Euclid and beyond: towards a long-term history of deductivity

Mark J. Schiefsky, Malcolm D. Hyman

The deductive organization of Euclid's Elements serves as a model for mathematical and scientific texts in a variety of subjects from antiquity through the early modern period, not only in the West but also in the Islamic world and beyond. The study of this tradition demands formal methods that will explicitly and unambiguously represent the deductive structures of given texts. This paper explores the creation of an ontology of proposition use within deductive texts and demonstrates graph theoretical models for visualizing and interpreting the structure of this ontology. Since the approach is language-independent, it is broadly applicable to any text in any language that makes use of the explicit deductive form exemplified by the Elements.

1 Introduction

The model of deductive reasoning set out in Euclid's *Elements* and other ancient Greek mathematical works has proven to be a remarkably durable and influential mode of thought. Its application yielded a set of results that won widespread agreement across a range of disparate cultural traditions, from the medieval Latin West to the Islamic world and, eventually, early modern China. And it served as an important model for other sciences until the Renaissance and beyond. We emphasize that Euclidean-style reasoning is by no means the only kind of deductive argument in Greek mathematics, nor in Greek thought more broadly considered. Yet, with the possible exception of Aristotelian syllogistic, it would be hard to point to any deductive tradition that has exerted a comparable influence. Understanding the reasons for its development and widespread acceptance are therefore matters of primary importance for the history of science.

In its original context, the ability of the Euclidean method to command general assent rested on a number of factors: a set of shared practices for diagrammatic reasoning, an extensible lexicon of technical terms and formulaic phrases, shared knowledge consisting of certain agreed starting points (the definitions, postulates, and axioms), standard forms of argument, and a 'toolbox' of general results having wide applicability.[1] The remarkable ability of the Greek mathematical tradition to generate new results is a consequence of the creative combination of these elements. Documenting and analyzing the development of such elements and the relations between them is the primary goal of a long-term history of deductivity as we conceive of it here.

It is important to emphasize that we are concerned with texts that possess a form rendering explicit their deductive structure. These texts take the form of a chain of propositions (as well as definitions, postulates, and axioms). This type of text originated in the Greek mathematical tradition. Although Indian mathematics, for instance, is in many respects extremely sophisticated, formal deductive structures are not used to justify mathematical statements [6, 12].

[1] The notion of the mathematical toolbox was introduced by K. Saito and taken up by Netz [4]. The role of the diagrams in Euclidean inferences is explored by Manders [2] (originally written in 1995).

The attempt to write a long-term history of deductivity presents a number of daunting challenges: the need to identify, analyze, and coordinate technical terminology in multiple languages; the analysis of images (especially mathematical diagrams) as well as text; and, in general, a volume of material that far exceeds the capacity of a single scholar or group of scholars. In this paper we argue that information technology offers a promising approach to meeting these challenges. In particular, we show how such an approach can be used to analyze the deductive organization of the *Elements* by representing it as an ontology of propositions and relations, which can be analyzed using graph-theoretical methods.

2 The visualization of deductive organization

The description and analysis of the complex networks of deductive relationships between different propositions in the *Elements* is crucial for understanding the deductive organization of the work [3]. Such networks can be treated as ontologies in which the objects are propositions linked by a single relation, 'is used in'; they can also be treated and visualized as directed graphs, as in figure 1. Here an arrow from A to B indicates that A is used in B: thus proposition 1 is used in proposition 2, 2 is used in 3, and 3 and 4 are used in 5. The modes of reference to propositions in the *Elements* vary from nearly word-for-word quotation to use without any explicit reference in the text; the determination of proposition use is thus often a matter of scholarly interpretation. This study is based on the analysis of Neuenschwander [5], who provides an authoritative census of proposition use in books 1–4.[2]

Once the relationships between propositions have been determined, standard software can be used to visualize them as directed graphs.[3] Such visualizations can reveal a good deal about deductive organization. Thus the important place of 1.45 in the deductive structure of book 1 is suggested by its placement

[2] Mueller [3, 52] accepts Neuenschwander's analysis of proposition usage in books 1–4 as definitive. Cf. [7, 1:513–517].

[3] The graphs in this paper were produced using GraphViz, open source graph visualization software available at `http://www.graphviz.org`. Graphs are specified in the DOT language.

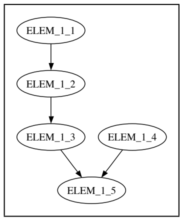

Figure 1: Deductive relationships between propositions 1–5 of book 1.

