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Lecture 15: Stars

1 Introduction

There are at least 100 billion stars in the Milky Way. Not everything in the night sky is a star – there are also planets and moons as well as nebula ("cloudy" objects including distant galaxies, clusters of stars, and regions of gas) – but it's mostly stars. These stars are almost all just points with no apparent angular size even when zoomed in with our best telescopes. An exception is Betelgeuse (Orion's shoulder). Betelgeuse is a red supergiant 1000 times wider than the sun. Even it only has an angular size of 50 milliarcseconds: the size of an ant on the Prudential Building as seen from Harvard square. So stars are basically points and everything we know about them experimentally comes from measuring light coming in from those points.

Since stars are pointlike, there is not too much we can determine about them from direct measurement. Stars are hot and emit light consistent with a blackbody spectrum from which we can extract their **surface temperature** T_s. We can also measure how bright the star is, as viewed from earth . For many stars (but not all), we can also figure out how far away they are by a variety of means, such as parallax measurements.[1] Correcting the brightness as viewed from earth by the distance gives the **intrinsic luminosity**, L, which is the same as the power emitted in photons by the star. We cannot easily measure the mass of a star in isolation. However, stars often come close enough to another star that they orbit each other. For such stars, we can measure their mass using Kepler's laws. Finally, by looking at details of the stellar spectra we can find evidence of metals. To an astronomer, **metallicity** is the amount of any element heavier than helium in a star.

With temperature and luminosity data, we can make a scatter plot of stars in the galaxy to see if there are any patterns. This was done by Hertzsprung and Russell first around 1910:

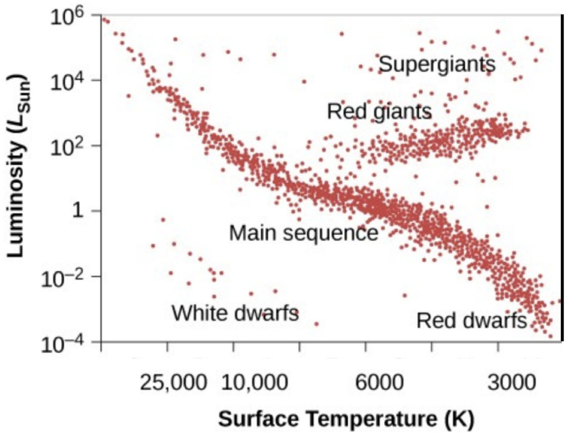


Figure 1. Hertzsprung–Russell diagram of luminosity and surface temperature.

We see from the Hertzsprung–Russell (HR) diagram that most stars fall in two swaths: a diagonal swath, known as the **main sequence** (90% of stars), which includes stars like our sun, and a horizontal swath with luminosities around 100 times brighter than our sun called the **horizontal branch** (labeled "Red giants" in the figure). A first question from this data is "why do stars fall along swaths rather than being distributed throughout the L/T plane?" The basic answer is the

1. Unfortunately, we cannot use the redshift of the blackbody spectrum to find distance, since a redshifted blackbody spectrum looks like a blackbody spectrum at a different temperature. This follows from dimensional analysis, since the Planck spectrum only depends on T through the combination $\frac{h\nu}{k_B T}$ so rescaling ν can be compensated by rescaling T.

Vogt-Russel theorem: all stars with the same chemical composition and mass are the same. In particular, if the chemical composition is the same, then the mass indexes a one-parameter family of points in *any* plot, including a HR diagram. The main sequence swath describes stars with hydrogen-burning cores, like our sun. The red giants composing the horizontal branch are burning helium. There is another region heading upwards from the horizontal branch not clearly distinguishable in this figure known as the **asymptotic giant branch**. It contains supergiants which burn heavy elements. Finally, there is a smattering of stars in the lower left called white dwarfs.

We'll first discuss how the branches on the HR diagram all fit together in the standard picture of stellar evolution. Then we'll do some simple stellar thermodynamics. Stellar physics is generally quite complicated, with lots of coupled equations, for convection, radiation, etc. These equations must be solved numerically to make precise quantitative predictions, but analytic approximations can be made as well. In fact, we can get a basic picture of how things work using just some basic physical reasoning and order-of-magnitude estimates. Some stars, namely white dwarfs and neutron stars, are simple enough that we will be able to describe them accurately using quantum statistical mechanics.

2 Overview of stellar evolution

In this section we discuss the standard narrative for star formation. Much of stellar physics is calibrated relative to the sun. The sun is denoted with the symbol $\odot$. The mass of the sun is $M_\odot = 2\times 10^{30}$ kg and the radius of the sun is $R_\odot = 7\times 10^5$km.

After the big bang, the universe cooled into around 75% hydrogen and 25% helium, with trace amounts of lithium. Some patches of the universe started out denser and hotter than others (we think this is due to primordial quantum density fluctuations during inflation), and the denser patches became more dense over time due to gravitational attraction. When a region of gas becomes dense enough, the density increase accelerates rapidly, a process called a **Jeans instability**. After the instability is reached the gas continues to compress until stars form, and if the temperature gets high enough, hydrogen fusion begins.

Hydrogen fusion in stars refers to a collection of processes whose net effect is to turn hydrogen H into ^{4}He. This fusion generates thermal pressure, keeping the stars from collapsing further under their own gravity. Thus for most of its lifetime, a star is in **hydrostatic equilibrium** and equilibrium statistical mechanics can be used.

High temperatures and densities are required for fusion. If the amount of matter that coalesced to form a star is not greater than around 0.08 $M_\odot$, the star will not burn and instead is just a ball of hydrogen, like Jupiter. These failed stars are called **brown dwarfs**. The main difference between a brown dwarf and a planet is how they formed: planets form from supernova remnants while brown dwarfs form from collapse of interstellar gas.

The region of the star where temperatures are high enough for hydrogen to fuse into helium is called the **core**. The core of the sun is a ball with radius around $0.25R_\odot$. For stars that are lighter than around $0.4M_\odot$ convection currents move the helium out of the core and allow hydrogen to fall in. The relatively low mass of these stars, right above the threshold for hydrogen fusion, make them small and cool. They are called **red dwarfs**. 73% of the stars in the Milky Way are red dwarfs. These stars burn hydrogen slowly, in principle until it is all used up. However, it takes hundreds of billions of years for a red dwarf to burn all its fuel and since the universe is only 15 billion years old, none of red dwarfs have run out of hydrogen yet.

In stars with masses above $0.4M_\odot$, like our sun (about 20% of stars), the density in their cores gets high enough to prevent much convection. Thus the helium produced there basically stays put. After the hydrogen in the core is all burned up (so only helium is left), the thermal pressure from fusion stops and the star starts to contract again. The subsequent increase in density and temperature allows fusion to commence in the shell slightly outside of the core. Helium produced from this shell falls into the core, increasing the temperature further. As heat flows out from the core, it pushes the outer layers farther away. The net effect is that 1) the core contracts to around 1/3 its original size and its temperature goes up from $15\times 10^6 K$ to $100\times 10^6 K$ and 2) the outer layers are pushed out (to $\sim 100 R_\odot$) and cool. The increased core temperature can allow helium to fuse into carbon, nitrogen and oxygen. Thus after core hydrogen is used up, the equilibrium configuration of the star is qualitatively different: it has a core of helium and metals and is larger with a cooler surface temperature. By the Vogt-Russel theorem, these stars form a different line in the HR diagram. The star has left the main sequence to become a **red giant**.

In the red giant phase the star continues to burn the remaining hydrogen, expanding and cooling, and fuses helium in the core. This stage is relatively fast, and less than 0.5% of stars are red giants. Eventually all the helium in the core is used up and core nuclear fusion stops. Then, once again, the thermal pressure stops and the star's core can contract again under its own weight and heat up, allowing helium fusion to continue in shells around it. It has entered the asymptotic giant branch. Again matter is pushing outward. This time the matter contains H, He, C, N and O. There is a lot of convection in this phase, causing the heavy elements to be dredged up from the core in a series of bursts. The result is the formation of a set of shells of matter called a **planetary nebula**:

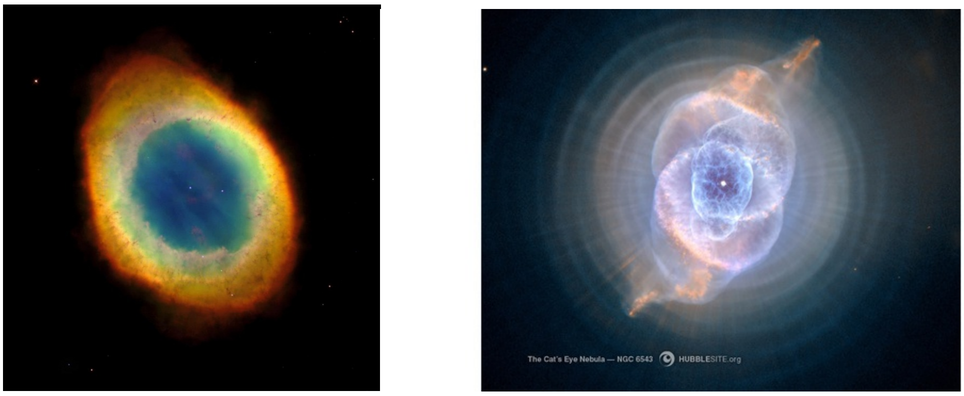


Figure 2. The ring nebula (left) and cats eye nebula (right) are planetary nebula. The dot in the middle of each is a white dwarf. Planetary nebula have nothing to do with planets.

