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RULING OUT DIVERGENT SPECULATIVE BUBBLES*

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The growth in the empirical literature on testing for *divergent* (in mean) speculative bubbles indicates that existing theoretical arguments for ruling out such price paths are not sufficiently compelling. Here we strengthen the case for ruling out explosive or implosive bubbles *a priori* by filling in two gaps. First, we demonstrate why divergent stochastic bubbles arising from purely extrinsic uncertainty [as in Blanchard (1979)] cannot be equilibria in a monetary optimizing model. Second, we show why existing arguments for ruling out implosive price bubbles are insufficient in some cases, and we provide stronger arguments.

1. Introduction

This paper is concerned with ruling out divergent (explosive or implosive) asset price bubbles. We refer to a bubble as divergent if the asset price series exhibits a higher order of non-stationarity than any of the underlying fundamentals. [Diba and Grossman (1984) provide this characterization.] Among all the various types of bubbles identified by theorists, divergent bubbles are in some sense a narrow and restrictive class.[1] However, such bubbles are very important in the empirical literature, mainly because divergent bubbles have clearly identifiable characteristics.[2] Non-divergent bubbles, on the other hand,

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[1] See, for example Azariadis (1981), Cass and Shell (1983), Diamond and Dybvig (1983), Farmer and Woodford (1984), and Waldo (1985). Tirole (1985) provides an example of an equilibrium in which the price level falls at the rate of interest even though the money supply is constant. We would not characterize his example as a divergent bubble [in the sense of Diba and Grossman (1984)], because the deflation is caused by population growth (which is also a fundamental).

[2] Econometricians have tested for asset price bubbles across a broad range of markets and historical episodes, including the gold market, the stock market, the foreign exchange market, and the domestic currency market during episodes of hyperinflation. See, for example, Blanchard and Watson (1982), Calvo (1978), Diba and Grossman (1984), Flood and Garber (1980), Meese (1986), West (1984a, b), and Woo (1984). Some of these studies investigate non-explosive as well as explosive bubbles.

are obviously difficult to distinguish empirically from other omitted variables. This is not to say that it is easy to positively identify divergent bubbles, either. Indeed, anticipated future disturbances can generate price paths which are qualitatively indistinguishable from divergent bubble paths. [Hamilton and Whiteman (1985) point out that bubbles and omitted variables are observationally equivalent, and present a particular formalization of that idea. Flood and Garber (1980) have also noted the observational equivalence of divergent bubbles and anticipated disturbances.]

Our purpose here is to strengthen the case for interpreting all empirical 'evidence' of divergent asset price bubbles as evidence of omitted variables. The present analysis builds on Obstfeld and Rogoff (1983), filling in some small but significant gaps. In our 1983 paper we demonstrated that explosive price bubbles are theoretically possible only in models of pure fiat money. Explosive bubbles are not equilibria if there is some (possibly very small) probability that the government will provide some (possible very small) real backing for the currency. One gap in our earlier analysis is that we did not analyze the case of bubbles arising from purely extrinsic uncertainty. Stochastic bubbles of this type were analyzed by Blanchard (1979). In Blanchard's aggregative model, there exist price bubble paths which are divergent in mean even though the bubble bursts with probability one. 'Extrinsic stochastic' bubbles might be appealing to those who believe that bubbles do exist in the real world, but they always burst eventually. We demonstrate here that bubbles very similar to those described by Blanchard can indeed arise in a micro-based model, but only *if* the monetary regime is one with pure fiat money. (The stochastic bubbles derived from our optimizing model do not burst with probability one. Instead, there is a positive probability that the real value of money will go to zero in finite time.) However, a fractional backing scheme, such as the one discussed in our previous paper, turns out to be sufficient to rule out explosive extrinsic stochastic bubbles.

Another problem we address is how to rule out implosive price level bubbles, equilibrium bubble paths along which the price level asymptotes to zero even though the growth rate of the money supply is constant. We strengthen existing arguments for ruling out implosive bubbles by providing a new necessary condition for equilibrium. In particular, we derive a condition on preferences sufficient to preclude implosive bubbles whenever the money growth rate is non-negative. Our condition is implied by the weak requirement that the derived utility from holding real balances to reduce transaction costs be bounded from above. Almost any reasonable transactions technology will obey this requirement. [See, for example, Feenstra (1984).][3]

[3] Brock (1974, pp. 760–764) gives a condition on $v(\cdot)$ sufficient to rule out implosive equilibrium price paths when money growth is non-negative. Our 1983 paper gives the impression that when the money stock is constant, no condition on preferences beyond a standard Inada condition is needed to preclude imploding bubbles, but this is not the case. The condition we derive here is somewhat more general than Brock's (see footnote 10 below) and is necessary as well as sufficient in the case where the money growth rate $\mu = 0$.

