilde{h}_n$ for the composite functor

$$(\mathsf{Top}, \mathsf{Top})_{*/} \xrightarrow{\widetilde{h}_*} \mathsf{Ab}^{\mathbb{Z}} \xrightarrow{n\text{-th graded piece}} \mathsf{Ab},$$

and similarly for $\widetilde{h}^n$.

Exercise 2.2. ([Str11], Problem 21.18) Show that the axioms imply the following Mayer-Vietoris sequence: if $\widetilde{h}^*$ is a cohomology theory and

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "\ell"] & D \end{tikzcd}$$

is a homotopy pushout square, then there is a natural long exact sequence

$$\cdots \longrightarrow \widetilde{h}^n(D) \longrightarrow \widetilde{h}^n(B) \oplus \widetilde{h}^n(C) \longrightarrow \widetilde{h}^n(A) \xrightarrow{\delta} \widetilde{h}^{n+1}(D) \longrightarrow \cdots$$

[1] In the category of abelian groups (and graded abelian groups) the categorical product and coproduct coincide.

The representability theorem of Edgar H. Brown ([Bro62]) says that any generalized cohomology theory, defined on the category Top of compactly generated weak Hausdorff spaces, gives rise to a sequence of spaces $\{E_n\}_{n\in\mathbb{Z}}$ such that the functor $\widetilde{h}^n(-)$ is naturally isomorphic to the functor $[-, E_n]$. That is to say, E_n *represents* the functor $\widetilde{h}^n(-)$. In more general and precise language:

Theorem 2.3 ([Bro62], Theorem 1)**.** *Let $F : \mathsf{Top} \to \mathsf{Set}_{*/}$ be a functor from the category of topological spaces weakly equivalent to CW complexes to the category of pointed sets and set maps. If F satisfies Axioms 1, 3, and 4, then there exists a CW complex Y, unique up to homotopy, such that $F(-)$ is naturally isomorphic to $[-, Y]$. Furthermore, there is an element $u \in F(Y)$ such that this natural isomorphism is given by $[X, Y] \cong F(X)$ via $f \mapsto f^*u$.* □

The proof ends up being by induction on the skeleta building up the CW-complex, and then finding a space Y that represents F for all spheres. It is not terribly difficult; see for instance Lecture 14 of [Pet17]. In the case that $F = \widetilde{h}^n$ is a generalized cohomology theory, we denote the representing space by E_n if left unstated.

Let's extract one more crucial bit of information from Axioms 2.1:

Proposition 2.4. *If $\widetilde{h}^*$ is a generalized cohomology theory, then the boundary map δ induces an isomorphism*

$$\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X),$$

where ΣX is the (reduced) suspension of X.

Proof. We use the Mayer-Vietoris sequence (Exercise 2.2): let A be X, C and B be the two reduced cones on X forming D, the suspension ΣX. It is easy to see that these fit into a homotopy pushout diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & CX \arrow[d, hook, "+"] \\ CX \arrow[r, hook, "-"] & \Sigma X. \end{tikzcd}$$

As CX is contractible, $\widetilde{h}^n(CX)$ vanishes in non-zero degree, and the long exact sequence associated to the homotopy pushout square induces isomorphisms

$$\widetilde{h}^n(X) \xrightarrow[\delta]{\simeq} \widetilde{h}^{n+1}(\Sigma X).$$

□

One would think that since there is a "suspension isomorphism" from $\widetilde{h}^n(X)$ to $\widetilde{h}^{n+1}(\Sigma X)$, there would be a corresponding map relating the representing spaces E_n and E_{n+1}. Indeed, this is true: recall that the functors $\Sigma(-)$ and $\Omega(-)$ are adjoint on the pointed homotopy category $\mathsf{hTop}_{*/}$. Thus we have a commutative diagram of isomorphisms

$$\begin{tikzcd} \widetilde{h}^n(X) \arrow[d, "\cong"] & \cong & \mathsf{hTop}_{*/}(X, E_n) \arrow[d, "\cong"] \arrow[rr, bend left] & & \\ \widetilde{h}^{n+1}(\Sigma X) & \cong & \mathsf{hTop}_{*/}(\Sigma X, E_{n+1}) & \cong & \mathsf{hTop}_{*/}(X, \Omega E_{n+1}). \end{tikzcd}$$

By Yoneda (since everything is natural), this gives a map $E_n \to \Omega E_{n+1}$, which by adjointness gives a map $\epsilon_n : \Sigma E_n \to E_{n+1}$. This motivates the following definition:

Definition 2.5. A SPECTRUM E is a sequence $\{E_n\}_{n\in\mathbb{Z}}$ of pointed CW-complexes, together with maps $\epsilon_n : \Sigma E_n \to E_{n+1}$ which are taken to be cellular.

