Lyapunov function as potential function: A dynamical equivalence*

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For a physical system, regardless of time reversal symmetry, a potential function serves also as a Lyapunov function, providing convergence and stability information. In this paper, the converse is constructively proved that any dynamics with a Lyapunov function has a corresponding physical realization: a friction force, a Lorentz force, and a potential function. Such construction establishes a set of equations with physical meaning for Lyapunov function and suggests new approaches on the significant unsolved problem namely to construct Lyapunov functions for general nonlinear systems. In addition, a connection is found that the Lyapunov equation is a reduced form of a generalized Einstein relation for linear systems, revealing further insights of the construction.

Keywords: Lyapunov function, potential function, stochastic process, generalized Einstein relation

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1. Introduction

The Lyapunov function[1–6] has been widely applied in engineering for stability analysis. Constructing such functions for general nonlinear systems is of great theoretical and practical interests,[3,5] but still a challenge that would require one's "divine inspiration" in application.[7] A known fact for this problem is that in an engineering system, a quantity has the meaning of energy or Hamiltonian is usually a Lyapunov function of the system, but two points remain unclear. What is the exact definition of this quantity in the whole phase space? Can every Lyapunov function have a corresponding physical meaning?

The Lyapunov function discussed in this paper is a natural generalization of its conventional version, similar to the definition in Ref. [8]. It enables us to do qualitative analysis of complex dynamical behaviors far from equilibrium (e.g., multi-stable states and periodic attractors) which are ubiquitous in real systems,[9,10] but beyond the scope of the conventional definition.

Research on a fundamental concept in physics, the potential function, has been motivated recently by uncovering local and global principles of complex dynamics in biology,[11–15] physics,[16–19] and control theory.[20,21] Efforts have also been made when it is difficult to obtain potential function, such as the development of quasi-potential methods.[22,23] One of the present authors proposed a general construction of potential functions for stochastic dynamics.[24,25] It was initially formulated during the study of the robustness of a stochastic switch.[26] The potential function in this framework plays the role of the "Hamiltonian", leading to the Boltzmann–Gibbs distribution as the steady state distribution of a stochastic dynamics.

There has been a growing interest in interdisciplinary study among control theory, physics, and biology.[27–30] We demonstrate in this paper that the Lyapunov function is equivalent to the potential function[24] of a system, a result connects engineering to physics and indicates new approaches for constructing Lyapunov functions.

2. Equivalence between Lyapunov function and potential function

To avoid unnecessary mathematical complication, here we only consider smooth dynamics. The present results can be directly extended to more general cases. The definition of the conventional Lyapunov function for a smooth dynamical system is given by

$$\dot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

Let $\boldsymbol{q}^*$ be a fixed point for the system and $L : \mathcal{O} \to \mathbb{R}$ a continuously differentiable function defined on an open set $\mathcal{O}$ containing $\boldsymbol{q}^*$. Then, L satisfying the following conditions is called a

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conventional Lyapunov function:

$$\begin{aligned}&\text{(i)}\ \ \dot{L}(\boldsymbol{q})=\frac{\mathrm{d}L}{\mathrm{d}t}|_{\boldsymbol{q}}\leqslant 0,\quad \text{for all}\quad \boldsymbol{q}\in\mathcal{O};\\ &\text{(ii)}\ \ L(\boldsymbol{q}^{*})=0\quad\text{and}\quad L(\boldsymbol{q})>0,\quad\text{if}\quad \boldsymbol{q}\neq\boldsymbol{q}^{*}.\end{aligned}$$

To generalize this local definition, we have the intuition that the value of the function should decrease along the trajectories, reaching a local minima at attractors and keeping constant on them. Therefore, the criterion (i) is essential and should be satisfied in the whole phase space. Boundary conditions are also essential for attractors. Here, we take a weaker form of (ii): $\nabla L(\boldsymbol{q}^{*})=0$ for all fixed points $\boldsymbol{q}^{*}$, and provide a definition.

Lyapunov function Let ψ be a continuously differentiable scalar function from the phase space to $\mathbb{R}$. Then ψ satisfying the following conditions is called a Lyapunov function.

$$\begin{aligned}&\text{(i)}\ \ \dot{\psi}(\boldsymbol{q})=\frac{\mathrm{d}\psi}{\mathrm{d}t}|_{\boldsymbol{q}}\leqslant 0,\quad\text{for all}\quad \boldsymbol{q}\ \text{in the phase space;}\\ &\text{(ii) Conditions for attractors, here we use}\\ &\quad \nabla\psi(\boldsymbol{q}^{*})=0\quad\text{for all}\quad \boldsymbol{q}^{*},\quad \dot{\boldsymbol{q}}=\boldsymbol{f}(\boldsymbol{q}^{*})=0.\end{aligned}$$

The Lyapunov function is equivalent to the potential function obtained from the following proposed canonical form.