Two caveats regarding our analysis should be stressed. First, we use a representative agent framework; some types of bubble can arise only when agents are heterogeneous. However, the empirical literature we are addressing is not explicitly based on heterogeneous agent models, either. Second, we repeat that our analysis cannot be used to rule out bubbles which do not diverge in mean (and therefore do not violate any transversality or boundary condition).

2. Explosive stochastic bubbles

Blanchard (1979) has suggested a class of stochastic bubbles driven by extrinsic uncertainty. [See also Shiller (1978), or Flood and Garber (1980).] Along the explosive price paths analyzed by Blanchard, there is a non-zero probability each period that the price level will return to its stationary saddle path value. In this section, we show how explosive stochastic price level bubbles can arise in Brock's (1974, 1975) maximizing model of pure fiat money. Like non-stochastic bubbles, these too can be ruled through the fractional backing scheme discussed in our earlier paper.

The economy is one in which individuals receive y units of the perishable consumption good each period. Let c_t denote the representative individual's consumption at time t, M_t her nominal money holdings, and $\beta \equiv (1+\delta)^{-1}$ her subjective discount rate. If the operator $\mathrm{E}_t\{\cdot\}$ yields mathematical expectations conditional on time t information, then the infinitely lived individual agent's problem is to maximize

$$\mathrm{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\left[u(c_t)+v(M_t/P_t)\right]\right\}, \tag{1}$$

subject to

$$M_t - M_{t-1} = P_t(y-c_t) + H_t; \tag{2}$$

the individual agent treats the paths of the price level $\{P_t\}_{t=0}^{\infty}$ and nominal transfers $\{H_t\}_{t=0}^{\infty}$ as exogenous. In eq. (1), $u(\cdot)$ and $v(\cdot)$ are increasing, strictly concave, and have the usual smoothness properties. Further, $v(\cdot)$ satisfies the Inada conditions

$$\lim_{m\to 0} v'(m) = \infty, \qquad \lim_{m\to\infty} v'(m) = 0. \tag{3}$$

Initial individual money holdings M_{-1} are given.

The Euler equation characterizing an optimal path for the individual is

$$\frac{u'(c_t) - v'(M_t/P_t)}{P_t} = \beta \mathrm{E}_t\left\{ \frac{u'(c_{t+1})}{P_{t+1}} \right\}. \tag{4}$$

In equilibrium, desired consumption c_t must equal aggregate output y in each period, and nominal money demand must equal the money supply. The latter is assumed to be non-stochastic and (for simplicity) constant at level M. Accordingly, $H_t \equiv 0$. Let m_t stand for real balances M_t/P_t. Multiplying (4) by M and substituting y for c_t, we obtain

$$m_t[u'(y) - v'(m_t)] = \beta u'(y)\mathrm{E}_t\{m_{t+1}\}. \tag{5}$$

In a rational expectations equilibrium, the stochastic process $\{m_t\}_{t=0}^{\infty}$ must satisfy (5), where the conditional expectation $\mathrm{E}_t\{\cdot\}$ is taken with respect to the actual probability distribution of the price level.

The model as stated contains no intrinsic uncertainty. Under certainty, there is a unique positive stationary level of real balance $\overline{m}$ satisfying (5) [so $v'(\overline{m}) = (1-\beta)u'(y)$]. When real balances are constant at $\overline{m}$, the economy is on its saddle path. Following Blanchard (1979), we introduce extrinsic uncertainty as follows: Assume that if real balances are $\overline{m}$, then agents expect next period's price level to be $\overline{P} = M/\overline{m}$ with probability one. If the economy is on an explosive bubble path with $m_t < \overline{m}$, then agents expect that the price level will return ('crash') to its saddle path level with probability π, but will continue on the bubble path with probability $1-\pi$. By (5), this means that given m_t, the value of m_{t+1} that prevails if a crash does not occur satisfies

$$m_t[u'(y) - v'(m_t)] = \beta u'(y)[\pi\overline{m} + (1-\pi)m_{t+1}]. \tag{6}$$

Before the foregoing price process can define an equilibrium path for the pure fiat money economy, a final assumption is needed: If $m_t = 0$, then $m_{t+1} = 0$ with probability one. This assumption turns out to be necessary for the existence of stochastic price bubble paths, but it also implies that such bubble paths do not burst with probability one as in Blanchard's model.

