The Journal of Risk and Uncertainty, 24:3; 195–212, 2002

Horizon Length and Portfolio Risk*

CHRISTIAN GOLLIER† gollier@cict.fr
Université de Toulouse, GREMAQ and IDEI, Place Anatole France, F-31042 Toulouse Cedex, France

RICHARD J. ZECKHAUSER richard_zeckhauser@Harvard.edu
Kennedy School of Government, Harvard University, Cambridge, MA 02138, USA

Abstract

In this paper, we compare the attitude towards current risk of two expected-utility-maximizing investors who are identical except that the first investor will live longer than the second one. It is often suggested that the young investor should take more risks than the old investor. We consider as a benchmark the case of complete markets with a zero risk-free rate. We show that a necessary and sufficient condition to assure that younger is riskier is that the Arrow-Pratt index of absolute tolerance (T) be convex. If we allow for a positive risk-free rate, the necessary and sufficient condition is T convex, plus $T(0) = 0$. It extends the well-known result that rational investors can behave myopically if and only if the utility function exhibits constant relative risk aversion.

Keywords: dynamic portfolio, convex risk tolerance, time horizon, asset demand

JEL Classification: D81

How should the length of one's investment horizon affect the riskiness of his portfolio? This question confronts startup companies choosing which ventures to pursue before they go public, ordinary investors building a nest egg, investment managers concerned with contract renewals and executives seeking strong performance before their stock options come due, among others.

Portfolio decisions are the focus of this analysis. However, the horizon-risk relationship extends beyond finance. Thus, students may vary their strategy on grades—say how venturesome a paper to write—over the course of the grading period; presidents may adjust the risk of their political strategies from early through mid-term and then as they approach an election.

Popular treatments suggest that short horizons often lead to more conservative strategies. Thus, the decisions of corporate managers, judged on their quarterly earnings, are said to focus too much on safe, short-term strategies, with underinvestment say in risky R&D projects. Privately-held firms, it is widely believed, secure substantial benefit from their ability to focus on longer-term projects. Mutual fund managers, who get graded regularly, are also alleged to focus on strategies that will assure a satisfactory short-term return, with

*Paper previously entitled "Should young people be less risk-averse?". This paper was written with the support of FNEGE and TIAA-CREF. We thank B.J. Pratt, E. Renault, J.-C. Rochet, P.A. Samuelson, M. Smith, and Ph. Weil for valuable conversations.
†To whom correspondence should be addressed.

long-term expectations sacrified.[1] However, as we will see, at the individual level, there is no strong empirical evidence that younger people take more portfolio risks.

Economists and decision theorists, speculators and bettors, have long been fascinated by the problem of repeated investment. Long ago Bernoulli provided the first motivation of utility theory when he confronted the St. Petersburg Paradox, whose components can be reformulated as payoffs from an infinite series of actuarially fair, double-till-you-lose bets. Successful speculators must manage their money effectively even when making a series of bets that are actuarially favorable. They must determine how much to allocate to each gamble given its odds, future prospects, and the time horizon.[2]

In recent years, this class of problems has been pursued in two different literatures, one in utility theory the other in dynamic investment strategies. A recurring theme in both is that the opportunity to make further investments affects how one should invest today. These two literatures have not merged, in part because a key question has gone unresolved. How should the length of the investment horizon affect the riskiness of one's investments? Some special cases have yielded results, notably logaritmic and power utility functions. And with any particular utility function and set of investment opportunities actual calculations, perhaps using a simulation, could answer that question within a dynamic programming framework. But the central theoretical question of the link between the structure of the utility function and the horizon-riskiness relationship remained unresolved. This paper attempts to resolve that question.

In the formal literature, the horizon-riskiness issue has received the greatest attention addressing portfolios appropriate to age. Samuelson (1989a) and several others have asked: "As you grow older and your investment horizon shortens, should you cut down your exposure to lucrative but risky equities?" Conventional wisdom answers affirmatively, stating that long-horizon investors can tolerate more risk because they have more time to recoup transient losses. This dictum has not received the imprimatur of science, however. As Samuelson (1963, 1989a) in particular points out, this "time-diversification" argument relies on a fallacious interpretation of the Law of Large Numbers: repeating an investment pattern over many periods does not cause risk to wash out in the long run.

Moreover, early models of dynamic risk-taking, such as those developed by Samuelson (1969) and Merton (1969) find no relationship between age and risk-taking. This is hardly surprising, since the models, like most in continuous time finance, employ utility functions exhibiting harmonic absolute risk aversion (HARA). This choice is hardly innocuous: If the risk-free rate is zero, myopic investment strategies are rational iff the utility function is HARA (Mossin, 1968). Given positive risk-free rates, empirically the normal case, only constant relative risk aversion (CRRA) utility functions allow for myopia. A myopic investor bases each period's decision on that period's initial wealth and investment opportunities, and maximizes the expected utility of wealth at the end of the period. The value function on wealth exhibits the same risk aversion as the utility function on consumption; the future is disregarded.

Judging by empirical evidence, constant relative risk aversion does not seem to be a reasonable assumption. Given CRRA, the optimal share of wealth invested in various assets would be independent of wealth level. However, as Kessler and Wolff (1991) show, the portfolios of households with low wealth contain a disproportionately large share of risk free assets. Measuring by wealth, over 80% of the lowest quintile's portfolio was in liquid assets, whereas the highest quintile held less than 15% in such assets.[3]

Guiso, Jappelli, and Terlizzese (1996) test the relation between age and risk-taking in a cross-section of Italian households. Their empirical results show that young people, contrary to the suggested strategy, actually hold the smallest proportion of risky assets in their portfolios. The share of risky assets increases by 20% to reach its maximum at age 61 (Guiso et al., 1996, p. 165).

Bodie, Merton, and Samuelson (1992) evaluate the advantage of young people, with more periods to live and work, who can adjust their labor supply in response to uncertain investment returns. This ability, they find, induces younger individuals with a CRRA utility function to take on greater investment risk than older ones. King and Leape (1987) find young people hold less diversified portfolios than old; they identify the accretion of financial information over the lifetime as a possible explanation.

