AIP | The Journal of Chemical Physics

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Citation: The Journal of Chemical Physics **141**, 044125 (2014); doi: 10.1063/1.4890968
View online: http://dx.doi.org/10.1063/1.4890968
View Table of Contents: http://scitation.aip.org/content/aip/journal/jcp/141/4?ver=pdfcov
Published by the AIP Publishing

Summing over trajectories of stochastic dynamics with multiplicative noise

Ying Tang,[1,2,a)] Ruoshi Yuan,[3] and Ping Ao[1,2,b)]
[1]Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China
[2]Key Laboratory of Systems Biomedicine Ministry of Education, Shanghai Center for Systems Biomedicine, Shanghai Jiao Tong University, Shanghai 200240, China
[3]School of Biomedical Engineering, Shanghai Jiao Tong University, Shanghai 200240, China

(Received 21 April 2014; accepted 11 July 2014; published online 30 July 2014)

We demonstrate that previous path integral formulations for the general stochastic interpretation generate incomplete results exemplified by the geometric Brownian motion. We thus develop a novel path integral formulation for the overdamped Langevin equation with multiplicative noise. The present path integral leads to the corresponding Fokker-Planck equation, and naturally generates a normalized transition probability in examples. Our result solves the inconsistency of the previous path integral formulations for the general stochastic interpretation, and can have wide applications in chemical and physical stochastic processes. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4890968]

I. INTRODUCTION

The path integral formulation is essential to applications in chemical and physical processes, such as sampling the rare events and determining the most probable path.[1–7] The classical Onsager-Machlup function[8] and its generalization[9–12] are applied successfully to the system with additive noise.[2,4] However, for the system with multiplicative noise, previous works on constructing the path integral for the general stochastic interpretation[13–15] (called the α-interpretation) are still controversial. While the uniqueness of the action function in the path integral is claimed,[13] it contradicts to the fact that different interpretations lead to corresponding different processes in the Langevin formulation.[16] Besides, though the path integral in Refs. 14 and 15 depends on the stochastic interpretation, we find that the obtained transition probability is incomplete for the general stochastic interpretation when applying their path integral formula to the geometric Brownian motion.

This controversy on the path integral formulation for the system with multiplicative noise can be traced back to an ambiguity in choosing the integration method for the overdamped Langevin dynamics, which leads to different stochastic interpretations.[16] In this paper, we provide an alternative way to construct the path integral formulation for the overdamped Langevin equation under the α-interpretation. Our main result, Eq. (9), leads to transition probabilities directly obeying the conservation law for general stochastic interpretations, which is exemplified by the Ornstein-Uhlenbeck process and the geometric Brownian motion. The present path integral formulation depends on α, and it can also generate the corresponding α-interpretation Fokker-Planck equation.[17] As our formulation of path integral starts from the corresponding Langevin equation of the equivalent Stratonovich's form, our result is consistent with ordinary calculus, and thus is convenient for applications.

Generally, the choice of the stochastic interpretation depends on the system under study. Ito's is widely used in mathematical economics.[18] Stratonovich's is popular in physics[19] in order to correctly describe gauge-invariant quantum dynamics.[20] The fluctuation theorems[21–30] can give accurate estimations on free energy changes only under anti-Ito's.[31] The experiments also show different preferable stochastic interpretations. The experiment conducted by the analog simulator shows that Stratonovich's is preferable,[32] while force measurements[33] and drift measurements[34] on a Brownian particle near a wall suggest that the system favors anti-Ito's,[35–39] to ensure the Boltzmann-Gibbs distribution for the final steady state. Thus, the present path integral formulation for the general stochastic interpretation is practical for applications to see which stochastic interpretation is preferable, and can be tested experimentally.

This paper is organized as follows. In Sec. II, we provide the path integral formulation and discuss its relation with the previous path integral frameworks. In Sec. III, we generate the corresponding Fokker-Planck equation from the present path integral formulation. In Sec. IV, we obtain the transition probabilities for the Ornstein-Uhlenbeck process and the geometric Brownian motion under the general stochastic interpretation. In Sec. V, we summarize our work. In Appendix A, we list the previous path integral frameworks for the general stochastic interpretation and apply them to the geometric Brownian motion to show the difference with our result. In Appendix B, we develop an equivalent form of the path integral formulation in the main text. In Appendix C, we generalize our path integral formula to multidimensional case.

II. PATH INTEGRAL FORMULATION

For convenience, we start from the one-dimensional overdamped Langevin equation with multiplicative noise

$$\dot{x} = f(x) + g(x)\xi(t), \tag{1}$$

[a)]Electronic mail: jamestang23@gmail.com
[b)]Electronic mail: aoping@sjtu.edu.cn

where x denotes the position, $\dot{x}$ denotes its time derivative, $f(x)$ is the drift term, and $g(x)\xi(t)$ models the stochastic force. Here, $\xi(t)$ is a Gaussian white noise with $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = \epsilon\delta(t-t')$ and the average is taken with respect to the noise distribution. The positive constant ϵ describes the strength of the noise, corresponding to k_BT in physical systems. For this Langevin equation, an ambiguity in choosing the integration method leads to different stochastic interpretations and a general notation is the α-interpretation.[17] The values $\alpha = 0$, $\alpha = 1/2$, and $\alpha = 1$ correspond to Ito's, Stratonovich's, and anti-Ito's, respectively. If starting from the second order Langevin equation with a variable friction coefficient, different approximation methods can also give the overdamped Langevin equation with various stochastic interpretations.[39,40]