2.1. A physical point of view on dynamical system

The evolution of a deterministic dynamical system described by a set of differential equations can be considered as a massless particle moving along the trajectories inside the phase space. We can assume the particle is charged as well. From a physical point of view, it is natural to explain the motion of this particle as a consequence of an underlying driving force where $\boldsymbol{F}_{\text{driving}}=m\ddot{\boldsymbol{q}}=0$. This force can be separated generally in physics into a dissipative and a conservative part $\boldsymbol{F}_{\text{driving}}=\boldsymbol{F}_{\text{conservative}}+\boldsymbol{F}_{\text{dissipative}}=0$. Without loss of generality, we use a frictional force to represent the dissipative part $\boldsymbol{F}_{\text{dissipative}}=-\boldsymbol{S}\dot{\boldsymbol{q}}$ and a Lorentz force together with an energy induced force as the conservative part $\boldsymbol{F}_{\text{conservative}}=e\dot{\boldsymbol{q}}\times\boldsymbol{B}+[-\nabla\psi(\boldsymbol{q})]$, thus $-\boldsymbol{S}\dot{\boldsymbol{q}}+e\dot{\boldsymbol{q}}\times\boldsymbol{B}-\nabla\psi(\boldsymbol{q})=0$ where $\boldsymbol{S}$ is symmetric and semi-positive definite.

The semi-positive definite requirement for $\boldsymbol{S}$ guarantees the resistance of the frictional force whose valid values are restricted to the negative half space. The potential function ψ here serves as an indicator demonstrating the influence of the other two forces onto the energy of the system. It is apparent that the energy induced force is equal and opposite everywhere to the resultant of the other forces as such

$$\boldsymbol{S}\dot{\boldsymbol{q}}+e\boldsymbol{B}\times\dot{\boldsymbol{q}}=-\nabla\psi(\boldsymbol{q}).\tag{2}$$

The work done by the frictional force is then the reduced amount of ψ.

However, a problem occurs for systems whose dimension is higher than 3, since the cross product $\boldsymbol{B}\times\dot{\boldsymbol{q}}$ is undefined. In order to generalize Eq. (2) to be valid for arbitrary n-dimensional systems, we introduce a generalized form of this vector-valued cross product $\boldsymbol{B}\times\dot{\boldsymbol{q}}$ as $\boldsymbol{T}\dot{\boldsymbol{q}}$, where $\boldsymbol{T}$ is an antisymmetric matrix. This definition is consistent with the three-dimensional case, since $\boldsymbol{B}\times\dot{\boldsymbol{q}}=\boldsymbol{T}\dot{\boldsymbol{q}}$ when $T_{ij}=-\varepsilon_{ijk}B_k$ and ε_{ijk} is the Levi–Civita symbol.

Hence, from the above heuristic reasoning, by setting $e=1$, we reach a general equation

$$[\boldsymbol{S}(\boldsymbol{q})+\boldsymbol{T}(\boldsymbol{q})]\,\dot{\boldsymbol{q}}=-\nabla\psi(\boldsymbol{q}),\tag{3}$$

where $\boldsymbol{S}(\boldsymbol{q})$ is symmetric ($\boldsymbol{S}=\boldsymbol{S}^{\mathrm{T}}$) and semi-positive definite and $\boldsymbol{T}(\boldsymbol{q})$ is antisymmetric ($\boldsymbol{T}=-\boldsymbol{T}^{\mathrm{T}}$). Equation (3) is referred to as the Canonical Form which induces $n(n-1)/2$ equations (utilizing the matrix-valued cross product defined later)

$$\nabla\times\{[\boldsymbol{S}(\boldsymbol{q})+\boldsymbol{T}(\boldsymbol{q})]\,\boldsymbol{f}(\boldsymbol{q})\}=0.\tag{4}$$

Note that $\nabla\times\nabla\psi(\boldsymbol{q})=[(\partial_i\partial_j-\partial_j\partial_i)\psi(\boldsymbol{q})]_{n\times n}=0$.

In Refs. [21] and [31], a reciprocal form is frequently mentioned (known as standard form)

$$\dot{\boldsymbol{q}}=-[\boldsymbol{D}(\boldsymbol{q})+\boldsymbol{Q}(\boldsymbol{q})]\,\nabla\psi(\boldsymbol{q}),\tag{5}$$

where $\boldsymbol{D}(\boldsymbol{q})$ is symmetric and semi-positive definite, and $\boldsymbol{Q}(\boldsymbol{q})$ is antisymmetric. Interestingly, there is a connection with physical meaning between $\boldsymbol{S}(\boldsymbol{q})$, $\boldsymbol{T}(\boldsymbol{q})$ and $\boldsymbol{D}(\boldsymbol{q})$ when noise exists. For deterministic systems, it is hold under the weak noise limit. This connection is characterized by the generalized Einstein relation[24,25]

$$[\boldsymbol{S}(\boldsymbol{q})+\boldsymbol{T}(\boldsymbol{q})]\boldsymbol{D}(\boldsymbol{q})[\boldsymbol{S}(\boldsymbol{q})-\boldsymbol{T}(\boldsymbol{q})]=\boldsymbol{S}(\boldsymbol{q}).\tag{6}$$