This paper considers properties of utility functions. It examines their implications for the horizon-risk relationship, where investors seek to maximize the expected utility of terminal wealth. Our holy grail is the set of necessary and sufficient conditions on u that guarantee an unambiguous effect of age on portfolio risk. To forewarn the reader of our results, if the risk-free rate is zero and if markets are complete, younger people will take less portfolio risk if and only if the Arrow-Pratt measure of absolute risk tolerance $T(z) = -u'(z)/u''(z)$ is concave (this measure is the inverse of the measure of absolute risk aversion). Concave absolute risk tolerance is a condition involving the second, third and fourth derivatives of the utility function.

This work is also linked to the literature on compound risks. Samuelson (1963), looking at how many replications of an actuarially favorable gamble to take, opened the utility literature in this area, which considers more generally the effects of taking one gamble on the willingness to take other independent gambles. The multiple-period portfolio problem, in essence, considers successive sets of independent gambles. In each period the investor can shift his portfolio, depending on past results and the new time horizon.

A number of recent articles examine portfolio choice given independent risks. Pratt and Zeckhauser (1987) investigate the interaction between independent risks in a static model. They introduce the concept of proper risk aversion. Said crudely, risk aversion is proper if adding an undesirable risk to wealth has a negative impact on the attitude towards other risks. This concept is not directly useful to our problem, since future risks that will be undertaken are inherently desirable (otherwise they would not be undertaken). Moreover, Pratt and Zeckhauser do not consider problems with dynamic structures. The new concepts of standard risk aversion (Kimball, 1993) and risk vulnerability (Gollier and Pratt, 1996) both directly address portfolio composition with independent risks. These approaches all relate to required conditions on the third and fourth derivatives of the utility function.

Though our analysis focuses primarily on investment risks, it touches briefly on other background risks, such as labor income. Some of these risks are likely to prove more significant for a young than an old individual. For example, if both work, wages when old are more distant for the young individual, hence likely to be less certain. If work is only conducted when young, then only a young person faces undesirable background risk for wages. If utility is vulnerable to risk (Gollier and Pratt, 1996), the risk on human capital increases aversion to other independent risks. This will inhibit risk taking while young.

In real world circumstances, there may be significant transaction costs—e.g. taxes, brokerage costs, bid-ask spreads—associated with realloacting one's portfolio. Such costs reduce flexibility, and could significantly alter the desirable compositions of short- versus long-run portfolios. Gollier, Lindsey, and Zeckhauser (1997) identify sufficient conditions to assure flexibility promotes risk taking (relative risk aversion constant and less than 1 works). To draw a firm distinction between investigations of flexibility and horizon length, that paper only consider two-periods models, and this paper assumes perfect flexibility.

Our analysis considers a two-period model. The paper is organized as follows: In Section 2, we address a world with complete markets at every period, with a zero risk-free rate, consumption only at the end of one's second period and a smooth utility function. This model serves as a benchmark for the remainder of the paper. The proof of the necessity and sufficiency of convex risk tolerance is straightforward in that case. An incomplete market framework with two assets is considered in Section 3. Section 4 introduces a positive risk-free rate, whereas the possibility of intermediate consumption is considered in Section 5. Section 6 is devoted to the discussion of various extensions. Section 7 concludes.

1. A benchmark: complete markets

Investors choose their portfolio to maximize the expected utility of their terminal wealth. Young investors invest and live for two periods, old investors merely for one. Each investor is endowed with total discounted wealth w at the beginning of period 1, and has utility function u, which is assumed to be twice differentiable, increasing and concave. The risk free rate is zero. At the end of any period t, $t = 1, 2$, the realization of random variable $\tilde{s}_t$ is revealed. We assume that $\tilde{s}_1$ and $\tilde{s}_2$ are independent, but not necessarily identically distributed. F_t denotes the cumulative distribution function of $\tilde{s}_t$.

At the beginning of every period t, each surviving investor takes a decision $\theta_t \in \Theta_t$ that yields a payoff $\phi(\tilde{s}_t, \theta_t)$ at the end of the period. The problem of the old investor in period $t = 1$ is written as follows:

$$\max_{\theta \in \Theta_1} Eu(w + \phi(\tilde{s}_1, \theta)), \tag{1}$$

where E is the expectation operator. By backward induction, the problem of the young investor in period $t = 1$ is

$$\max_{\theta \in \Theta_1} Ev(w + \phi(\tilde{s}_1, \theta)), \tag{2}$$

where the value function v is defined as

$$v(z) = \max_{\theta \in \Theta_2} Eu(z + \phi(\tilde{s}_2, \theta)). \tag{3}$$

Variable z hereafter denotes the wealth available at the beginning of the second period, i.e., $z = w + \phi(s_1, \theta_1)$. The young investor is less risk-averse than the old one if v is less concave than u in the sense of Arrow-Pratt. The concept of less risk aversion is useful for many comparative statics problems. For example, Pratt (1964) showed that agent A will always accept more lotteries than agent B if and only if he is less risk-averse than B. Determining whether v is less concave than u only requires considering the second period problem. Since the comparative statics of a decrease in risk aversion (in period 1) are well-known, we focus the analysis on the second period problem here and in the remainder of the paper. This allows us to remove unnecessary time indices.

In this section, we assume a complete set of contingent claims markets. Investors must choose the optimal bundle $\theta = c(\tilde{s})$ of contingent claims to purchase. The exogenous (probability-adjusted) contingent price of a unit claim for state s is $\pi(s) \geq 0$. Since the risk free rate is zero, the price schedule must have an expectation equaling unity: $E\pi(\tilde{s}) = 1$. Let $c(s)$ be the demand for the contingent claim associated with event s. Problem (3) can now be rewritten as

$$v(z) = \max \quad Eu(c(\tilde{s})) \tag{4}$$

$$\text{s.t.} \quad E\pi(\tilde{s})c(\tilde{s}) = z. \tag{5}$$

Namely, the problem of the individual endowed in period 2 with wealth z is to maximize the expected utility of his final consumption under budget constraint (5). The standard first-order condition for (4) is

$$u'(c(s)) = \lambda\pi(s) \tag{6}$$

for any s. Parameter λ is the Lagrange multiplier associated with the budget constraint. It is a function of z. Let $T(c) = -u'(c)/u''(c)$ and $T_v(z) = -v'(z)/v''(z)$ denote the index of absolute risk tolerance for function u and function v, respectively. The following lemma is instrumental for our result.

Lemma 1. $T_v(z) = E\pi(\tilde{s})T(c(\tilde{s}))$, *where function* $c(s)$ *is the solution of program* (4).