Planetary nebula are colorful because UV radiation from the core ionizes the atoms in the gas. This makes them some of the most beautiful objects in the sky. They have nothing to do with planets ("planetary" is a historical misnomer – at low resolution, they look a bit like round, green or blue planets).

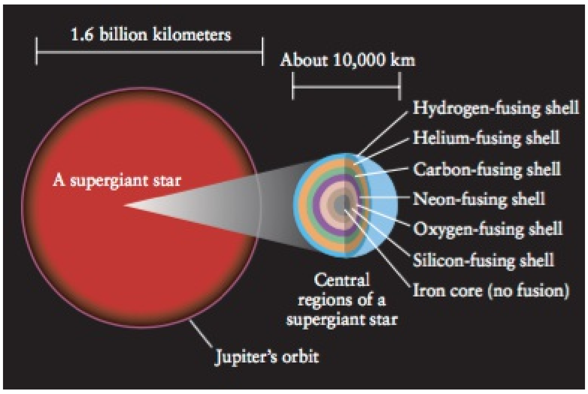
Two things can now happen to the core of an asymptotic giant branch star, depending on its mass. If the star's mass is not large enough for its weight to overcome the electron degeneracy pressure in the core, the core cannot contract enough for fusion to continue. In this case, the core becomes a star called a **white dwarf** held together by gravity and stabilized by electron degeneracy pressure. About 4% of stars are white dwarfs. The white dwarf is a kind of star that cannot be understood without quantum statistical mechanics. A main-sequence star is stable against gravitational collapse because of the thermal pressure from active nuclear fusion. In a white dwarf, there is no nuclear fusion and little thermal and radiation pressure, so only degeneracy pressure can be stabilizing the star.

In Section 4 we will show that the degeneracy pressure limits the mass of a white dwarf. The limit is called the **Chandrasekhar limit**, $M_{\mathrm{WD}} \lesssim 1.4 M_\odot$. The corresponding bound on the original star's mass is $M \lesssim 8 M_\odot$. If $M \gtrsim 8 M_\odot$ then the electron degeneracy pressure in the core cannot prevent further contraction. These high-mass stars continue to collapse leading to additional stages of nuclear fusion. The next set of reactions include carbon fusion, which produces heavier elements like Ne, Na and Mg, and neon fusion, which produces more O and Mg. Once the carbon and neon are used up, the cycle advances: core contraction, shell expansion, and core temperature/density increase, leading to oxygen fusion. Then silicon fusion. Eventually iron is produced. Iron is the endpoint of nuclear fusion since elements above iron have endothermic fusion reactions and exothermic fission reactions[2]. Stars where these processes are happening are called **supergiants**.

Each stage of evolution of a supergiant generates a new shell of core material. This leads to a

2. After iron, the Coulombic repulsion of the protons which scales like Z^2 begins to outweigh the nuclear attraction which scales like Z, with Z the atomic number.

star that has different elements in different places, like shells of an onion:



(1)

All this nuclear activity makes supergiant stars very bright. Some of the brightest stars in the sky are supergiants, including Betelgeuse, Rigel, and Antares.

Once iron is formed and fusion stops, the core is only around the size of the earth and the supergiant is the size of our solar system. The thermal pressure stops abruptly and the gravitational pressure then commences to collapse the star. This happens very rapidly, raising the temperature to 5 billion K in about a tenth of a second. After around $0.25s$ the core can have a density of $4 \times 10^{17} \frac{\text{kg}}{m^3}$. That's like packing the entire mass of the earth into Jefferson Hall. The high temperature unleashes high energy photons which break apart the nuclei into protons and neutrons. The high density of electrons and protons allows them to fuse:

$$p^+ + e^- \to n + \nu \tag{2}$$

As the neutrinos produced by this fusion leave the star, the pressure drops further allowing additional contraction. The star begins collapsing catastrophically, at as much as 1/4 the speed of light. Eventually, the core of neutrons cannot collapse further due to the strong interactions and the neutron degeneracy pressure. The incoming waves of matter then bang into the neutron core and bounce outward. This leads to an enormous explosion called a **core-collapse supernova**. After the explosion, the core may be obliterated. Or there may be a remnant **neutron star** (made of neutrons stabilized by degeneracy pressure) or, if the core mass is too large, a **black hole**.

Here's a summary of the story:

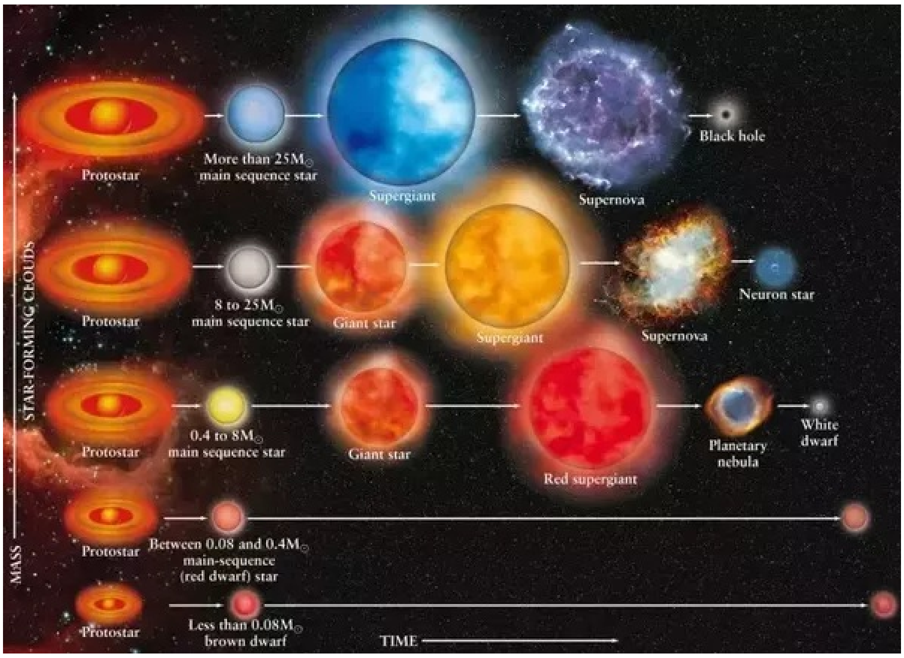


Figure 3. The fate of a star depends on its mass.

During a supernova, elements heavier than iron can be produced. Although the production of these elements by fusion is not energetically favorable, there is so much energy around in a supernova that it can still happen – like thermal motion pushing a ball up a hill. That being said, it is not clear if enough heavy elements can be produced this way to explain their abundances. In particular, it seems hard to get enough gold. Interestingly, the recent 2017 discovery of a neutron star merger through gravitational waves suggests that neutron star mergers may be another important source of gold, potentially resolving this deficit (see Section 6).

After the supernova the explosion remnants are expelled off into the galaxy. Eventually they coalesce along with more primordial hydrogen to form a new generation of stars. These new stars, called **population I stars**, include our sun. They are still mostly hydrogen, but do have some heavy elements too. Second generation (population I) stars have higher metallicities than first generation (population II) stars (yes, the notation is backwards) and are often surrounded by other objects with high metallicity, such as planets. Thus astronomers searching for planets in other solar systems focus on population I stars.

3 Stellar thermodynamics

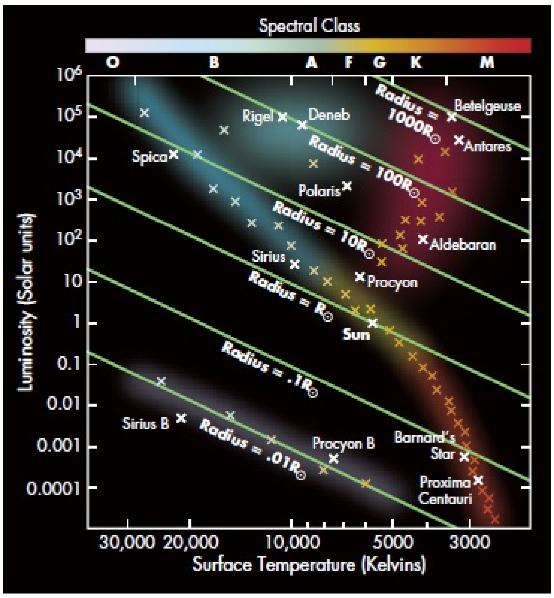
In this section, we will try to back up some of the qualitative arguments from the overview with equations. We will discuss some of the equations of stellar structure and discuss polytropes and the Eddington model, which is a very good approximation to stars in the main sequence. This will allow us to compute the density profile in the stars and the core temperature, and we will use some of the equations we derive here in the calculation of the Chandrasekhar limit in Section 4.

3.1 Luminosity

Let's begin with luminosity and temperature, as shown in the HR diagram. Knowning the intrinsic luminosity and surface temperature, we can find the radius R with the Stefan-Boltzmann law for the flux $\Phi = \sigma T_s^4$:

$$L = \text{surface area} \times \text{flux} = 4\pi R^2 \Phi = 4\pi R^2 \sigma T_s^4 \tag{3}$$

with $\sigma = 5.6 \times 10^{-8} \frac{W}{m^2 K^4}$ is Stefan's constant. So stars with the same R should fall along diagonal lines in the log-log HR diagram:



(4)

We that see that most the the main sequence stars have radii R within two orders of magnitude of the sun: $0.1R_{\odot} < R < 10R_{\odot}$. Red giants are bigger, and the white dwarfs are smaller, hence their names.

