Unique Equilibrium Selection, Potential, and Noisy Perturbations: A Continuous-Time Approach*

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November 20, 2015

Abstract

In this paper, we propose an equilibrium-selection approach that perturbs static games with noise. Before playing the original game, players commonly observe noisy shocks and gradually adjust their actions without observing any information about their opponents’ behavior. We prove equilibrium uniqueness under an arbitrary noise level with mild smoothness conditions. Moreover, in the case of potential games, the unique equilibrium approximates the global maximizer of the potential as the noise level approaches zero.

1 Introduction

The existence of multiple equilibria is pervasive in many strategic interactions. Examples abound, from currency attacks, technology adaption, banking, and imperfect competition, to public goods provision. The lack of unique prediction is undesirable for economists, creating an obstacle against making comparative statics or policy suggestions. Many papers have emphasized that the lack of a unique equilibrium relies on the simplicity of complete information game formulation that assumes players’ common knowledge about the payoff functions; when players’ higher-order beliefs about payoffs are slightly perturbed from the common-knowledge case, this can lead to unique equilibrium action (Rubinstein 1989; Carlson and van Damme 1993). This insight has led to a vast literature on incomplete information games and higher-order beliefs, including the global game approach that has been successfully applied in various applications (e.g., Morris and Shin 2003).

In this paper, we propose a different approach that perturbs a complete information game by introducing payoff uncertainty but without relying on higher-order belief uncertainty. We consider a situation in which players gradually adjust their actions during the “preparation phase” before the final outcome of the game is realized, and their actions are mutually unobservable during the phase. Specifically, we consider a finite-horizon continuous-time model in which players commonly observe exogenous noise processes that perturb their payoffs. Each player controls her own action subject to (possibly very small) adjustment costs. This model is interpreted as a dynamic elaboration of a static game benchmark in which players make simultaneous and once-time decisions about their actions.

*We thank Drew Fudenberg, Stephen Morris and Daisuke Oyama for their helpful comments. This paper subsumes a part of our earlier working paper “Continuous Revision Games.”

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We prove that the equilibrium of this model is unique under mild technical conditions, such as smoothness of payoff functions. In contrast to the global-game approach and other methods in the equilibrium selection literature, our result does not require any substantial restrictions on the nature of the underlying games such as strategic complementarities. The proof of the uniqueness result is based on forward-backward stochastic differential equations (FBSDE, in short), using the seminal result by Delarue (2002), combined with the stochastic maximum principle.

Intuitively, the commonly observable noise in our model plays the role of a public correlation device that coordinates players’ behaviors. The presence of the noise allows players to choose contingent control plans at each point of time conditional on noise realizations. On the other hand, without any noise, each player just might as well commit to a particular control during the entire preparation phase; This situation is reduced to a one-shot game that can have multiple equilibria in general. In the equilibrium of our model, players can infer the opponents’ past adjustment processes by observing noise realization histories. This allows us to show the uniqueness result by solving the equilibrium strategies backward from the deadline. (See Section 3 for more discussions, and Appendix A for the uniqueness result in a model with discrete and short periods.)

Perhaps surprisingly, the noise level can be arbitrarily small to obtain the uniqueness result. As a particular case of our model, in Section 4 we consider games that admit potential functions (Monderer and Shapley, 1996) that are adapted to our set-up. This class of games has been used in many applications such as network games and Cournot competition with differentiated goods. We show that, when the frictions (i.e., the levels of noise and adjustment costs) are small, the unique equilibrium outcome distribution of our model approximates the maximizer of a potential function. This potential maximizer selects a particular one of possible multiple Nash equilibria in the static game benchmark. Thus, our selection result provides a rationale for the refinement criteria, i.e., the potential maximizer, used in applications. Our result deals with an infinite action space, complementing the existing ones that are mostly focused on finite action spaces (See Section 5 for the related literature).

We also show that the equilibrium is characterized as a unique solution to PDE. While this does not have a closed-form solution in general, it is possible to solve for it under special cases in applications: stochastic contest (Section 3.1) and team competition (Section 3.2).

2 Model Setup

We consider a continuous-time game over a finite horizon \([0, 1]\) played by a finite set of players \(N = \{1, 2, \ldots, n\}\). Player \(i\)'s payoff is given by

\[
U^i((A^i_j + \sigma^j B^i_j)_{j \in N}) - \int_0^1 c^i(a^i_t) dt, \tag{1}
\]

where player \(i\) chooses \(a^i_t\) from a bounded interval \(A^i\) at each moment \(t \in [0, 1]\), and \(A^i_t = A^i_0 + \int_0^t a^i_s ds\). We call \(A^i_t\) as player \(i\)'s action at time \(t\), and \(A^i_0\) is interpreted as a status-quo action from which it is costly to change. The first term \(U^i : \mathbb{R}^n \to \mathbb{R}\) is player \(i\)'s terminal payoff which depends on the profile of players’ actions at 1 perturbed by Brownian shock \(B^i_1 = (B^i_j)_{j \in N}\). The parameter \(\sigma^i\) controls the volatility of the Brownian shock for \(i \in N\). Note that player \(i\)'s terminal payoff is affected by other players’ controls as well. We assume that Brownian shock is independent across players for simplicity, but this assumption can be

\footnote{Recent examples that use infinite action spaces include networks (Boucher, 2015) and auctions (Jehiel and Lamy, 2015).}
relaxed with additional restrictions on the correlation structure.\textsuperscript{2} We sometimes use the vector notation by $A_i = (A_i^t)_{t \in \mathbb{N}}$ and $\sigma = (\sigma^t)_{t \in \mathbb{N}}$. The second term in equation (1) represents an instantaneous adjustment cost $c^i : \mathcal{A}^i \rightarrow \mathbb{R}$ associated with controls. We impose the following conditions on $U^i$ and $c^i$:\textsuperscript{3}

Assumption 1.

1. $U^i$ belongs to $C^{3,b}$.
2. $c^i$ belongs to $C^{2,b}$ such that $(c^i)' > 0$ is strictly away from 0.
3. $[\min_x \frac{\partial U_i(x)}{\partial x_i}, \max_x \frac{\partial U_i(x)}{\partial x_i}] \subseteq (c^i)'(\mathcal{A}^i)$.$^{4}$

We assume that each player cannot observe opponents’ control processes $a_t^j$ but can observe the evolution of the Brownian shock $B_t$. Formally, a control process of player $i$ is a $\mathcal{A}^i$-valued process that is adapted to the filtration $\mathcal{F}^B$ induced by the process $(B_t)_{t \in [0,1]}$. This means that $a_t^i$ is a function from the histories of $B_t$ to the control domain. A Nash equilibrium is a profile of control policies $A$ under which each player maximizes her expected total payoff (1) over feasible control policies.