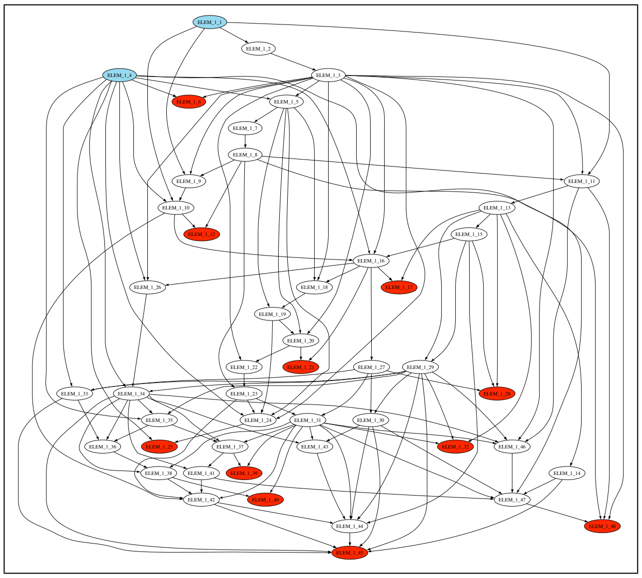

Figure 2: Deductive relationships in book 1. Proposition 1.45 is at the bottom center; starting points are colored in light blue and end points in red.

at the bottom center of the graph of deductive relationships for that book (fig. 2). Proposition 1.45 shows how to construct a parallelogram 'in' a given angle (i. e. with a given angle between two of its sides) and equal in area to a given rectilineal area. Its importance lies in the fact that it enables any rectilineal area to be represented as a rectangle; the further step of showing that any rectilineal area can be represented as a *square* is taken in 2.14, for the proof of which 1.45 is essential. 1.45 is thus crucial to showing that any rectilineal figure can be 'squared'. From the graph it is evident that 1.45 draws on seven earlier propositions (14, 29, 30, 33, 34, 42, and 44), each of which itself depends on a number of earlier propositions.[4]

Graphs of deductive relationships can also reveal the general character of the deductive organization of particular books or sequences of propositions. Thus the shape of the graph for book 2 (fig. 3) is dramatically different from that for book 1 (fig. 2). This difference results in part from the different number of starting points (propositions that make no use of any earlier proposition).[5] In the graphs, starting points are colored light blue and end points (propositions from which nothing is deduced in the book in question) are red. There are exactly two starting points among the propositions of book 1, 1.1 and 1.4, in relation to the 48 propositions of the book. In book 2, however, there are 19 starting points (most drawn from book 1) for only 14 propositions proved. Books 3 and 4 have a similar ratio of starting points to propositions proved; the data are summarized in the following table:

[4] Mueller argues that the need to prove 1.45 is the most important consideration in determining the content and order of propositions in book 1, in the sense that the analysis of the conditions of solution of the problem posed in 1.45 leads naturally back to the earlier propositions and theorems in the book [3, 16–27]. This is not to deny, of course, the fundamental importance of other results proved in book 1, especially 1.47 (the Pythagorean theorem).

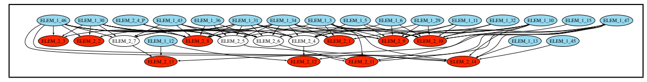

Figure 3: Deductive relationships in book 2.

Book	Starting points	Propositions
1	2	48
2	19	14
3	25	37
4	34	16

A further major difference evident from these graphs is the length of the longest path from proposition to proposition; for book 2 the longest path is 2 (e. g. the path joining 1.30, 2.7, and 2.13), whereas the graph for book 1 contains much longer paths (e. g. the path of length 10 joining 1.1, 1.2, 1.3, 1.5, 1.7, 1.8, 1.23, 1.31, 1.43, 1.44, 1.45).

3 A graph-theoretical approach

These impressions, derived from inspection of the graphs, can be made more precise by applying some graph theoretical methods. In the terminology of graph theory, networks of deductive relationships are collections of *vertices* (i. e. propositions) and *edges* (i. e. the lines connecting propositions) in which each edge expresses a one-way or directed relationship ('is used in') between vertices. With each graph is associated a unique *adjacency matrix* (A). This is a square $n \times n$ matrix, where n is the number of vertices, whose values are defined as follows: if vertex i is joined to vertex j by an edge in the graph then $A(i,j) = 1$; otherwise $A(i,j) = 0$. For example, the adjacency matrix for the graph of deductive relationships between the first five propositions of book 1 (fig. 1) is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For a given row i of the adjacency matrix, the row vector $A(i,)$ indicates which propositions proposition i is used in. Thus, the vector corresponding to row 1 of the adjacency matrix $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ has value 1 only in position 2, because proposition 1 is used only in proposition 2. Similarly, for a given column j of the matrix, the column vector $A(,j)$ indicates which

[5] By 'starting point' we mean simply a proposition that is not based on a prior *proposition*; we do not consider the postulates, definitions, or common notions in this study.

propositions are used in proposition j. The adjacency matrix thus gives a simple method for indexing the use of propositions throughout the *Elements*.