Eq. (6) leads to a diagram that shows the positive (feasible) realizations of m_t consistent with intertemporal rational expectations equilibrium. Define $A(m) \equiv m[u'(y) - v'(m)]$ and $B(m) \equiv \beta u'(y)[\pi\overline{m} + (1-\pi)m]$. The difference equation (6) can then be written as

$$A(m_t) = B(m_{t+1}). \tag{7}$$

Fig. 1 depicts the dynamics dictated by (7). As in Obstfeld and Rogoff (1983), fig. 1 applies to the case where $\lim_{m\to 0} mv'(m) = 0$, which admits explosive

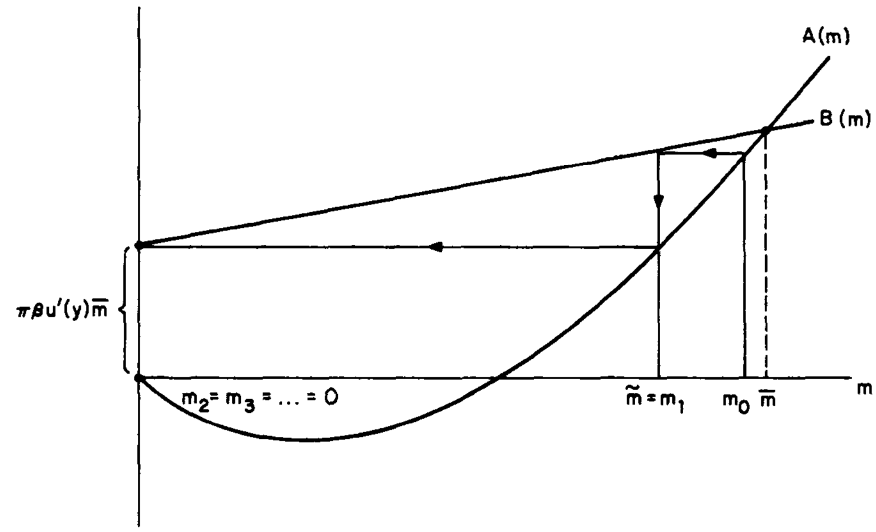

Fig. 1

deterministic bubbles under a pure fiat money regime.[4] One path that satisfies (7) is the saddle path, $m_t = \bar{m}$ for all t.

There are stochastic bubble paths that also satisfy (7). For example, consider the path beginning at m_0 in the figure. The initial price level is $P_0 = M/m_0$. At time zero agents expect a $t = 1$ price level of $\bar{P}$ (with probability π) or $P_1 = M/m_1$ (with probability $1 - \pi$).[5] If no crash occurs at $t = 1$, the price level P_1 is equal to $\tilde{P}$, defined by

$$\frac{u'(y) - v'(M/\tilde{P})}{\tilde{P}} = \frac{\pi\beta u'(y)}{\bar{P}}. \tag{8}$$

At $\tilde{P}$, agents expect real balances to jump to their saddle path value with probability π, or to be zero with probability $1 - \pi$. Eq. (8) implies that individuals have no incentive to reduce their real balances at $m_1 = \tilde{m}$, despite the fact that their money holdings may lose their real value forever. (The

[4] It is easy to see that stochastic as well as deterministic explosive price level bubbles are not equilibria when $\lim_{m \to 0} mv'(m) > 0$. Obstfeld and Rogoff (1983) show that this case is economically unreasonable, since it implies that $v(0) = -\infty$.

[5] Note that the inflation rate conditional on no crash occurring must rise with π to offset the possibility of a sharp capital gain on real balances.

possible capital loss is compensated by the current utility of money in reducing transaction costs and the possible capital gain in the event of a crash.)[6] If P subsequently jumps to $\bar{P}$, the economy remains on the saddle path forever. If P subsequently becomes infinite, the Euler equation becomes (and remains) $[u'(y)-v'(m_t)]/P_t = \beta u'(y)/P_{t+1}$, which is satisfied [given that $\lim_{m\to\infty} mv'(m)=0$] if $P=\infty$ and $m=0$ in all future periods.