3.2 Core temperature

Although the surface temperature is what we readily see, it is the core temperature that is critical to allow nuclear fusion to occur. There are many ways to estimate the core temperature. Consider for example, a model that treats the star as two parts, a core, where fusion is occurring, and an outer shell where fusion is not occurring. The core has some pressure P that keeps the shell from collapsing in. The pressure of the shell is due to its weight as set by the gravitational pull of the core. Since $P = \frac{F}{A}$ we get

$$P_{\text{grav}} = \frac{F_{\text{grav}}}{4\pi r_{\text{core}}^2} = G\frac{M_{\text{shell}} M_{\text{core}}}{4\pi r_{\text{core}}^4} \tag{5}$$

The pressure in the core is a thermal pressure. If the core is made up of a gas of hydrogen the ideal gas law gives

$$P_{\text{therm}} = \frac{N_{\text{core}}\, k_B T_{\text{core}}}{V_{\text{core}}} = \frac{M_{\text{core}}}{m_p} \frac{1}{\left(\frac{4}{3}\pi r_{\text{core}}^3\right)} k_B T_{\text{core}} \tag{6}$$

Setting $P_{\text{grav}} = P_{\text{therm}}$ gives

$$k_B T_{\text{core}} = \frac{1}{3} \frac{G M_{\text{shell}} m_p}{r_{\text{core}}} \tag{7}$$

Using dimensional analysis, we can estimate that $\frac{M_{\text{shell}}}{r_{\text{core}}} \approx \frac{M_{\odot}}{R_{\odot}}$. We then find

$$T_{\text{core}} = \frac{G M_{\odot} m_p}{3 k_B R_{\odot}} = 7.7 \times 10^6 K \tag{8}$$

Current best theoretical models put T_{core} around $15 \times 10^6 K$, so this back-of-the-envelope calculation is off by about a factor of 2. You get similar answers using the virial theorem ($E_{\text{grav}} = -2E_{\text{kin}}$), as you did on a problem set, or by equating the escape velocity to the average thermal velocity.

How hot does the core have to be for hydrogen fusion to occur? The first stage in hydrogen fusion is

$$p^+ + p^+ \to D^+ + e^+ + \nu \tag{9}$$

where D^+ is the deuteron (^{2}H nucleus). This process occurs through the weak interaction and is generally much much slower than the rates for $D + D \to^4$ He which occur through the strong interaction. For hydrogen fusion to occur, the protons have to be able to approach closely enough to fuse. Classically, they can only approach each other until their kinetic energy is used up fighting the potential energy barrier:

$$\frac{e^2}{4\pi\varepsilon_0 r_{\min}} = \frac{1}{2} m_p v^2 \tag{10}$$

Taking the velocity to be that of a thermal ideal gas, $\frac{1}{2} m_p v^2 = \frac{3}{2} k_B T$, and asking for the protons to get within the a proton radius of each other, $r_{\min} = r_p \approx 10^{-15} m \approx \frac{h}{m_p c}$, Eq. (10) then gives an estimate of the temperature

$$T_{\min}^{\text{classical}} = \frac{1}{3k_B} \frac{e^2}{4\pi\varepsilon_0 r_p} = 4.2 \times 10^9 K \tag{11}$$

This temperature is 300 times higher than we estimated for the sun's core temperature, suggesting that fusion should not occur. Of course, it is not necessary for the average proton to get over the barrier; we can exploit the far tail. Even then, the probability of having enough energy is given by the Boltzmann factor $P \sim e^{-300}$, which amounts to zero collisions in the Sun. According to this classical estimate, protons in the sun simply do not collide.a

Fortunately, it is not necessary for protons to actually pass over the Coulomb barrier. Instead, they can tunnel through it quantum mechanically. To estimate the temperature where tunnelling can occur, we can use that a particle moving with velocity v has a wavefunction that extends to the size of its de Broglie wavelength $\lambda = \frac{h}{m_p v}$. Replacing $r_{\min}$ in Eq. (10) by λ leads to $\frac{e^2}{4\pi\varepsilon_0}\frac{m_p v}{2\pi\hbar} = \frac{1}{2}m_p v^2$. Taking $v = v_{\rm rms} = \sqrt{\frac{3k_BT}{m_p}}$ this leads to

$$T_{\min} = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{m_p}{3\pi^2\hbar^2 k_B} = 19.7\times 10^6 K \tag{12}$$

This is now within a factor of 2-3 of our estimate for the sun's core temperature. Thus, the chance of proton's overlapping enough to possibly allow fusion is order 1 and many protons in the sun can pass this threshold. A more accurate calculation puts the hydrogen fusion ignition temperature at $10\times 10^6 K$, so again our back-of-the-envelope calculation is not bad.

Going form a lower bound on the temperature for fusion to occur to the rate for fusion is very difficult. The reason is that 3 things must happen

1. protons must tunnel through electromagnetic Coulomb barrier
2. after tunneling, the strong force must to hold them together long enough for
3. the weak force to allow fusion to occur

Despite these complications involving all the forces of nature simultaneously, the calculations can be done and seem to be in excellent agreement with observations of the sun.

One prediction of hydrogen fusion in the sun, from Eq. (9), is that a boatload of neutrinos should be produced. We first measured these **solar neutrinos** in the 1960s, and only around 1/3 of the predicted flux was observed. This was called the **solar neutrino problem**. At the time the missing neutrinos were attributed to theorists bungling the (extremely complicated) nuclear physics calculations. It turns out the theorists' calculations (mostly John Bahcall) were actually nearly perfect. The missing neutrinos were due to quantum mechanical oscillations among electron, muon and tauon neutrinos, possible if and only if neutrinos have mass. When muon neutrinos were observed around the year 2000 with exactly the right solar flux, the solar neutrino problem was solved and neutrino mass was unambiguously established!

After deuterium is formed, nuclear fusion proceeds rapidly to form ^{4}He, as the $D + D \to^4$ He reactions involve the strong force, which has cross sections orders of magnitude larger than the $p + p \to D$ cross section. Careful calculations along these lines indicate the following table of ignition threshold temperatures for various nuclear reactions:

process	**reactions**	**minimum T**
hydrogen fusion	H→He	$10\times 10^6 K$
helium fusion	He → C,O	$100\times 10^6 K$
carbon fusion	C→O, Ne, Mg, Na	$500\times 10^6 K$
neon fusion	Ne → O, Mg	$1200\times 10^6 K$

(13)

Working backwards from these numbers and using Eq. (8) we see that for helium fusion to occur, the core of the sun would have to shrink by a factor of 10 of so, so this is what happens as a star enters the red giant phase.

3.3 Stellar structure equations

We want to do a little better than the rough estimates above. Stars are equilibrium objects and governed by a set of relatively simple equations called the **equations of stellar structure**. We will discuss two of these and show how they can be used to compute the density and temperature distributions within a star.

The mass $m(r)$ within the sphere at distance r is determined by integrating the density

$$m(r) = \int_0^r dr' \, 4\pi r'^2 \rho(r') \tag{14}$$

Taking $\frac{d}{dr}$ of both sides leads to the 1st stellar structure equation called the conservation-of-mass equation:

$$\boxed{\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)} \qquad \text{(mass conservation)} \tag{15}$$

Next, we know that the gravitational force keeping a star together must balance the internal pressure pushing it apart. The force acting on the spherical shell at distance r that has mass $dm = 4\pi r^2 \rho dr$ is determined by the mass $m(r)$ within that shell. So,

$$F_g = -G \frac{m(r) \times 4\pi r^2 \rho(r) dr}{r^2} \tag{16}$$

The pressure $P(r)$ is also a function of r. There is pressure pushing out from $r - dr$ and pushing in from $r + dr$, so the net force on the shell is

$$F_p = 4\pi r^2 \Big[P(r + dr) - P(r - dr) \Big] = 4\pi r^2 \frac{dP}{dr} dr \tag{17}$$

Setting the internal pressure F_p equal to the gravitational pressure F_g gives the 2nd stellar structure equation called the equation of hydrostatic equilibrium

$$\boxed{\frac{dP(r)}{dr} = -\frac{G m(r) \rho(r)}{r^2}} \qquad \text{(hydrostatic equilibrium)} \tag{18}$$

There are two more stellar structure equations, related to energy production and transport in the star. These depend on the rate of heat generation, the ability of the star to absorb radiation (its opacity), and the rates of conductivity and convection. We're going to skip these last two equations because they involve more complicated physics and because we can actually learn a lot from just the first two.

Multiplying both sides of the hydrostatic equilibrium equation by $\frac{r^2}{\rho(r)}$ and differentiating we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)} \frac{dP(r)}{dr} \right) = -G \frac{dm}{dr} = -4\pi G r^2 \rho(r) \tag{19}$$

where the mass conservation equation was used in the second step. This single equation couples the pressure and density.

To solve Eq. (19) we need to know something about the pressure or density. For example, if the density is constant, its solution is

$$P(r) = P_c - G \frac{\rho^2}{6} r^2 \tag{20}$$

with P_c the pressure at $r = 0$. Thus the pressure decays quadratically from its central value in the constant density approximation. This is not a great approximation, but a decent start.