We motivate our model as an elaboration of a static game in which each player $i$ chooses an action $A_i^1 \in \mathcal{A}^i$ simultaneously and the payoff is equal to

$$U^i((A_i^t)_{t \in \mathbb{N}}) - c^i(A_i^1 - A_i^0).$$ (2)

In this static model, in general an equilibrium is not unique. In sharp contrast, as we will show below, our dynamic model produces a unique equilibrium even if the noise level $\sigma$ is arbitrarily small. Our primary analysis focuses on the limit case of $\sigma \rightarrow 0$, where $\sigma \cdot B_t$ represents tiny noisy perturbation to the static game payoff (2). One could also interpret this perturbation as a kind of “trembling hands” because the shock directly influences the players’ actions.

Alternatively, one could think of the magnitude of $\sigma$ as being not small, and associate the noise $B_t$ with specific economic factors depending on applications. Our model describes a situation in which players dynamically adjust their positions when commonly observable exogenous shocks enter their payoff functions in an additive manner. Some examples are as follows:

Example 1.

1. \textit{Lobbying activity in legislation}: The congress votes on a bill on a fixed future date. Two lobbying parties, $l$ and $r$, who are for and against the bill, respectively, try to influence politicians on the issue before the voting date. Assume that politicians determine their attitudes based on total lobbying activities adjusted by the public support to the bill, $(A_i^1 + B_i^1)_{t \leq \tau}$, where $B_t$ represents the public opinion and stochastically changes over time (See Section 3.1 for the analysis of this class of games).

\textsuperscript{2}Moreover, our results still hold when the Brownian shock has non-zero drift terms.

\textsuperscript{3}Here $C^{k,b}$ denotes the set of functions that are continuously differentiable up to the $k$'th order with bounded derivatives.

\textsuperscript{4}This is a technical condition that ensures that each player chooses an interior solution to the local optimization problem. This enables us to apply the PDE solution existence result. Even without this condition, as is clear from the proof, we can show that an equilibrium is unique if it exists.
2. **Hedge funds competing for investor flows**: There are hedge funds that trade different asset classes. Hedge fund $i$'s performance is its total effort $\int_0^t a_i^t dt$ plus the overall asset-class performance $B^i_t$. At the end of the game, investors adjust their allocation of their money among the hedge funds based on their final performances $\left( \int_0^t a_j^t dt + B^j_1 \right)_{j \in N}$, which affects the payment $U^i$ received by hedge fund $i$.

3. **Cross-boarder Cournot competition**: Two firms are located in different countries competing in the global market. Firm $i$’s output level at $t \in [0,1]$ is $\int_0^t a_i^s ds$ plus a country-specific shock $B^i_t$ caused by observable macroeconomic fluctuations of inflation, currency, and weather. The prices, and thus the firms’ profits, are determined in the global market at the end of the game, based on the total output levels of the two firms.

4. **Learning about economic fundamentals**: The noise process $B^i_t$ can be interpreted as Bayesian updating about unobservable economic fundamentals when players observe the same signals. By the Kalman filtering, under Gaussian priors and signals, the posterior means of the fundamentals evolve as Brownian motions with time-varying variances (While we assumed $\sigma^i$ to be time-invariant for simplicity, this restriction is not necessary for our results). Specifically, consider competition by investors in which each one has its own project with unknown quality, and adjusts its investment scale $A^i_t$ accordingly before project completion. $B^i_t$ denotes the expected quality of investor $i$’s project quality under the public belief at $t$. Assuming that (up to the logarithm transformation) the expected output of $i$ depends on the sum of total investment scale $A^1$ and the expected quality $B^i_1$, the example fits our framework.

## 3 Equilibrium Uniqueness

In this section, we establish the existence of a unique Nash equilibrium. To characterize the players’ incentives, we first define payoff-relevant state

$$X^i_t := \int_0^t a^i_u du + \sigma^i B^i_t, \quad X_t := (X^i_t)_{i \in N}. \quad (3)$$

Note that the value of the state is not directly observable by the players. For each $i$, let $U^i_t := \frac{\partial U^i}{\partial x^i}$ denote the partial derivative of $U^i$ with respect to the $i$’th argument. We introduce adjoint variables which represent the marginal benefits by raising state variables:

**Definition 1.** Given a control policy profile $a = (a^i)_{i \in N}$, adjoint variables $Y(a) = (Y^i(a))_{i \in N}$ are defined by

$$Y^i_t(a) = \mathbb{E} \left[ U^i_t(X_1) \bigg| \mathcal{F}^B_t \right].$$

Note that $Y^i_t(a)$ is a bounded martingale by Assumption 1. Thus we can apply the martingale representation theorem and ensure the existence of $Z^i(a) = (Z^{i,k}(a))_{k \in N}$, which is a $\mathcal{F}^B$-adapted stochastic process, such that

$$dY^i_t(a) = Z^i_t(a) \cdot dB_t \quad \text{with a terminal condition} \quad Y^i_1(a) = U^i_1(X_1). \quad (5)$$

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As in the corporate finance literature, the effort choice could be interpreted as (the negative of) how much cash flow the managers steal from their funds.
The following lemma follows by the stochastic maximum principle\(^6\) that provides a necessary condition for the optimal control policy in terms of the adjoint variable.

**Lemma 1.** Fix any control policy profile \(a = (a^i)_{i \in N}\). Then \(a^i\) is optimal against \((a^j)_{j \neq i}\) only if
\[
a^i_t = f^i(Y^i_t(a)) \quad \text{almost surely for almost all } t \in [0, 1]
\]
where \(f^i\) is defined as
\[
f^i(y) = \arg \max_{a^i \in A^i} \left\{ ya^i - c^i(a^i) \right\}.
\]

Therefore, if a Nash equilibrium \(a\) exists, the equations (3), (4), (5), and (6) must be jointly satisfied. These define a multidimensional FBSDE, and we can apply the result of Delarue (2002) to ensure the unique existence of its solution in this setting. This implies that a Nash equilibrium is unique if it exists. To show the equilibrium existence, we construct a particular one by solving the PDEs stated below. Finally, this strategy profile is confirmed to be the unique equilibrium, as stated in Theorem 1. See the Appendix for the details of the proof.

**Theorem 1.** There is a unique Nash equilibrium. The corresponding control policy of \(i\) takes the form \(a^i_t = f^i(y^i(t, x))\), where \(y = (y^i)\) is a unique solution to the PDE
\[
\frac{\partial y^i}{\partial t} + \sum_j f^j(y^j) \frac{\partial y^i}{\partial x^j} + \frac{1}{2} \sum_j (\sigma^j)^2 \frac{\partial^2 y^i}{\partial x^j \partial x^j} = 0,
\]
with terminal conditions \(y^i(1, x) = U^i(x)\).