For the overdamped Langevin equation under the α-interpretation, by modifying the drift term, we have its Stratonovich's form[16]

$$\dot{x} = f(x) + \left(\alpha - \frac{1}{2}\right) g'(x)g(x) + g(x)\xi(t), \tag{2}$$

where the superscript prime denotes the derivative to x. The advantage of using this Stratonovich's form is that ordinary calculus rule can be simply applied.[41] Then, this equation can be transformed to be a Langevin equation with a additive noise by a change of variable $q = H(x)$ with $H'(x) = 1/g(x)$,[13]

$$\dot{q} - h(q) = \xi(t), \tag{3}$$

where we have introduced an auxiliary function,

$$h(q) = \frac{f(H^{-1}(q))}{g(H^{-1}(q))} + \left(\alpha - \frac{1}{2}\right) g'(H^{-1}(q)). \tag{4}$$

To get the transition probability for Eq. (3), we first discretize the time into N segments: $t_0 < t_1 < \cdots < t_{N-1} < t_N$ with $\tau = t_n - t_{n-1}$ small and let $q_n = q(t_n)$. For the sake of consistency, as we have chosen the equivalent Stratonovich's form, the corresponding discretized Langevin equation needs the mid-point discretization

$$q_n - q_{n-1} - \left[\frac{h(q_n) + h(q_{n-1})}{2}\right]\tau = W_n - W_{n-1}, \tag{5}$$

where $W(t)$ is the Wiener process given by $dW(t) = \xi(t)dt$. Thus, the Jacobian for the variable transformation between $q(t)$ and $W(t)$ is

$$J \approx \exp\left[-\frac{\tau}{2}\sum_{n=1}^{N-1}\frac{dh(q_n)}{dq_n}\right]. \tag{6}$$

With the conditional probability functional of the Wiener process[11,16]

$$P(W_Nt_N|W_0t_0) = \int_{W_0}^{W_N}\mathcal{D}W \exp\left(-\int_{t_0}^{t_N}\frac{1}{2}\dot{W}^2dt\right), \tag{7}$$

where the Wiener measure is defined as $\int_{W_0}^{W_N}\mathcal{D}W \doteq \lim_{N\to\infty}\frac{1}{2\pi\tau}\prod_{n=1}^{N-1}\int\frac{dW_n}{2\pi\tau}$. By changing the measure from $\mathcal{D}W$ to $\mathcal{D}q$, the path integral formulation for Eq. (3) is obtained

$$P(q_Nt_N|q_0t_0) = \int_{q_0}^{q_N}\mathcal{D}q \exp\left\{-\int_{t_0}^{t_N}\left[\frac{1}{2\epsilon}(\dot{q}-h)^2 + \frac{1}{2}\frac{dh}{dq}\right]dt\right\}, \tag{8}$$

where $\int_{q_0}^{q_N}\mathcal{D}q \doteq \lim_{N\to\infty}\frac{1}{\sqrt{2\pi\tau\epsilon}}\prod_{n=1}^{N-1}\int\frac{dq_n}{\sqrt{2\pi\tau\epsilon}}$. The integral of the action function on the exponent obeys ordinary calculus due to the mid-point discretization and the last term comes from the Jacobian. We mention that another full derivation of Eq. (8) from Eq. (3) can be found in Ref. 42.

By changing the variable reversely: $x = H^{-1}(q)$ with $dx/dq = g(x)$, we get the path integral for Eq. (1) under the α-interpretation,

$$P(x_Nt_N|x_0t_0) = \int_{x_0}^{x_N}\mathcal{D}x \exp\left\{-\int_{t_0}^{t_N}\left[\frac{1}{2g^2\epsilon}\left(\dot{x} - f - \left(\alpha - \frac{1}{2}\right)g'g\right)^2 + \frac{g}{2}\left(\frac{f}{g} + \left(\alpha - \frac{1}{2}\right)g'\right)'\right]dt\right\}, \tag{9}$$

where $\int_{x_0}^{x_N}\mathcal{D}x \doteq \lim_{N\to\infty}\frac{1}{\sqrt{2\pi\tau\epsilon}g(x_N)}\prod_{n=1}^{N-1}\int\frac{dx_n}{\sqrt{2\pi\tau\epsilon}g(x_n)}$. The action function is dimensionless because ϵ has the same dimension as energy. Though the Jacobian term comes from the measure transformation and does not belong to the conventional action part, it is usually included in the action function for applications.

The path integral formulation for the case with multiplicative noise may not be absolute continuous in mathematical definition.[43] However, we will show that the semi-classical method can still be applied with the present path integral to get a correct transition probability in examples, including the case with multiplicative noise. For the case with additive noise, Eq. (9) degenerates to be

$$P(x_Nt_N|x_0t_0) = \int_{x_0}^{x_N}\mathcal{D}x \exp\left\{-\int_{t_0}^{t_N}\left[\frac{1}{2g^2\epsilon}(\dot{x}-f)^2 + \frac{1}{2}f'\right]dt\right\}. \tag{10}$$

We remark that though we use the equivalent Stratonovich's form above, the path integral Eq. (9) is for the Langevin equation (1) under the α-interpretation. Our result Eq. (9) can also be obtained by generalizing the result in Ref. 42 to be applicable to the α-interpretation. We will also use other types of equivalent Langevin equations with the corresponding stochastic interpretation to construct the path integral formulation in Appendix B. The chosen stochastic interpretation in turn assigns a corresponding discretized scheme for the path integral. For example, the mid-point discretization should be used for the action function under the Stratonovich's form, and the α-type discretization (at the point $\alpha x_n + (1-\alpha)x_{n-1}$ in each interval) is needed for the α-form. These forms with the consistent stochastic interpretation are equivalent. Thus, one can choose any specific form in applications for the convenience of numerical calculations. This indicates that a particular numerical method can be adopted with adding the corresponding drift term to achieve the physically preferable stochastic interpretation.