This is basically the definition of spectrum given in Part III of [Ada74], and it is the first one I encountered. What gets confusing is that there are dozens of models of spectra, nearly all of which give the same homotopy category, but which are pretty different on the point-set level. These notes are at their core my attempt to make sense of all of this.

Example 2.6. We can extract a spectrum out of a space X in a simple way: the SUSPENSION SPECTRUM $\Sigma^\infty X$ will have n-th space given by $\Sigma^n X$, the n-th suspension of X. The structure maps

$$\epsilon_n : \Sigma\Sigma^n X \longrightarrow \Sigma^{n+1} X$$

are just the identity maps.

Once the models of spectra present in this text reach a number greater than one, I'll refer to a spectrum as given in Definition 2.5 as a *sequential spectrum*. Often, that terminology refers to a coordinate-free definition with a similar flavor to the above; I'll ignore that for now. The point of spectra is that they generalize both cohomology theories and spaces, in that there are functors from both categories into the soon-to-be-defined homotopy category $\mathsf{hSpectra}$:

$$\mathsf{CohomThy} \xrightarrow{\text{Br. Rep.}} \mathsf{hSpectra} \xleftarrow{\Sigma^\infty(-)} \mathsf{Top}.$$

For this to be the case, we need to have a good category $\mathsf{Spectra}$, with a model structure (Appendix A) that gives us some desired properties on the stable homotopy category[2] $\mathsf{hSpectra}$.

As an aside, I should note that it is possible to develop quite a bit of stable homotopy theory without much categorical machinery. One needs only content oneself with developing material in lesser generality and with more attention paid to point-set details. For instance, [Ada74] says little about the category theory underlying all of this. Nonetheless, I find the language of categories and model categories unifying and enlightening, and thus do not shy away from it.

3 The stable homotopy category

Here, I'll go over the properties that one might *want* from the stable homotopy category. Main references for this section are Cary Malkiewich's wonderful document [Mal14], and of course Part III of [Ada74].

Throughout, I'll precociously denote the stable homotopy category by $\mathsf{hSpectra}$. Recall that in the classical (pointed) homotopy category $\mathsf{hTop}_{*/}$, there is an adjunction

$$\begin{tikzcd} \mathsf{hTop}_{*/} \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathsf{hTop}_{*/}. \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

[2]Shame on me for using the words "stable homotopy category" before I should have...

However, not every space is the suspension of another space; not every space is the loop space of another space. That is to say, these functors are not *adjoint equivalences*; the stable homotopy category will rectify that. In some sense, stability comes from the fact that in $\mathsf{hSpectra}$, every object is the suspension of another. By the Eckmann-Hilton argument, this implies that every mapping object $\mathsf{hSpectra}(X, Y)$ will be an abelian group. Neat!

Recall now the suspension spectrum functor $\Sigma^\infty(-) : \mathsf{Top} \to \mathsf{hSpectra}$ defined in Example 2.6. As spectra seem to be an "enrichment" of spaces, we might expect a forgetful (i.e., right adjoint) functor to $\Sigma^\infty(-)$. This is indeed the case: there is a functor $\Omega^\infty(-) : \mathsf{hSpectra} \to \mathsf{Top}$ (read "loops-infinity") that is right-adjoint to $\Sigma^\infty(-)$. The basic properties of $\mathsf{hSpectra}$ can be summarized in bullet points:

1. The stable homotopy category is closed symmetric monoidal and "tensor triangulated", with tensor product $\wedge$ and "shift functor" Σ.

2. It comes equipped with a lax monoidal functor $H(-) : \mathsf{Ab} \to \mathsf{hSpectra}$.

3. There is a faithful functor $\Sigma^\infty(-) : \mathsf{hTop} \to \mathsf{hSpectra}$ such that the diagram

$$\begin{tikzcd} \mathsf{hTop} \arrow[r, "\Sigma"] \arrow[d, "\Sigma^\infty"'] & \mathsf{hTop} \arrow[d, "\Sigma^\infty"] \\ \mathsf{hSpectra} \arrow[r, "\Sigma"] & \mathsf{hSpectra} \end{tikzcd}$$

   of functors commutes. This functor is furthermore such that $\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty(X) \wedge \Sigma^\infty(Y)$, where the first $\wedge$ is taking place in hTop, and the second in $\mathsf{hSpectra}$.

4. The unit object for the tensor product $\wedge$ in $\mathsf{hSpectra}$ is the stable sphere $\mathbb{S} = \Sigma^\infty S^0$.

5. There is a collection of functors $\pi_n : \mathsf{hSpectra} \to \mathsf{Ab}$ such that every distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

   gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Z) \longrightarrow \pi_{n-1}(X) \longrightarrow \cdots$$

6. There is a functor $\mathsf{CohomThy} \to \mathsf{hSpectra}$, given by Brown Representability, that is faithful and essentially surjective (but not full).