Proof: As is well-known, we have that

$$v'(z) = \lambda.$$

Fully differentiating the above equality, inverting and multiplying by $-v'(z)$ yields

$$T_v(z) = -\frac{v'(z)}{v''(z)} = -\frac{\lambda}{\frac{d\lambda}{dz}}. \tag{7}$$

The value of the right-hand side of (7) is obtained by fully differentiating condition (6). It yields $u''(c(s))\frac{dc(s)}{dz} = \pi(s)\frac{d\lambda}{dz}$. Replacing $\pi(s)$ by $u'(c(s))/\lambda$ yields in turn

$$\frac{dc(s)}{dz} = -T(c(s))\frac{1}{\lambda}\frac{d\lambda}{dz}. \tag{8}$$

Now, by the budget constraint (5), we obtain

$$1 = E\pi(\tilde{s})\frac{dc(\tilde{s}))}{dz} = -\frac{1}{\lambda}\frac{d\lambda}{dz}E\pi(\tilde{s})T(c(\tilde{s})).$$

Combining the above condition with condition (7) concludes the proof. □

The absolute risk tolerance of an individual is a martingale with respect to risk-adjusted probabilities $dG(s) = \pi(s)dF(s)$. Cox and Leland (1982) and He and Huang (1994) obtain the same property in a continuous-time framework. G is a well-defined distribution function with $dG \geq 0$ and $\int dG(s) = \int \pi(s)dF(s) = 1$. The assumption that the risk-free rate is zero is central in this interpretation of T_v. If $\hat{E}$ denotes the expectation operator with respect to distribution G of $\tilde{s}$, we get

$$T_v(z) = \hat{E}T(c(\tilde{s})),$$

whereas the budget constraint is rewritten as $\hat{E}c(\tilde{s}) = z$. By Jensen's inequality, $T_v(z)$ is larger than $T(z) = T(\hat{E}c(\tilde{s}))$ if and only if T is convex. This proves the following result.

Proposition 1. *Consider the two-period investment problem with a complete set of markets at every period and a zero risk-free rate. Young investors are more* (*resp. less*) *risk-averse than old investors if the absolute risk tolerance of final wealth is concave* (*resp. convex*)*. If absolute risk tolerance is not concave*, *one can find a distribution of state prices such that young investors are not more risk-averse than old investors.*

This means that for investors with concave absolute risk tolerance, the longer the horizon the less risky the portfolio of contingent claims that is purchased. The converse property also holds: young investors are less risk-averse than old ones, independent of the state price distribution if and only if absolute risk tolerance is convex.

In the remainder of this paper, we will examine the robustness of this result with respect to different potential extensions such as introducing incompletness in the market structure, allowing for a positive risk free rate, or allowing for an intermediary consumption.

2. The standard portfolio problem

Most investment contexts do not offer complete contingent claims markets. With the "standard portfolio problem", there are two assets available in each period. The first offers a zero sure return. The return $\tilde{s}$ on the second asset is random; first-period decisions and outcomes are independent of second-period returns.

With such an investment program, the existence of the second-period investment opportunity exerts two effects on first-period risk taking, which we label the flexibility and background risk effects. In general, the investor will adjust optimal risk exposure in the second period to the outcome in the first. With decreasing absolute risk aversion, for example, the better the first-period outcome the more of the risky asset is purchased in the second period. The opportunity to adjust one's portfolio is an advantage; this flexibility effect always reduces aversion to current risks.

Beyond this, the presence of a second-period risk can be analyzed as a "background risk" with respect to the first-period choice problem. With a zero risk-free rate available, the opportunity to make a second-period investment is weakly advantageous, and is only neutral with a CARA utility function or when none of the risky asset is ever purchased, i.e., when $E\tilde{s} = 0$. There is a lively literature on the effect of a background risk on attitudes towards another risk, but it does not address the case of favorable risks.[4] The critical question is when can the flexibility effect be assured to overcome (be weaker than) a potential negative background risk effect.

2.1. The model

Let $\theta = \alpha$ denote the demand for the risky asset. It depends upon the level z of wealth available at the beginning of the second period. The investor selects $\alpha(z)$ to solve program (9).

$$v(z) = \max_a Eu((z-a) + a(1+\tilde{s})) = \max_a \; Eu(z + a\tilde{s}). \tag{9}$$

The problem is to establish the relationship between the degree of concavity of u and the degree of concavity of v. If $\tilde{s}$ is a two-state random variable, markets are complete, and we know from Proposition 1 that the concavity (resp. convexity) of T is necessary and sufficient for a positive (resp. negative) effect of age on risk-taking. We examine the robustness of these results to the introduction of additional states. The first-order condition on $\alpha(z)$ is written as

$$E\tilde{s}u'(z + \alpha(z)\tilde{s}) = 0. \tag{10}$$

Fully differentiating this condition yields

$$\frac{d\alpha}{dz} = -\frac{E\tilde{s}u''(z+\alpha\tilde{s})}{E\tilde{s}^2u''(z+\alpha\tilde{s})}. \tag{11}$$

As is well-known, $\alpha(z)$ is increasing, constant or decreasing depending upon whether absolute risk aversion is decreasing, constant or increasing. The envelope theorem yields

$$v'(z) = Eu'(z + \alpha\tilde{s}). \tag{12}$$

Fully differentiating again this equality yields

$$v''(z) = Eu''(z+\alpha\tilde{s}) + \frac{d\alpha}{dz}E\tilde{s}u''(z+\alpha\tilde{s}). \tag{13}$$

Combining conditions (11), (12) and (13) allows us to write

$$\begin{aligned} -\frac{v''(z)}{v'(z)} &= -\frac{Eu''(z+\alpha\tilde{s})}{Eu'(z+\alpha\tilde{s})} + \frac{d\alpha}{dz}\frac{-E\tilde{s}u''(z+\alpha\tilde{s})}{Eu'(z+\alpha\tilde{s})}. \\ &= -\frac{Eu''(z+\alpha\tilde{s})}{Eu'(z+\alpha\tilde{s})} + \frac{[E\tilde{s}u''(z+\alpha\tilde{s})]^2}{E\tilde{s}^2u''(z+\alpha\tilde{s})Eu'(z+\alpha\tilde{s})}. \end{aligned} \tag{14}$$

Notice that by the Cauchy-Schwarz inequality, $[E\tilde{s}u''(z+\alpha\tilde{s})]^2$ is smaller than $E\tilde{s}^2u''(z+\alpha\tilde{s})Eu''(z+\alpha\tilde{s})$. Combining this property with Eq. (14) implies that $-v''/v'$ is positive: a longer time horizon length never transforms risk-averse investors into risk-lovers.

