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Lecture 8: Fourier transforms

1 Strings

To understand sound, we need to know more than just which notes are played – we need the shape of the notes. If a string were a pure infinitely thin oscillator, with no damping, it would produce pure notes. In the real world, strings have finite width and radius, we pluck or bow them in funny ways, the vibrations are transmitted to sound waves in the air through the body of the instrument etc. All this combines to a much more interesting picture than pure frequencies. For example, the spectrum of a violin looks like this:

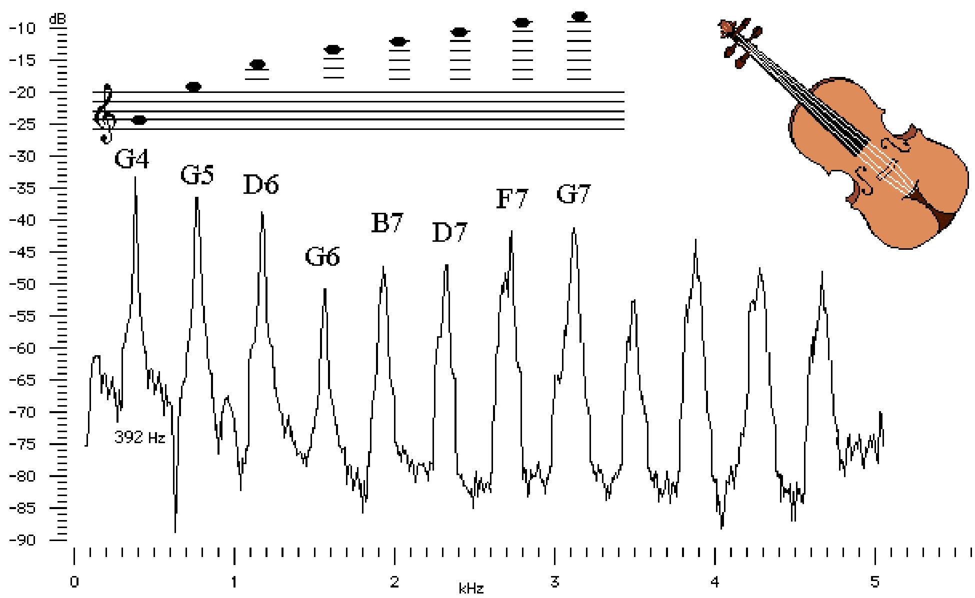


Figure 1. Spectrum of a violin

This figure shows the intensity of each frequency produced by the violin (the vertical axis is in decibels, which is a logarithmic measure of sound intensity; we'll discuss this scale in Lecture 10). We know the basics of this spectrum: the fundamental and the harmonics are related to the Fourier series of the note played. Now we want to understand where the shape of the peaks comes from. The tool for studying these things is the Fourier transform.

2 Fourier transforms

In the violin spectrum above, you can see that the violin produces sound waves with frequencies which are arbitrarily close. The way to describe these frequencies is with **Fourier transforms**.

Recall the Fourier exponential series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n x}{L}} \tag{1}$$

where

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) e^{-i\frac{2\pi n x}{L}} \tag{2}$$

To check this, we plug Eq. (1) into Eq. (2) giving

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left[\sum_{m=-\infty}^{\infty} c_m e^{i\frac{2\pi m x}{L}}\right] e^{-i\frac{2\pi n x}{L}} = \frac{1}{L}\sum_{m=-\infty}^{\infty} c_m \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, e^{i\frac{2\pi m x}{L}} e^{-i\frac{2\pi n x}{L}} \tag{3}$$

Then using the mathematical identity

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, e^{i(m-n)x\frac{2\pi}{L}} = L\delta_{mn} \tag{4}$$

we get

$$c_n = \frac{1}{L}\sum_{m=-\infty}^{\infty} c_m L \delta_{nm} = c_n \tag{5}$$

as desired. That is, we have checked Eq. (2).