When integrating the action function, we should keep using the stochastic calculus rule consistent with its discretized scheme. If we apply the path integral here, we need to use Stratonovich's calculus. When the path integral in Refs. 14 and 15 is applied, the α-type integration rule[41] is required. Taking the Ornstein-Uhlenbeck process as an example, we will show in Appendix B that both their path integral with the α-type integration and Eq. (10) with ordinary calculus can automatically give a normalized transition probability. Even so, when applying their path integral to the geometric Brownian motion, the transition probabilities obtained by both ordinary calculus and the α-type integration are incomplete.

For systems with additive noise, our result demonstrates that the classical Onsager-Machlup function[8] with Ito's integration[16] and the effective action[9–12] with Stratonovich's calculus are equivalent.[44] For the system with multiplicative noise, the consistency of the path integral and the stochastic calculus has been noted in Ref. 13. However, only the Langevin equation under Ito's interpretation is considered at the beginning. Thus, their result is only for Ito's interpretation and not for the α-interpretation, corresponding to $\alpha = 0$ in Eq. (9). This explains why their result shows the uniqueness of the path integral. Besides, the path integral formulations on manifolds have been discussed previously,[30,42,45–47] nevertheless, their results are mainly for Stratonovich's interpretation. These results in one dimension with zero curvature are consistent with Eq. (9) when $\alpha = 1/2$. The explicit difference between our result with $\alpha = 1/2$ and the result in Ref. 42 comes from different ways of representing the measure, and can be fixed by normalization. Furthermore, considering the general stochastic interpretation, the path integral formulated by Wissel[48] is also different from our result.

III. THE FOKKER-PLANCK EQUATION

In this section, we first derive the Fokker-Planck equation from the path integral for the system with additive noise. Then, by the variable transformation, we obtain the Fokker-Planck equation of the α-interpretation for Eq. (1). According to Eq. (8), the transition probability in each interval is

$$\begin{aligned} P(q_n t_n | q_{n-1} t_{n-1}) &= \frac{1}{\sqrt{2\pi\tau\epsilon}} \exp\left\{ -\frac{\tau}{2\epsilon}\left[\frac{\Delta q_n}{\tau} \frac{1}{2}(h(q_n) + h(q_{n-1}))\right]^2 \right. \\ &\quad \left. - \frac{\tau}{2}\frac{dh}{dq}(q_{n-1})\right\}, \end{aligned} \tag{11}$$

where $\Delta q_n = q_n - q_{n-1}$. The normalization condition is satisfied: $\int P(q_n t_n | q_{n-1} t_{n-1}) dq_n = 1$. Then, by the moment generating function[49]

$$A_k(q_{n-1}) = \frac{(-1)^k}{\tau k!} \int dq_n (q_n - q_{n-1})^k P(q_n t_n | q_{n-1} t_{n-1}), \tag{12}$$

we calculate out the first two of $A_k(q_{n-1})$ as $A_k(q_{n-1}) \approx O(\tau)$ for $k > 2$,

$$\begin{aligned} A_1(q_{n-1}) &= -h(q_{n-1}) + O(\tau), \\ A_2(q_{n-1}) &= \frac{\epsilon}{2} + O(\tau). \end{aligned} \tag{13}$$

By taking the limit $\tau \to 0$, we get the Fokker-Planck equation for Eq. (3),

$$\partial_t \rho(q,t) = -\partial_q[h(q)\rho(q,t)] + \frac{\epsilon}{2}\partial_q^2[\rho(q,t)]. \tag{14}$$

In order to derive the Fokker-Planck equation for Eq. (1), we change the variable inversely: $x = H^{-1}(q)$ with $dx/dq = g(x)$. With the aid of the corresponding transformation for the moments of the Fokker-Planck equation,[50] we have

$$\begin{aligned} \partial_t \rho(x,t) &= -\partial_x[(f(x) + \alpha g'(x) g(x))\rho(x,t)] \\ &\quad + \frac{\epsilon}{2}\partial_x\left[g^2(x)\rho(x,t)\right], \end{aligned} \tag{15}$$

which is the same as the conventional α-interpretation Fokker-Planck equation.[17]

We emphasize that when taking the partial derivative to x in the Fokker-Planck equation, we can always use ordinary calculus regardless of the interpretation adopted for the Langevin equation. In the Langevin dynamics, the expansion for a smooth function of x is in orders of different time scales: dW, dt, $dW^2\, dW dt$, etc. Then, dW^2 should be counted up to the order of dt[16], which leads to different stochastic interpretations. However, on the level of the Fokker-Planck equation, the expansion is in orders of the space coordinates: dx, dx^2, etc. Thus, there is no ambiguity for the stochastic interpretation for a given Fokker-Planck equation.

IV. EXAMPLES

A. The Ornstein-Uhlenbeck process

This process can be described by the Langevin equation[16]

$$\dot{x} = -kx + \sqrt{D}\xi(t), \tag{16}$$

where k, D are positive constants. The temperature has been set to be a unit in examples for convenience.