It is easy to see that the stochastic bubble equilibria just described cannot occur if the government provides a trivial amount of real backing for the currency. Suppose that the government owns a portion of the aggregate endowment y and rebates its share to the public in the form of lump sum transfers. As in Obstfeld and Rogoff (1983), the government can guarantee a minimal real redemption value for money by promising to back each currency unit by a claim to a small fraction of its endowment. This suffices to rule out as equilibria the bubble paths considered above, since $m=0$ (and $\pi=0$) is no longer an equilibrium.[7]

As we stressed in the introduction, our analysis applies only to divergent bubbles and does not necessarily apply to other types of non-uniqueness problems. It is possible, for example, to develop versions of Brock's model where the steady state is locally stable (the absolute value of the root of the approximating linear difference equation is less than one in the neighborhood of the steady state).[8] If the equilibrium is stable instead of saddle point stable (as in the example presented here), then there is a continuum of initial price levels consistent with equilibrium. All paths tend toward the steady state and none violates any transversality or boundary condition. At present, there is no general method for ruling out such bubbles. However, stable bubble paths do not exhibit the explosive behavior which many empirical studies are aimed at detecting. Indeed, it is not obvious how one could ever positively identify non-explosive bubbles, since they are difficult to distinguish from other omitted variables.

We have not discussed implosive price bubbles in this section because their analysis rests on the same principles as the analysis of implosive nonstochastic bubbles. That subject is taken up next.

[6] If π did not become zero once real balances were zero at $t=2$, there would have to be some probability that real balances would become negative in period $t=3$. Otherwise the equilibrium condition (7) could not be satisfied. But the price level cannot be negative with free disposal. Thus, the stochastic bubble equilibria described here do not crash with probability one as in Blanchard's (1979) aggregative model.

[7] William Brock and Mark Gertler, in unpublished notes, have reached conclusions similar to those reached here. In particular, they show how extrinsic bubbles can be ruled out in a stochastic growth model.

[8] For example, locally stable steady state equilibria can arise when monetary policy is endogenous [as in Black (1974)] or when the instantaneous utility function is not separable [see Gray (1984) or Obstfeld (1984)].

3. Ruling out implosive price level paths

Implosive price bubbles have attracted less attention than explosive bubbles, perhaps because it is commonly thought that they are ruled out by transversality conditions. This presumption is not strictly correct since, as we illustrate below, there are certain apparently plausible cases where implosive bubbles can be equilibria. Fortunately, we are able to show that implosive bubbles cannot arise provided the underlying transactions technology obeys a very weak (and very reasonable) restriction.

Our analysis will be based on a fully deterministic version of the previous section's model. Because the precise rate of money growth is now central to the discussion, we no longer restrict it to be zero as in section 2. A continuous time framework is more convenient for the analysis, and in this case the Euler equation for real balances is

$$\dot{m}_t = [\mu + \delta - v'(m_t)]\, m_t, \tag{9}$$

where μ is the instantaneous growth rate of the nominal money supply and δ is the instantaneous subjective discount rate [see Brock (1974)]. To simplify notation, the utility function has been normalized so that $u'(y) = 1$. It is assumed for now that $\mu \geq 0$.

The positive steady state level of real balances $\overline{m}$ is given by $v'(\overline{m}) = \mu + \delta$. To rule out as a possible equilibrium any solution to (9) with $m_0 > \overline{m}$, it is sufficient to show that such a path violates the transversality condition

$$\lim_{t \to \infty} \mathrm{e}^{-\delta t}/P_t = 0. \tag{10}$$

Since (10) is necessary but not generally sufficient for an equilibrium when $\mu \geq 0$, violation of (10) is sufficient, but not always necessary, to rule out an implosive path. The appendix shows why Euler equation (9) and condition (10) are jointly necessary for an equilibrium under the present assumptions.[9] The following result provides the basis of our analysis:

Theorem 1. A necessary and sufficient condition for an Euler path starting with $m_0 > \overline{m}$ to violate the transversality condition (10) is that the integral (11) below be convergent:[10]

$$\int_{m_0}^{\infty} \frac{v'(m)\mathrm{d}m}{[\mu + \delta - v'(m)]\, m}. \tag{11}$$

[9] See Brock (1974), Gray (1984), or Obstfeld and Rogoff (1983) for intuitive interpretations of the transversality condition.