Our aim in this paper is to compare $-v''(z)/v'(z)$ to $-u''(z)/u'(z)$. Our two effects emerge from condition (14). The flexibility effect is expressed by the second term in the right-hand side of (14), which is negative.[5] The background risk effect corresponds to the first term in the right-hand side of (14). Future risk $\tilde{y} \equiv \alpha\tilde{s}$ can be interpreted as a background risk with respect to the independent current risk. If α would be fixed and independent of the realization of the first-period risk (i.e., $d\alpha/dz = 0$), the degree of risk aversion in period 1 would equal $-Eu''(z+\tilde{y})/Eu'(z+\tilde{y})$. This can be either larger or smaller than $-u''(z)/u'(z)$.

2.2. *A sufficient condition for a longer time horizon to reduce risk taking*

We now show that the concavity of absolute risk tolerance is sufficient to guarantee that younger people should purchase less of the risky asset in the standard portfolio problem. Consider a particular wealth level z. Without loss of generality, let us normalize the demand of the risky asset with wealth z to be unity ($\alpha(z) = 1$). Let F denote the cumulative distribution function of $\tilde{s}$. Let us also define $\tilde{x}$ as the random variable with cumulative distribution function G, with

$$dG(x) = \frac{u''(z+x)\,dF(x)}{\int u''(z+s)\,dF(s)}.$$

We verify that dG is positive under risk aversion, and that $\int dG(x) = 1$. Using this change of variable and conditions (10) and (14), $-v''(z)/v'(z)$ is larger than $-u''(z)/u'(z)$ if

$$E\tilde{x}T(z+\tilde{x}) = 0 \Longrightarrow \frac{1}{T(z)} \leq \frac{1}{ET(z+\tilde{x})} - \frac{(E\tilde{x})^2}{E\tilde{x}^2 ET(z+\tilde{x})},$$

or, equivalently,

$$E\tilde{x}T(z+\tilde{x}) = 0 \Longrightarrow T(z)\left[1 - \frac{(E\tilde{x})^2}{E\tilde{x}^2}\right] \geq ET(z+\tilde{x}). \tag{15}$$

It is noteworthy that this way of presenting the problem directly yields the result due to Mossin (1968) that myopia is optimal if absolute risk tolerance is linear in wealth. Suppose in accordance that $T(w) = a + bw$. Let μ_k denote the k-th moment of $\tilde{x} : \mu_k = E\tilde{x}^k$. It is easy to check that condition (15) holds with an equality, implying Mossin's result:

$$a\mu_1 + b(z\mu_1 + \mu_2) = 0 \Longrightarrow (a+bz)\left[1 - \frac{\mu_1^2}{\mu_2}\right] = a + b(z+\mu_1).$$

Condition (15) is a necessary and sufficient condition for younger people to purchase less of the risky asset, independent of the initial wealth and of the distribution of the risky asset (with a positive expected excess return). We know that this condition is satisfied when $\tilde{x}$ has a two-point support if and only if absolute risk tolerance is concave.[6] In the next lemma, we show that it is enough to look at this condition for $\tilde{x}$ with a three-point support to get it for any random variable.

Lemma 2. *Condition* (15) *holds for any* $\tilde{x}$ *if it holds for any* $\tilde{x}$ *with a three-point support.*

Proof: We look for the characteristics of the $\tilde{x}$ which are the most likely to violate condition (15). To do so, we solve the following problem for any scalar υ:

$$\begin{aligned} \min_{dG\geq 0} \quad & T(z)(1-\upsilon\int x dG(x)) - \int T(z+x)\,dG(x) \\ \text{subject to} \quad & \int xT(z+x)\,dG(x) = 0; \\ & \int x(1-\upsilon x)\,dG(x) = 0; \\ & \int dG(x) = 1. \end{aligned} \tag{16}$$

If the solution of this problem is positive for any υ, condition (15) would be proven. Observe that the above problem is a standard linear programming problem. Therefore the solution contains no more than three x that are such that $dG(x) > 0$, since they are three equality constraints in the program. Thus, condition (15) would hold for any $\tilde{x}$ if it holds for any $\tilde{x}$ with three atoms. This concludes the proof of Lemma 2.2. Notice that Pratt and Zeckhauser (1987) and Kimball (1993) use the same kind of technique. □

Going from two-point supports to three-point supports raises an interesting difficulty. Take any three-point $\tilde{x}$ that satisfies the first-order condition $E\tilde{x}T(z+\tilde{x}) = 0$. It is easy to check that we can find a pair of two-point random variables $(\tilde{y}_1, \tilde{y}_2)$ and a probability $\lambda \in [0, 1]$ such that $\tilde{y}_i$ satisfies the first-order condition $E\tilde{y}_iT(z+\tilde{y}_i) = 0$, $i = 1, 2$, and $\tilde{x}$ is the compound lottery $(\lambda, \tilde{y}_1; 1-\lambda, \tilde{y}_2)$. Suppose that absolute risk tolerance be concave. By Proposition 1, it implies that

$$T(z)\left[1 - \frac{(E\tilde{y}_i)^2}{E\tilde{y}_i^2}\right] \geq ET(z+\tilde{y}_i), \tag{17}$$

for $i = 1, 2$. Observe that

$$\lambda ET(z+\tilde{y}_1) + (1-\lambda)T(z+\tilde{y}_2) = ET(z+\tilde{x}). \tag{18}$$

The expectation operator is linear in probabilities. Thus, the weighted sum of the right-hand side of inequalities (17) for $i = 1$ and $i = 2$ equals the right-hand side of inequality (15). We would be done if the same operation could be applied with the left-hand sides, i.e., if the left-hand sides would also be the expected value of a function of the random variable. This is not the case, and that explains why we cannot extend the result from two-point supports to three-point supports and, therefore, to all random variables. It happens that $(E\tilde{x})^2/E\tilde{x}^2$ is a convex function of the vector of probabilities of $\tilde{x}$. It implies that[7]

$$\lambda\frac{(E\tilde{y}_1)^2}{E\tilde{y}_1^2} + (1-\lambda)\frac{(E\tilde{y}_2)^2}{E\tilde{y}_2^2} \geq \frac{(E(\lambda\tilde{y}_1 + (1-\lambda)\tilde{y}_2))^2}{E\left(\lambda\tilde{y}_1^2 + (1-\lambda)\tilde{y}_2^2\right)} = \frac{(E\tilde{x})^2}{E\tilde{x}^2}. \tag{19}$$