We apply Eq. (9),

$$\begin{aligned} &P(x_N t_N | x_0 t_0) \\ &= \int_{x_0}^{x_N} \mathcal{D}x \exp\left\{ -\frac{1}{2D}\int_{t_0}^{t_N} (\dot{x} + kx)^2 dt + \frac{k\Delta t}{2} \right\}, \end{aligned} \tag{17}$$

where $\Delta t = t_N - t_0$. We then get the transition probability by the semi-classical method[51] using ordinary calculus rule in calculation of the action function

$$\begin{aligned} &P(x_N t_N | x_0 t_0) \\ &= \sqrt{\frac{k}{D\pi(1 - e^{-2k\Delta t})}} \exp\left[-\frac{k(x_N - e^{-k\Delta t}x_0)^2}{D(1 - e^{-2k\Delta t})} \right]. \end{aligned} \tag{18}$$

Note that we have directly obtained a normalized transition probability. If we apply the path integral in Refs. 14 and 15 with the consistent α-type integration, we can have the same

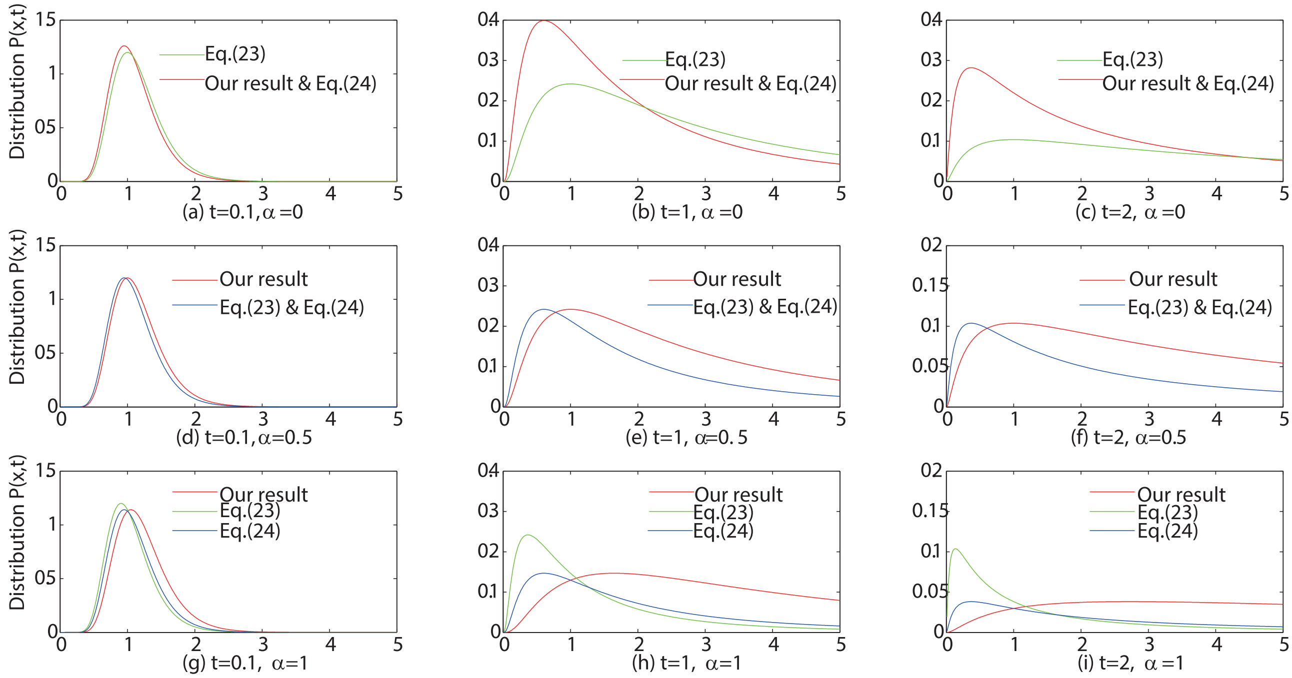

FIG. 1. (a)–(i) The evolution of the probability distributions for the geometric Brownian motion Eq. (19) with $k = 1$, $\sigma = 1$ and the delta function $\delta(x_0 - 1)$ as the initial distribution under different stochastic interpretations: $\alpha = 0$, $\alpha = 1/2$, and $\alpha = 1$. Different colors denote results from different path integral formulations. The three graphs in the first line correspond to Ito's interpretation ($\alpha = 0$). In this case, our result Eqs. (22) and (24) coincide. The graphs in the second line correspond to Stratonovich's interpretation ($\alpha = 1/2$). In this case, Eqs. (23) and (24) coincide. In the third line, the graphs show the result under anti-Ito's interpretation ($\alpha = 1$).

transition probability through a similar procedure. We also note that their path integral with ordinary calculus cannot lead to the correct result Eq. (18).