[10] Brock (1974, p. 760) gives a sufficient condition for the violation of (10) when $\mu \geq 0$, that $v'(m) \leq m^{\lambda}$ ($\lambda < 0$) for m sufficiently large. It is straightforward to confirm that for any function which meets Brock's condition (e.g., any member of the constant relative risk aversion family), (11) converges.

Comment. Theorem 1 is useful because the integral expression (11) is easier to interpret than the transversality condition (10). See Theorem 2 below.

Proof. Along an Euler path with $m_0 > \bar{m}$, (9) implies that

$$m_t = m_0 \exp\left((\mu + \delta)t - \int_0^t v'(m_s)\mathrm{d}s\right). \tag{12}$$

Because $m_t = \mathrm{e}^{\mu t} M_0 / P_t$, condition (10) is equivalent to

$$\lim_{t \to \infty} \exp\left(-\int_0^t v'(m_s)\mathrm{d}s\right) = 0, \tag{13}$$

which holds if and only if

$$\lim_{t \to \infty} \int_0^t v'(m_s)\mathrm{d}s = \infty. \tag{14}$$

Now change variables from s to m in (14), using the fact that, by (9), $\mathrm{d}m = [\mu + \delta - v'(m)]m\,\mathrm{d}s$ along the path we are studying. Then (14) holds along this path if and only if the utility of money function $v(\cdot)$ satisfies

$$\int_{m_0}^{\infty} \frac{v'(m)\mathrm{d}m}{[\mu + \delta - v'(m)]m} = \infty. \tag{15}$$

It follows that (10) is violated if and only if (11) converges. This completes the proof.

As was noted above, convergence of (11) is sufficient to rule out implosive bubbles when $\mu \geq 0$. But when $\mu = 0$, transversality condition (10) is sufficient, as well necessary, for an equilibrium (see the appendix). In this case, therefore, divergence of (11) is necessary and sufficient for an imploding price level path satisfying (9) to be an equilibrium.

The following is a case in which $\mu = 0$ and (11) diverges, so that implosive Euler paths are equilibria.

Example.[11] Suppose that $\mu = 0$ and that for large m, $v'(m) = 1/\log(m)$. After substitution for $v'(m)$ in (11), we obtain the inequality

$$\int_{m_0}^{\infty} \frac{\mathrm{d}m}{m[\delta \log(m) - 1]} > \int_{m_0}^{\infty} \frac{\mathrm{d}m}{\delta m \log(m)}, \tag{16}$$

which follows because $\delta \log(m) > 1$ when $m_0 > \bar{m}$ and m is given by (9). It is

[11] Guillermo Calvo and Roque Fernandez suggested this example.

therefore enough to prove that the right-hand side of (16) is divergent. But

$$\int_{m_0}^{\infty} \frac{\mathrm{d}m}{m \log(m)} = \log[\log(\infty)] - \log[\log(m_0)] = \infty. \tag{17}$$

Thus (10) is satisfied along this Euler path.

Fortunately, equilibrium implosive price paths are not a problem when $\mu \geq 0$. The following calls into question the *economic* relevance and interpretation of any implosive price path along which transversality condition (10) holds:

Theorem 2. If (10) holds along an implosive price path, then $\lim_{m \to \infty} v(m) = \infty$.

Proof. By the concavity of $v(\cdot)$, for $m > \bar{m}$, $[v(m) - v(\bar{m})]/(m - \bar{m}) \geq v'(m)$. Therefore

$$\int_{m_0}^{\infty} \frac{[v(m) - v(\bar{m})]\mathrm{d}m}{(m - \bar{m})m[\mu + \delta - v'(m)]} \geq \int_{m_0}^{\infty} \frac{v'(m)\mathrm{d}m}{[\mu + \delta - v'(m)]m}, \tag{18}$$

where the right-hand side of (18) equals the integral (11). If there exists a finite B such that for all m, $v(m) \leq B$, then the left-hand side of (18) is bounded by

$$\int_{m_0}^{\infty} \frac{[B - v(\bar{m})]\mathrm{d}m}{(m - \bar{m})m[\mu + \delta - v'(m)]}. \tag{19}$$

The integral (19) is convergent because $\mu + \delta - v'(m)$ approaches $\mu + \delta$ monotonically as $m \to \infty$ and $\mu + \delta - v'(m_0) > 0$. Therefore (11) can diverge only if $v(m)$ is unbounded as $m \to \infty$. So if $v(m)$ is bounded, Theorem 1 tells us that (10) cannot hold.