This gives the other $n(n+1)/2$ equations. Equations (4) and (6) provide totally n^2 equations. There are n^2 unknowns in $[\boldsymbol{S}(\boldsymbol{q})+\boldsymbol{T}(\boldsymbol{q})]$ as well. For a chosen $\boldsymbol{D}(\boldsymbol{q})$ with proper boundary conditions, we can solve Eqs. (4) and (6) to obtain $[\boldsymbol{S}(\boldsymbol{q})+\boldsymbol{T}(\boldsymbol{q})]$. Then, $\psi(\boldsymbol{q})$ can be derived from Eq. (3).

Equations (3)–(6) can be derived through the treatment of continuous stochastic processes introduced in Ref. [24], where $\boldsymbol{D}(\boldsymbol{q})$ is the diffusion matrix and $\boldsymbol{S}(\boldsymbol{q})$ is the friction matrix. Results for discrete processes without detailed balance are recently proposed as well.[32] Generalized Einstein relation (6) demonstrates the connection between friction and diffusion. If the detailed balance condition is satisfied ($\boldsymbol{T}=0$ or $\boldsymbol{Q}=0$) and the diffusion is position independent, this relation reduces to the Einstein relation.[33] When detailed balance is broken ($\boldsymbol{T}(\boldsymbol{q})\neq 0$[34,35]), this relation still holds and the potential function can be obtained. Note that we now have two different views considering the results presented in this paper. From the stochastic point of view, we know that for a deterministic system, there also exists an undefined (by its differential equations) random driving force (determined by the diffusion matrix $\boldsymbol{D}(\boldsymbol{q})$, the strength is zero, under weak noise limit) that has the common origin with its frictional force (if the system is conserved, the friction is zero), like the stochastic cases discussed in Refs. [36] and [37]. For stochastic systems, under the treatment,[24] the following results can be applied as well.

2.2. A constructive proof

For a dynamical system, any potential function $\psi(\boldsymbol{q})$ in Eq. (3) is a Lyapunov function. Conversely, explicit construction of $\boldsymbol{S}(\boldsymbol{q})$ and $\boldsymbol{T}(\boldsymbol{q})$ can be given for any Lyapunov function of the system. To achieve this conclusion, we provide an auxiliary concept.

Matrix-valued cross product The matrix-valued cross product of two vectors $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ is given by $\boldsymbol{x}\times\boldsymbol{y}=(x_iy_j-x_jy_i)_{n\times n}=\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}-\boldsymbol{y}\boldsymbol{x}^{\mathrm{T}}$, where T denotes the transpose of a matrix. The output is no longer a vector but an antisymmetric matrix.

From Eq. (3), $\dot{\boldsymbol{q}}=\boldsymbol{f}(\boldsymbol{q}^*)=0\Rightarrow\nabla\psi(\boldsymbol{q}^*)=0$. Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi(\boldsymbol{q}) &= \dot{\boldsymbol{q}}^{\mathrm{T}}\nabla\psi(\boldsymbol{q})=-\dot{\boldsymbol{q}}^{\mathrm{T}}\left[\boldsymbol{S}(\boldsymbol{q})+\boldsymbol{T}(\boldsymbol{q})\right]\dot{\boldsymbol{q}}\\ &= -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{S}(\boldsymbol{q})\dot{\boldsymbol{q}}\leqslant 0,\end{aligned}$$

we find that ψ satisfies $\dot{\psi}\leqslant 0$ for all $\boldsymbol{q}\in\mathbb{R}^n$. Hence, ψ is a Lyapunov function.

Conversely, for any Lyapunov function $\psi(\boldsymbol{q})$ of a given system $\dot{\boldsymbol{q}}=\boldsymbol{f}(\boldsymbol{q})$, by setting (here $\boldsymbol{I}$ denotes the identity matrix)

$$\boldsymbol{S}(\boldsymbol{q})=-\frac{\nabla\psi\cdot\boldsymbol{f}}{\boldsymbol{f}\cdot\boldsymbol{f}}\boldsymbol{I}, \tag{7}$$

$$\boldsymbol{T}(\boldsymbol{q})=-\frac{\nabla\psi\times\boldsymbol{f}}{\boldsymbol{f}\cdot\boldsymbol{f}}, \tag{8}$$

then utilizing $\nabla\psi\cdot\boldsymbol{f}=\nabla\psi\cdot\dot{\boldsymbol{q}}=\dot{\psi}\leqslant 0$, we know $\boldsymbol{S}(\boldsymbol{q})$ is symmetric and semi-positive definite and $\boldsymbol{T}(\boldsymbol{q})$ is antisymmetric by the definition of the matrix-valued cross product. Since