It is also possible to use the adjacency matrix to compute both the length of the longest path through the graph and the number of paths connecting any two vertices.[6] For book 1, this calculation yields a length of 20 for the longest path (between propositions 1 and 45); the maximum number of paths linking any two propositions is a remarkably high 558, again between propositions 1 and 45. As well as confirming the importance of proposition 45 in the deductive structure of book 1, these figures provide a quantitative measure of the distinctiveness of book 1 in comparison to other books. The following table sets out the maximum path length and maximum number of paths between any two propositions in the case of the first four books:

Book	Longest path	Max. no. of paths
1	20	558
2	2	2
3	6	9
4	5	5

4 The analysis of indirect proposition use

These graph theoretical methods provide a convenient approach to studying the indirect use of propositions. By 'indirect use' we mean that in determining which propositions are used by a given proposition, we should include not only those that are directly used, but also those that are implied by the propositions directly used. For example, in fig. 1, while propositions 3 and 4 of book 1 are both directly used in proposition 5, propositions 1 and 2 are indirectly used in 5 as well, since 2 is used in 3 and 1 in 2; thus the set of propositions directly and indirectly used by proposition 5 is: 1, 2, 3, 4. In terms of the graphical representation of deductive relationships, considering indirectly as well as directly used propositions is equivalent to taking all propositions lying on all paths through the graph that lead to the given proposition.[7]

When indirect as well as direct use is considered, it turns out that of all the propositions in book 1, 1.45 makes use of the largest number of propositions (34), a further indication of its centrality in the deductive structure of the book. Plotting the number of directly and indirectly used propositions against the proposition number gives a sense of the degree to which each new proposition builds on what has already been established. Figure 4 shows such a plot for book 1. The progression

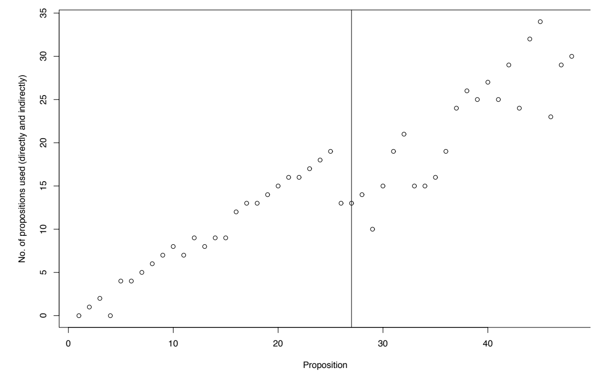

Figure 4: Direct and indirect usage of propositions in book 1. The vertical line marks proposition 1.27.

is very close to linear up to 1.25, then drops off abruptly and is much less regular through the remainder of the book. This pattern reflects the fact that the deductive organization of 1.1–25 is very tight: with each new proposition, there is roughly the same increase in the number of propositions used, indicating that each new proposition makes use of most of the results previously established. Proposition 1.26, which is the final result on triangle congruence, makes use only of 1.3, 1.4, and 1.16; proposition 1.27 marks a new start in the book, the beginning of the theory of parallels. Figure 5 shows the result when books 2–4 are added to the picture; it is clear that while there are some stretches where the plot is approximately linear, there is no sequence comparable to 1.1–25 in length. Here is a further indication of the distinctive character of book 1, and especially the first part of the book, in comparison to other books. Still, within each book, there is an overall tendency for the number of propositions used to increase with the proposition number. Moreover the last proposition of book 4, 4.16, makes use of the largest number of propositions (72) of any proposition in books 1–4. Whatever variation there may be in the organization of particular sequences of propositions, *overall* the trend in the first four books is towards the use of increasing numbers of previously established results.