To derive the Fourier transform, we write

$$k_n = \frac{2\pi n}{L} \tag{6}$$

where n is still an integer going from $-\infty$ to $+\infty$. For arbitrary L, k_n can get arbitrarily big in the positive or negative direction. However, at fixed L, the lowest non-zero k_n cannot be arbitrarily small: $|k_n| > \frac{2\pi}{L}$. Then, we define

$$\tilde{f}(k_n) = \frac{L c_n}{2\pi} = \frac{1}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) e^{-i k_n x} \tag{7}$$

The factor of 2π in this equation is just a convention. Now we can take $L \to \infty$ so that k_n can get arbitrarily close to zero. This gives

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \tag{8}$$

where now k can be any real number. This is the Fourier transform. It is a continuum generalization of the c_n's of the Fourier series.

The inverse of this comes from writing Eq. (1) as a integral. From Eq. (6), we find $dk_n = \frac{2\pi}{L}\Delta n$. This leads to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ikx}\Delta n = \sum_{n=-\infty}^{\infty} c_n e^{ikx}\frac{L}{2\pi} dk_n = \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx} \tag{9}$$

where we have used Eq. (7) and taken $L \to \infty$ in the last step.

So we have

$$\boxed{\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \qquad \Longleftrightarrow \qquad f(x) = \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx}} \tag{10}$$

We say that $\tilde{f}(k)$ is the **Fourier transform** of $f(x)$. The factor of 2π is just a convention. We could also have defined $f(x)$ with the 2π in it. The sign on the phase is also a convention (that is, we could have defined $\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{ikx}$ instead). Keep in mind that different conventions are used in different places and by different people. There is no universal convention for the 2π factors. All conventions lead to the same physics.

The Fourier transform of a function of x gives a function of k, where k is the **wavenumber**. The Fourier transform of a function of t gives a function of ω where ω is the angular frequency:

$$\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{-i\omega t} \tag{11}$$

3 Example

As an example, let us compute the Fourier transform of the position of an underdamped oscillator:

$$f(t) = e^{-\gamma t}\cos(\omega_0 t)\theta(t) \tag{12}$$

where the **unit-step function** is defined by

$$\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leqslant 0 \end{cases} \tag{13}$$

This function insures that our oscillator starts at time $t = 0$. If didn't include, the amplitude would blow up as $t \to -\infty$.

We first write

$$f(t) = e^{-\gamma t}\cos(\omega_0 t)\theta(t) = \frac{1}{2} e^{-\gamma t} e^{i\omega_0 t}\theta(t) + \frac{1}{2} e^{-\gamma t} e^{-i\omega_0 t}\theta(t) \tag{14}$$

So we can Fourier transform the simpler exponential function. Starting with the first term, we find

$$\begin{aligned}
\tilde{f}_{+\omega_0}(\omega) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} dt\, e^{-\gamma t} e^{-i(\omega - \omega_0)t}\theta(t) \\
&= \frac{1}{4\pi}\int_0^{\infty} dt\, e^{(-\gamma - i\omega + i\omega_0)t} \\
&= \frac{1}{4\pi}\frac{1}{-\gamma - i(\omega - \omega_0)} e^{(-\gamma - i\omega + i\omega_0)t}\Big|_0^{\infty} \\
&= \frac{1}{4\pi}\frac{1}{\gamma + i(\omega - \omega_0)}
\end{aligned}$$

In the last step we have used that the $t = \infty$ endpoint vanishes due to the $e^{-\gamma t}$ factor and that at the $t = 0$ endpoint the exponential is 1. The second term in Eq. (14) is the first term with $\omega_0 \to -\omega_0$. Thus the full Fourier transform is

$$\tilde{f}(\omega) = \frac{1}{4\pi}\left[\frac{1}{\gamma + i(\omega - \omega_0)} + \frac{1}{\gamma + i(\omega + \omega_0)}\right] = \frac{1}{2\pi i}\frac{\omega - i\gamma}{(\omega - i\gamma)^2 - \omega_0^2} \tag{15}$$