I'll elaborate on all of these, perhaps out of order, in due time. There may be properties missing from this summary, but we'll get to them all eventually.

3.1 Tensor triangulated categories

3.2 A Note on Stability

4 Spectra beget cohomology theories

I mentioned earlier that there is a faithful functor $\mathsf{CohomThy} \to \mathsf{hSpectra}$ that is essentially surjective; it follows that there should be a full functor $\mathsf{hSpectra} \to \mathsf{CohomThy}$ that is a left inverse

to this first one, up to natural isomorphism. In this section, we'll describe how to get a homology and cohomology theory from a spectrum. If that spectrum is a ring spectrum (that is, a monoid object in the symmetric monoidal category $\mathsf{hSpectra}$), then there are a whole slew of products on the corresponding co/homology theories. We'll be able to do this just from the wanted properties of the stable homotopy category, i.e., purely axiomatically[3].

For motivation, let's look to the most *ordinary* of cohomology theories, singular cohomology! The functors $H^n(-;G)$ are represented by the CW-complexes $K(G,n)$. For each group G, there is a corresponding Eilenberg–MacLane spectrum HG, which is given level-wise by $HG_n = K(G,n)$. The suspension maps are those adjoint to the homotopy equivalences $K(G,n+1) \to \Omega K(G,n)$. One might expect to be able to recover the cohomology groups $H^*(X;G)$ by mapping the suspension spectrum of X into the Eilenberg–MacLane spectrum HG and taking the "n-th graded bit"; that is essentially what we will do. In the homology case, we have to work with co-represented functors, so things will be appropriately dualized.

Definition 4.1. Let E be a spectrum. We can define a (reduced) cohomology theory $\widetilde{E}^*$ and a (reduced) homology theory $\widetilde{E}_*$ associated to E via

$$\begin{aligned}\widetilde{E}^n(X) &:= [\Sigma^\infty X, \Sigma^n E] \\ &= \mathsf{hSpectra}(\Sigma^\infty X, \Sigma^n E); \\ \widetilde{E}_n(X) &:= \pi_n(\Sigma^\infty X \wedge E) \\ &= \mathsf{hSpectra}(\mathbb{S}^n, \Sigma^\infty X \wedge E).\end{aligned}$$

4.1 Verification of the axioms

4.2 Ring spectra and multiplicative cohomology theories

5 Models for the category of spectra

5.1 Sequential spectra

5.2 Symmetric spectra

5.3 Orthogonal spectra

5.4 The stable homotopy category for sequential spectra

6 Homotopy theory in higher categories

A Everything I've ever needed to know about model categories

This is really "everything I know about model categories", mostly because this section will expand over time as I learn things. Here is what I know:

[3] One can do this in greater generality, perhaps working with presentable stable monoidal $(\infty,1)$-categories. I should learn about that.

A.1 Background/Motivation/History

People like categorifying things, or so it seems. The theory of model categories came out of an attempt to axiomatize all things nice about the category Top of compactly-generated, weak Hausdorff topological spaces[4] and its corresponding homotopy category hTop. Recall that hTop is the category with objects the same as those in Top, but where morphisms are homotopy classes of maps:

$$\mathsf{hTop}(X, Y) = [X, Y].$$

We say that a map of topological spaces $f : X \to Y$ is a WEAK EQUIVALENCE if it induces isomorphisms $\pi_* f : \pi_* X \to \pi_* Y$ on all homotopy groups. On the other hand, it is a HOMOTOPY EQUIVALENCE if there exists a map $g : Y \to X$ such that

$$f \circ g \simeq \mathbb{1}_Y \quad \text{and} \quad g \circ f \simeq \mathbb{1}_X.$$

A famous early success of algebraic topology was the celebrated theorem of J. H. C. Whitehead:

Theorem A.1 ([Whi49])**.** *A map $f : X \to Y$ of topological spaces is a weak equivalence if and only if it is a homotopy equivalence.* □

Note A.2. Technically, Whitehead stated this for X and Y CW-complexes, and f a cellular map. Since I've assumed that Top is precisely those spaces with the weak homotopy type of a CW-complex, this doesn't matter.