From conditions (17) and (19), we conclude that the concavity of absolute risk tolerance implies that

$$T(z)\left[1-\frac{(E\tilde{x})^2}{E\tilde{x}^2}\right] \geq T(z)\left[1-\left\{\lambda\frac{(E\tilde{y}_1)^2}{E\tilde{y}_1^2}+(1-\lambda)\frac{(E\tilde{y}_2)^2}{E\tilde{y}_2^2}\right\}\right] \geq ET(z+\tilde{x}) \quad (20)$$

for all three-point $\tilde{x}$ that is distributed as $(\tilde{y}_1, \lambda; \tilde{y}_2, 1-\lambda)$, with $E\tilde{y}_i T(z+\tilde{y}_i) = 0$. Thus, we conclude that condition (15) is satisfied, i.e., that younger people take less risk if absolute risk tolerance is concave.

Proposition 2. *Consider the two-period investment problem with a zero yield risk-free asset and another risky asset. Young investors are more risk-averse than old investors if the absolute risk tolerance of final wealth is concave.*

But the reverse is not true here. When markets are incomplete, the convexity of T is not sufficient for younger people to be less risk-averse. This is due to the convexity of $\frac{(E\tilde{x})^2}{E\tilde{x}^2}$ in the vector of probabilities. This is confirmed by the following counterexample. Take $T(w) = w^{-2}$, which is convex, and $z = 1$. Let $\tilde{x}$ be distributed as $(-0.5, 0.10659; 1, 0.83266; 10, 0.06075)$.[8] It is easily verified that $E\tilde{x}T(z+\tilde{x}) = 0$, but

$$T(z)\left[1-\frac{(E\tilde{x})^2}{E\tilde{x}^2}\right] = 0.723 > 0.635 = ET(z+\tilde{x}).$$

Condition (15) is satisfied for this three-point distribution, although absolute risk tolerance is convex, not concave.

To sum up, the concavity of absolute risk tolerance is sufficient for younger people to purchase less of the risky asset. The age of the investor does not influence the optimal portfolio composition when absolute risk tolerance is linear. But the convexity of absolute risk tolerance does not imply that younger people purchase more of the risky asset.

2.3. A sufficient condition for time horizon to increase risk taking

Proposition 2 is not helpful in confirming the often suggested advice to take a less risky position when the time horizon shortens. It just confirms that the convexity of absolute risk tolerance is necessary for this guideline rule to be optimal. The objective of this section is to provide some sufficient conditions for this guideline. A sufficient condition that we already mentioned is that the distribution of returns has a two-point-support, together with $T'' > 0$. This is not very helpful either because few real world risks are binary. Exceptions are some forms of unemployment risks because of the presence of large fixed costs in hiring and layoff procedures, together with some insurable risks.

Another sufficient condition may be obtained by using the same kind of technique as in the previous section. Absolute risk aversion is denoted $A(z) = 1/T(z)$. Let $\tilde{y}$ have a cumulative distribution function H that is defined by

$$dH(y) = \frac{u'(z+y)\,dF(y)}{\int u'(z+s)\,dF(s)}.$$

We verify again that dH is positive, and that $\int dH(x) = 1$. Using this change of variable together with conditions (10) and (14), $-u''(z)/u'(z)$ is larger than $-v''(z)/v'(z)$ if

$$E\tilde{y} = 0 \Longrightarrow A(z) \geq EA(z+\tilde{y}) - \frac{(E\tilde{y}A(z+\tilde{y}))^2}{E\tilde{y}^2 A(z+\tilde{y})} \tag{21}$$

Suppose that absolute risk *aversion* is concave. Then, $EA(z+\tilde{y})$ is smaller than $A(z)$ by Jensen's inequality. It directly implies condition (21).

Proposition 3. *Consider the two-period investment problem with a zero yield risk-free asset and another risky asset. Young investors are less risk-averse than old investors if one of the following two conditions is satisfied*:

- *the future risk is binary and the absolute risk tolerance of final wealth is convex*; *or*
- *absolute risk aversion is concave.*

The above propositions provide qualitative results. We would like to know the quantitative magnitude of the duration or age effect on risk taking, and could determine that if we knew the first four derivatives of the utility function. For now, consider an illustration for the case of $u(z) = z + \ln z$ which yields a convex risk tolerance, with $\tilde{s} = (-1, 2; 1/2, 1/2)$. After some tedious computations, we get $T(5) = 30.0$, whereas $T_v(5) = 64.1$: the young investor is more than twice as risk-tolerant as the old investor. This implies that if the expected excess return of the risky asset is small, the young will invest twice as much in it as will the old. This is an example of a utility function for which the time horizon effect is large.

2.4. *Is T concave or convex?*

To apply the results above we must know whether absolute risk tolerance is concave or convex with respect to wealth. First principles do not tell us, and there is no empirical evidence on this particular question. However, we can look at some of the theoretical and empirical implications of the sign of T'' on risk choices, savings and portfolio decisions.

If T is convex:

1. The demand for risky assets is convex with wealth, at least when the variance of returns is small with respect to the expected excess return. In fact, we observe that investment in stocks as a function of wealth is highly convex. But this can be due to many other factors, such as fixed costs to participate in stock markets, or common factors, such as sophistication or risk-taking propensities, that increase income and stock investments.
2. The marginal propensity to consume (MPC) out of wealth is decreasing, at least under certainty. The MPC does decrease with wealth.
3. The equity premium is an increasing function of wealth inequality (see Gollier, 2001a). Thus, the convexity of T could contribute to explain the equity premium puzzle.

If T is concave:

1. Risk aversion is proper (see Pratt and Zeckhauser, 1987), standard (see Kimball, 1993) and risk vulnerable (see Gollier and Pratt, 1996). This is proven by Hennessy and Lapan

(1998). These properties have intuitively appealing implications; for example, the presence of a zero-mean background risk raises the aversion to other independent risks.
2. The certainty equivalent value of a lottery is a concave function of the payoff vector (see Hennessy and Lapan, 1998).This result provides a simple way to test the concavity of T using experimental data.