$$\begin{aligned}&[(\boldsymbol{z}\cdot\boldsymbol{x})\boldsymbol{y}+(\boldsymbol{z}\times\boldsymbol{y})\boldsymbol{x}]_i\\ &=\sum_k z_kx_ky_i+\sum_k(z_iy_k-z_ky_i)x_k=[(\boldsymbol{x}\cdot\boldsymbol{y})\boldsymbol{z}]_i,\end{aligned}$$

we can obtain

$$(\boldsymbol{f}\cdot\boldsymbol{f})\nabla\psi=(\nabla\psi\cdot\boldsymbol{f})\boldsymbol{f}+(\nabla\psi\times\boldsymbol{f})\boldsymbol{f}, \tag{9}$$

by letting $\boldsymbol{x}=\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{q})$ and $\boldsymbol{z}=\nabla\psi(\boldsymbol{q})$. Then

$$\begin{aligned}\left[\boldsymbol{S}(\boldsymbol{q})+\boldsymbol{T}(\boldsymbol{q})\right]\dot{\boldsymbol{q}} &= -\frac{(\nabla\psi\cdot\boldsymbol{f}+\nabla\psi\times\boldsymbol{f})\boldsymbol{f}}{\boldsymbol{f}\cdot\boldsymbol{f}}\\ &= -\nabla\psi(\boldsymbol{q}).\end{aligned}$$

This demonstrates that any Lyapunov function for a given system will satisfy Eq. (3).

Explicit construction for the chosen $\boldsymbol{D}(\boldsymbol{q})$ and $\boldsymbol{Q}(\boldsymbol{q})$ fulfilling (5) and (6) can be provided under the former configuration of $\boldsymbol{S}(\boldsymbol{q})$ and $\boldsymbol{T}(\boldsymbol{q})$ employing the matrix-valued cross product,

$$\boldsymbol{D}(\boldsymbol{q}) = -\left[\frac{\boldsymbol{f}\cdot\boldsymbol{f}}{\nabla\psi\cdot\boldsymbol{f}}\boldsymbol{I}+\frac{(\nabla\psi\times\boldsymbol{f})^2}{(\nabla\psi\cdot\boldsymbol{f})(\nabla\psi\cdot\nabla\psi)}\right], \tag{10}$$

$$\boldsymbol{Q}(\boldsymbol{q}) = \frac{\nabla\psi\times\boldsymbol{f}}{\nabla\psi\cdot\nabla\psi}. \tag{11}$$

The proof is straightforward based on the relation $\forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$

$$(\boldsymbol{x}\times\boldsymbol{y})^3=\left[(\boldsymbol{x}\cdot\boldsymbol{y})^2-(\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{y}\cdot\boldsymbol{y})\right](\boldsymbol{x}\times\boldsymbol{y}). \tag{12}$$

One can check the semi-positive definite property of $\boldsymbol{D}(\boldsymbol{q})$ by considering $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{D}(\boldsymbol{q})\boldsymbol{q}$ and completing the square.

Note that the construction is not unique. As the formerly presented construction starts from (3), we can symmetrically provide another one from (5). Since $[\boldsymbol{D}(\boldsymbol{q})+\boldsymbol{Q}(\boldsymbol{q})]\nabla\psi=-\dot{\boldsymbol{q}}$, then we have

$$\begin{aligned}\boldsymbol{D}(\boldsymbol{q}) &= -\frac{\boldsymbol{f}\cdot\nabla\psi}{\nabla\psi\cdot\nabla\psi}\boldsymbol{I},\\ \boldsymbol{Q}(\boldsymbol{q}) &= -\frac{\boldsymbol{f}\times\nabla\psi}{\nabla\psi\cdot\nabla\psi},\\ \boldsymbol{S}(\boldsymbol{q}) &= -\left[\frac{\nabla\psi\cdot\nabla\psi}{\boldsymbol{f}\cdot\nabla\psi}\boldsymbol{I}+\frac{(\boldsymbol{f}\times\nabla\psi)^2}{(\boldsymbol{f}\cdot\nabla\psi)(\boldsymbol{f}\cdot\boldsymbol{f})}\right],\\ \boldsymbol{T}(\boldsymbol{q}) &= \frac{\boldsymbol{f}\times\nabla\psi}{\boldsymbol{f}\cdot\boldsymbol{f}}.\end{aligned} \tag{13}$$

These different constructions will not affect our equivalence result. Earlier work[21,31] has different motivations, special cases of the results presented in this work are covered, e.g., all the systems discussed in Ref. [31] are unified by our standard form (5) with different choices of $\boldsymbol{D}(\boldsymbol{q})$ and $\boldsymbol{Q}(\boldsymbol{q})$. Among them, the example of Hamiltonian system (the phase space is $2l$ dimensional, $\boldsymbol{q}=(\hat{\boldsymbol{q}},\hat{\boldsymbol{p}})$. The matrices $\boldsymbol{S}$ and $\boldsymbol{T}$ are $2l\times 2l$. By energy conservation, $\boldsymbol{S}=0$, then

$$\boldsymbol{T}=\begin{pmatrix}0 & \boldsymbol{I}_l\\ -\boldsymbol{I}_l & 0\end{pmatrix},$$

ψ serves as the Hamiltonian, where $\boldsymbol{I}_l$ is the l-dimensional identity matrix.) shows that the generalization of the vector-valued cross product in (3) is appropriate.