As mentioned before, the spectrum plotted for an audio signal is usually $|\tilde{f}(\omega)|^2$. Let's see what this looks like. We'll take $\omega_0 = 10$ and $\gamma = 2$. The function and the modulus squared $|\tilde{f}(\omega)|^2$ of its Fourier transform are then:

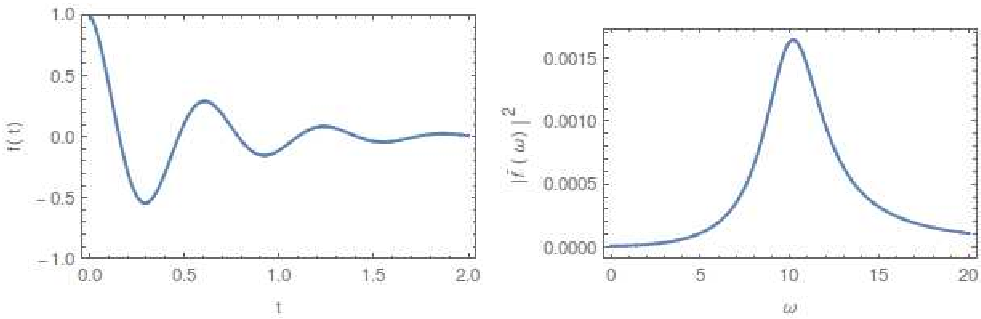


Figure 2. An underdamped oscillator and its power spectrum (modulus of its Fourier transform squared) for $\gamma = 2$ and $\omega_0 = 10$.

We now can also understand what the shapes of the peaks are in the violin spectrum in Fig. 1. The widths of the peaks give how much each harmonic damps with time. The width at half maximum gives the damping factor γ.

4 Fourier transform is complex

For a real function $f(t)$, the Fourier transform will usually not be real. Indeed, the imaginary part of the Fourier transform of a real function is

$$\mathrm{Im}[\tilde{f}(k)] = \frac{\tilde{f}(k) - \tilde{f}(k)^\star}{2i} = \frac{1}{2i}\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} - \int_{-\infty}^{\infty} dx\, f(x) e^{ikx}\right] \tag{16}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\sin(kx) \equiv \tilde{f}_s(k) \tag{17}$$

This is a Fourier sine transform. Thus the imaginary part vanishes only if the function has no sine components which happens if and only if the function is even. For an odd function, the Fourier transform is purely imaginary. For a general real function, the Fourier transform will have both real and imaginary parts. We can write

$$\tilde{f}(k) = \tilde{f}_c(k) + i\tilde{f}_s(k) \tag{18}$$

where $\tilde{f}_s(k)$ is the Fourier sine transform and $\tilde{f}_c(k)$ the Fourier cosine transform. One hardly ever uses Fourier sine and cosine transforms. We practically always talk about the complex Fourier transform.

Rather than separating $\tilde{f}(k)$ into real and imaginary parts, which amounts to Cartesian coordinates, it is often helpful to write it as a magnitude and phase, as in polar coordinates. So we write

$$\tilde{f}(k) = A(k) e^{i\phi(k)} \tag{19}$$

with $A(k) = |\tilde{f}(k)|$ the **magnitude** and $\phi(k)$ the phase.

The energy in a frequency mode only depends on the amplitude: $I = A(\omega)^2$. When one plots the spectrum as in audacity, what is being shown is $A(\omega)^2$. This corresponds to the intensity or power in a particular mode, as we will see in Lecture 10. Power is useful in doing a frequency analysis of sound since it tells us how loud that frequency is. But looking at the amplitude is not the only thing one can do with a Fourier transform. Often one is also interested in the phase.