In sum, there are arguments suggesting that T is convex, and equally appealing arguments indicating that it is concave. In sum, broad implications flow from the shape of T, but only direct and detailed empirical study will tell us what shape to expect.

3. Positive risk-free rate

A zero risk-free rate was invoked in Sections 2 and 3 to eliminate a wealth effect that would introduce noise with respect to our main message. Over the period 1926–1987, the arithmetic mean annual return of inflation-adjusted U.S. Treasury Bills was 0.5% (geometric mean is 0.4%), whereas it was 1.9% (geometric mean is 1.7%) for inflation-adjusted intermediate-term U.S. Government Bonds.[9] Although these rates are well below those economists usually discuss, they still suggest that a positive risk-free rate should be considered. Let $R \geq 1$ denote one plus the risk-free rate. In the complete market framework discussed in Section 2, the problem was written as

$$v_R(z) = \max \quad Eu(c(\tilde{s})) \tag{22}$$

$$\text{s.t.} \quad E\pi_R(\tilde{s})c(\tilde{s}) = z. \tag{23}$$

Function v_R is the value function of the young investor under a positive risk-free rate $R = E\pi_R(\tilde{s})$. Define $\pi(s) = \pi_R(s)/R$. By definition, $E\pi(\tilde{s}) = 1$. Multiplying budget constraint (23) by R makes it equivalent to

$$E\pi(\tilde{s})c(\tilde{s}) = Rz.$$

It is thus seen that

$$v_R(z) = v(Rz), \tag{24}$$

where v is the value function under $R = 1$ that has been examined in Section 2. The same equivalence holds in the standard portfolio problem. From condition (24), one can state that a positive interest rate changes the attitude towards current risks (that makes z risky) in two ways. First, as mentioned above, there is a wealth effect expressed by the fact that $ER\tilde{z} > E\tilde{z}$. Second, there is a magnification effect. The risk borne today will be magnified by investing the part of the payoff from the current risk in the risk-free asset during the next period.

Differentiating condition (24) twice yields

$$T_{v_R}(z) = \frac{1}{R}T_v(Rz), \tag{25}$$

where T_{v_R} is the absolute risk tolerance of the young investor. The two effects of a positive risk-free rate are apparent in Eq. (25). The wealth effect takes the form of absolute risk tolerance measured at $Rz > z$. The magnification effect is equivalent to absolute risk tolerance divided by factor R.

Proposition 4. *In the complete markets model, if the risk-free rate can take any nonnegative value, the young investor will always be less risk-averse than the old investor if and only if the absolute risk tolerance of the utility function u is convex and superhomogeneous.*[10]

Proof: The proof of sufficiency proceeds as follows: Proposition 1 and the convexity of T yield

$$T_{v_R}(z) = \frac{1}{R}T_v(Rz) \geq \frac{1}{R}T(Rz).$$

The superhomogeneity of T yields in turn

$$T_{v_R}(z) \geq T(z).$$

For the necessity of convex absolute tolerance, take $R = 1$ and apply Proposition 1. For an outline of the proof of the necessity of superhomogeneity, suppose by contradiction that T is not superhomogeneous. Then, there exists z and $R > 1$ such that $T(z) > \frac{1}{R}T(Rz)$. Consider a distribution of $\tilde{s}$ with an arbitrary small expectation. Then, $\alpha(Rz) \cong 0$ and $\alpha'(Rz) \cong 0$, so that $T_v(Rz) \cong T(Rz)$. Thus, $T(z) > \frac{1}{R}T_v(Rz) = T_{v_R}(z)$, a contradiction. □

Superhomogeneity of T guarantees that a young investor who put his end-of-period-1 wealth in the risk-free asset is less risk-averse in period 1 than the old investor who simply consumes his end-of-period-1 wealth. If the young investor does not invest all his wealth in the risk-free asset in period 2, the convexity of T reinforces the magnification effect. Samuelson (1989b) showed that the same age-phased property holds in the case $u(z) = (z - S)^{1-\gamma}/(1 - \gamma)$, $1 \neq \gamma > 0$, or $u(z) = \ln(z - S)$, for any (minimum-consumption) constant $S > 0$. Since absolute risk tolerance is linear and superhomogeneous. In these cases, age-phasing occurs only due to the risk-free rate effect. A counterexample is obtained with the exponential utility function, which yields $T_{v_R}(z) = \frac{1}{R}T(z)$. In that case, only the magnification effect is at work, with young investors reducing their risky investments by factor R in relation to the strategy of old investors.

Notice that a positive function which is superhomogeneous must be increasing. Therefore, unlike the results presented in Sections 2 and 3 in which the convexity of T alone was relevant, here we need the absolute risk tolerance to be convex and increasing. If we limit the analysis to utility functions that are defined over R^+ (Remark: this excludes the cited Samuelson example), then the superhomogeneity condition can be simplified. Indeed, if a function T defined over R^+ is positive and convex, superhomogeneity is equivalent to the condition that this function evaluated at 0 is zero. We can thus rewrite the above result as follows.

Corollary 1. *Consider the complete markets model with a utility function defined over R^+. In the absence of intermediate consumption and with a nonnegative risk-free rate, the* *young investor is always less risk-averse than the old investor if and only if T is convex and $T(0) = 0$.*

Mossin (1968) showed that introducing a positive risk-free rate in the standard portfolio problem reduces the set of utility functions for which myopia is rational. Namely, with $R > 1$, myopia is rational if $T(z)$ is not only linear in z, but also proportional in z, i.e., $T(0) = 0$. The same restriction applies here.

A side result of this analysis is obtained by examining the problem of access to markets. Consider two young investors who live for two periods. The two investors have access to the risk-free asset market, but only one of the two has access to the market for contingent claims. Should he be more or less risk-averse than the other in period 1? It depends upon whether $v_R(z) = v(Rz)$ is more or less concave than $u_R(z) = u(Rz)$, or, in other words, whether $\frac{1}{R}T_v(Rz)$ is larger or smaller than $\frac{1}{R}T(Rz)$. From Proposition 1, we directly get the result that the investor who will have access to the risky asset market in the future takes more risk today if and only if T is convex.