Note that the players cannot directly observe the current state variables \(X_t = (X^i_t)\). However, in an equilibrium, they can correctly infer the current state variables by the observation of the history of noise \(B_t\) because each player knows the opponents’ control policies. This allows the equilibrium control policies to be written as “Markovian,” that is, \(a^i_t\) depends only on \((t, X_t)\).\(^7\) While the PDE does not have a closed-form solution in general, we can solve it under several special cases as we show in the following subsections.

It is important to note that the equilibrium uniqueness result holds even if the noise levels \(\sigma^i\) are arbitrarily small. On the other hand, when the noise levels are zero, then the players receive no observation during the course of play. They then choose constant actions over time (due to the strict convexity of the control costs), and thus this case is essentially reduced to the static game in (2). Therefore, multiple equilibria are possible without noise. The presence of noise changes the situation substantially by inducing players to choose control choice plans contingent on every noise realization. Because the \(B_t\)-process is publicly observable, at each point of time \(t\), players can back out the control actions that have been taken by the others, and compute the current values of \(\int_0^t a^j_t \, dt\).\(^8\) When \(t\) is very close to 1, the remaining changes in actions \(\int_t^1 a^j_t \, dt\) by the opponents are considerably small because of the boundedness of control ranges. Thus, each player’s control incentive is not influenced too much by her belief about the opponents’ controls during \([t, 1]\), and therefore, the optimal control at \(t\) is “almost” unique. Then, intuitively speaking, we seem to be able to pin-down their optimal actions at each \(t\) backward from the endpoint 1. In Appendix A, we verify this intuition by formally considering a discrete period model and showing the uniqueness of equilibrium when the period length is sufficiently small (Proposition 3).

\(^6\)See Pham (2009) for example.

\(^7\)Based on the obtained control policies, we can also derive a system of PDEs that characterize the equilibrium continuation values.

\(^8\)Here it is important that the players choose pure strategies in order to conduct this inference. We do not consider mixed strategies because the players never have incentives to randomize due to the strict convexity of the control costs.
3.1 Application: Stochastic Contest

As an illustration of our framework, we consider a contest model in which two players compete for a fixed prize. Contest models are broadly used in applications. In addition to lobby activities in Example 1.1, there are many examples including internal labor markets and R & D races (See Konrad (2009) for a survey). It is known that a contest game in general can have multiple equilibria (e.g., Chowdhurya and Shereometab (2011)). In contrast, our model always predicts a unique Nash equilibrium by Theorem 1.

There are two players 1 and 2. Player \( i = 1, 2 \) chooses effort \( a^i_t \) at each point of time, and \( c^i(a^i_t) \) denotes the effort cost. The winner is determined stochastically, depending on the difference in players’ total effort levels \( A^1_t - A^2_t + \bar{B}_t \) that is adjusted by the exogenous noise term \( \bar{B}_t := \sigma^1 B^1_t - \sigma^2 B^2_t \). By using a so-called difference-form contest success function (e.g., Che and Gale (2000)), we assume that players terminal payoffs at \( t = 1 \) are given by

\[
U^1 = F(A^1_t - A^2_t + \bar{B}_t) = F(X^1_t - X^2_t), \quad U^2 = 1 - U^1
\]

for some continuously differentiable and increasing function \( F : \mathbb{R} \rightarrow [0, 1] \). Here \( F(\cdot) \) denotes the probability that player 1 wins the contest, and we normalize the payoff of winning the prize to be 1.

First, we show that the equilibrium control policies can be analytically solved when the cost functions are the same. Because the marginal payoff in own state is the same for each player \( U^1_1 = U^2_2 \), their control levels are the same at each point i.e., \( a^1_t = a^2_t \). Therefore, the process \( X^1_t - X^2_t \) follows \( d(X^1_t - X^2_t) = dB_t \), and the equilibrium effort level takes the form

\[
a^i_t = f \left( \mathbb{E}[F'(X^1_t - X^2_t)|F^B_t] \right) = f \left( \frac{\int_{-\infty}^{\infty} F'(x) \exp \left[ \frac{-(x-X^1_t+X^2_t)^2}{4(1-t)\bar{\sigma}^2} \right] dx}{2(1-t)\bar{\sigma}\sqrt{2\pi}} \right),
\]

where \( \bar{\sigma} \) denotes the variance of \( \bar{B}_t = \sigma^1 B^1_t - \sigma^2 B^2_t \), and \( f := f^1 = f^2 \). For example, under the quadratic cost \( c^i(a^i) = \bar{c}(a^i)^2 / 2 \), \( f(y) = \frac{y}{\bar{\sigma}} \).

Next, we conduct the numerical analysis by using the logit function and quadratic cost functions

\[
U^i(X^1, X^2) = \frac{\exp[\gamma X^1]}{\exp[\gamma X^1] + \exp[\gamma X^2]}, \quad c^i(a^i) = \bar{c}(a^i)^2 / 2
\]

where \( \gamma > 0 \) parameterizes the noisiness of the contest. By numerically solving the PDE in Theorem 1, we can calculate the optimal control policy \( a^i \) as a function of \( (t, X^1, X^2) \). An interesting observation from the simulations is that equilibrium effort level \( a^i(t, X^1, X^2) \) is monotone in \( t \) (Figure 1). In particular, it is increasing in \( t \) in competitive situations, i.e. \( X^1 \approx X^2 \). To understand this, the benefit of making an effort in such a situation is more effective when \( t \) is closer to the deadline, because the remaining size of the Brownian noise is smaller. On the other hand, consider the case where \( |X^1 - X^2| \) is large. Then, the winner of the contest is almost determined near the deadline, so that players tend to stop choosing high effort levels, which makes \( a^i \) decreasing in \( t \).

\footnote{Lang et al. (2014) and Seel and Strack (2015) consider continuous-time contest models over a finite horizon during which each player does not observe any information about their opponents. Each player solves a stopping problem of a privately observable state variable before the deadline.}

\footnote{The simulations suggest that \( a^i \) tends to be increasing in \( t \) especially when \( \gamma \) is small. To interpret this, under more random contests (smaller \( \gamma \)), marginal effects of raising efforts on the winning probability is higher. Therefore the players become competitive even when the difference \( |X^1 - X^2| \) is large.}
and the equilibrium control policies are given by (see Appendix B.4 for the detail).

In this subsection, we consider a situation in which the players are divided into two groups (or, teams) and the two compete each other. It is shown that the PDE in Theorem 1 is reduced to be one-dimensional and admits a closed-form solution. We assume player $i$’s terminal payoff takes the form

$$U^i(X) = \theta^i h(\sum_{j \in N} \phi^j X^j)$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$, and $\theta^i, \phi^i \in \mathbb{R}$ are constants.