For a visual example, we can take the Fourier transform of an image. Suppose we have a grayscale image that is 640×480 pixels. Each pixel is a number from 0 to 255, going from black (0) to white (255). Thus the image is a function $f(x, y)$ with $0 \leqslant x < 640$, $0 \leqslant y < 480$ which takes values from 0 to 255. We can then Fourier transform this function to a function $\tilde{f}(k_x, k_y)$:

$$\tilde{f}(k_x, k_y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \mathrm{dy}\, f(x, y) e^{-i k_x x} e^{-k_y y} \tag{20}$$

The 2D Fourier transform is really no more complicated than the 1D transform – we just do two integrals instead of one. So what we do we get? Here's an example

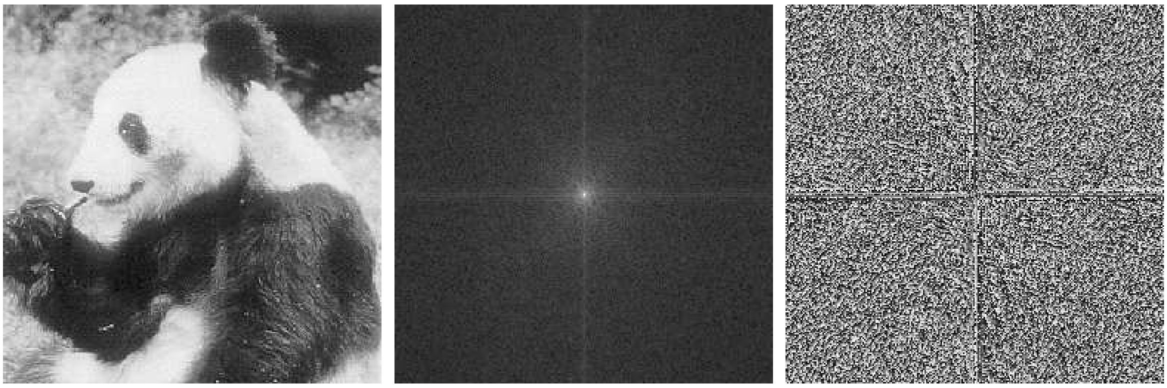


Image $f_{\text{panda}}(x, y)$ Magnitude, $A_{\text{panda}}(k_x, k_y)$ Phase $\phi_{\text{panda}}(k_x, k_y)$

Figure 3. Fourier transform of a panda. The magnitude is concentrated near $k_x \sim k_y \sim 0$, corresponding to large-wavelength variations, while the phase looks random.

We can do the same thing for a picture of a cat:

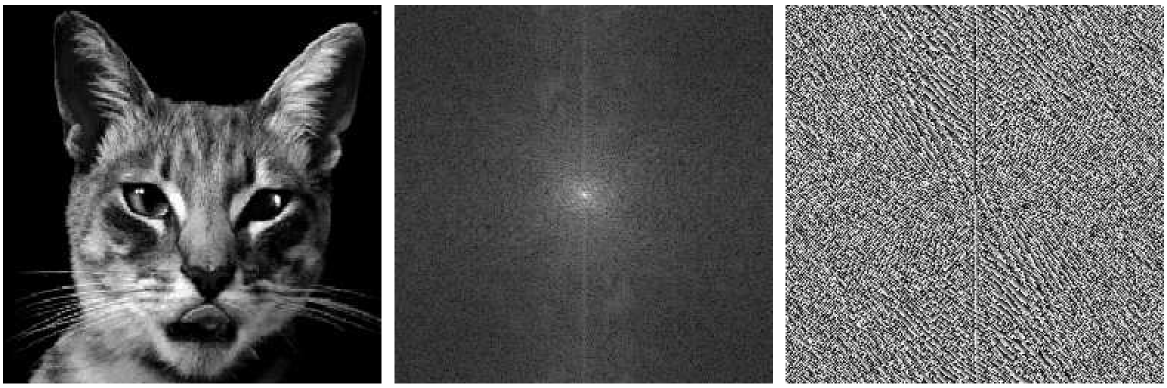
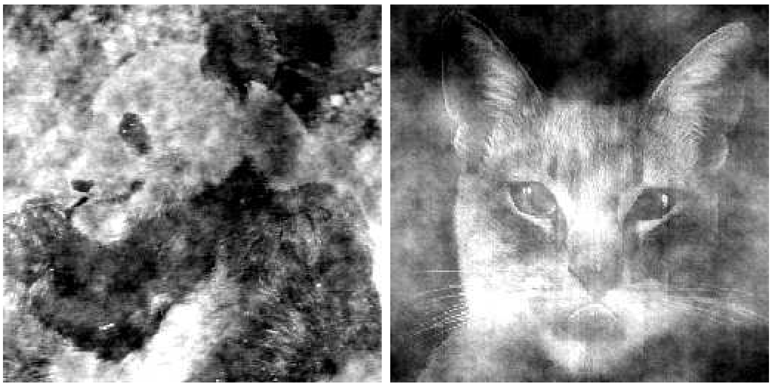