3. Examples

The equivalence of Lyapunov function and potential function establishes a bridge between engineering and physics, therefore problems from one field may find out solution from the other field. A dynamical view of a physical system: consider the dynamics of a charged massless particle moving in the presence of friction, electrostatic and magnetostatic fields, a force balance equation (2) is derived. The electrostatic potential $\psi(\boldsymbol{q})$ serves as a Lyapunov function of the dynamics.

For a system that can be mapped onto the physical setting of Eq. (2), a Lyapunov function can be obtained by measuring the electrostatic potential. Note that naively constructed energy-like function is generally not a Lyapunov function.[40] A physical view of a dynamical system: a set of Eqs. (4) and (6) for Lyapunov function is established. In application, we have mainly examined three typical classes of dynamics: fixed points, limit cycles, and chaotic systems.

3.1. Sum of squares

For the linear system $\dot{\boldsymbol{q}} = \boldsymbol{F}\boldsymbol{q}$, a typical construction of Lyapunov function in engineering[21] is

$$L(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{q}. \tag{14}$$

We consider it as a potential function, then the friction matrix $\boldsymbol{S}(\boldsymbol{q})$ and Lorentz force matrix $\boldsymbol{T}(\boldsymbol{q})$ can be obtained as

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{q}) &= -\frac{\boldsymbol{q}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}^2\boldsymbol{q}}{\boldsymbol{q}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{q}}\boldsymbol{I}, \\ \boldsymbol{T}(\boldsymbol{q}) &= -\frac{(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{q})\times(\boldsymbol{F}\boldsymbol{q})}{\boldsymbol{q}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{q}}. \end{aligned} \tag{15}$$

We can see that an additional requirement for this construction to be valid is $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}^2\boldsymbol{q} \leq 0$. Besides, we can provide a more straightforward setting. Since $-\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{q} = -\boldsymbol{\nabla}L(\boldsymbol{q})$, we have $\boldsymbol{S}(\boldsymbol{q}) = -\left(\boldsymbol{F}^{\mathrm{T}}+\boldsymbol{F}\right)/2$ and $\boldsymbol{T}(\boldsymbol{q}) = -\left(\boldsymbol{F}^{\mathrm{T}}-\boldsymbol{F}\right)/2$. In this configuration, we require $\left(\boldsymbol{F}^{\mathrm{T}}+\boldsymbol{F}\right)$ to be semi-negative definite.

Moreover, for the nonlinear system (1), if it has a Lyapunov function of the form

$$L(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{f}(\boldsymbol{q})\cdot\boldsymbol{f}(\boldsymbol{q}), \tag{16}$$

the explicit expression for the matrices are

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{q}) &= -\frac{(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{f})\cdot\boldsymbol{f}}{\boldsymbol{f}\cdot\boldsymbol{f}}\boldsymbol{I}, \\ \boldsymbol{T}(\boldsymbol{q}) &= -\frac{(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{f})\times\boldsymbol{f}}{\boldsymbol{f}\cdot\boldsymbol{f}}, \end{aligned} \tag{17}$$

where $\boldsymbol{J}(\boldsymbol{q})$ denotes the Jacobian matrix of $\boldsymbol{f}(\boldsymbol{q})$. The requirement for this setting will be $\left[\boldsymbol{J}(\boldsymbol{q})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{q})\right]\cdot\boldsymbol{f}(\boldsymbol{q}) \leq 0$. Besides, we know $-\boldsymbol{J}(\boldsymbol{q})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{q}) = -\boldsymbol{\nabla}L(\boldsymbol{q})$. Then the friction matrix and Lorentz force matrix are

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{q}) &= -\frac{1}{2}\left[\boldsymbol{J}(\boldsymbol{q})^{\mathrm{T}}+\boldsymbol{J}(\boldsymbol{q})\right], \\ \boldsymbol{T}(\boldsymbol{q}) &= -\frac{1}{2}\left[\boldsymbol{J}(\boldsymbol{q})^{\mathrm{T}}-\boldsymbol{J}(\boldsymbol{q})\right]. \end{aligned} \tag{18}$$

Therefore, if $\left[\boldsymbol{J}(\boldsymbol{q})^{\mathrm{T}}+\boldsymbol{J}(\boldsymbol{q})\right]$ is semi-negative definite, equation (16) is a potential function of the system.