4. Intermediate consumption

Thus far we have assumed that the investor's utility function applies solely to terminal wealth; he has a pure investment problem. In real world contexts, investors consume a portion of their lifetime wealth each period. Introducing intermediate consumption raises several interesting questions. Kimball (1990) and others have addressed the problem of optimal saving given exogenous future uncertainty. In this paper, we extend this approach by considering future risks that are endogenous. Allowing for intermediate consumption makes young people potentially more willing to take risks than in the pure investment problem because current risks can be attenuated by spreading consumption over time. For example, in the case of the absence of future risk opportunity, if complete consumption-smoothing is optimal, a \$1 loss on current investment will be split into a fifty cent reduction in current consumption and a fifty cent reduction in future consumption. Given a concave utility on consumption, that has a smaller effect on total utility than a straight \$1 reduction in final consumption in the investment problem.

We consider a model in which consumption is chosen at the end of each period, after having observed the realization of the random variable characterizing that period's risk.[11] For tractability and to allow for time-consistency of decision, we consider an expected utility model with a time-separable utility function u on consumption. If parameter β is the discount factor, the dynamic structure of the problem is described by the value function v_R that is defined as follows:

$$v_R(z) = \max_{c,\theta\in\Theta} u(c) + \beta Eu(R(z-c) + \phi(\tilde{s},\theta)). \tag{26}$$

Using the definition of function v as in (3), this can be rewritten as:

$$\begin{aligned} v_R(z) &= \max_{c} u(c) + \beta \max_{\theta\in\Theta} Eu(R(z-c) + \phi(\tilde{s},\theta)) \\ &= \max_{c} u(c) + \beta v(R(z-c)). \end{aligned} \tag{27}$$

The first-order condition on c obtains:

$$u'(c) = \beta R v'(R(z-c)). \tag{28}$$

It yields $c'(z) = \beta R^2 v''(R(z-c))/[u''(c) + \beta R^2 v''(R(z-c))]$. The envelope theorem gives $v'_R(z) = \beta R v'(R(z-c)) = u'(c)$. Differentiating this equality and rearranging terms yields

$$T_{v_R}(z) = T(c(z)) + \frac{1}{R} T_v(R(z - c(z))). \tag{29}$$

If T is convex, one can apply Proposition 1 to write:

$$T_{v_R}(z) \geq T(c(z)) + \frac{1}{R} T(R(z - c(z))).$$

Using again the convexity of T together with Jensen's inequality yields

$$\begin{aligned} T_{v_R}(z) &\geq \frac{R+1}{R}\left[\frac{R}{R+1} T(c(z)) + \frac{1}{R+1} T(R(z - c(z)))\right] \\ &\geq \frac{R+1}{R} T\left(\frac{R}{R+1} z\right). \end{aligned}$$

If we assume that the return on the risk-free asset is larger than -1, i.e., $R \geq 0$, the left-hand side of the last inequality is larger than $T(\frac{R+1}{R}\frac{R}{R+1}z) = T(z)$ if and only if T is subhomogeneous. This proves the sufficiency of the following proposition. The proof of its necessity is easily obtained by contradiction, and is left to the reader.

Proposition 5. *In the complete markets model, if intermediate consumption is allowed, the young investor is always less risk-averse than the old investor if and only if the absolute risk tolerance of the utility function u is convex and subhomogeneous.*

When intermediate consumption is introduced, the subhomogeneity of T must be added to its convexity in order to get an unambiguous comparative static property. This reverses the superhomogeneity condition that proves necessary in the investment problem with a positive risk-free rate. There is a simple intuition for this reversal. As an illustration, consider the above model with $\beta = R = 1$ and $\tilde{s} = 0$ with probability 1. Then $v_R(z) = 2u(z/2)$: consumption is perfectly smoothed in the two periods, with no precautionary saving in the absence of any future risk. Value function $v_R(z)$ thus has the same degree of concavity as $u(z/2)$. There reduces by 50% the risk borne in period 1. Namely, the ability to split the current risk makes the young investor more willing to accept risk than the old investor. This is a correct interpretation of the well-known concept of "time-diversification". As superhomogeneity was necessary to take care of the magnification effect of a positive risk-free rate in the investment problem, the condition of subhomogeneity is necessary to take care of the reduction effect of time-diversification with intermediate consumption. Convexity of T assures that the existence of the future risk does not reverse this effect. This reversal has already been observed by Samuelson (1989b) in the case of $u(z) = (z - S)^{1-\gamma)}/(1-\gamma)$, $S > 0$, $\gamma > 0$, in which only the magnification/reduction effect exists.

The limit case is obtained with T being linear and homogeneous, i.e., the CRRA case. Not surprisingly, this is the standard assumption in dynamic risk-taking models.

5. Extensions

Our analysis addresses cases where utility functions are smooth (twice differentiable) and all risks relate to the performance of one's portfolio. In extensions developed elsewhere (Gollier, 2001c, Chapter 11), we show that horizon length and risk can interact in much the same way when marginal utility is not differentiable. We then present results when there are background risks, including shocks at retirement, and shocks to labor income over one's lifetime. Limiting the analysis to HARA utility functions, the question is to determine whether background risk can convexify the absolute risk tolerance of the indirect utility function. In brief, we show that background risks can push the relationship between horizon length and portfolio risk in either direction.

A crucial assumption of this paper is about the absence of predictability in future assets returns. It is intuitive that the effect of the time horizon on the optimal risk-taking depends on the statistical relationship between current returns and future returns. In particular, in case of mean-reversion, longer time horizons should induce more risk taking. As shown by Kim and Omberg (1996), this is optimal in the CRRA case only if relative risk aversion is larger than unity. Gollier (2001b), using the same technique as developed in this paper, explores the underlying mechanisms that lead to this result.

6. Conclusion

Time and uncertainty are inextricably entwined in investment decisions. Economists have developed useful instruments to treat either uncertainty or time in decision processes. When models entail both time and uncertainty, policy prescriptions are available only under very specific conditions on preferences. Namely, analytical results obtain only when absolute risk tolerance is linear in wealth. When it is linear, no relationship emerges between age and the attitude towards risk. In this paper, we have extended the constellation of issues relating time, uncertainty, and risk preference to a range of other cases. In particular, we have determined the qualitative effect of age on risk-taking in relation to the properties of the utility function for consumption or wealth. The concavity or convexity of absolute risk tolerance proves critical to our positive results. However, its implications for risk-taking depend critically on the economic environment, e.g., whether markets are complete or the risk-free rate is zero.