This theorem tells us that the class $\mathscr{W}$ of weak equivalences in Top become isomorphisms upon passage to the homotopy category hTop. Furthermore, any map $f : X \to Y$ in the point-set category that induces an isomorphism in the homotopy category must have been a weak equivalence to begin with. Thus the category hTop is in some sense the result of "localizing" Top at the class $\mathscr{W}$ of weak equivalences, as can be made precise:

Proposition A.3. *Let $\mathsf{UncleFunctor} : \mathsf{Top} \to \mathsf{hTop}$ be the canonical functor sending Top into its homotopy category[5], and suppose $F : \mathsf{Top} \to \mathscr{C}$ is a functor taking weak equivalences to isomorphisms. Then there is a unique functor $F_{\mathsf{h}} : \mathsf{hTop} \to \mathscr{C}$ and a natural isomorphism $\alpha : F \Rightarrow F_{\mathsf{h}} \circ \mathsf{UncleFunctor}$:*

$$\begin{tikzcd}
\mathsf{Top} \arrow[rr, "F"] \arrow[dr, "\mathsf{UncleFunctor}"'] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathscr{C} \\
& \mathsf{hTop} \arrow[ur, "F_{\mathsf{h}}"'] &
\end{tikzcd}$$

Furthermore, the map

$$(-) \circ \mathsf{UncleFunctor} : \mathscr{C}^{\mathsf{hTop}} \to \mathscr{C}^{\mathsf{Top}}$$

between functor categories is fully faithful.

Proof. The functor F_{h} acts as F on objects. On morphisms, pick a representing map $f : X \to Y$ for a homotopy class $[f]$, and send $[f]$ to $Ff \in \mathscr{C}(FX, FY)$. We must check that this is well-defined: suppose g and f are homotopic maps from X to Y in Top. As such, they have a common homotopy

[4] We will sometimes refer to this category as the "point-set category", to distinguish it from the homotopy category.

[5] Thanks to Andrew Gordon for the naming convention.

inverse, which we will call $h : Y \to X$. Back in Top, these maps are all weak equivalences, so they will become isomorphisms in $\mathscr{C}$ under the functor F. The point is that Ff and Fg are both inverse to Fh in $\mathscr{C}$, so the standard argument about uniqueness of inverses applies here, showing that this functor is well-defined.

The part about $(-) \circ \mathsf{UncleFunctor}$ being fully faithful follows immediately. $\square$

There are some other nice, distinguished classes of morphisms in Top, that one learns about early on in one's topological career[6]. The first is likely the class of *fibrations*, whose definition we recall below. The next class, dual in more ways than I first realized, are the *cofibrations*, also recalled below. This treatment of the material is essentially due to J. P. May, in his wonderful book [May99].

Definition A.4. A map $p : E \to B$ is called a FIBRATION if for any space Y, map $f : Y \to E$, and map $h : Y \times I \to B$, there is a map $\widetilde{h} : Y \times I \to E$ lifting h such that the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B \end{tikzcd}$$

commutes. This property is called the HOMOTOPY LIFTING PROPERTY or HLP.

Remark A.5. You will sometimes see such a map called a "Hurewicz fibration" in the literature, and a "Serre fibration" to mean the same thing, but where Y is restricted to be a CW-complex. This distinction need not concern us, because we work in the category Top of compactly-generated weak Hausdorff spaces.

Definition A.6. A map $i : A \to X$ is said to be a COFIBRATION if for every space Y, map $f : X \to Y$, and map $h : A \to Y^I$, there is an extension $\widetilde{h} : X \to Y^I$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{h}"] & Y \end{tikzcd}$$

commutes, where $p : Y^I \to Y$ is the map sending $\gamma : [0,1] \to Y$ to $\gamma(0)$. This is called the HOMOTOPY EXTENSION PROPERTY, because the maps $h : A \to Y^I$ and $\widetilde{h} : X \to Y^I$ are adjoint to homotopies $h' : A \times I \to Y$ and $\widetilde{h}' : X \times I \to Y$.

You can prove without too much difficulty (or you can look in [May99], Chapters 6 and 7) that the classes of fibrations and cofibrations are closed under pullbacks and pushouts, respectively. Suppose now we are in the following situation: we are given a fibration $p : E \to B$, a cofibration

[6] Topology career? Topological career? I don't know which one is right/sounds best.

$i : A \to X$, and a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] & B. \end{tikzcd}$$

If either i or p are also weak equivalences, then there exists a lift $X \to E$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

commutes. I don't want to write this out in full at the moment, so I won't[7].

There are a bunch of other properties of these three classes of morphisms in the category Top; I'll not say them. Rather, I can say that the theory of model categories, developed by Quillen, serves to encapsulate precisely those nice properties of these classes and axiomatize them, thus giving an axiomatic approach to homotopy theory. Model categories will do a few nice things for us:

- You can "do homotopy theory" in any model category.
- There is a way to formally invert weak equivalences (localize) that sidesteps any size considerations.
- We have a really nice presentation of derived functors in this framework.

A.2 Model categories: the basics

First, some references. The standard source, and really the only one you'd ever need, is Mark Hovey's monograph [Hov98]. It covers all the basic definitions, major examples, and more, in a manner that is fairly readable (if at times a bit terse).

B Higher category theory for lower-categorical minds

C Homological algebra out the wazoo

[7] My book, my rules!

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