B. The geometric Brownian motion

This process is popular in mathematical finance and recently attracts more interest in physical society.[52] It can be given by the following Langevin equation:[18]

$$\dot{x} = kx + \sigma x \xi(t), \tag{19}$$

where k, σ are positive constants. We apply Eq. (9),

$$P(x_N t_N | x_0 t_0) = \int_{x_0}^{x_N} \mathcal{D}x \exp\left\{-\int_{t_0}^{t_N} \frac{[\dot{x} - kx - (\alpha - 1/2)\sigma^2 x]^2}{2\sigma^2 x^2} dt\right\}. \tag{20}$$

In the action function, to calculate the path-dependent term

$$\mathbf{S}_0 = \frac{1}{2\sigma^2} \int_{t_0}^{t_N} \frac{\dot{x}^2}{x^2} dt, \tag{21}$$

we make a variable transformation $y = \ln x$ and then $\dot{y} = \dot{x}/x$ by ordinary calculus. Thus, through the semi-classical method,[51] we finally reach the result

$$P(x_N t_N | x_0 t_0) = \frac{1}{\sqrt{2\pi \Delta t}\sigma x_N} e^{-\frac{1}{2\sigma^2 \Delta t}[\ln(x_N/x_0) - (k + (\alpha - 1/2)\sigma^2)\Delta t]^2}, \tag{22}$$

where the pre-factor $1/x_N$ comes from the measure transformation for $\int_{x_0}^{x_N} \mathcal{D}x$. This transition probability agrees with Eq. (A9) for the general stochastic interpretation and the result in Ref. 18, where two special cases Ito's and Stratonovich's were discussed.

When applying the path integral in Refs. 14 and 15, we derive two kinds of transition probabilities obtained by two integration rules: ordinary calculus and the α-type integration. The detailed calculation can be found in Appendix A. First, with ordinary calculus we have

$$\hat{P}(x_N t_N | x_0 t_0) = \frac{1}{\sqrt{2\pi \Delta t}\sigma x_N} e^{-\frac{1}{2\sigma^2 \Delta t}[\ln(x_N/x_0) - (k - \alpha\sigma^2)\Delta t]^2 - \alpha k \Delta t}. \tag{23}$$

After normalization, i.e., when $\alpha k \Delta t$ is eliminated, this formula still differs from Eq. (22).

Second, if we use the α-type integration,[41] we have

$$\hat{P}(x_N t_N | x_0 t_0) = \frac{1}{\sqrt{2\pi \Delta t}\sigma x_N} e^{-\frac{1}{2\sigma^2 \Delta t}[\ln(x_N/x_0) - (k - \sigma^2/2)\Delta t]^2 - \alpha k \Delta t}. \tag{24}$$

After normalization, Eq. (24) is still different from Eq. (22) except under Ito's interpretation. The comparison of the transition probabilities, Eqs. (22)–(24) is shown in Fig. 1.

V. CONCLUSION

From the overdamped Langevin equation with multiplicative noise, we have constructed the path integral formulation for the general α-interpretation. It is convenient for applications as ordinary calculus can be applied. The corresponding α-interpretation Fokker-Planck equation has been generated, and thus the three widely used descriptions in stochastic process are connected. For the system with additive noise, our result demonstrates the equivalence of the effective action in Ref. 4 and the classical Onsager-Machlup function with their corresponding stochastic integration. For the system with multiplicative noise, the present path integral provides a complete formulation for general stochastic interpretations directly obeying the conservation law.

For the high dimensional case with a non-singular diffusion matrix, our method can be used similarly to develop the path integral formulation. The fluctuation theorem based on the path integral here can be obtained. The forward and the reverse dynamical processes should be defined in accordance with the α-interpretation Fokker-Planck equation. Whether or not the fluctuation theorem is related to the stochastic interpretation is an interesting topic to be explored. The influence of our result on the numerical side need to be illustrated. The connection between our path integral formulation and the operator orderings[53] in quantum mechanics also remains to be discovered.

ACKNOWLEDGMENTS

Stimulating discussions with Hong Qian, Alberto Imparato, David Cai, Andy Lau, Xiangjun Xing, and Bo Yuan are gratefully acknowledged. We thank Jianhong Chen for the critical comments. This work is supported in part by the National 973 Project No. 2010CB529200 and by the Natural Science Foundation of China (NSFC) Project Nos. NSFC61073087 and NSFC91029738.

APPENDIX A: THE PREVIOUS PATH INTEGRAL FOR THE GENERAL STOCHASTIC INTERPRETATION

In this appendix, we list two kinds of previous path integral formulations for Eq. (1) under the general stochastic interpretation. The first one is[13]

$$P(x_N t_N|x_0 t_0) = \int_{x_0}^{x_N} \mathcal{D}x \exp\left\{-\int_{t_0}^{t_N}\left[\frac{1}{2g^2\epsilon}\left(\dot{x} - f + \frac{1}{2}g'g\right)^2 + \frac{g}{2}\left(\frac{f}{g} - \frac{1}{2}g'\right)'\right]dt\right\}, \tag{A1}$$

where $\int_{x_0}^{x_N}\mathcal{D}x \doteq \lim_{N\to\infty}\frac{1}{\sqrt{2\pi\tau\epsilon g(x_N)}}\prod_{n=1}^{N-1}\int\frac{dx_n}{\sqrt{2\pi\tau\epsilon g(x_n)}}$ and the superscript prime denotes the derivative to x. According to their generation on the path integral, the action function obeys ordinary calculus.

Note that their path integral is independent of α and thus the authors claim the uniqueness of their action function. If applying their path integral with ordinary calculus,[13] we notice that the transition probability is always the same for any stochastic interpretation. However, it is known that for the geometric Brownian motion different stochastic interpretations lead to the corresponding different results.[18] We further find that they just develop the path integral formula for the Langevin equation under Ito's interpretation. On the contrary, we start from the Langevin equation under the α-interpretation and then the action function is not unique but α-dependent.

