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# THE CONSISTENT APPLICATION OF BOUNDARY CONDITIONS IN RATIONAL EXPECTATIONS MODELS

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When boundary conditions are applied consistently, real variables are invariant to the unit of account in both backward looking and forward looking solutions to monetary models of inflation with rational expectations.

Choosing an appropriate solution to monetary models of inflation that postulate either perfect foresight or rational expectation formation is a problem that continues to receive much attention. Blanchard (1979) noted that any weighted average of the so called 'backward looking' and 'forward looking' solutions is also a solution to such models. He went on to examine economic considerations for placing all of the weight on the forward solution; he did not find them compelling.

More recently, Flood and Garber (1978) have come up with a new argument in favor of the forward looking solution (in cases in which the model is forward stable). They suggest as a reasonable economic principle that the equilibrium values of real variables should be invariant to changes in the unit of account. For example, changing the unit of account from dollars to dimes should not affect the time path of real money balances. They contend that the backward looking solution (and therefore any solution placing weight upon it) violates this principle.

Below, we specify explicit boundary conditions and apply them consistently as we change the unit of account. We show that if the boundary conditions are consistently applied, then contrary to the results of Flood and Garber, the time path of real money balances is invariant to a change in the unit of account in both forward and backward looking solutions. This re-establishes the force of Blanchard's analysis.

Flood and Garber consider the following simple monetary model of inflation:

$$m_t - p_t = -\alpha(p_{t+1} - p_t), \qquad \alpha > 0, \tag{1}$$

where $m$ is the logarithm of the nominal money supply, and $p$ is the logarithm of

* This paper represents the views of the authors and should not be interpreted as reflecting the views of the Federal Reserve System.

the price level. Perfect foresight has been imposed, as for our purposes it is not essential to make the model explicitly stochastic.

Eq. (1) can be solved by iterating backward with

$$p_t = \psi p_{t-1} - \alpha^{-1} m_{t-1}, \qquad \psi \equiv (1+\alpha)/\alpha > 1, \tag{2}$$

or by iterating forward with

$$p_t = \psi^{-1} p_{t+1} + (1+\alpha)^{-1} m_t . \tag{3}$$

The backward and forward looking solutions are then defined as

$$p_t^{\mathrm{BL}} = \lim_{T\to\infty} (\psi^T p_{t-T} - \alpha^{-1}(m_{t-1} + \ldots + \psi^{T-1} m_{t-T})), \tag{4}$$

and

$$p_t^{\mathrm{FL}} = \lim_{T\to\infty} (\psi^{-T} p_{t+T} + (1+\alpha)^{-1}(m_t + \ldots + \psi^{-T+1} m_{t+T-1})), \tag{5}$$

and as Blanchard noted,

$$p_t^{\lambda} = \lambda p_t^{\mathrm{BL}} + (1-\lambda) p_t^{\mathrm{FL}} \tag{6}$$

must also be a solution to (1) for any $\lambda$.

The first term on the right-hand side of (4) or (5) is the solution to the homogeneous part of (1). It shows how an initial or terminal position will affect the solution at time $t$. For the backward and forward looking solutions to be uniquely defined, boundary conditions must be specified, [1]

$$\lim_{T\to\infty} \psi^T p_{t-T} = 0 \qquad \text{(backward looking)}, \tag{7}$$

or

$$\lim_{T\to\infty} \psi^{-T} p_{t+T} = 0 \qquad \text{(forward looking)}. \tag{8}$$

Eq. (8) is known in the rational expectations literature as Sargent's 'no speculative bubbles' condition. These terms are explicitly suppressed in the analysis of Flood and Garber and also Blanchard.

In general, one would not expect the infinite sums in both (4) and (5) to converge. The present model is forward stable in the sense that the root $\psi^{-1}$ is less than one. The weights on the $m$'s in the backward solution grow without bound, so the infinite sum in (4) will not converge if, say, the money supply is held constant throughout time. However, there do exist money supply processes for which both

[1] It should be noted that the choice of boundary conditions is not a settled issue; see, for example, Canzoneri (1979) and the references therein. Our present discussion focuses upon the indeterminancy embodied in (6).

infinite sums converge, and the indeterminancy embodied in (6) cannot be resolved on these grounds alone. [2,3]

Flood and Garber suppress the homogeneous terms and consider only the infinite summation terms on the right-hand sides of (4) and (5). The forward looking summation has weights that sum to unity. They reason that only it will generate a path for real balances invariant to the unit of account.