For the two-player case, this model accommodates the stochastic contest in the previous subsection by setting $\theta^1 = \phi^1 = 1$ and $\theta^2 = \phi^2 = -1$. More generally, if $sgn(\phi^i) = sgn(\theta^i)$ for every $i$, the model can be seen as competition between two teams. Here players with positive $\phi^i$ and $\theta^i$, denoted by $N_+$, are grouped into the same team, and the remaining players form another team. $h(\cdot)$ represents the final outcome from the competition that depends on the value of $\sum_{j \in N} \phi^j X^j$, which is the difference between the first team’s total output $\sum_{i \in N_+} \phi^i A_1^i$ and the second team’s $\sum_{i \in N \setminus N_+} |\phi^i| A_1^i$, perturbed by the noise $\bar{B}_1 := \sum_{i \in N} \phi^i B_1^i$. Here $|\theta^i|$ represents the fraction of $i$’s final surplus.

We also assume that players’ cost functions are quadratic, so that $\hat{c}^i(a) = \frac{c}{2} a^2$. Then we can reduce the PDE system (7) to the following one-dimensional problem

$$\frac{\partial y}{\partial t} + \left( \sum_{i} \phi^i \theta^i \right) \frac{\partial y}{\partial \bar{x}} + \frac{\sigma^2}{2} \frac{\partial^2 y}{\partial \bar{x}^2} = 0, \quad y(1, \bar{x}) = h'(\bar{x}),$$

where payoff relevant states $(X^i)_{i \in N}$ are aggregated into a one-dimensional value $\tilde{X} = \sum \phi^i X^i$, and the equilibrium control policies are given by $a_1^i = \phi^i \theta^i y(t, \tilde{X}_t)$.

The PDE has a closed-form solution given by

$$y(t, \bar{x}) = -\frac{\int e^{-\frac{(\bar{x} - \bar{x}')^2}{4K(1-t)} + \frac{h(\bar{x}')}{2\sigma}} d\bar{x}'}{K \int e^{-\frac{(\bar{x} - \bar{x}')^2}{4K(1-t)} + \frac{h(\bar{x}')}{2\sigma}} d\bar{x}'}$$

where $\bar{\sigma} := \frac{\sigma^2}{2}$ and $K := \sum \phi^i \theta^i$. In particular, for almost all $\bar{x}$, $y(t, \bar{x}) \rightarrow -\frac{\bar{x} - k(\bar{x}, 1-t)}{K(1-t)}$ holds as $\bar{\sigma} \rightarrow 0$ where $k(\bar{x}, 1-t) := \arg \max_{\bar{x}' \in \mathbb{R}} -\frac{(\bar{x} - \bar{x}')^2}{2(1-t)} + Kh(\bar{x}')$. That is, in the small noise limit, the equilibrium strategy can be computed through the static optimization problem

$$\max_{\bar{x}' \in \mathbb{R}} -\frac{(\bar{x} - \bar{x}')^2}{2(1-t)} + Kh(\bar{x}')$$. The derivations here are based on the literature on Burgers equations (see Appendix B.4 for the detail).
4 Potential Maximizer

We have shown that the model predicts a unique Nash equilibrium even if the static game benchmark admits multiple equilibria, and this is true even if the noise or cost are arbitrarily small. This motivates us to use our model to “select” among multiple equilibria. We do so by using the concept of a potential function introduced by Monderer and Shapley (1996) that is adapted to our setting.

**Definition 2.** A potential function is a continuously differentiable function $G : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\frac{\partial G(x)}{\partial x_i} = U_i(x)$ holds for any $x$ and $i$.

When there is a potential function, players incentives have the same directions as increasing the potential. Note that this implies neither an equilibrium is unique nor Pareto efficient. It is helpful to observe the equivalence of local potential maximizers and Nash equilibria in the static benchmark game (without adjustment costs) based on the finding in Monderer and Shapley (1996). Here we say that $x^*$ is a local maximizer of $G$ if $x^*i \in \arg \max_x G(x_i, x^{-i})$ holds for each $i$.

**Proposition 1.** Suppose that there is a potential function $G$. Then $a_1 = (a_1^i)_{i \in N}$ is a Nash equilibrium of the static benchmark in (2) where $c^i = 0$ for each $i$ if and only if $A_1$ is a local maximizer of $G$.

In general there can be many local maximizers of a potential function, and therefore there can be multiple equilibria in the static benchmark case (without adjustment costs$^{11}$).

As a simple way of identifying a potential function, consider the following form:

$$U^i(x) = U(x) + \phi^i(x^i) + \psi^i(x^{-i}).$$

for $U : \mathbb{R}^n \rightarrow \mathbb{R}$, $\phi^i : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi^i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ for each $i = 1, ..., n$. The common utility term $U(x)$ captures the pure coordination incentive among the players, while the $\phi^i$ terms distort their incentives from maximizing $U$. The $\psi^i$ functions represent externalities that do not influence their behavior. It is clear to see that $U(x) + \sum_i \phi^i(x^i)$ serves as a potential function.

Below we list some examples of games with potential functions.

**Example 2** (Network game). Consider the following quadratic form$^{12}$:

$$U^i(x) = \alpha_i x_i - \frac{1}{2}(x^i)^2 + \delta \sum_j G_{ij} x_i x^j$$

where $\alpha_i \geq 0$. $\delta$ is the parameter of network effects; their actions admit strategic complementarities (resp. substitutes) if $\delta > 0$ (resp. $\delta < 0$). The matrix $G$ represents an undirected network among the players by which $G_{ij} = 1$ if $i$ and $j$ are connected, and $G_{ij} = 0$ otherwise. This class of games is often studied in the networks literature, and multiple equilibria is one of the important issues.$^{13}$

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$^{11}$Clearly, there can be multiple equilibria even when adjustment costs $c$ are present in the static benchmark case.

$^{12}$In the network game literature, the domain of the payoff functions is not necessarily the whole space of $\mathbb{R}^n$ but often its bounded subset. Similarly, we can truncate the $U$-function at the tail of the domain. By this modification, the boundedness of $U$ required in Assumption 1 is satisfied.

$^{13}$See Bramoullé and Kranton (2015) for a survey. For instance, Bramoullé and Kranton (2007) emphasize the relevance of multiplicity of equilibria in a public good provision game. Bramoullé et al. (2014) is the first paper that applies the potential function approach to this setting, which provides a condition for equilibrium uniqueness.
Example 3 (Cournot competition). The quadratic formulation in (9) accommodates Cournot competition, in which $x_i$ represents the product output of firm $i$. For each $i, j$, $G_{ij} = 1$ if the products of firm $i$ and $j$ are substitutes. Firm $i$'s price is given by a linear demand curve $p_i = \bar{p} - \delta \sum_j G_{ij} x_i x_j$, and the firm receives the net production cost revenue $x_i p_i - \kappa x_i$, where $\kappa > 0$ denotes a constant marginal cost.