Alternatively, we might postulate that a star is like an ideal gas dominated by adiabatic convection, so $PV^\gamma = \text{constant}$, with $\gamma = \frac{5}{3}$ for a monatomic gas. Then

$$P \propto \left(\frac{1}{V} \right)^\gamma \propto \rho^{5/3} \tag{21}$$

This is called the **adiabatic convection model**. We can then plug this in to Eq. (19) and solve. Better approximations come from understanding the sources of stellar pressure. Pressure in a star can be gas pressure, radiation pressure, or degeneracy pressure. We discuss the first two here and degeneracy pressure in Section 4.

For the gas pressure, we treat the star as an ideal gas, so $P_{\text{gas}} = \frac{N}{V} k_B T$. To use this in Eq. (19) we must relate $\frac{N}{V}$ to the mass density ρ. That is, we need to know how many independent ideal gas particles there are for every m_p of mass. The answer depends on whether the gas is ionized or not. Recall that ionization energies of atoms are in the 10 eV range (13eV for H and 79eV for He). The cores of stars are around $10^7 K \sim$ keV, thus we can safely assume full ionization: all the electrons are stripped form the atoms. If the gas is 75% hydrogen and 25% ^{4}He by mass (the cosmological abundance), then for every $^4He^{2+}$ nucleus there are 12 H^+ nuclei and $2 + 12 = 14$ free electrons. So $\frac{N}{V\rho} = \frac{1 + 12 + 14}{4m_p + 12m_p} = \frac{1}{0.59m_p}$. In general, we write $\frac{N}{V} = \frac{\rho}{\mu m_p}$ with a new parameter μ so that

$$P_{\text{gas}} = \frac{\rho}{\mu m_p} k_B T \tag{22}$$

With ionized hydrogen and helium we found $\mu = 0.59$. Including the metal content of the sun, this goes up slightly, to $\mu = 0.62$.

Radiation pressure is determined by blackbody radiation:

$$P_{\text{rad}} = \frac{4\sigma}{3c} T^4 \tag{23}$$

with σ Stefan's constant. Note that matter and radiation have different scalings with temperature, so bigger, hotter stars have relatively more radiation pressure than smaller, cooler stars.

3.4 Polytropic Stellar Models

To determine how much gas and radiation pressure are present, we need the other stellar structure equations related to energy production and transport. It turns out that for hydrogen burning stars, these equations imply that the ratio $\frac{P_{\text{rad}}}{P_{\text{gas}}}$ is roughly independent of r throughout the star. This observation was made by Eddington in 1926. If $\frac{P_{\text{rad}}}{P_{\text{gas}}}$ is constant in a star, then both P_{rad} and P_{gas} are proportional to the total pressure $P_{\text{tot}} = P_{\text{rad}} + P_{\text{gas}}$. Writing $P_{\text{rad}} = \beta P_{\text{tot}}$ then $P_{\text{gas}} = (1 - \beta) P_{\text{tot}}$ and so $\frac{P_{\text{rad}}}{P_{\text{gas}}} = \frac{\beta}{1-\beta}$. We can then solve Eqs. (22) and (23) for the temperature in terms of density

$$\frac{4\sigma}{3c} T^4 = \frac{\beta}{1-\beta} \frac{\rho}{\mu m_p} k_B T \quad \Rightarrow \quad T = \left(\frac{3ck_B}{4\mu m_p \sigma} \frac{\beta}{1-\beta}\right)^{1/3} \rho^{1/3} \tag{24}$$

then

$$P = \frac{1}{1-\beta} P_{\text{gas}} = \frac{1}{1-\beta} \frac{k_B}{\mu m_p} \left(\frac{3ck_B}{4\mu m_p \sigma} \frac{\beta}{1-\beta}\right)^{1/3} \rho^{4/3} = \left(\frac{3ck_B^4 \beta}{4\mu^4 m_p^4 \sigma (1-\beta)^4}\right)^{1/3} \rho^{4/3} \tag{25}$$

This is called the **Eddington Solar Model**. The Eddington solar model gives an excellent approximation to our sun and most main sequence stars.

The scaling $P \sim \rho^{4/3}$ is similar to the adiabatic convection model $P \sim \rho^{5/3}$ in Eq. (21). Other models also give power laws (we show in Section 4 that a degenerate ultrarelativistic electron gas as in a white dwarf gives $P \sim \rho^{4/3}$). So in many cases, the equation of state amounts to

$$P = K\rho^{1+\frac{1}{n}} \tag{26}$$

for some n and some K. Such models are called **polytropic models**. For the adiabatic convection model, $n = \frac{3}{2}$; for the Eddington model, $n = 3$ and

$$P = K\rho^{4/3}, \quad K = \left(\frac{3ck_B^4 \beta}{4\mu^4 m_p^4 \sigma (1-\beta)^4}\right)^{1/3} \qquad \text{(Eddington model polytrope)} \tag{27}$$

It's helpful to make ρ and r dimensionless. Writing ρ_c for the density at $r = 0$, we can introduce a dimensionless radius ξ and a dimensionless function $\theta(\xi)$ with the transformations

$$\rho(r) = \rho_c [\theta(\xi)]^n, \quad \xi = \frac{r}{r_0}, \qquad r_0 = \sqrt{\frac{(n+1)K}{4\pi G \rho_c^{1-\frac{1}{n}}}} \tag{28}$$

The choice of r_0 is fixed so that when we plug these relations into Eq. (19) with the polytropic form in Eq. (26) we get a non-dimensional equation

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n} \tag{29}$$

This is called the **Lane-Emden Equation** and is the basis of much stellar modeling.

The Lane-Emden equation can only be solved analytically for only a few values of n, but it's easy to solve numerically for any n by integrating. Assuming the density is non-singular (doesn't go to ∞)at $r=0$ with core density ρ_c gives the boundary conditions $\theta(0)=1$ and $\theta'(0)=0$. We then find numerical solutions for various n

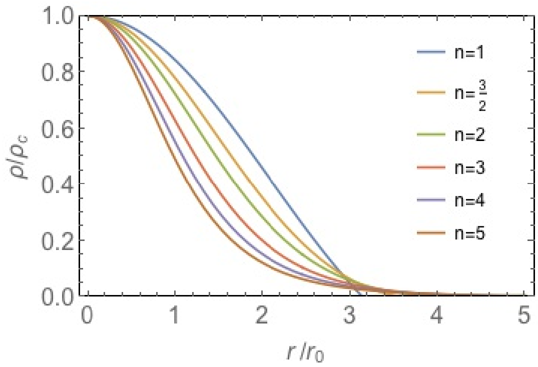


Figure 4. Density profile of a star, $\rho=\theta(\xi)^n$ with $\xi=\frac{r}{r_0}$ according for polytropes of various n.

We see that the profiles are actually qualitatively similar. For $n<5$ the curves all cross $\rho=0$, which tells us the size of the star. That is, we define $\xi_{\max}$ as the scale where $\theta(\xi_{\max})=0$, then the radius of the star is, $R=r_0\xi_{\max}$. For $n\geqslant 5$ there is no zero-crossing ($\xi_{\max}=\infty$) and the model is not a good one for any real star. In the Eddington model, $n=3$, the zero crossing $\theta(\xi_{\max})=0$ occurs at $\xi_{\max}=6.9$.

The density determines the total mass. We can write this in terms of a dimensionless integral:

$$M=4\pi\int_0^R \rho(r)r^2dr=4\pi\int_0^{\xi_{\max}}\rho_c\theta(\xi)^n(r_0\xi)^2d(r_0\xi)=4\pi\rho_c\frac{R^3}{\xi_{\max}^3}\int_0^{\xi_{\max}}[\theta(\xi)]^n\xi^2d\xi=C_nR^3\rho_c \tag{30}$$

where

$$C_n=\frac{4\pi}{\xi_{\max}^3}\int_0^{\xi_{\max}}[\theta(\xi)]^n\xi^2d\xi \tag{31}$$

is a dimensionless number that we can evaluate numerically for each n. For example, for $n=3$ we get $C_3=0.0077$. Values for $\xi_{\max}$ and C_n for some values of n are given in Table 1. These are computed in a mathematica notebook on the canvas site:

n	1	$\frac{3}{2}$	2	3	4	5
$\xi_{\max}$	π	3.7	4.4	6.9	15.0	∞
C_n	12.6	9.3	7.0	0.077	1.5	∞

Table 1. Values for the zero crossing ξ_n and the mass integral C_n for various polytropes.