The second previous path integral for the general stochastic interpretation is[14,15]

$$\hat{P}(x_N t_N|x_0 t_0) = \int_{x_0}^{x_N}\mathcal{D}x \exp\left\{-\int_{t_0}^{t_N}{}_{\alpha}\frac{1}{2g^2\epsilon}(\dot{x} - f + \alpha g'g)^2 dt - \alpha\int_{t_0}^{t_N} f' dt\right\}, \tag{A2}$$

where $\int_{x_0}^{x_N}\mathcal{D}x \doteq \lim_{N\to\infty}\frac{1}{\sqrt{2\pi\tau\epsilon g(x_N)}}\prod_{n=1}^{N-1}\int\frac{dx_n}{\sqrt{2\pi\tau\epsilon g(x_n)}}$. The symbol $\int_{t_0\,\alpha}^{t_N}$ means the integrand obeys the α-type integration:[41] for a smooth function $F(x(t))$,

$$dF(x) \approx F'(x)dx + \frac{1-2\alpha}{2}F''(x)g^2(x)dt. \tag{A3}$$

The symbol of integral without subscript is the ordinary integral, i.e., $\int_{t_0\,\frac{1}{2}}^{t_N} = \int_{t_0}^{t_N}$. Thus, we omit the subscript 1/2 of $\int_{t_0\,\frac{1}{2}}^{t_N}$ in this paper. The integral for the Jacobian term does not specify the stochastic interpretation and always obeys ordinary calculus.

We notice that it is necessary to use the α-type integration for the action function, as their corresponding discretization is at the point $\alpha x_n + (1-\alpha)x_{n-1}$ in each interval. For the case with additive noise, Eq. (A2) becomes

$$\hat{P}(x_N t_N|x_0 t_0) = \int_{x_0}^{x_N}\mathcal{D}x \exp\left\{-\int_{t_0}^{t_N}{}_{\alpha}\frac{1}{2g^2\epsilon}(\dot{x} - f)^2 dt - \alpha\int_{t_0}^{t_N} f' dt\right\}. \tag{A4}$$

We find that for the Ornstein-Uhlenbeck process the transition probability calculated by Eq. (A4) directly with ordinary calculus is not normalized automatically. On the contrary, the corresponding α-type integration directly leads to a normalized transition probability. Therefore, the consistent stochastic integration is necessary to calculate the action function.

However, for the geometric Brownian motion, we will show in the following that the transition probabilities by applying Eq. (A2) with both ordinary calculus and the α-type integration are not consistent with the known result.[18] Before that, we first provide the transition probability for the geometric Brownian motion under the general stochastic interpretation by generalizing the derivation in Ref. 18.

For the one-dimensional geometric Brownian motion under the α-interpretation, the equivalent equation under Ito's interpretation is

$$\dot{x} = (k + \alpha\sigma^2)x + \sigma x\xi(t). \tag{A5}$$

The Ito's formula,[16] i.e., $\alpha = 0$ in Eq. (A3), tells that for $F(x) = \ln x$,

$$dF(x) \approx \left(k + \alpha\sigma^2 - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t). \tag{A6}$$

As a result, the solution to Eq. (A5) given initial condition x_0 at time t_0 is

$$x(t) = x_0 \exp\left\{\left[k + \left(\alpha - \frac{1}{2}\right)\sigma^2\right]t + \sigma W(t)\right\}. \tag{A7}$$

By the distribution function of $x(t)$

$$P\left(\frac{W(\Delta t)}{\sqrt{\Delta t}} \le \frac{[\ln(x_N/x_0) - (k + (\alpha - 1/2)\sigma^2)\Delta t]}{\sigma\sqrt{\Delta t}}\right), \tag{A8}$$

we get the transition probability

$$P(x_N t_N | x_0 t_0) = \frac{1}{\sqrt{2\pi\Delta t}\sigma x_N} e^{-\frac{1}{2\sigma^2\Delta t}[\ln(x_N/x_0) - (k + (\alpha - 1/2)\sigma^2)\Delta t]^2}. \tag{A9}$$

It is consistent with Eq. (22) for the general stochastic interpretation and the previous result in Ref. 18, where Ito's and Stratonovich's cases were discussed.

Now, we apply Eq. (A2) and have the action function

$$\hat{\mathbf{S}} = \frac{1}{2\sigma^2}\int_{t_0}^{t_N}{}_{\alpha}\left\{\frac{\dot{x}^2}{x^2} - 2(k - \alpha\sigma^2)\frac{\dot{x}}{x} + (k - \alpha\sigma^2)^2\right\}dt + \alpha k\Delta t. \tag{A10}$$

We then use two different integration rules separately. First, we use ordinary calculus. For the first term of the action function, we make a variable transformation $y = \ln x$ and thus $\dot{y} = \dot{x}/x$. Then, with the semi-classical method on a free particle,[51] we have Eq. (23).

Second, if we use the α-type integration for the action function, then

$$\frac{1}{2\sigma^2}\int_{t_0}^{t_N}{}_{\alpha}\left(\frac{\dot{x}}{x}\right)dt = \frac{1}{2\sigma^2}\left[\ln\left(\frac{x_N}{x_0}\right) + \left(\frac{1}{2} - \alpha\right)\sigma^2\Delta t\right]. \tag{A11}$$

Besides, for the variable transformation $y = \ln x$, we should have $\dot{y} = \dot{x}/x - (1/2 - \alpha)\sigma^2$ and finally obtain Eq. (24) after similar procedure.