Consider, however, the complete backward looking solution (4), and suppose that the time path $(m_t^{\$}, p_t^{\$})$ satisfies the equilibrium condition (4). Is the path for real money balances, $m_t - p_t$, altered if units are changed from dollars to dimes? Since $m_t^{\mathrm{d}} = m_t^{\$} + \ln 10$ and $p_t^{\mathrm{d}} = p_t^{\$} + \ln 10$, the answer is 'no' if $(m_t^{\$} + \ln 10,$ $p_t^{\$} + \ln 10)$ also satisfies (4). This will be the case if

$$1 = \lim_{T\to\infty} (\Psi^T - \alpha^{-1}(1 + \psi + \psi^2 + \ldots + \psi^{T-1})), \tag{9}$$

and this condition is easily verified. [4]

The argument we have just given does not depend upon any particular set of boundary conditions; eqs. (7) and (8) are consistent with what we did, but they were never referred to explicitly. What we did implicitly assume was that the boundary conditions did not change when we changed the unit of account. Suppose we adopt (7) in dollar terms as the proper boundary condition. Since $\psi^T \to \infty$ as $T \to \infty$, (7) implies

$$\lim_{T\to\infty} p_{t-T}^{\$} = 0, \tag{10}$$

[2] The simplest example is

$$m_t = 0 \quad \text{for} \quad t < t_0,$$
$$= \bar{m} \quad \text{for} \quad t \geqslant t_0,$$

for any arbitrary selection time $t_0$. Flood and Garber give the example

$$m_t = \bar{m}(\psi + 1)^t \quad \text{for} \quad t < t_0,$$
$$= \bar{m} \quad \text{for} \quad t \geqslant t_0.$$

[3] We also note in passing that for the economies we have in mind, there is likely to be an asymmetry between backward and forward iterated solutions. In (2), $p_{t-1}$ and $m_{t-1}$ are historically given data; at time $t$, they are fixed and known. One can simply iterate (2) back to any arbitrary selected date $t_0$ and call the resulting expression a backward solution. It is not necessary for the infinite sum in (4) to converge for the backward solution to have meaning. Convergence of the forward solution is, however, a meaningful way of pinning down our expectations about the future price level.

[4] Verification,

$$\psi^T - \alpha^{-1}(1 + \psi + \ldots + \psi^{T-1}) = \psi^T - \alpha^{-1}(1 - \psi)^{-1}(1 - \psi^T) = \psi^T - (-1)(1 - \psi^T) = 1,$$

where use has been made of the fact that $\psi \equiv (1 + \alpha)/\alpha$. Taking the limit as $T$ goes to $\infty$, we have eq. (9).

and

$$\lim p^{\mathrm{d}}_{t-T} = \lim(p^{\$}_{t-T} + \ln 10) = \ln 10. \qquad (10')$$

In other words, a consistent application of the boundary condition (7) implies that the dollar price goes to one while the dime price goes to ten. And this makes the 'initial' condition (at $-\infty$) for real money balances invariant to changes in the unit of account.

Flood and Garber suppressed the term $\lim_{T\to\infty} \psi^T p_{t-T}$ in the backward looking solution, and, noting that the weights on the $m$'s sum to infinity, concluded that $(m^{\$}_t + \ln 10, p^{\$}_t + \ln 10)$ could not satisfy eq. (4). Implicitly, they were imposing the condition

$$\lim_{T\to\infty} \psi^T p^{\$}_{t-T} = 0 \qquad (11)$$

before the change in units, and

$$\lim_{T\to\infty} \psi^T p^{\mathrm{d}}_{t-T} = 0 \qquad (11')$$

after the change. (11) has the dime price starting at 10, while (11′) has it starting at 1. Clearly, the 'initial' condition for real balances has been changed with the change in units, and this is why their result differs from ours.

This returns the debate to the issues raised in Blanchard.

## References

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