3.2. Limit cycle

For a planar system

$$\begin{cases} \dot{q}_1 = -q_2 + q_1(1-q_1^2-q_2^2), \\ \dot{q}_2 = q_1 + q_2(1-q_1^2-q_2^2), \end{cases} \tag{19}$$

we can construct a potential function by indicating a diffusion matrix $\boldsymbol{D} = \boldsymbol{I}$.[38] A Mexican hat shape potential function is then derived $\psi(q) = q^2(q^2-2)/4$ with $q^2 = q_1^2+q_2^2$. Meanwhile, we obtain

$$\begin{aligned} \boldsymbol{S}(q) &= \frac{(1-q^2)^2}{(1-q^2)^2+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \boldsymbol{T}(q) &= \frac{(1-q^2)}{(1-q^2)^2+1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{20}$$

One can check that

$$\begin{aligned} & [\boldsymbol{S}(q)+\boldsymbol{T}(q)]\,\dot{\boldsymbol{q}} \\ &= \frac{(1-q^2)}{(1-q^2)^2+1}\begin{pmatrix} (1-q^2) & 1 \\ -1 & (1-q^2) \end{pmatrix} \\ &\quad\times\begin{pmatrix} -q_2+q_1(1-q^2) \\ q_1+q_2(1-q^2) \end{pmatrix} \\ &= \begin{pmatrix} (1-q^2)q_1 \\ (1-q^2)q_2 \end{pmatrix} = -\boldsymbol{\nabla}\psi. \end{aligned}$$

This potential function serves also a Lyapunov function of the system. It is obvious from the Mexican hat shape ψ that $q=0$ is a repelling fixed point and $q=1$ is an attracting limit cycle.

For further illustration, a thorough study of potential function near fixed points whether stable or not is given in Ref. [39]. Explicit Lyapunov functions have been constructed for the well-known dissipative gyroscopic system[40] and competitive Lotka–Volterra systems.[29] All rotationally symmetric limit cycles are handled in Ref. [38]. For more general van der Pol type oscillators, potential function is exactly obtained.[35] We also show that potential functions can be constructed in continuous dissipative chaotic systems and can be used to explain for the different origins of chaotic attractor and strange attractor.[41]

4. Discussion

4.1. Singularity

From Eqs. (10) and (11), we observe that points with $\dot{\psi}(q) = \boldsymbol{\nabla}\psi\cdot\dot{\boldsymbol{q}} = \boldsymbol{\nabla}\psi\cdot\boldsymbol{f} = 0$ may cause singularity. Actually, it can be proved that:

Theorem The union of all the ω-limit sets[2] for solutions starting from all points $\boldsymbol{q}$ in the phase space is denoted by $Inv = \bigcup_{\boldsymbol{q}}\omega(\boldsymbol{q})$. Then, $\dot{\psi}(\boldsymbol{q}) = 0$ for all $\boldsymbol{q} \in Inv$.

Proof We apply proof by contradiction. Consider a point q_0 with $\dot{\psi}(q_0) < 0$. Since ψ and $\dot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q})$ are continuously differentiable functions then $\dot{\psi} = \boldsymbol{\nabla}\psi\cdot\dot{\boldsymbol{q}}$ is continuous. Thus, there exists an open set $\mathcal{O}$ as such for any $\boldsymbol{q}\in\mathcal{O}$, $\dot{\psi}(\boldsymbol{q}) < c < 0$ where c is a negative constant.

Suppose $\boldsymbol{q}_0 \in Inv$, then there is a solution $\boldsymbol{q}(t)$ and a sequence t_i, $i \in \mathbb{N}$, $\boldsymbol{q}(t_i) \in \mathcal{O}$ and $\lim_{i\to\infty} \boldsymbol{q}(t_i) = \boldsymbol{q}_0$. Since ψ is continuously differentiable and satisfies $\dot{\psi} \leqslant 0$, the trajectory L from $\boldsymbol{q}(t_i)$ to $\boldsymbol{q}(t_{i+1})$ must have a segment $\Delta L \subseteq L \cap \mathcal{O}$ inside $\mathcal{O}$, we have

$$\begin{aligned}\psi(\boldsymbol{q}(t_{i+1})) - \psi(\boldsymbol{q}(t_i)) &= \int_L \dot{\psi} \mathrm{d}t = \int_{L\setminus\mathcal{O}} \dot{\psi} \mathrm{d}t + \int_{L\cap\mathcal{O}} \dot{\psi} \mathrm{d}t \\ &\leqslant c||\Delta L|| < 0.\end{aligned}$$