APPENDIX B: PATH INTEGRAL FORMULATION OF THE EQUIVALENT α-FORM

In this appendix, we provide another way to develop the path integral formulation starting from the one-dimensional Langevin equation under the α-interpretation. Instead of modifying the drift term and writing down its equivalent Langevin equation in Stratonovich's form, we directly use Eq. (1). The α-type chain rule, Eq. (A3), should be applied to do the variable transformation, $q = H(x)$,

$$dH(x) \approx H'(x)dx + \frac{1 - 2\alpha}{2}H''(x)g^2(x)dt, \tag{B1}$$

where the superscript prime denotes the derivative to x. Then, with $H'(x) = 1/g(x)$ and $H''(x) = -g'(x)/g^2(x)$ Eq. (1) can be transformed to be a Langevin equation with additive noise

$$\dot{q} - h(q) = \xi(t), \tag{B2}$$

with

$$h(q) = \frac{f(H^{-1}(q))}{g(H^{-1}(q))} + \left(\alpha - \frac{1}{2}\right)g'(H^{-1}(q)). \tag{B3}$$

For the sake of consistency, as we have chosen the α-interpretation, the corresponding discretized Langevin equation should be

$$q_n - q_{n-1} - [\alpha h(q_n) + (1 - \alpha)h(q_{n-1})]\tau = W_n - W_{n-1}. \tag{B4}$$

Thus, the Jacobian for the variable transformation between $q(t)$ and $W(t)$ becomes

$$J \approx \exp\left[-\alpha\tau\sum_{n=1}^{N-1}\frac{dh(q_n)}{dq_n}\right]. \tag{B5}$$

Then, with the property of Wiener process and the Chapman-Kolmogorov equation,[13] the transition probability is obtained

$$P(q_N t_N | q_0 t_0) = \int_{q_0}^{q_N}\mathcal{D}q \exp\left\{-\int_{t_0}^{t_N}{}_{\alpha}\frac{1}{2\epsilon}[\dot{q} - h(q)]^2 dt - \alpha\int_{t_0}^{t_N}\frac{dh(q)}{dq}dt\right\}. \tag{B6}$$

To get the transition probability $P(x_N t_N | x_0 t_0)$ for Eq. (1), we change the variable reversely,

$$P(x_N t_N | x_0 t_0) = \int_{x_0}^{x_N}\mathcal{D}x \exp\left\{-\int_{t_0}^{t_N}{}_{\alpha}\frac{1}{2g^2\epsilon}(\dot{x} - f)^2 dt - \alpha\int_{t_0}^{t_N} g\left[\frac{f}{g} + \left(\alpha - \frac{1}{2}\right)g'\right]' dt\right\}. \tag{B7}$$

The different forms of the present path integral come from the equivalent Langevin equation under different stochastic interpretations. With the consistent calculus rule, all the forms are equivalent. The corresponding integration rule should be applied with the specific form of the present path integral. Therefore, one can conveniently choose the path integral form with the integration rule for the problem considered.

For additive noise cases, Eq. (B7) becomes

$$P(x_N t_N | x_0 t_0) = \int_{x_0}^{x_N}\mathcal{D}x \exp\left\{-\int_{t_0}^{t_N}{}_{\alpha}\frac{1}{2g^2\epsilon}(\dot{x} - f)^2 dt - \alpha\int_{t_0}^{t_N} f' dt\right\}, \tag{B8}$$

which is the same as Eq. (A4). This demonstrates that the present path integral formulation and that in Refs. 14 and 15 show no difference for the additive noise cases. It should be emphasized that though Eq. (B8) explicitly contains an α-term, it is independent of α after integration. Because the integral $\int_{t_0}^{t_N}{}_{\alpha}$ should be done through the α-type integration, which will eliminate the α-term on the exponent.

APPENDIX C: PATH INTEGRAL FORMULATION IN HIGH DIMENSION

In this appendix, we follow the method in the main text to implement the path integral formalization for the m-dimensional Langevin equation with multiplicative noise

$$\dot{x}^i = f^i(\mathbf{x}) + g^{ij}(\mathbf{x})\xi^j(t), \tag{C1}$$

where x^i denotes the ith component of the m-dimensional position vector, $\dot{x}^i$ denotes its time derivative, $f^i(\mathbf{x})$ is the ith component of the drift term. The noise $\xi^j(t)$ is the jth component of the l-dimensional Gaussian white noise with $\langle \xi^j(t)\rangle = 0$, $\langle \xi^s(t)\xi^j(t')\rangle = \epsilon\delta^{sj}\delta(t-t')$ with the average taken with respect to the noise distribution, and $g^{ij}(\mathbf{x})\xi^j(t)$ models the stochastic force. Here, $g^{ij}(\mathbf{x})$ is an $m \times l$ tensor and $D^{ik}(\mathbf{x}) = g^{ij}(\mathbf{x})g^{kj}(\mathbf{x})/2$ ($k = 1, \ldots, m$) denotes the $m \times m$ diffusion matrix. The Einstein's summation rule has been used. We mention that $l > m$ means the dimension of the noise may be larger than the dimension of the system. However, here we discuss the case where the tensor g_{ij} is invertible for convenience of calculation, and thus $(g^{-1})^{ki}g^{ij} = g^{ij}(g^{-1})^{jk} = \delta_{ik}$.