Now, let's specialize to the Eddington model, with $n=3$. So we use $\xi_3=0.69$ and $C_3=0.0077$. Plugging in Eqs. (28) and Eq. (27) to $R=r_0\xi_{\max}$ gives an expression for R in terms of the core density ρ_c:

$$R=r_0\xi_{\max}=\sqrt{\frac{K}{\pi G\rho_c^{2/3}}}\xi_{\max}=\xi_{\max}\left(\frac{3ck_B^4\beta}{4\pi^3\mu^4G^3m_p^4\sigma(1-\beta)^4}\right)^{1/6}\left(\frac{1}{\rho_c}\right)^{1/3}\qquad (n=3) \tag{32}$$

Since $R \sim \rho_c^{-1/3}$ for $n=3$ and $M \sim R^3 \rho_c$ by Eq. (30) we see that, for $n=3$, the mass is conveniently independent of core density ρ_c:

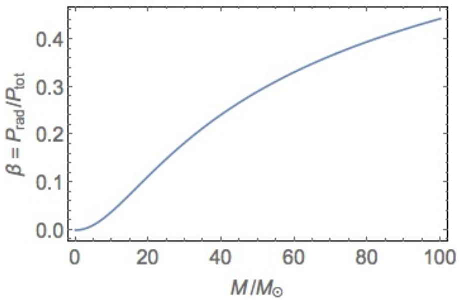
$$M = C_3 R^3 \rho_c = C_3 \xi_{\max}^3 \left(\frac{K}{\pi G}\right)^{3/2} \qquad (n=3) \tag{33}$$

Plugging in $\xi_{\max}=0.69$, $C_3=0.77$ and K from Eq. (27), we get

$$M = 0.077 \times (0.69)^3 \times \sqrt{\frac{3 c k_B^4 \beta}{4\pi^3 G^3 \mu^4 m_p^4 \sigma (1-\beta)^4}} = 17.9 M_\odot \frac{\sqrt{\beta}}{(1-\beta)^2 \mu^2} \tag{34}$$

Thus remarkably, we get a prediction for the mass of the sun within striking distance of the sun's actual mass from this model alone; we did not have to specify the radius R or the core density ρ_c.

Recall that μ refers to the chemical composition of a star. The sun has $\mu = 0.62$ so we'll use this value; using this and setting $M = M_\odot$ we can solve Eq. (34) for $\beta_\odot = 4.6 \times 10^{-4}$. Recalling that $\beta = \frac{P_{\text{rad}}}{P_{\text{tot}}}$, this indicates that there is very little radiation pressure in the sun. As the mass increases, so does β:



(35)

Thus radiation pressure is more important for heavier stars.

Plugging in $R = R_\odot$ and $M = M_\odot$ to Eq. (33) we find the sun's core density to be

$$\rho_c = \frac{M_\odot}{C_3 R_\odot^3} = 76.2 \times 10^3 \frac{\text{kg}}{m^3} \tag{36}$$

Our result is about half as dense as a more accurate model predicts, $\rho_{c\odot} = 156 \times 10^3 \frac{\text{kg}}{m^3}$, but not bad. Note that the density of the sun's core is around 100 times greater than the average solar density, $\rho_{\text{avg}} = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} = 14 \times 10^2 \frac{\text{kg}}{m^3}$.

We can invert Eq. (33) to find

$$K = \pi G \left(\frac{M}{C_n \xi_{\max}^3}\right)^{2/3} \tag{37}$$

Then the core pressure is

$$P_c = K \rho_c^{4/3} = \pi G \left(\frac{M_\odot}{C_3 \xi_{\max}^3}\right)^{2/3} \left(\frac{M_\odot}{C_3 R_\odot^3}\right)^{4/3} = \pi G \frac{M_\odot^2}{\xi_{\max}^2 C_3^2 R_\odot^4} = 1.24 \times 10^{16}\,\text{atm} \tag{38}$$

More accurate models give $P_c = 2.38 \times 10^{16}$ atm, so again, we are about a factor of 2 off.

The temperature in the Eddington model is given by Eq. (22) with $P_{\text{gas}} = (1-\beta)P$:

$$T(r) = (1-\beta) \frac{\mu m_p}{k_B} \frac{P(r)}{\rho(r)} \tag{39}$$

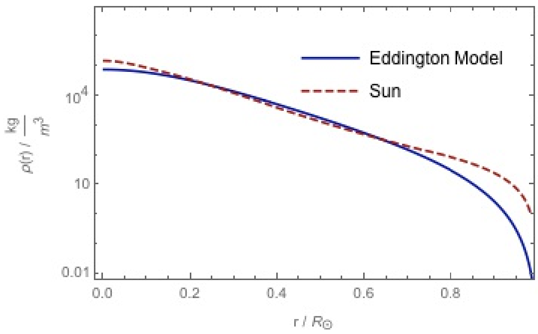
So the core temperature is

$$T_c = (1-\beta) \frac{\mu m_p}{k_B} \frac{P_c}{\rho_c} = (1-\beta) \frac{\mu}{k_B} \pi G \frac{M_\odot m_p}{\xi_{\max}^2 C_3 R_\odot} = 12.2 \times 10^6 K \tag{40}$$

which is quite close to the more accurate value of $T_c = 15.8 \times 10^6\,K$.

The advantage of a complete model like this one, in constrast to simple dimensional analysis, is that it predicts the shape of the density, temperature and pressure profiles. The density is the $n=3$ curve in Fig. 4. Putting in ρ_c, we can plot our prediction and compare to a more complete and accurate model (the "Standard Solar Model" labeled "Sun" below). We get:

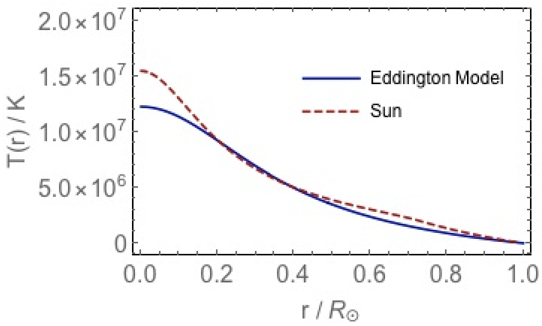
$$\rho(r) = \tag{41}$$



Other than the factor of 2 difference in ρ_C, which you can't really see in a log plot, this isn't bad.

The temperature scales like $\rho^{1/3}$ from Eq. (24), so we can compute its profile too

$$T(r) = T_c \left(\frac{\rho(r)}{\rho_c}\right)^{1/3} = \tag{42}$$



Given the rate for hydrogen fusion, which is strongly temperature and density dependent, these profiles (or more accurate ones) can be used to determine the lifetime (and age) of the sun and other stars.

Whether or not the Eddington Model is correct, we can draw some general lessons from this exercise. Since *some* equation of state exists for the star, there is some unique solution to the stellar structure equations and therefore some precise relation between the radiation pressure and the thermal gas pressure in a star. This leads to a set of one-parameter families of solutions: given a single dimensionful parameter, such as the mass, then the temperature, size, luminosity, pressure and density are all determined. This is the origin of the Vogt-Russel theorem mentioned in the introduction. Each *type* of star, with its appropriate nuclear physics and equations of state, leads to a 1D curve in the HR diagram. Thus there is the main sequence, described reasonably well by the Eddington Solar Model. For the horizontal branch (red giants), different physics is relevant, but those stars all lie on a curve. For the asymptotic giant branch, the stars line on a different curve. The width of the curves in the HR diagram are due to the stars aging – over time, their chemical composition slowly changes and the physics slowly varies, until the star explodes when the star moves to a different HR branch.

4 White dwarfs

We have discussed thermal pressure in a star and radiation pressure. Both are important in main sequence and giant stars. A third type of pressure, degeneracy pressure, is critical to determining the endpoint of stellar evolution. In particular, white dwarf stars, which have no nuclear reactions going on, are prevented from collapse due to electron degeneracy pressure alone.

Despite their lack of nuclear fusion, white dwarfs are very hot. Remember, they are remnants of the cores of giant stars that have collapsed so far that only degeneracy pressure keeps them from imploding all the way down to a black hole. The collapse to form the white dwarf starts from a very hot core, with temperatures at least $10^7\,K$ and then gets hotter as it collapses. Thus the *surface* temperature of a white dwarf is comparable the *core* temperature of a main sequence star $\sim 10^7$ K. This much much hotter than the surface temperature of the sun $\sim 5800\,K$. White dwarfs do not have active nuclear fusion, but the thermal motion of the charged particles (electrons and protons) causes them to radiate electromagnetically. They are so hot that the blackbody spectrum in the visible region is essentially flat, which is why they are white. Without fusion, gravity makes the white dwarf contract until it becomes extremely dense: a typical white dwarf has a density of $\rho \approx 10^{10} \frac{\text{kg}}{m^3}$. This is more than 1 million times denser than the sun $\rho_\odot \approx 1400 \frac{\text{kg}}{m^3}$. A typical white dwarf has the mass of the sun and the size of the earth.

The chemical composition of white dwarfs can vary, depending on what collapsed to form them. There are helium white dwarfs, from the collapse of a star that never got hot enough to fuse helium. Most white dwarfs are mostly carbon and oxygen, from red giants or supergiants that have fused helium but could not compress enough to fuse carbon. They do not have much hydrogen, since hydrogen is all burned up at earlier stages of stellar evolution. At a temperature of $T = 10^7 K$ the typical thermal energy is $k_B T = 850\,\text{eV}$. This is well above the typical ionization energies of atoms ($\sim$10eV), so white dwarfs should be thought of as fully ionized. Thus for each proton there is one free electron. The elements that can compose a white dwarf: helium, carbon or oxygen, all have roughly equal numbers of neutrons and protons, so there is around 1 electron for every $2m_p$ of stellar mass. With a density of $\rho = 10^{10} \frac{\text{kg}}{m^3}$ the number density of electrons is then

$$n_e = \frac{\rho}{2m_p} = 2 \times 10^{36} \frac{1}{m^3} \tag{43}$$

For example, recall from Lecture 13, that the Fermi energy for an non-relativistic electron gas is $\varepsilon_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$. For the white dwarf, this evaluates to $\varepsilon_F = 450\,\text{keV}$ corresponding to a Fermi temperature of $T_F = \frac{\varepsilon_F}{k_B} = 5 \times 10^9 K$. This is much much higher than typical white dwarf temperature of $10^7 K$, so we can assume the white dwarf is *completely degenerate*. Note also that ε_F is the same order as the electron rest mass $m_e c^2 = 511$ keV. Thus the electron gas in a white dwarf can be relativistic.