Image $f_{\text{cat}}(x, y)$ Magnitude, $A_{\text{cat}}(k_x, k_y)$ Phase $\phi_{\text{cat}}(k_x, k_y)$

Figure 4. Fourier transform of a cat. The magnitude is concentrated near $k_x \sim k_y \sim 0$, but maybe not as much as the panda, since that cat has smaller wavelength features. Phase still looks random.

Now let's Fourier transform back. Of course for the cat and panda we get back the orignal image. But what happens if we combine the magnitude for the panda with the phase for the cat, and vice versa?



$A_{\rm cat}(k_x, k_y)$ and $\phi_{\rm panda}(k_x, k_y)$ $A_{\rm panda}(k_x, k_y)$ and $\phi_{\rm cat}(k_x, k_y)$

Figure 5. We take the inverse Fourier transform of function $A_{\rm cat}(k_x, k_y)e^{i\phi_{\rm panda}(k_x,k_y)}$ on the left, and $A_{\rm panda}(k_x, k_y)e^{i\phi_{\rm cat}(k_x,k_y)}$ on the right.

It looks like the phase is more important than the magnitude for reconstructing the original image. The importance of phase is critical for many engineering applications, such as signal analysis. It is also relevant for image compression technologies.

5 Filtering

One thing we can do with the Fourier transform of an image is remove some components. If we remove low frequencies, less than some ω_f say, we call it a **high-pass filter**. A lot of background noise is at low frequencies, so a high-pass filter can clean up a signal. If we throw out the high frequencies, it is called a **low-pass filter**. A low pass filter can be used to smooth data (such as a digital photo) since it throws out high frequency noise. A filter that cuts out both high and low frequencies is called a **band-pass filter**.