No plausible transactions technology can yield the result that the derived utility of money increases without bound for a fixed consumption level even if $v'(m) > 0$ for all finite m. We therefore conclude that implosive bubbles cannot reasonably occur in this model when $\mu \geq 0$.[12]

What about cases where $\mu < 0$? These are relevant to discussions of the 'optimum quantity of money'. It turns out that imploding bubbles can be

[12] The condition given in Theorem 2 is necessary, but not sufficient, for an implosive bubble path to be an equilibrium path when $\mu \geq 0$. The transversality condition (10) is violated for all $v(\cdot)$ in the constant relative risk aversion class (see footnote 10), and some of these are unbounded as $m \to \infty$. Of course if $\lim_{m \to \infty} v(m) = \infty$, the individual's objective function (1) may be unbounded for price paths $\{P_t\}$ such that $P_t \to 0$. This would require reformulation of the individual's problem in terms of an 'overtaking' principle.

equilibria in these cases even if (10) does not hold.[13] The reason why (10) is no longer necessary is explained by Brock (1974, p. 764, and 1975, p. 145), but it is worth repeating. Transversality condition (10) ensures that an agent cannot increase her utility by reducing her nominal balances permanently by a dollar, provided this is feasible (see the appendix). But this reduction is simply not feasible when nominal balances are declining toward zero along the initial Euler path. The dollar would eventually have to be repurchased to keep money holdings non-negative, and the cost of this transaction would, by (9), nullify the initial gain in instantaneous utility. (This problem goes away if monetary growth, though currently negative, goes to zero at some finite time in the future.)

To summarize, a very weak preference restriction prevents imploding bubbles when $\mu \geq 0$, but this restriction is inapplicable when $\mu < 0$. Note, however, that a government promise to place a floor on the money price of output by unlimited purchases of goods with money at a sufficiently low P always prevents the emergence of imploding bubbles. And the government will never have to exercise its guarantee. Placing a ceiling on the real value of money is the mirror image of the threatened intervention that precludes exploding bubbles in Obstfeld and Rogoff (1983). Wallace (1981) studies a similar scheme in an overlapping generations economy.

4. Conclusion

In this paper we have provided further microeconomic justification for ruling out divergent price bubbles, stochastic and otherwise. Given the strong a priori grounds for ruling out such bubbles, it would appear that it is not meaningful to test empirically for their presence. How can one then interpret the existing empirical literature that tests for divergent bubbles? Consider once again the model of section 2, and assume there are no bubbles. Assume instead that in period t, agents know there will be an election at the end of period $t+3$. Furthermore, agents know that if the current government stays in power, the money supply will remain forever at its current level $\overline{M}$; but the money supply will be $2\overline{M}$ from period $t+4$ onward if the opposition is elected at the end of $t+3$. The opposition party is expected to win with an exogenous probability π. What does the path of the price level look like if, ex post, the incumbent party wins the election and the money supply never changes? It is easy to show that the price level will follow a path such as the one depicted in fig. 2.

[13] Given our assumption that $v'(\cdot) > 0$, $\mu + \delta > 0$ is required for the existence of equilibrium. Brock (1974, 1975) discusses the possibility that $v'(\cdot) \leq 0$ for sufficiently large m. Note that if $v'(m) = 0$ for some finite m, (10) cannot hold along an implosive Euler path for $\mu \geq 0$.