We begin by considering the limit where the electrons are ultrarelativistic ($\varepsilon \gg m_e c^2$). We will justify this approximation after doing a more difficult calculation using a relativistic, but not necessarily ultrarelativistic, electron gas in Section 5.

For an electron gas, the allowed momenta are the usual

$$\vec{p} = \hbar \frac{\pi}{L} \vec{n}, \quad \vec{n} = \text{triplet of whole numbers} \tag{44}$$

If the gas is ultrarelativistic the dispersion relation is

$$\varepsilon = c\,|\vec{p}| = c\hbar \frac{\pi}{L} n \tag{45}$$

We get the density of states by the usual replacement

$$2\sum_n \rightarrow 2 \times \frac{1}{8} \int 4\pi n^2 dn = \frac{V}{c^3 \pi^2 \hbar^3} \int \varepsilon^2 d\varepsilon \tag{46}$$

so

$$g(\varepsilon) = \frac{V}{c^3 \pi^2 \hbar^3} \varepsilon^2 \tag{47}$$

Then

$$N = \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon = \frac{V}{c^3 \pi^2 \hbar^3} \int_0^{\varepsilon_F} \varepsilon^2 d\varepsilon = \frac{V}{3c^3 \pi^2 \hbar^3} \varepsilon_F^3 \tag{48}$$

and thus the Fermi energy is

$$\varepsilon_F = c\hbar \left(3\pi^2 \frac{N}{V}\right)^{1/3} \tag{49}$$

Next, we compute the total energy of the white dwarf. Assuming complete degeneracy, we need to integrate the momenta up to p_F:

$$E = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon) d\varepsilon = \frac{V}{c^3\pi^2\hbar^3}\int_0^{\varepsilon_F} \varepsilon^3 d\varepsilon = \frac{V}{4c^3\pi^2\hbar^3}\varepsilon_F^4 \tag{50}$$

With the total energy, we can now compute the pressure

$$P_{\text{degen}} = -\frac{\partial E}{\partial V} = \frac{c\hbar}{12\pi^2}\left(\frac{3\pi^2 N}{V}\right)^{4/3} \tag{51}$$

This is the **degeneracy pressure**. It is entirely due to the electrons being in excited states due to the Pauli exclusion principle (and not to thermal motion, since we are working in the $T=0$ limit).

Scaling out the mass (recalling that there are 2 electrons for each m_p of mass, so $\rho = 2m_p n_e$), we can write

$$P_{\text{degen}} = \frac{c\hbar}{12\pi^2}\left(\frac{3\pi^2}{2m_p}\right)^{4/3}\rho^{4/3} \tag{52}$$

This gas is therefore a polytrope as in Eq. (26) with index $n=3$, just like the Eddington model, and with $K = \frac{c\hbar}{12\pi^2}\left(\frac{3\pi^2}{2m_p}\right)^{4/3}$. By the way we have implicitly used that the mass density ρ has the same shape as the electron density n_e, which follows from local charge neutrality.

Conveniently, we have already studied this polytropic form. Recall that the Lane-Emden equation describes hydrostatic equilibrium: the gravitational attraction is exactly counterbalanced by pressure. So using the Lane-Emden solution, we should be able to immediately find the Chandrasekhar mass. In the Lane-Emden equation we use $\rho(r) = \rho_c\theta(\xi(r))^3$ where

$$\xi = \frac{r}{\sqrt{\frac{K}{\pi G\rho_c^{2/3}}}} \leqslant \xi_{\max} = 6.89 \tag{53}$$

so that

$$R = \xi_{\max}\sqrt{\frac{K}{\pi G\rho_c^{2/3}}} = \frac{\xi_{\max}}{2}\left(\frac{3c^2\hbar^3}{16\pi\rho_c^2 m_p^4 G_N^3}\right)^{1/6} \tag{54}$$

Plugging this into Eq. (33) gives a bound on the mass that can be supported by degeneracy pressure alone

$$M \leqslant C_3 R^3\rho_c = \frac{C_3\xi_{\max}^3}{32}\sqrt{\frac{3c^3\hbar^3}{\pi G_N^3 m_p^4}} = 0.77\sqrt{\frac{c^3\hbar^3}{G_N^3 m_p^4}} = 1.41 M_\odot \tag{55}$$

This is known as the **Chandrasekhar limit**. If a white dwarf has a mass heavier than this limit, its gravitational attraction will overwhelm its degeneracy pressure and it will collapse.

Most white dwarfs in our galaxy whose masses can be measured (those in binary systems), have masses around $0.5M_\odot$. One of the most massive white dwarfs known is Sirius B. It is the companion of Sirius A, the "dog star", a main sequence star that is the brightest in the sky. To find the Sirius binary system, follow Orion's belt. Because Sirius B is in a binary system, we can measure its mass to be $1.02 M_\odot$. This is, of course, within the Chandrasekhar bound.

While the ultrarelativistic limit has let us compute the Chandrasekhar limit relatively quickly, the radius R has completely dropped out of the result. This is an artifact of the $n=3$ polytropic form (it also dropped out in the Eddington model in Eq. (34)). Thus we cannot compute the density of the star and verify that we are in the ultrarelativistic limit. Moreover, it is also questionable to apply the ultrarelativistic limit for the entire energy integral because the lowest energy states are necessarily non-relativistic. We next re-calculate the bound not assuming the ultrarelativistic limit. This will confirm and justify Eq. (55).

5 Complete Chandrasekhar limit calculation

The calculation from Section 4 assumed the electrons were ultrarelativistic. This let us reuse results from the $n=3$ polytrope that describes our sun to quickly get the Chandrasekhar bound. In this section, we perform a more complete calculation, not assuming an ultrarelativistic dispersion relation. This will allow us to justify the ultrarelativistic limit from Section 4 and also allow us to determine the radius and density of a white dwarf.

If you are exhausted and already believe the ultrarelativistic limit, feel free to only skim this section. The main result is shown in Fig. 6. Please at least make you understand what is being plotted in this figure.

5.1 Relativisitic case

For a relativistic but not necessarily ultrarelativistic electron, the dispersion relation is

$$\varepsilon(p)=\sqrt{m_e^2c^4+c^2p^2} \tag{56}$$

with m_e the electron mass. Because of this awkward square root, we will integrate using the momentum rather than the energy. So we replace

$$2\sum_n \to 2\times\frac{1}{8}\int 4\pi n^2 dn=\frac{V}{\pi^2\hbar^3}\int p^2 dp \tag{57}$$

That is, the density of momentum states is

$$g(p)=\frac{V}{\pi^2\hbar^3}p^2 \tag{58}$$

Correspondingly, instead of the Fermi energy, it will be useful to us the **Fermi momentum** p_F, defined so that

$$N=\int_0^{p_F} g(p)\,dp=\frac{V}{\pi^2\hbar^3}\int_0^{p_F} p^2 dp=\frac{V}{3\pi^2\hbar^3}p_F^3 \tag{59}$$

Thus,

$$p_F=\hbar\left(3\pi^2\frac{N}{V}\right)^{1/3} \tag{60}$$

The Fermi energy is related to the Fermi momentum by

$$\varepsilon_F=\varepsilon(p_F)-\varepsilon(0)=\sqrt{m_e^2c^4+c^2p_F^2}-m_ec^2 \tag{61}$$

Next, we compute the total energy of the white dwarf. Assuming complete degeneracy, we need to integrate the momenta up to p_F:

$$E=\int_0^{p_F}\sqrt{m_e^2c^2+p^2}\,g(p)dp=\frac{Vc}{\pi^2\hbar^3}\int_0^{p_F}\sqrt{m_e^2c^2+p^2}\,p^2dp \tag{62}$$

Changing variables to $x=\frac{p}{m_e c}$ gives

$$E=\frac{Vm_e^4c^5}{\pi^2\hbar^3}\int_0^{x_F}\sqrt{1+x}x^2\,dx=\frac{Vm_e^4c^5}{8\pi^2\hbar^3}f(x_F) \tag{63}$$

where

$$x_F=\frac{p_F}{m_ec}=\frac{\hbar}{m_ec}\left(3\pi^2\frac{N}{V}\right)^{1/3} \tag{64}$$

and

$$f(x)=8\int_0^{x_F}\sqrt{1+x}x^2\,dx=\sqrt{1+x^2}(x+2x^3)-\ln\left(x+\sqrt{1+x^2}\right) \tag{65}$$

With the energy, we can now compute the pressure

$$P_{\text{degen}}=-\frac{\partial E}{\partial V}=\frac{m_e^4c^5}{8\pi^2\hbar^3}\left[-f(x_F)-f'(x_F)V\frac{\partial x_F}{\partial V}\right] \tag{66}$$

This simplifies to

$$\boxed{P_{\text{degen}}=\frac{m_e^4c^5}{8\pi^2\hbar^3}\left[\frac{1}{3}x_F^3\sqrt{1+x_F^2}(2x_F^3-3)+\ln\left(x_F+\sqrt{1+x_F^2}\right)\right]} \tag{67}$$

This is the formula for the degeneracy pressure for a relativistic electron gas.