The following proposition shows that the equilibrium outcome approximates the maximizer of a potential function as the frictions vanish.

**Proposition 2.** Suppose that there exists a potential function $G$ that admits its maximum $G^* := \max_x G(x)$. Then, for any $\epsilon > 0$ there exist constants $\bar{\sigma}, \bar{c}, \bar{A} > 0$ such that

$$
\mathbb{E}[G(X_1)] + \epsilon \geq G^*
$$

holds under the unique equilibrium if $\sigma^i \leq \bar{\sigma}$, $\sup_a c^i(a) \leq \bar{c}$, and $\text{diam}(A_i) \geq \bar{A}$ for each $i$.

In the proof, we construct a hypothetical optimization problem by a single agent who controls everyone’s action to maximize the potential subject to the adjustment costs. This is constructed in a way that the equilibrium of the game, which is unique by Theorem 1, coincides with the optimal policy in such a problem. Thus adjustment costs are small, then the optimal control achieves the global maximizer of the potential $G$.

5 Related Literature

In Iijima and Kasahara (2015, IK15 henceforth), we analyze a similar but distinct situation under which players can directly observe state variables $X_t$, and show a unique equilibrium result under certain conditions. We comment on three key differences between the two papers. First of all, the role of noise in equilibrium uniqueness is strikingly different. In the current paper, it serves as a public coordination device, and the model is reduced to be a static one without noise. In IK15, without noise the model is reduced to a dynamic game with perfect action observability; Noise is mainly used to eliminate discontinuities in the continuation payoff functions. Second, the equilibrium selection problem is not a focus of IK15, in which we only characterize the small friction limit in a team production problem with symmetric players. The current paper uses the potential function approach to characterize the equilibrium selection in more general and challenging situations that involve players’ conflicting preferences over different equilibria. Third, their proof technique in IK15 relies on the BSDEs based on the measure change argument. This approach is not applicable in the current problem, so that we use FBSDE that involves non-trivial forward equation using the stochastic maximum principle.

This paper shares the sprit with the literature on global games (Carlson and van Damme 1993, Morris and Shin 2003) in that both approaches introduce small payoff shock to a complete information game that leads to equilibrium uniqueness. In the global game approach, it is crucial that players receive private information so that the payoff functions are not commonknowledge. In contrast, our model does not use such belief heterogeneity but requires players’ flexible adjustments during the preparation phase. An advantage of our approach is that the uniqueness result holds under any terminal payoff $U^i$ subject to a technical smoothness condition, while the uniqueness in global games relies on strategic complementarities. It is also

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14 Under the terminology of the differential game literature, the equilibrium concept in IK15 corresponds to closed-loop equilibria, as opposed to the open-loop equilibria in the current paper.
known that dynamic versions of the global game typically admits multiple equilibria (see, for example, Angeletos and Pavan (2013)).

Carbonell-Nicolau and McLean (2014) study potential games with infinite action spaces as in our paper, and show that potential maximizers are robust against action trembles. There are also several approaches that favor the potential maximizer in finite-action potential games. Frankel et al. (2003) consider global games with strategic complementarities, and show that the unique equilibrium in the small noise limit maximizes the potential. Hofbauer and Sorger (1999) show that the potential maximizer is globally accessible and absorbing under the perfect foresight dynamics (Matsui and Matsuyama 1995).\footnote{Hofbauer and Sorger (2002) find a formal connection between the perfect foresight dynamics and open-loop equilibria in a class of infinite horizon deterministic differential games. This class of differential games can have multiple open-loop equilibria, and the perfect foresight dynamics can admit multiple paths.} Ui (2001) shows that the potential maximizer is robust against incomplete information in the sense of Kajii and Morris (1997). It is not known whether such a robust equilibrium is unique in general. For symmetric games, Blume (1993) shows that the long-run distribution of myopic logit response dynamics is concentrated on the potential maximizer as the noise level becomes small.

Caruana and Einav (2008), Kamada and Kandori (2014), and Calcagno et al. (2014) study models with a preparation phase before the game is played. The players’ actions and/or payoff relevant states are directly observable without any noise in their set-ups, and thus the properties of equilibria are somewhat different from ours. In particular, Kamada and Kandori (2014) show a large set of equilibria even if their one-shot game benchmark admits a unique equilibrium. Calcagno et al. (2014) study finite-action coordination games but obtains the selection of Pareto efficient equilibrium, which does not in general agree with the potential maximizer criteria.

Technically, the model in this paper can be seen as a class of stochastic differential games. While open-loop equilibria have been studied in the applied mathematics literature, such as Hamadène (1998, 1999) for linear-quadratic games, none of them has either established equilibrium uniqueness or worked out in specific economic applications.

6 Conclusion

In this paper, we propose an equilibrium-selection approach that perturbs static games with noise. Before playing the original game, players commonly observe noisy shocks and gradually adjust their strategies without any information about their opponents’ behaviors. We prove equilibrium uniqueness under an arbitrary noise level with mild smoothness conditions. In the case of potential games, the unique equilibrium approximates the global maximizer of the potential as the noise level approaches zero.

While we provided the potential maximizing result for games that have “exact” potential functions of Monderer and Shapley (1996), it remains an interesting open question to investigate more general potential functions.\footnote{See, for example, Morris and Ui (2005), Oyama et al. (2008), Oyama and Terceix (2009), and Nora and Uno (2014).} Our solution technique relies on FBSDE and PDE. Although these technical tools are extensively studied in the fields of stochastic control and mathematical finance, they have not attracted much attention in the areas of dynamic games until recently. The results in this paper and the companion paper IK15 demonstrate the usefulness of these techniques in characterizing dynamic game equilibria in specific economic applications.
A Discrete-Time Limit

In this subsection, we consider a discrete-time model that approximates our continuous-time model and show the equilibrium uniqueness result if each period length is small enough. The model is parameterized by \( n \in \mathbb{N} \), which denotes the number of periods. The only difference from the original model is that we restrict the players’ control policies such that \( a^i_t \) is required to be constant within each period \([\frac{k}{n}, \frac{k+1}{n})\) where \( k = 0, 1, \ldots, n - 1 \). A Nash equilibrium is defined in a based on these restricted spaces of control policies.