5 Further perspectives

In conclusion we note that this method could easily be applied to the large number of texts in the Western mathematical and scientific tradition that present their results in Euclidean form, down to and including works such as Newton's *Principia*. Such a study would allow for a precise characterization of the development of deductive methods over the long term; it would no doubt reveal significant variation as well as general trends. Moreover, since there is nothing language-specific about an ontology of propositions, the same method could be used to characterize the differences between the various Greek, Latin, Arabic, and even Chinese versions of the *Elements*. The ontologies with which we are concerned could readily be expressed in standard ontology languages such as OWL (Web Ontology Language) or CL

[6] These calculations are based on the following properties of an adjacency matrix A [1, 217–222]. (1) The i, jth element of the matrix A^n gives the number of paths of length n between vertices i and j. (2) Let I be the $n \times n$ identity matrix (i. e. the $n \times n$ matrix in which all elements on the diagonal are 1 and all other elements are 0). Then the i, jth element of the matrix $(I - A)^{-1}$ gives the total number of paths between i and j. The matrix computations in this paper were performed using the R statistical language, freely available online at `http://www.r-project.org`.

[7] Let A be an adjacency matrix of direct proposition usage and I the corresponding adjacency matrix of direct and indirect proposition usage. Let $I(,n)$ denote the nth column vector of I, and similarly for A. Then for any n, $I(,n) =$ union of $A(,n)$ and $I(,m)$ for all m such that $A(m,n) = 1$.

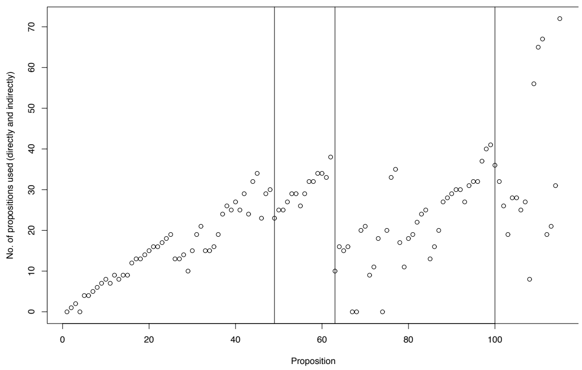

Figure 5: Direct and indirect usage of propositions in books 1–4. Vertical lines mark the beginning of books 2, 3, and 4.

(Common Logic), allowing the application of a wide variety of reasoning and visualization tools. Our case study demonstrates that an ontology-based approach can productively be applied to the analysis of the structure of knowledge in particular formal domains, the study of its development over time, and the comparison of knowledge across cultural and linguistic boundaries.

References

[1] Gary Chartrand. *Introductory Graph Theory*. Dover, New York, 1985.

[2] Kenneth Manders. The Euclidean diagram (1995). In Paolo Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 80–133. Oxford University Press, Oxford and New York, 2008.

[3] Ian Mueller. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. MIT Press, Cambridge, MA, 1981.

[4] Reviel Netz. *The Shaping of Deduction in Greek Mathematics*. Cambridge University Press, Cambridge, 1999.

[5] Erwin Alfred Neuenschwander. *Die ersten vier Bücher der Elemente Euklids: Untersuchungen über den mathematischen Aufbau, die Zitierweise und die Entstehungsgeschichte*. Zürich: Universitätsdruckerei H. Stürtz, 1973. Offprint from *Archive for History of Exact Sciences*, Volume 9, Number 4/5, 1973, pp. 325–380.

[6] Kim Plofker. *Mathematics in India*. Princeton University Press, Princeton and Oxford, 2009.

[7] Bernard Vitrac. *Les Éléments. Traduction et commentaires par Bernard Vitrac*. Presses Universitaires de France, Paris, 1990–2001. 4 vols.

Contact

Prof. Dr. Mark J. Schiefsky
Harvard University, Department of the Classics
204 Boylston Hall, Cambridge, MA 02138 USA
Tel.: +1 617-495-9301
Fax.: +1 617-496-6720
Email: mjschief@fas.harvard.edu

Dr. Malcolm D. Hyman
Max-Planck-Institut für Wissenschaftsgeschichte
Boltzmannstr. 22, 14195 Berlin
Tel.: +49 (0)30-22667-129
Fax: +49 (0)30-22667-299
Email: hyman@mpiwg-berlin.mpg.de

Bild

Mark J. Schiefsky is Professor in the Department of the Classics at Harvard University. His research centers on the history of medicine, mathematics, and related sciences in Greco-Roman antiquity, and on the use of information technology for the analysis of sources in the history of science.

Bild

Malcolm D. Hyman is a Research Scientist at the Max Planck Institute for the History of Science in Berlin. His research interests center on the application of computer technology and models derived from cognitive science to the history of knowledge.