Hence $\{\psi(\boldsymbol{q}(t_n))\}$ is a strictly decreasing sequence. By the continuity of ψ, $\lim_{i\to\infty} \psi(\boldsymbol{q}(t_i)) = \psi(\boldsymbol{q}_0)$, thus

$$\psi(\boldsymbol{q}(t_i)) > \psi(\boldsymbol{q}_0), \quad i \in \mathbb{N}. \tag{21}$$

Let $\boldsymbol{q}_0(t)$ be the solution starting at $\boldsymbol{q}_0$. Since $\dot{\psi}(\boldsymbol{q}_0) < 0$, the point $\boldsymbol{q}_0$ cannot be a fixed point. Given $s > 0$, since $\boldsymbol{q}_0 \in \mathcal{O}$, we have $\psi(\boldsymbol{q}_0(s)) < \psi(\boldsymbol{q}_0)$ that is $\exists \varepsilon > 0$, $\psi(\boldsymbol{q}_0(s)) + \varepsilon < \psi(\boldsymbol{q}_0)$. For the condition ψ is continuously differentiable, $\exists \gamma > 0$, if $|\boldsymbol{q}'(s) - \boldsymbol{q}_0(s)| < \gamma$, then $|\psi(\boldsymbol{q}'(s)) - \psi(\boldsymbol{q}_0(s))| < \varepsilon/2$. Since $\boldsymbol{f}$ is continuously differentiable, then the solutions are continuously dependent on the initial conditions, $\exists \delta > 0$, if $|\boldsymbol{q}'(0) - \boldsymbol{q}_0| < \delta$, then $|\boldsymbol{q}'(s) - \boldsymbol{q}_0(s)| < \gamma$.

From our assumption $\lim_{i\to\infty} \boldsymbol{q}(t_i) = \boldsymbol{q}_0$, we know $\exists n \in \mathbb{N}$, $|\boldsymbol{q}(t_n) - \boldsymbol{q}_0| < \delta$. Here we can replace $\boldsymbol{q}'(0)$ with $\boldsymbol{q}(t_n)$ and obtain

$$\psi(\boldsymbol{q}(t_n + s)) < \psi(\boldsymbol{q}_0(s)) + \frac{\varepsilon}{2} < \psi(\boldsymbol{q}_0(s)) + \varepsilon < \psi(\boldsymbol{q}_0). \tag{22}$$

As we know, $\exists m \in \mathbb{N}, m > n$ and $t_m > t_n + s$, then $\psi(\boldsymbol{q}(t_m)) > \psi(\boldsymbol{q}_0) > \psi(\boldsymbol{q}(t_n + s))$, which conflicts with $\dot{\psi} \leq 0$. Hence, $\boldsymbol{q}_0 \notin Inv$.

Therefore, points from limit sets may lead to singularity. Discussion on two typical types of limit set: fixed point and limit cycle, is presented below. Such singularity is usually canceled by the numerator (in the limit cycle example, there is a fixed point at $q = 0$, but $\boldsymbol{S}(q)$ and $\boldsymbol{T}(q)$ are always nonsingular), but may be unavoidable for other cases, reflecting the nature of the dynamics.

All fixed points $\boldsymbol{q}^* \in Inv$, leading to the singularity of $\boldsymbol{D}(\boldsymbol{q})$. Since $\nabla\psi(\boldsymbol{q}^*) = 0$ and $\dot{\boldsymbol{q}}(\boldsymbol{q}^*) = 0$, this type of singularity is similar to that of $\boldsymbol{S}(\boldsymbol{q})$ and $\boldsymbol{T}(\boldsymbol{q})$ without invalidating the construction.

Points on limit cycles belong to the set Inv where $\dot{\psi} = 0$. From the former example, we find that while approaching to the limit cycle $q \to 1$, the potential gradient goes to zero $\nabla\psi = q(q^2 - 1) \to 0$; the Lorentz force goes to zero in the same order $\boldsymbol{T} : (1 - q^2)$ of the potential gradient and changes sign at the limit cycle; and the friction goes to zero at a higher order $\boldsymbol{S} : (1 - q^2)^2$ than that of the potential gradient. The dynamics at the limit cycle is no longer dissipative but conserved in this limit. Hence, as the total energy of the system, the value of the Lyapunov function should be equal on limit cycles where the system is conserved. This is guaranteed by (a) from the definition.

Intuitively, a Lyapunov function for a deterministic dynamical system satisfies $\dot{\psi} \leqslant 0$, which indicates all possible places in the phase space when $t \to \infty$ as the evolution converges. Therefore, such a function with $\dot{\psi} < 0$ except a set of singular points (e.g., points in $Inv = \bigcup_{\boldsymbol{q}} \omega(\boldsymbol{q})$) is usually what we seek, providing a strongest prediction on the evolution result.