The findings presented here illuminate the essential link between horizon length and risk aversion. Some of their most important applications, however, may lie somewhat afield. For example, they may help us to revisit the role of liquidity constraints in dynamic models in finance, to consider how the frequency of market openings affects optimal risk-taking, to examine the effect of return predictability on dynamic portfolio strategies, or to better understand the advantages for accurate description of adding a second period, with rebalancing, to otherwise static investment models.

Notes

1. The experience of the U.S. mutual fund Twentieth Century Giftrust is instructive. It requires monies to be left with it for 10 years at least. Managers of the fund suggest that thanks to this 10-year no-withdrawal rule the fund was able to return 24 percent annually since 1985, nearly 10 points better than the S & P500. (See *Newsweek* 6/19/95, page 60.)
2. With n periods, utility function $u(w_n)$ on terminal wealth and initial wealth w_0, he must lay out a contingent strategy of how much to put at risk each period. This dynamic programming problem becomes tractable with an analytical solution only for HARA utility functions.
3. If individuals have CRRA utility functions but differ in risk aversion, given the extraordinary relative performance of equities in the postwar world, empirically we will find that those with lesser risk aversion will hold more stocks and hold greater wealth.
4. Pratt and Zeckhauser (1987) introduce the concept of properness for situations where background risk is undesirable. Kimball (1993) develops standardness, when background risk increases expected marginal utility. Gollier and Pratt (1995) consider background risks with a non-positive mean. None of these restrictions is satisfied by background risk $\alpha\tilde{s}$ under consideration here.
5. Gollier, Lindsey, and Zeckhauser (1997) examine the effect of flexibility on the acceptance of risk.
6. This can easily be checked directly with condition (15).
7. The proof of this claim is obtained by easy manipulations of (19). Denoting $a_i = E\tilde{y}_i$ and $b_i = E\tilde{y}_i^2 > 0$, condition (19) is equivalent to

   $$2a_1a_2 \leq a_1^2\frac{b_2}{b_1} + a_2^2\frac{b_1}{b_2}.$$

   This is in turn equivalent to

   $$\left[a_1\sqrt{\frac{b_2}{b_1}} + a_2\sqrt{\frac{b_1}{b_2}}\right]^2 \geq 0,$$

   which is always true.
8. These probabilities are the solution to max version of program (16) when the cumulative distribution function G has its support in $(-0.5,1,10)$.
9. See Ibbotsen and Sinquefield (1989), page 74.
10. A function g is superhomogeneous (resp. subhomogeneous) if and only if $g(Rz) \geq$ (resp. $\leq$) $Rg(z)$ for all z and all $R \geq 1$.
11. See Bodie, Merton, and Samuelson (1992) where labor supply, not consumption, adjusts to counterbalance poor investment outcomes.

References

Bodie, Z., R.C. Merton, and W.F. Samuelson. (1992). "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model," *Journal of Economic Dynamics and Control* 16, 427–449.

Cox, J. and H. Leland. (1982). "Notes on Dynamic Investment Strategies," *Proceedings of Seminar on the Analysis of Security Prices, Center for Research in Security Prices* (CRSP), Graduate School of Business, University of Chicago.

Gollier, C. (1993). "Economic Theory of Risk Exchanges: A Review." In G. Dionne (ed.), *Contributions to Insurance Economics.* Dordrecht: Kluwer Academic Press, pp. 3–23.

Gollier, C. (2001a). "Wealth Inequality and Asset Pricing," *Review of Economic Studies* 68, 181–203.

Gollier, C. (2001b). "Optimal Dynamic Portfolio with First-Order and Second-Order Predictability," University of Toulouse.

Gollier, C. (2001c). *The Economics of Risk and Time*. Cambridge: MIT Press.

Gollier, C., J. Lindsey, and R.J. Zeckhauser. (1997). "Investment Flexibility and the Acceptance of Risk," *Journal of Economic Theory* 76, 219–242.

Gollier, C. and J.W. Pratt. (1996). "Risk Vulnerability and the Tempering Effect of Background Risk," *Econometrica* 64, 1109–1124.

Guiso, L., T. Jappelli, and D. Terlizzese. (1996). "Income Risk, Borrowing Constraints, and Portfolio Choice," *American Economic Review* 86, 158–172.

He, H. and C.-F. Huang. (1994). "Consumption-Portfolio Policies: An Inverse Optimal Problem," *Journal of Economic Theory* 62, 257–293.

Hennessy, D.A. and H.E. Lapan. (1998). "On the Nature of Certainty Equivalent Functionals, Discussion Paper," Iowa State University.

Huberman, G. and S. Ross. (1983). "Portfolio Turnpike Theorems, Risk Aversion, and Regularly Varying Utility Functions," *Econometrica* 51, 1345–1361.

Ibbotson, R.G. and R.A. Sinquefield. (1989). *Stocks, Bonds, Bills, and Inflation: Historical Returns (1926–1987)*. New York: Dow Jones-Irwin.

Kessler, D. and E.N. Wolff. (1991). "A Comparative Analysis of Household Wealth Patterns in France and in the United States," *Review of Income and Wealth* 37, 249–266.

Kim, T.S. and E. Omberg. (1996). "Dynamic Nonmyopic Portfolio Behavior," *Review of Financial Studies* 9, 141–161.

Kimball, M.S. (1990). "Precautionary Saving in the Small and in the Large," *Econometrica* 58, 53–73.

Kimball, M.S. (1993). "Standard Risk Aversion," *Econometrica* 61, 589–611.

King, M.A. and J.I. Leape. (1987). "Asset Accumulation, Information, and the Life Cycle," NBER (Cambridge, MA) Working Paper 2392.

Merton, R.C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics* 51, 247–257.

Mossin, J. (1968). "Optimal Multiperiod Portfolio Policies," *Journal of Business* 215–229.

Pratt, J.W. and R. Zeckhauser. (1987). "Proper Risk Aversion," *Econometrica* 55, 143–154.

Samuelson, P.A. (1963). "Risk and Uncertainty: The Fallacy of the Law of Large Numbers," *Scientia* 98, 108–113.

Samuelson, P.A. (1989a). "The Judgement of Economic Science on Rationale Portfolio Management: Indexing, Timing, and Long-Horizon Effects," *Journal of Portfolio Management* Fall 1989, 3–12.

Samuelson, P.A. (1989b). "A Case at Last for Age-Phased Reduction in Equity," *Proceedings of the National Academy of Sciences* 86, 9048–9051.