Note that the payoff is given by

\[
U^i((A^0_i + \frac{1}{n} \sum_{t=0}^1 a^i_t + \sigma^i B^i_{t}), i \in \mathbb{N}) - \frac{1}{n} \sum_{t=0}^1 c^i(a^i_t).
\]

where each \((B^i_t)_{0 \leq t \leq 1}\) is a standard Brownian motion, which is observable to every player. We maintain the same Assumption 1 about \( U^i \) and \( c^i \).

Then the following uniqueness result is obtained, whose proof is contained in Appendix B.

Proposition 3. Consider the discrete-time version of the model. Then, for large enough \( n \), there is at most one Nash equilibrium.

At this point we do not know whether there always exists a Nash equilibrium in the discrete-time model. In contrast to the continuous-time model, we cannot appeal to PDE technique to construct a particular equilibrium.

B Proofs

B.1 Preliminaries

B.1.1 FBSDE

Fix a probability space \((\Omega, \mathcal{G}, \mathbb{Q})\) in which \( \{\mathcal{G}_t\}_{0 \leq t \leq 1} \) satisfies the usual conditions, and \((Z_t)_{0 \leq t \leq 1}\) is a standard \( d \)-dimensional \( \mathcal{G}_t \)-Brownian motion. (Markovian) FBSDE is a stochastic system of equations that takes the form

\[
\begin{align*}
    dX_t &= F(t, X_t, W_t, \beta_t)dt + \Sigma dZ_t, \quad X_0 = x_0 \\
    dW_t &= G(t, X_t, W_t, \beta_t)dt + \beta_t dZ_t, \quad W_1 = U(X_1)
\end{align*}
\]

(10)

for some \( F : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n, \ G : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^d, \ U : \mathbb{R}^d \rightarrow \mathbb{R}^n, \ x_0 \in \mathbb{R}^d \) and \( \Sigma \in \mathbb{R}^{d \times d} \) is an invertible matrix.

A solution \((X, W, \beta)\) to FBSDE is a \( \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \) valued and \( \mathcal{G} \)-adapted process that satisfies (10) and

\[
    \mathbb{E} \left[ \int_0^1 (|X_t|^2 + |W_t|^2 + |\beta_t|^2) \right] < \infty.
\]

The following result is from Delarue (2002).

Lemma 2 (Markovian Quadratic FBSDE). Consider FBSDE (10). Assume that there exists a constant \( K > 0 \) such that the following conditions are satisfied (for all \( t, x, x', w, w', b, b' \)):

1. \( F \) is continuous in \( x \) such that

\[
|F(t, x, w, b) - F(t, x, w', b')| \leq K(|w - w'| + |b - b'|)
\]

\[
(x - x') \cdot (F(t, x, w, b) - F(t, x', w, b)) \leq K |x - x'|^2
\]

\[
|F(t, x, w, b)| \leq K(1 + |x| + |w| + |b|)
\]
2. \( G \) is continuous in \( w \) such that
\[
|G(t, x, w, b) - G(t, x', w, b')| \leq K(|x - x'| + |b - b'|)
\]
\((w - w') \cdot (G(t, x, w, b) - G(t, x, w', b)) \leq K|w - w'|^2
\]
\[|G(t, x, w, b)| \leq K(1 + |x| + |w| + |b|)
\]

3. \( U \) is Lipschitz continuous, i.e.,
\[
|U(x) - U(x')| \leq K(|x - x'|)
\]
\[|U(x)| \leq K(1 + |x|).
\]

Then there exists a unique solution \((X, W, \beta)\) of the FBSDE, which takes the form \(W_t = w(t, X_t), \beta_t = b(t, X_t)\) for function \(w : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^n\) that is uniformly Lipschitz continuous in \(x\), and \(b : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^{d \times n}\).

**B.1.2 Uniformly Parabolic PDEs**

The following lemma on uniformly parabolic PDEs is directly taken from Ladyzenskaja et al. (1995, Theorem 7.1, p.596). It gives a set of sufficient conditions that ensures a unique solution to a system of second-order PDEs. In the statement below, \( u = (u^i)^{n}_{i=1} : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^n \) is the function to be determined, and subscripts represent partial derivatives, i.e., \( u_t = (u^i_t)^{n}_{i=1} \), \( u_{ij} = (u^{ij}_{kl})^{n}_{k,l=1} \), \( u_k = (u^k)^{n}_{i=1} \) and \( u_x = (u^k_{x})_{k=1}^{d} \). Let \( \mathbb{H} \) denote the set of \( N \)-dimensional functions \( u(x, t) \) that are continuous, together with all derivatives of the form \( D_t^r D_x^s \) for \( 2r + s \leq 3 \). A function is called \( C^{3,b} \) if it has continuous and bounded derivatives up to the third order.

**Lemma 3.** Consider the following \( n \)-dimensional PDE system of \( u = (u^n)^{N}_{n=1} : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^n \).
\[
u_t - \sum_{k,d} \alpha^{kl}(x, t, u)u_{kl} + \sum_{k} \delta^k(x, t, u, u_x)u_k + \gamma(x, t, u, u_x) = 0, \quad u(x, T) = U(x),
\]

where \( \alpha^{kl} : \mathbb{R}^d \times [0, 1] \times \mathbb{R}^n \to \mathbb{R} \), \( \delta^k : \mathbb{R}^d \times [0, 1] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R} \), \( \gamma : \mathbb{R}^d \times [0, 1] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n \), and \( U : \mathbb{R}^d \to \mathbb{R}^n \) for each \( i, j = 1, ..., d \) are exogenous functions.

Then, there exists a unique solution to the above system in \( \mathbb{H} \) if the following conditions are satisfied:

1. There exist \( M_1, M_2 \geq 0 \) such that
\[
\forall i = 1, ..., n, \quad \gamma^i(x, t, u, 0) \geq -M_1 |u|^2 - M_2.
\]

2. There exist constants \( M_3, \mu, \nu > 0 \) such that
   
   (a) (Uniform Parabolicity)
   \[
   \forall \xi \in \mathbb{R}^d, \quad \nu \leq \frac{1}{|\xi|^2} \sum_{k,d} \alpha^{kl}(t, x, u)\xi^k \xi^l \leq \mu,
   \]

   (b) \[
   \forall i = 1, ..., d, \quad |\delta^k(x, t, u, p)| \leq \mu(1 + |p|)^2
   \]
\(|\gamma(x,t,u,p)| \leq M_3 \frac{|p|}{|p|}(1 + |p|)^2\)

\(\forall k,l = 1, \ldots, d, \forall i = 1, \ldots, n, \left| \frac{\partial \alpha^{kl}(x,t,u,p)}{\partial x^k}(x,t,u,p), \frac{\partial \alpha^{kl}(x,t,u,p)}{\partial u^i}(x,t,u,p) \right| \leq \mu\)

3. The first derivatives of \(\alpha^{kl}, \delta^k, \gamma\) with respect to \((t,x,u,p)\) are continuous. And \(\frac{\partial^2 \alpha^{kl}}{\partial u^i \partial u^j}, \frac{\partial^2 \alpha^{kl}}{\partial x^k \partial u^i}, \frac{\partial^2 \alpha^{kl}}{\partial x^k \partial u^l}, \frac{\partial^2 \alpha^{kl}}{\partial x^k \partial u^m}\) are continuous.