Here are some examples

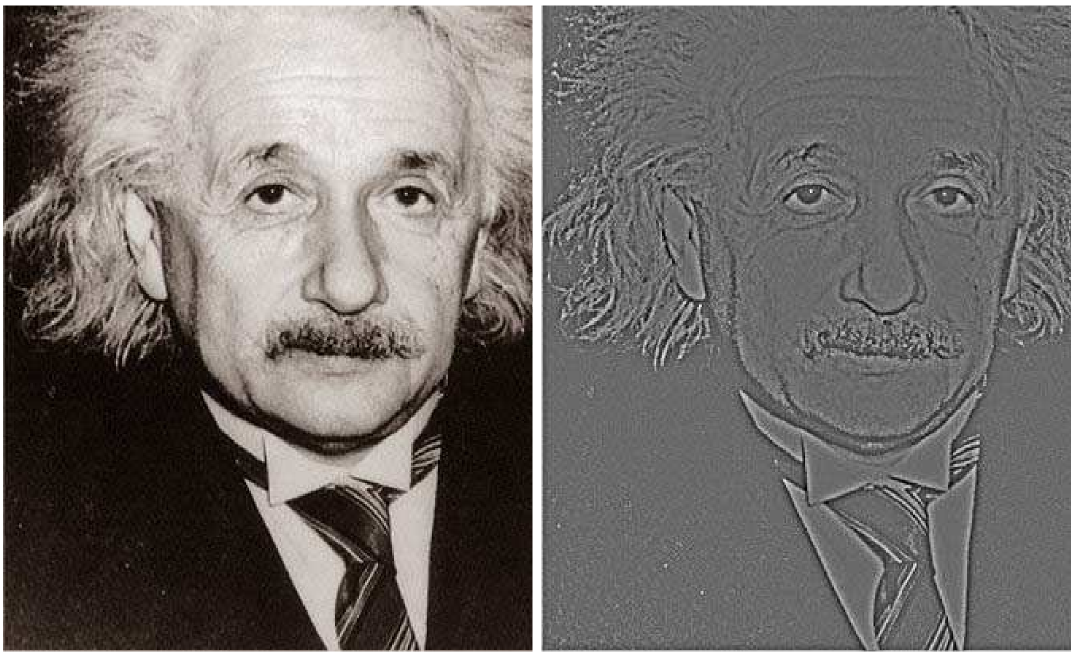


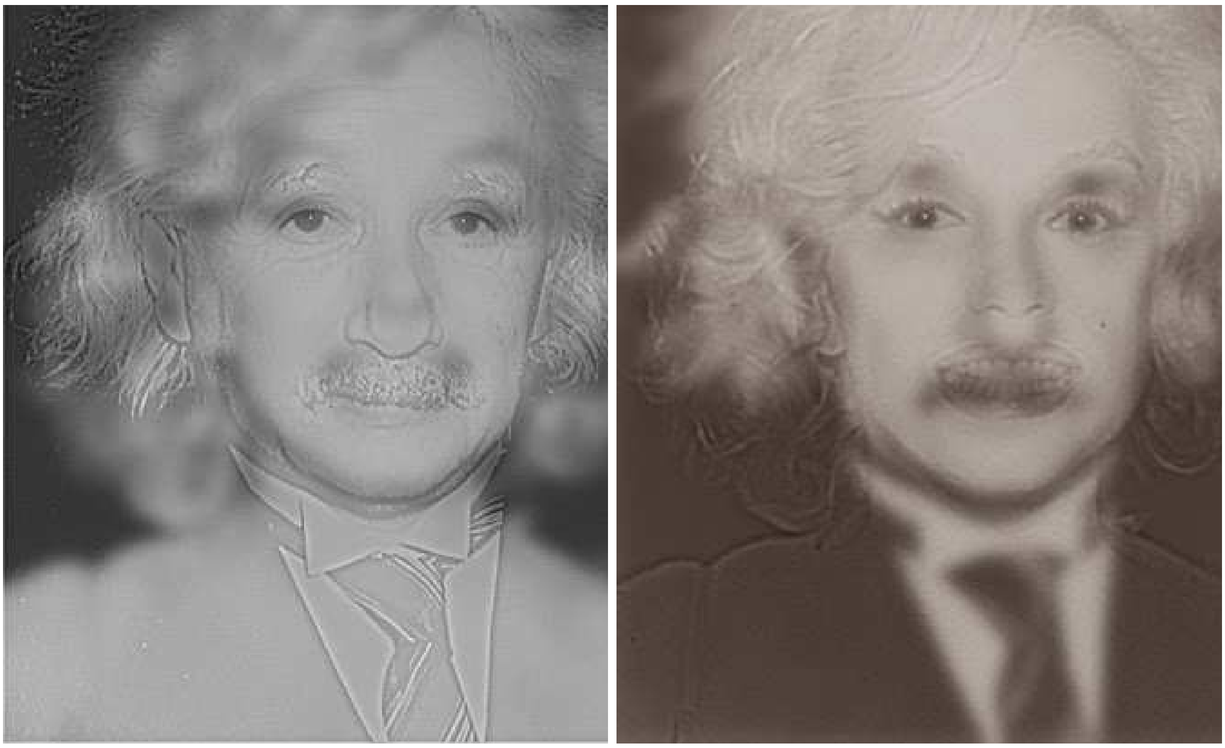
photo of Einstein Photo after high-pass filter