As a check, we can expand in the ultra-relativistic limit, $x_F \gg 1$ giving

$$P_{\text{degen}}^{\text{ultra-rel}} = \frac{m_e^4 c^5}{12\pi^2\hbar^3} x_F^4 = \frac{c\hbar}{12\pi^2}\left(\frac{3\pi^2 N}{V}\right)^{4/3} \tag{68}$$

in agreement with Eq. (51). We can also work out the non-relativistic limit, by expanding to leading order in x_F:

$$P_{\text{degen}}^{\text{non-rel}} = \frac{m_e^4 c^5}{15\pi^2\hbar^3} x_F^5 = \frac{\hbar^2}{5m_e}(3\pi^2)^{2/3}\left(\frac{N}{V}\right)^{5/3} \tag{69}$$

As in the ultrarelativistic case, we can use $N = \frac{\rho}{2m_p}$ to write this as a polytrope

$$P_{\text{degen}}^{\text{non-rel}} = K\rho^{5/3}, \quad K = \frac{\hbar^2}{10 m_e m_p}\left(\frac{3\pi^2}{2m_p}\right)^{2/3} \tag{70}$$

So this polytrope has index $n = \frac{3}{2}$, like an adiabatic ideal gas.

5.2 Equilibrium

For the white dwarf to be in equilibrium, the outward pressure which tries to increase the volume must be compensated by the inward pressure from gravity. To compute the inward pressure, we need the volume dependence of the gravitational energy. If we assume uniform density, the gravitational energy is

$$E_{\text{grav}}^{\text{const dens.}} = -\int dr \frac{G}{r} \times \overbrace{\left(\frac{4}{3}\pi\rho r^3\right)}^{\text{mass inside shell at } r} \times \overbrace{(4\pi r^2\rho)}^{\text{mass of shell at } r} dr = -\frac{16}{15}\pi^2\rho^2 G R^5 = -\frac{3}{5}G\frac{M^2}{R} \tag{71}$$

where $M = \frac{4}{3}\pi R^3\rho$ was used in the last step.

Of course, ρ is not constant. To get an improved calculation of the gravitational energy, we can use the density profile from the Lane-Emden equation using one of the polytropic form limits. The energy integral can be reduced to a scaleless integral similar to Eq. (30):

$$E_{\text{grav}} = -\frac{16\pi^2}{3}G\int_0^R dr\, r^4\rho(r)^2 = -\frac{16\pi^2}{3}G\rho_c^2\frac{R^5}{\xi_{\max}^5}\int_0^{\xi_{\max}}[\theta(\xi)]^{2n}\xi^4 d\xi \tag{72}$$

Then using Eq. (30), $M = C_n R^3\rho_c$, we can write

$$E_{\text{grav}} = -D_n G\frac{M^2}{R}, \quad D_n = \frac{16\pi^2}{3\xi_{\max}^5 C_n^2}\int_0^{\xi_{\max}}[\theta(\xi)]^{2n}\xi^4 d\xi \tag{73}$$

For $n = 3$ (ultra-relativistic case) we have as before that $\xi_{\max} = 6.9$ and $C_3 = 0.077$ and now find $D_3 = 0.68$. You can work out the numbers for the non-relativistic case yourself, as we won't use them here. It's perhaps illuminating to compare the constant and ultrarelativistic density profiles. Normalizing to the same total mass M:

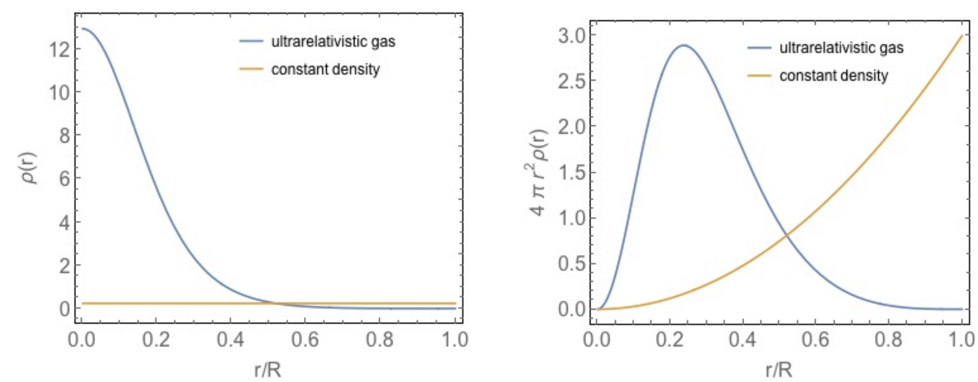


Figure 5. Density profiles for the constant-density assumption and an ultrarelativistic degenerate electron gas assumption (the $n = 3$ polytrope). Right shows $4\pi r^2\rho(r)$ normalized to integrate to 1.

Writing in general $E_{\rm grav} = -DG\frac{M^2}{R}$, the gravitational pressure is, using $V = \frac{4}{3}\pi R^3$ and $\frac{\partial V}{\partial R} = 3\frac{V}{R}$

$$P_{\rm grav} = -\frac{\partial E_{\rm grav}}{\partial V} = \frac{\partial}{\partial R}\left(DG\frac{M^2}{R}\right)\frac{\partial R}{\partial V} = -\left(DG\frac{M^2}{R^2}\right)\left(\frac{R}{3V}\right) = -\frac{DGM^2}{4\pi R^4} \tag{74}$$

Thus degeneracy pressure can hold off the collapse if

$$P_{\rm degen} \geqslant D\frac{GM^2}{4\pi R^4} \tag{75}$$

As a consistency check, we compare to the ultra-relativistic case. Using the ultra-relativistic form for $P_{\rm degen}$ in Eq. (68) and replacing $\frac{N}{V} = \frac{M}{2m_p}\frac{1}{\frac{4}{3}\pi R^3}$ and $D = D_3 = 0.68$ we need to solve

$$P_{\rm degen}^{\rm ultra-rel} = \frac{c\hbar}{12\pi^2}\left(3\pi^2\frac{M}{2m_p}\frac{3}{4\pi R^3}\right)^{4/3} \geqslant D_3\frac{GM^2}{4\pi R^4} \tag{76}$$

Here we see that R drops out and so

$$M \leqslant \frac{9\sqrt{3\pi}}{64D_3^{3/2}}\sqrt{\frac{c^3\hbar^3}{G_N m_p^4}} = 1.41\,M_\odot \tag{77}$$

Which is the same as we found using Eq (55).

Back to the general relativistic case, with the same replacement $\frac{N}{V} = \frac{3M}{8m_p R^3}$ we can write Eq. (64) as

$$x_F = \frac{\hbar}{m_e cR}\left(\frac{9\pi M}{8m_p}\right)^{1/3} \tag{78}$$

In terms of this function x_F, the equilibrium condition in Eq. (75) with Eq. (67) becomes:

$$\boxed{\frac{1}{3}x_F^3\sqrt{1+x_F^2}(2x_F^3-3) + \ln\left(x+\sqrt{1+x^2}\right) = D\left(\frac{8\pi^2\hbar^3}{m_e^4c^5}\right)\frac{GM^2}{4\pi R^4}} \tag{79}$$

5.3 Mass radius relation

To study Eq. (79) we first put in some numbers. Using $V = \frac{4}{3}\pi R^3$ and $M = 2m_p N$ we can write x_F in Eq. (64) as

$$x_F(R) = \frac{\hbar}{m_e c}\left(3\pi^2\frac{\frac{M}{2m_p}}{\frac{4}{3}\pi R^3}\right)^{1/3} = \frac{\hbar}{m_e cR}\left(\frac{9\pi M}{8m_p}\right)^{1/3} = 0.97\left(\frac{R_E}{R}\right)\left(\frac{M}{M_\odot}\right)^{1/3} \tag{80}$$

where $M_\odot = 1.98\times10^{30}$kg is the mass of the sun and $R_E = 6370$km is the radius of the earth. Plugging in m_e and the other constants, we can also write the right-hand side of Eq. (79) as

$$D\left(\frac{8\pi^2\hbar^3}{m_e^4c^5}\right)\frac{1}{4\pi}G\frac{M^2}{R^4} = 0.69D\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_E}{R}\right)^4 \tag{81}$$

Because $f(x)$ is an order-1 function and x_F and the right hand side of (79) are all the same order when $R \sim R_E$ and $M \sim M_\odot$, we expect that the solutions will also have this form.

Note that we are not making any claims about whether M is close to $M_\odot$ or R is close to R_E, we are just writing x_F in a suggestive way. In other words, it so happens that $M_\odot$ and R_E are close to combinations of constants that appear in x_F:

$$\sqrt{\frac{c^3\hbar^3}{G_N^3 m_p^4}} = 1.8M_\odot, \qquad \sqrt{\frac{\hbar^3}{cG_N m_e^2 m_p^2}} = 0.78\,R_E \tag{82}$$

For example, Sirius B, the white dwarf partner of the bright Sirius A, has a mass of $1.02 M_\odot$. Now we can calculate its radius by numerically solving Eq. (79). We get $R = 0.54 R_E$, confirming our estimates of density from the beginning of this section. Most white dwarfs are around $0.5 M_\odot$; for these, we find $R \approx R_E$.