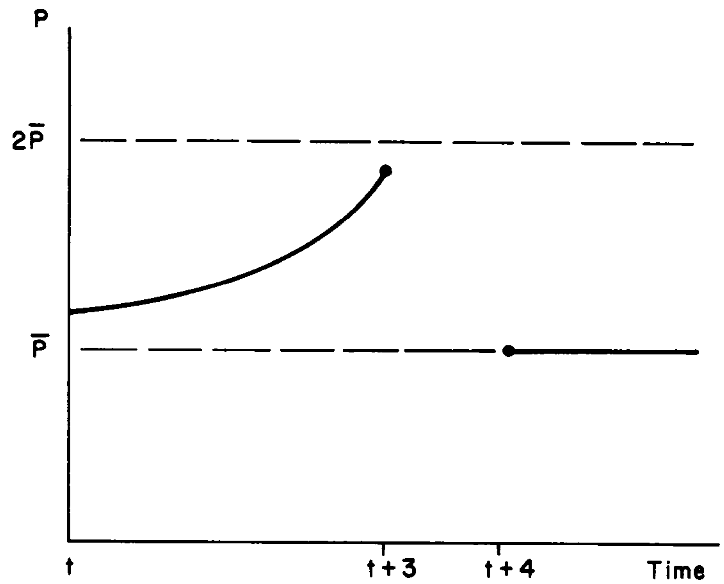

Fig. 2

An econometrician unaware of the election's significance for the stochastic properties of the money supply process might mistake the price path in fig. 2 for evidence of an extrinsic price bubble, one that bursts in period $t+4$. However, the analysis of this paper suggests that one should not adopt that interpretation. In the absence of any observed change in the money supply, a price path like the one in fig. 2 should be interpreted as evidence of a market fundamental that cannot be observed by the econometrician – an omitted variable.[14]

Appendix

This appendix establishes that when monetary growth is non-negative, the condition

$$\lim_{t\to\infty} \beta^t/P_t = 0 \tag{A.1}$$

is necessary for a price level path $\{P_t\}$ to be an equilibrium. (The discrete time version of the model is used to aid intuition.) Brock (1974) shows that the

[14]Again, see Hamilton and Whiteman (1985).

condition

$$\lim_{t\to\infty} \beta^t m_t = 0 \tag{A.2}$$

is *sufficient* for an equilibrium.[15] (A.1) is equivalent to (A.2), and thus necessary *and* sufficient, when monetary growth is zero. Clearly (A.2) implies (A.1) if the money stock is growing at a non-negative rate, but the converse is not true in general.

Consider the deterministic version of the first-order condition (4). Iterating this relation forward yields

$$u'(c_t)/P_t = \sum_{i=0}^{\infty} \beta^i v'(M_{t+i}/P_{t+i})/P_{t+i} + \lim_{T\to\infty} \beta^T u'(c_{t+T})/P_{t+T}. \tag{A.3}$$

If markets clear, so that $c_{t+i}=y$ and $M_{t+i}=M^s_{t+i}$ (the money supply) for all i, it must be true that

$$u'(y)/P_t = \sum_{i=0}^{\infty} \beta^i v'(M^s_{t+i}/P_{t+i})/P_{t+i} + \lim_{T\to\infty} \beta^T u'(y)/P_{t+T}. \tag{A.4}$$

Assume that

$$\lim_{T\to\infty} \beta^T/P_{t+T} > 0. \tag{A.5}$$

We will show that (A.5) implies $\{P_t\}$ is not an equilibrium, as a typical agent can raise her utility as follows: Consume $P_t e$ dollars at time t, reducing nominal balances by this amount in period t and all succeeding periods. Note that this is always a *feasible* deviation, for e sufficiently small, if M^s is not shrinking to zero over time. The deviation causes an immediate consumption gain that *exceeds* $u'(y+e)e$ (by concavity). It causes a lifetime loss of liquidity services *less than*

$$\sum_{i=0}^{\infty} \beta^i v'[(M^s_{t+i} - P_t e)/P_{t+i}](P_t e/P_{t+i}). \tag{A.6}$$

[15] It is easy to see that this sufficient condition never holds for imploding Euler paths with $\mu > 0$, and always holds when $\mu < 0$.

It is therefore certainly true that if there is a positive e such that

$$u'(y+e)/P_t > \sum_{i=0}^{\infty} \beta^i v'[(M^s_{t+i} - P_t e)/P_{t+i}]/P_{t+i}, \tag{A.7}$$

$\{P_t\}$ cannot be an equilibrium path. This conclusion is also reached by Brock (1974, p. 763). But (A.4) and (A.5) imply that

$$u'(y)/P_t > \sum_{i=0}^{\infty} \beta^i v'(M^s_{t+i}/P_{t+i})/P_{t+i}. \tag{A.8}$$

At $e=0$, $u'(y+e)$ is decreasing in e and $v'[(M^s_{t+i} - P_t e)/P_{t+i}]$ is increasing in e (for all i); but the continuity of these functions ensures that if (A.8) holds, we can find a positive e small enough that (A.7) holds as well. It follows by contradiction that (A.1) is a necessary condition of equilibrium when money growth is non-negative.

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