4.2. Lyapunov equation and generalized Einstein relation

Another connection between engineering and physics is revealed by the link between Lyapunov equation and the generalized Einstein relation (6). For linear systems $\dot{\boldsymbol{q}} = \boldsymbol{F}\boldsymbol{q}$, if any two eigenvalues λ_i and λ_j of $\boldsymbol{F}$ satisfy $\lambda_i + \lambda_j \neq 0$, then for any symmetric and positive definite matrix $\boldsymbol{R}$, there is a unique symmetric and invertible matrix $\boldsymbol{P}$ fulfilling the Lyapunov equation[6]

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{F} + \boldsymbol{R} = 0. \tag{23}$$

The system then has a Lyapunov function $L = \boldsymbol{q}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{q}$. If $\boldsymbol{P}$ is also positive definite, L will be reduced to a global strong Lyapunov function. Under such a configuration, with the restriction that the matrices $\boldsymbol{S}$ and $\boldsymbol{T}$ are position independent, we observe $[\boldsymbol{S} + \boldsymbol{T}] = -2\boldsymbol{P}\boldsymbol{F}^{-1}$, $\boldsymbol{S} = -\boldsymbol{P}\boldsymbol{F}^{-1} - \left(\boldsymbol{F}^{\mathrm{T}}\right)^{-1}\boldsymbol{P}$, since $\boldsymbol{F}$ has no zero eigenvalue. Here, we set $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{R}\boldsymbol{P}^{-1}/4$. Although from (23), we know the matrix $\boldsymbol{P}$ is dependent on the choice of $\boldsymbol{R}$. It can be proved that there is a one-to-one correspondence between symmetric and positive definite matrices $\boldsymbol{R}$ and $\boldsymbol{D}$. Through an equivalence transformation by multiplying invertible matrices on both sides of Eq. (23), we rewrite $[\boldsymbol{S} + \boldsymbol{T}]\boldsymbol{D}[\boldsymbol{S} - \boldsymbol{T}] - \boldsymbol{S} = 0$ as

$$\begin{aligned}&-2\boldsymbol{P}\boldsymbol{F}^{-1}\left[\frac{1}{4}\boldsymbol{P}^{-1}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{F} + \boldsymbol{R}\right)\boldsymbol{P}^{-1}\right] \\ &\times\left[-2\left(\boldsymbol{F}^{\mathrm{T}}\right)^{-1}\boldsymbol{P}\right] = 0.\end{aligned}$$

Hence, Lyapunov equation (23) is a reduced form of the generalized Einstein relation (6) for linear systems. This example clearly demonstrates that there are usually more than one Lyapunov function for a system with different $\boldsymbol{D}$. For a certain $\boldsymbol{D}$, relation (6) will guarantee the uniqueness of the Lyapunov function. It should be mentioned that recent work even showed that the condition $\lambda_i + \lambda_j \neq 0$ for $\boldsymbol{F}$ may not be necessary.[39]

4.3. Construction of Lyapunov function

Ordinary differential equations for a deterministic dissipative system indicate only a dissipation along the trajectories without specifying the speed of the dissipation. Thus, arbitrary speed is acceptable, leading to different Lyapunov functions. The Lyapunov function obtained thereof is merely a qualitative measure (a partial order, e.g., the function value of different stable fixed points are not comparable) but not quantitative. By indicating the diffusion matrix $\boldsymbol{D}(\boldsymbol{q})$ which is in fact the microscopic description of dissipation, the frictional force is determined. Hence, the dissipation along the trajectories is provided, defining the Lyapunov function as a unique quantitative measure. This measure has the role of the "Hamiltonian" in a Boltzmann–Gibbs distribution[25,40] on the final steady state (if the unique steady state exists) of the system's evolution.

The treatment in Ref. [24] defines a new stochastic integration,[34] named A-type. By adding white noise with diffusion matrix $\boldsymbol{D}(\boldsymbol{q})$ for a deterministic system, we obtain a set of stochastic differential equations (SDEs). Using A-type integration for these SDEs, the final steady state distribution is Boltzmann–Gibbs type $\rho(\boldsymbol{q}) \propto \exp[-\psi(\boldsymbol{q})]$, where $\psi(\boldsymbol{q})$ is the potential function defined by the canonical form (3). Therefore, $\psi(\boldsymbol{q}) \propto -\ln[\rho(\boldsymbol{q})]$, that is, the Lyapunov function can be numerically calculated by the long time sampling of trajectories of the stochastic process constructed. Besides, solving equations (3)–(6) directly is a way to get Lyapunov function. A gradient expansion method is provided in Ref. [24] as well. These are the approaches of constructing Lyapunov functions revealed by the equivalence result in this paper.

5. Conclusion

We have provided a constructive proof on the equivalence of the Lyapunov function and the potential function, establishing a set of physical equations for Lyapunov function. We have pointed out that for linear systems, the Lyapunov equation is a reduced form of the generalized Einstein relation. Finally, we have given the explanation of the non-uniqueness of Lyapunov functions through a stochastic view of deterministic systems and discussed the new approaches to obtain Lyapunov functions.

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