Using the value of $D = D_3 = 0.68$ we computed using the ultrarelativistic polytrope, we can solve Eq. (79) numerically to get the mass-radius relation:

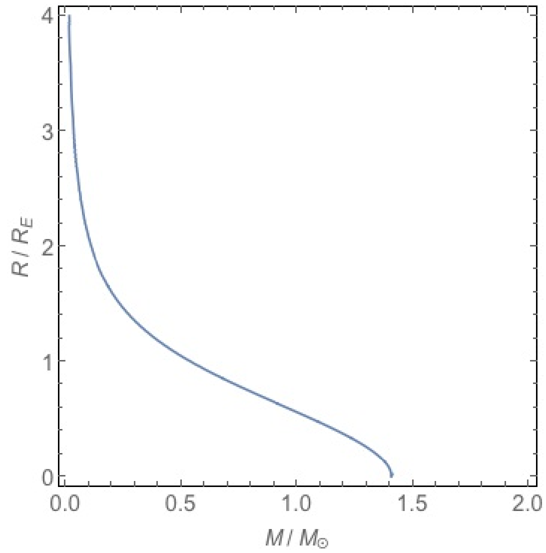


Figure 6. The relationship between mass and size of a white dwarf determined by equating the degeneracy pressure to the gravitational pressure. Axes are normalized to the sun's mass and the earth's radius.

We see from the plot that for $M \gtrsim 1.4 M_\odot$ there are no solutions. Having included full relativistic corrections, we can also check our assumptions. Note that the bound comes from small R. As $R \to 0$ the gas gets denser and denser, so the electrons become faster and faster. We can see this also from Eq. (78) where $x_F \sim \frac{1}{R}$ so as $R \to 0$, x_F gets large. Expanding at large x_F is exactly the ultrarelativistic limit. Therefore the bound is as in Eq. (77):

$$M \leqslant \frac{9\sqrt{3\pi}}{64 D_3^{3/2}} \sqrt{\frac{c^3 \hbar^3}{G_N m_p^4}} = 1.41 M_\odot \tag{83}$$

This is called the **Chandrasekhar limit**. For masses of white dwarfs greater than the Chandrasekhar limit, there is no solution to equating the degeneracy pressure with the gravitational pressure. The gravitational pressure is just too great, and the white dwarf will collapse (ultimately to a neutron star or a black hole).

6 Neutron stars

When a star's total mass is above around $8 M_\odot$, the mass around the core will be larger than the Chandrasekhar limit. In this case after helium fusion ends and the core temperature is not hot enough for carbon fusion to occur, the gravitational pressure will overwhelm the electron degeneracy pressure and the core will collapse past the white dwarf stage. As mentioned in Section 2, the further compression heats up the core, allowing the remaining stages of fusion to occur, making elements up to iron. After these stages finish, the star collapses again, reaching such high densities that the protons smash into the electrons producing neutrons and neutrinos. This $p^+ + e^- \to n + \nu$ reaction is energetically favorable because the neutrinos stream out of the star relieving some of the pressure. Ultimately, only neutrons are left.

Neutrons are fermions, and the core of neutrons in a neutron star is a Fermi gas, like a white dwarf. We can determine the characteristic size and mass of a neutron star by substituting in the appropriate scale, namely, replacing m_e by $m_n = m_p$. For a white dwarf, recall the characteristic mass and radius as in Eq. (82):

$$M_{\rm WD} \approx \sqrt{\frac{c^3\hbar^3}{G_N^3 m_p^4}} = 3.6\times 10^{30}{\rm kg} = 1.8 M_\odot, \qquad R_{\rm WD} \approx \sqrt{\frac{\hbar^3}{c G_N m_e^2 m_p^2}} = 4970\,{\rm km} = 0.78 R_E \tag{84}$$

We see that the mass scale does not depend on m_e, so it is unaffected by $m_e \to m_p$ and so should be around the same for a neutron star. More precisely, if we recall that in a white dwarf there is 1 electron for every $2m_p$ of mass so that $n = \frac{\rho}{2m_p}$ in a neutron star we have more simply $n = \frac{\rho}{m_p}$. This amounts to replacing $m_p \to \frac{m_p}{2}$ and $m_e \to m_p$ and thus $M_{\rm NS} \approx 4 M_{\rm WD}$ and so

$$M_{\rm NS} \lesssim 4\times 1.4 M_\odot = 5.6\,M_\odot \tag{85}$$

This is our estimate for the Chandrasekhar bound for neutron stars. The actual bound should be a bit lower since the binding energy of the neutrons, effects from general relativity, and rotational energy cannot be neglected. One estimate, called the **Tolman–Oppenheimer–Volkoff limit**, is $M_{\rm NS} \lesssim 3\,M_\odot$, but this estimate is controversial. Determining a precise upper bound on the mass of a neutron star is still an open theoretical question. The largest observed neutron stars to date is around $2.7 M_\odot$.

The characteristic size of a neutron star is determined from the size of the white dwarf with the $m_p \to \frac{m_p}{2}$ and $m_e \to m_p$ replacements:

$$R_{\rm NS} \approx \sqrt{\frac{4\hbar^3}{c G_N m_p^4}} = 5\,{\rm km} \tag{86}$$

Thus, although neutron stars have similar masses to white dwarfs, they are much much denser. The whole mass of the sun is being squeezed into $5\,{\rm km} = 3\,{\rm miles}$ – roughly the size of Cambridge. The density of such a star is

$$\rho_{\rm NS} \approx \frac{M_\odot}{\frac{4}{3}\pi R_{\rm NS}^3} = 3\times 10^{18}\frac{\rm kg}{m^3} \tag{87}$$

Compare this to the density of water $\rho_{\rm water} = 10^3\frac{\rm kg}{m^3}$, to the core of the sun $\rho_{c\odot} = 10^5\frac{\rm kg}{m^3}$, or to the density of a white dwarf, $\rho_{\rm WD} \approx 10^9\frac{\rm kg}{m^3}$. None of these are even close. In fact, the density of a neutron star is larger than the density of a proton: $\rho_{\rm proton} \approx \frac{m_p}{\frac{4}{3}\pi(10^{-15}m)^3} = 10^{17}\frac{\rm kg}{m^3}$! That is, while a white dwarf is like a gigantic metal, a neutron star is like a gigantic nucleus, with atomic number 10^{57}.

Such a giant nucleus has a lot of strange properties. By angular momentum conservation, a slowly rotating star will collapse to a extremely rapidly rotating neutron star: like an ice skater bringing her arms in by a factor of a million. Neutron stars can be spinning as fast as 1000 times per second. The spinning also concentrates the magnetic field of the star, to as much as 10^{15} Gauss. Compare this to the earth's magnetic field (0.5 G) or the fields in magnets at the Large Hadron Collider (5×10^4 G). Such large magnetic fields can act as a dynamo generating large electric fields near the star's surface. These enormous electric fields then create electron/positron pairs which are thrown out of the star, radiating electromagnetically. The result is a beam of light, spinning around as the neutron star spins, like an out-of-control lighthouse beacon. If the beam is aligned to hit us, we see this is a periodic signal. Such neutron stars are known as **pulsars**.

In July, 1054 AD, a Chinese astronomer observed a new "guest star", brighter than any other star in the sky. It lasted for about a month and then faded. Arab astronomers observed the same object, and perhaps Native American astronomers as well. We now know that this was a core-collapse supernova. 1000 years later, the supernova remnants are visible as the **crab nebula**, in the constellation Taurus. There is a neutron star in the middle of the crab nebula called the crab pulsar. The pulsar has a mass of $1.4 M_\odot$ and period of $0.3s$.

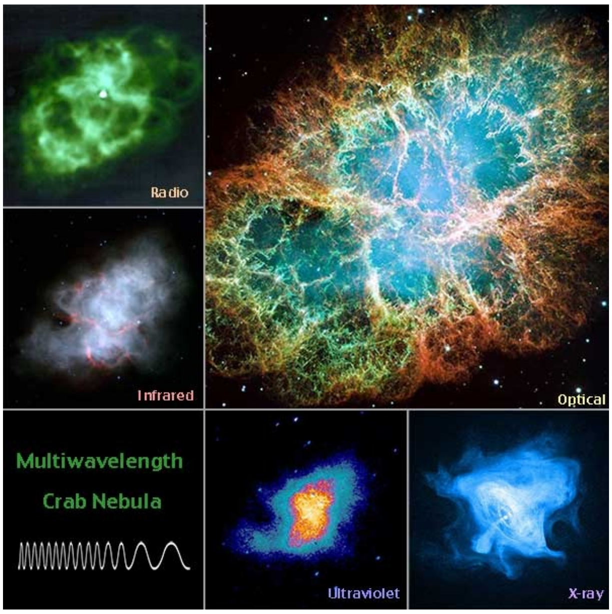


Figure 7. The crab nebula is around 10 light years across. In its center is a neutron star pulsar. The neutron star cannot be seen in the optical spectrum, but is clearly visible in the radio and x-ray bands.

In 2017, a binary neutron star system was discovered through its gravitational wave signal as the neutron stars merged. The neutron stars had masses around $1.5M_{\odot}$. One interesting mystery that such mergers might explain is where all the gold in the universe came from. It turns out it is very difficult to explain how gold might have come from supernovae. However, the nuclear physics of neutron star mergers seems like it gives a better explanation. This science is all very recent, and while early results seem promising, more data and more careful calculations and simulations are needed to draw any definite conclusions.