Following the derivation for the one-dimensional case, we next implement the path integral formalization. The Eq. (C1) under the α-interpretation has the equivalent Stratonovich's form

$$\dot{x}^i = f^i(\mathbf{x}) + \left(\alpha - \frac{1}{2}\right)\Delta f^i(\mathbf{x}) + g^{ij}(\mathbf{x})\xi^j(t), \tag{C2}$$

where the drift $\Delta f^i(\mathbf{x})$ is[17]

$$\Delta f^i(\mathbf{x}) = \sum_j \sum_k g^{kj}\partial_{x^k} g^{ij}. \tag{C3}$$

Now, the ordinary calculus can be applied to the variable transformation due to Stratonovich's calculus. Thus, by multiplying $(g^{-1})^{ki}$, and using the variable transformation $q^k = H^k(\mathbf{x})$ with $\partial_{x^i} q^k = \partial_{x^i} H^k = (g^{-1})^{ki}$, we get the Langevin equation with the additive noise

$$\dot{q}^k = h^k(\mathbf{q}) + \xi^k(t), \tag{C4}$$

where $h^k(\mathbf{q}) = (g^{-1})^{ki}[H^{-1}(\mathbf{q})] \cdot \{f^i[H^{-1}(\mathbf{q})] + (\alpha - 1/2)\Delta f^i [H^{-1}(\mathbf{q})]\}$, and H^{-1} is the inverse of the variable transformation with $\partial(H^{-1})^i/\partial q^k = g^{ik}$.

To get the transition probability for Eq. (C4), we discretize the time into N segments: $t_0 < t_1 < \cdots < t_{N-1} < t_N$ with $\tau = t_n - t_{n-1}$ small, and let $q_n^k = q^k(t_n)$. As we have chosen the equivalent Stratonovich's form, the corresponding discretized Langevin equation should be

$$q_n^k - q_{n-1}^k - \left[\frac{h\left(q_n^k\right) + h\left(q_{n-1}^k\right)}{2}\right]\tau = W_n^k - W_{n-1}^k, \tag{C5}$$

where $W^k(t)$ is the Wiener process given by $dW^k(t) = \xi^k(t)dt$. Thus, the Jacobian for the variable transformation between $\mathbf{q}(t)$ and $\mathbf{W}(t)$ is a multiplication of the Jacobian for each component of the vector

$$J \approx \prod_{k=1}^{m} \exp\left[-\frac{\tau}{2}\sum_{n=1}^{N-1}\frac{dh^k(\mathbf{q}_n)}{dq_n^k}\right]. \tag{C6}$$

Then, with the property of Wiener process and the Chapman-Kolmogorov equation,[16] the transition probability is obtained

$$\begin{aligned} P(\mathbf{q}_N t_N|\mathbf{q}_0 t_0) &= \int_{\mathbf{q}_0}^{\mathbf{q}_N}\mathcal{D}\mathbf{q}\exp\left\{-\int_{t_0}^{t_N}\frac{1}{2\epsilon}\sum_{k=1}^{m}\left[(\dot{q}^k - h^k(\mathbf{q}))^2\right.\right. \\ &\quad \left.\left. + \frac{dh^k(\mathbf{q})}{dq^k}\right]dt\right\}, \end{aligned} \tag{C7}$$

where $\int_{\mathbf{q}_0}^{\mathbf{q}_N}\mathcal{D}\mathbf{q} \equiv \lim_{N\to\infty}\frac{1}{(\sqrt{2\pi\tau\epsilon})^m}\prod_{k=1}^{m}\prod_{n=1}^{N-1}\int\frac{dq_n^k}{\sqrt{2\pi\tau\epsilon}}$. The integral on the exponent obeys ordinary calculus, and the last term comes from the Jacobian.

To get the transition probability $P(\mathbf{x}_N t_N|\mathbf{x}_0 t_0)$ for Eq. (C1) under the α-interpretation, we change the variable by $x^i = (H^{-1})^i(\mathbf{q})$ with $\partial(H^{-1})^i/\partial q^k = g^{ik}$,

$$\begin{aligned} P(\mathbf{x}_N t_N|\mathbf{x}_0 t_0) &= \int_{\mathbf{x}_0}^{\mathbf{x}_N}\mathcal{D}\mathbf{x}\exp\left\{-\int_{t_0}^{t_N}{}_{\frac{1}{2}}dt\left\{\frac{1}{2\epsilon}\left(\dot{x}^i - f^i\right.\right.\right. \\ &\quad \left. - \left(\alpha - \frac{1}{2}\right)\Delta f^i\right)^{\tau}(g^{-1})^{ki}(g^{-1})^{kj} \\ &\quad \times\left(\dot{x}^j - f^j - \left(\alpha - \frac{1}{2}\right)\Delta f^j\right) \\ &\quad \left.\left. + \frac{1}{2}g^{ik}\frac{d}{dx^i}\left[(g^{-1})^{kj}\left(f^j + \left(\alpha - \frac{1}{2}\right)\Delta f^j\right)\right]\right\}\right\}, \end{aligned} \tag{C8}$$

where $\int_{\mathbf{x}_0}^{\mathbf{x}_N}\mathcal{D}\mathbf{x} \equiv \lim_{N\to\infty}\frac{1}{(\sqrt{2\pi\tau\epsilon})^m}\prod_{k=1}^{m}\prod_{n=1}^{N-1}\int\frac{[g^{-1}(\mathbf{x}_n)]^{ki}dx_n^i}{\sqrt{2\pi\tau\epsilon}}$. The symbol $\int_{t_0}^{t_N}{}_{\frac{1}{2}}$ means that the integral on the exponent obeys Stratonovich's calculus. Note that Eq. (C8) when $\alpha = 1/2$ is consistent with the result in Ref. 42.

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