4. \(U(\cdot)\) is \(C^{3,b}\).

### B.2 Proof of Lemma 1

**Proof.** Since control policies do not affect the diffusion coefficients of state variables in the current setting, we can solve the problem by a simplified Hamiltonian \(H^i\) defined by

\[H^i(t,a^i,Y^i_t) := a^i Y^i_t - c^i(a^i).\]

By the stochastic maximum principle of Bismut (1973), player \(i\)’s optimal control policy \(\tilde{a}^i_t\) against \(a^{-i}\) must satisfy the local optimality condition almost surely for almost all \(t\):

\[\tilde{a}^i_t \in \arg \max_{a^i \in A^i} H^i(t,a^i,Y^i_t(a)) = \arg \max_{a^i \in A^i} \left\{ a^i Y^i_t - c_i(a^i) \right\} = f^i(Y^i_t(a)).\]

\(\Box\)

### B.3 Proof of Theorem 1

The proof consists of two steps:

1. There exists at most one equilibrium.
2. Construction of an equilibrium and derivation of PDE.

**Step 1: There exists at most one equilibrium.**

The argument in Section 3 suggests that, if an equilibrium \(a\) exists, its associated processes \((X_t,Y_t,Z_t)\) solve the following FBSDEs:

\[X^i_t = X^i_0 + \int_0^t f^i(Y^i_s)ds + \sigma_i B^i_t, \quad Y^i_t = U^i_t(X_1) - \int_t^1 Z^i_s \cdot dB_s,\]

where \(a^i_t = f^i(Y^i_t)\). Because \(f^i\) is Lipschitz continuous and bounded, the FBSDEs admit a unique solution by Lemma 2. Thus, there can be at most one equilibrium.

**Step 2: Construction of an equilibrium.**

We find a particular equilibrium strategy such that \(a^i_t\) depends only on \((t,X_t)\). This is constructed by solving the system of PDEs

\[y^i_t + \sum_j f^j(y^j_t)y^j_t + \frac{1}{2} \sum_j (\sigma^j)^2 y^j_{jj} = 0, \quad y^i(T,x) = U^i_t(x),\]  \hspace{1cm} (11)

where the subscripts denote partial derivatives.
Lemma 4. There exists a unique $C^{3,b}$ solution $(y^i)_{i \in \mathbb{N}}$ to PDEs (11).

Proof. This follows by applying Lemma 3. The PDEs correspond to the notations of Lemma 3 as

$$\alpha_{ii} = \frac{1}{2}(\sigma^i)^2, \alpha_{ij} = 0, \delta_i = f^i(y^i), \gamma = 0, \psi = (U^i_1).$$

By checking each of conditions for Lemma 3, we finish the proof of this lemma.  

Fix $\tilde{y} = (\tilde{y}^i)$ is the unique $C^{3,b}$ solution to PDEs (11). We construct a control policy profile $\tilde{a} = (\tilde{a}^i)$ which is defined by

$$\tilde{a}^i_t = f^i(\tilde{y}^i(t, X_t)).$$  \hspace{1cm} (12)

Lemma 5. Let $Y^i_t$ be the adjoint variable of player $i$ under $\tilde{a}^i$. Then the following system of equations is satisfied

$$X^i_t = X^i_0 + \int_0^t f^i(Y^i_s)ds + \sigma^i B^i_t, \quad Y^i_t = U^i_1(X_1) - \int_t^1 Z^i_s \cdot dB_s.$$

Proof. We construct the following $\mathcal{F}^B$-adapted processes $(X_t, Y_t, Z_t)$:

$$Y^i_t = \tilde{y}^i(t, X_t), \quad Y^i_t = U^i_1(X_1),$$

$$dX^i_t = f^i(Y^i_t)dt + \sigma^i dB^i_t, \quad Z^{i,j}_t = \sigma^j \tilde{y}^j(t, X_t)$$

Then, first note that $Y^i_1 = U^i_1(X_1)$. Second, by Ito’s formula, we have

$$dY^i_t = \left(\tilde{y}^i_t + \sum_j f^j(\tilde{y}^j_t)\tilde{y}^j_t + \sum_j (\sigma^j)^2 \frac{1}{2} \tilde{y}^j_{tt} \right) dt + \sum_j \sigma_j^j \tilde{y}^j dB^j_t$$

$$= Z^i_t \cdot dB_t,$$

where we omitted the arguments of $y^i(t, X_t)$. Therefore, $(X_t, Y_t, Z_t)$ is a unique solution to the above FBSDE.  

Lemma 6. The control policy profile $\tilde{a}$ defined by (12) is a Nash equilibrium.

Proof. Fix a control policy profile $(a^j)_{j \neq i}$ other than $i$. The optimization problem of player $i$ is the maximization of

$$\mathbb{E} \left[ U^i(X^i_1, (\tilde{X}^j_{t \neq i}) - \int_0^1 c^i(a^i_t)dt \right]$$

over all her control policies, where

$$dX^i_t = a^i_t dt + \sigma^i dB^i_t,$$

$$d\tilde{X}^j = f^j(\tilde{y}^j(t, \tilde{X}_t))dt + \sigma^j dB^j_t, \quad j \in \mathbb{N}$$

This is a Markov stochastic control problem. The standard verification argument of HJB equations applies (e.g., Theorem 4.4 in Fleming and Soner (2005)), and, thus, there exists an optimal control policy. Here choosing $a^i = f^i(\tilde{y}^i(t, \tilde{X}_t))$ leads to $\tilde{X}^i = X^i$. By Lemma 1, $\tilde{a}^i$ is the unique control policy that satisfies the necessary condition for optimality, and thus is an optimal control policy.  

The above results establish the proof of Theorem 1.
B.4 Detail of the Closed-Form Solution in Subsection 3.2

We define the new variable \( \tau := 1 - t \in [0, 1] \) and the function \( \hat{y}(\tau, \bar{x}) := -K(1 - \tau, \bar{x}) \). By (8), we have

\[
\frac{\partial \hat{y}}{\partial \tau} + \frac{\partial \hat{y}}{\partial \bar{x}} \hat{y} = \sigma K \frac{\partial^2 \hat{y}}{\partial \bar{x}^2}, \quad \hat{y}(0, \bar{x}) = -Kh'(\bar{x}).
\]

This is the one-dimensional viscous Burgers equation and its solution\(^{\text{17}}\) is given by

\[
\hat{y}(\tau, \bar{x}) = \int e^{\frac{(\bar{x} - \bar{x}')^2}{4\sigma K + 1}} \frac{h(\bar{x}')}{2\sigma} d\bar{x}'.
\]

Therefore,

\[
y(t, \bar{x}) = -\int e^{\frac{(\bar{x} - \bar{x}')^2}{4\sigma K + 1}} \frac{h(\bar{x}')}{2\sigma} d\bar{x}'
\]

As is shown in Evans (2010, Section 4.5.2), for almost all \( \bar{x} \), \( \hat{y}(\bar{x}, \bar{x}) \rightarrow \frac{\bar{x} - k(\bar{x}, \tau)}{\tau} \) holds as \( \bar{\sigma} \rightarrow 0 \), where \( k(\bar{x}, \tau) := \arg \max_x \frac{-(\bar{x} - x)^2}{2\tau} + Kh(x) \). Thus, for almost all \( \bar{x} \), \( y(t, \bar{x}) \rightarrow \frac{-\bar{x} - k(\bar{x}, 1 - t)}{K(1 - t)} \) holds as \( \bar{\sigma} \rightarrow 0 \).

B.5 Proof of Proposition 2

Proof. Consider a stochastic control problem that maximizes the potential function minus control costs:

\[
\max_a H(a) := \mathbb{E} \left[ G(X_1) - \int_0^1 \sum_i c^i(a^i_t) dt \right]. \tag{13}
\]

maximized over control policy profiles.

Lemma 7. Suppose that there exists a potential function \( G \). Then, the optimal control policy in the problem (13) coincides with the unique equilibrium.

Proof. As in the original model, we can solve the single agent control problem (13) by the stochastic maximum principle. Define the adjoint variable by

\[
\hat{Y}_t^i := \mathbb{E} \left[ \frac{\partial G(X_1)}{\partial x^i} \bigm| \mathcal{F}_t \right] = \mathbb{E} \left[ U^i_t(X_1) \bigm| \mathcal{F}_t \right],
\]

for each \( i \), where the equality uses the potential function assumption. Then this adjoint variable \( \hat{Y}_t^i \) is the same as the adjoint variable \( Y_t^i \) for the equilibrium strategy in the proof of Theorem 1. Therefore, the optimal control policy in the single agent problem (13) coincides with the equilibrium strategy profile of the original game. \( \square \)

Fix any initial point \( X_0 \) and \( \epsilon > 0 \). Let \( x^* \) be a maximizer of the potential function. Let each sup \(|c^i(a)| \) be sufficiently small so that \(|\int_0^1 \sum_i c^i(a^i_t)| \leq \epsilon/3 \) holds for any control policy profile \( a \). Because \( G \) is bounded and continuous, \( ||\mathbb{E} [G(x^* + \sigma B_1)] - G(x^*)|| \leq \epsilon/3 \) holds if \( \sigma^i \) is sufficiently small for each \( i \).

Consider the following control policy profile \( a^i_t \), which is deterministic and constant over time, defined by

\[
a^i_t = x^i - X^i_0.
\]

\(^{\text{17}}\)See for example Section 4.4.1.b in Evans (2010).
This completes the proof.

Then we have

\[
H(\tilde{a}^t) = \mathbb{E} \left[ G(X_0 + \int_0^t a_i^t dt + \sigma B_1) - \int_0^1 \sum_i c^i(a_i^t)dt \right]
= \mathbb{E} [G(x^* + \sigma B_1)] - \int_0^1 \sum_i c^i(a_i^0)dt.
\]

Therefore \( H(a^t) + \frac{\sigma^2}{2t} \geq G(x^*) \), which implies \( \max_a H(\tilde{a}) + \frac{\sigma^2}{2t} \geq G(x^*) \). Further, by the above lemma,

\[
\left| \max_a H(\tilde{a}) - \mathbb{E}[G(X_1)] \right| \leq \left| \mathbb{E}[\int_0^1 \sum_i c^i(a_i^*)] \right| \leq \epsilon/3.
\]

This proves the claim that \( \mathbb{E}[G(X_1)] + \epsilon \geq G(x^*) \).

\[\Box\]

### B.6 Proof of Proposition 3

**Proof.** Let \( L_c := \inf_{i,a^t}(c^i)''(a^t) \), \( L_u := \sup_{i,j,k,x} \left| \frac{\partial^2 U^i(x)}{\partial x_j \partial x_k} \right| \). Take \( n \) large enough so that \( \frac{L_c}{L_u} < n \).

We define the payoff relevant state variable process \( X_t \) as in the main model. Consider the last period \( t = \frac{n-1}{n} \) and take any history \( (B_s)_{0 \leq s \leq \frac{n-1}{n}} \). Given the opponents’ choices \( (a^t_j)_{j \neq i} \), player \( i \)'s optimal action \( a^t_i \) maximizes

\[
-c^i(a^t_i)\frac{1}{n} + \mathbb{E} \left[ U^i((x^j + a^t_i/n + \epsilon^t_j)_{j \in N}) \right]
\]

where each \( \epsilon^t_j \) independently follows the normal distribution \( N(0, \frac{\sigma^2}{n}) \). Thus each \( a^t_i \) must satisfy

\[
a^t_i = ((c^i)')^{-1} \left( \mathbb{E} \left[ \frac{\partial U^i}{\partial x^t}((x^j + a^t_i/n + \epsilon^t_j)_{j \in N}) \right] \right) =: g^t((a^t_j)_{j \in N}). (14)
\]

Letting \( g = (g^t)_{t \in N} \), it defines a map \( g : \prod_i \mathcal{A}^t \to \prod_i \mathcal{A}^t \). We claim that there is a unique fixed point of \( g \). To see this, note that the partial derivative of the \( g^t \) in \( a^t_i \), is

\[
\frac{\mathbb{E} \left[ \frac{\partial U^i}{\partial x^t}((x^j + a^t_i/n + \epsilon^t_j)_{j \in N}) \right]}{n(c^i)'' \left( \mathbb{E} \left[ \frac{\partial U^i}{\partial x^t}((x^j + a^t_i/n + \epsilon^t_j)_{j \in N}) \right] \right)},
\]

which is bounded by \( \frac{L_c}{nL_u} < 1 \) in absolute. Thus \( g \) is a contraction map.

In general, the same argument can be used to show that at any period \( t \) after any history, there can be at most one action profile \( (a^t_j)_{j \in N} \) that satisfies the players’ optimization incentives. This completes the proof.

\[\Box\]

### References