Figure 6. What a high-pass filter does to Albert Einstein.

photo of Einstein Photo after low-pass filter

Figure 7. What a low-pass filter does to Marylyn Monroe.

Now let's combine the two



high-pass Einstein +low pass Marylyn

low-pass Einstein +high-pass Marylyn

Figure 8. Combining filtered images

Take a look at these last two images from up close and from far away. What do you see? Why?

6 Dirac δ function

Another extremely important example is the Fourier transform of a constant:

$$\delta(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{-i\omega t} \tag{21}$$

Its Fourier inverse is then

$$1 = \int_{-\infty}^{\infty} d\omega \delta(\omega) e^{i\omega t} \tag{22}$$

This object $\delta(\omega)$ is called the **Dirac δ function**. It is enormously useful in a great variety of physics problems, especially in quantum mechanics, but also in waves.

To figure out what $\delta(\omega)$ looks likes, we use the fact that the Fourier transform of the inverse Fourier transform gives a function back. That is, for any smooth function $f(x)$

$$f(x) = \int_{-\infty}^{\infty} dk e^{ikx} \tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx} \int_{-\infty}^{\infty} dy f(y) e^{-iky} \tag{23}$$

$$= \int_{-\infty}^{\infty} dy \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ik(y-x)} f(y) \tag{24}$$

$$= \int_{-\infty}^{\infty} dy \delta(y-x) f(y) \tag{25}$$

where we used Eq. (21) in the last step. Setting $x=0$, we see that the δ-function satisfies

$$\int_{-\infty}^{\infty} dx \delta(x) f(x) = f(0) \tag{26}$$

for any smooth function $f(x)$. $\delta(x)$ also has the property that $\delta(x)=0$ for $x \neq 0$ (see Section 6.1 below), so that

$$\int_{-x_0}^{x_0} dx \delta(x) f(x) = f(0) \tag{27}$$

for any x_0.

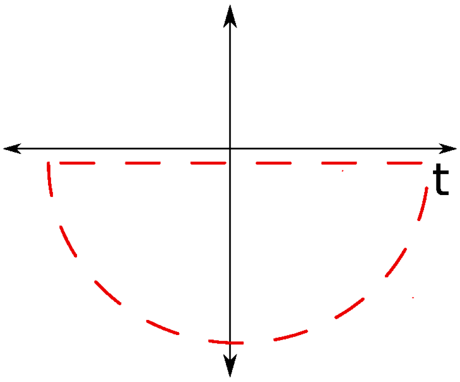
Eq. (26) and (27) uniquely define the δ-function. Indeed, the δ-function is no ordinary function. It is instead a member of a class of mathematical objects called **distributions**. While functions take numbers and give numbers (like $f(x)=x^2$), distributions only give numbers after being integrated.

You should think of $\delta(x)$ as zero everywhere except at $x=0$ where it is infinite. However, the infinity is integrable: $\int_{-x_0}^{x_0} \delta(x) = 1$ for any $x_0 > 0$.

From the physics point of view, we showed that if we have an amplitude which is constant in time $f(t)=1$ then the only frequency mode supported has 0 frequency. This makes sense – a constant has an infinite wavelength and never repeats. Conversely, if $\tilde{f}(\omega)=1$ it says that all frequencies are excited. This corresponds to **white noise**. The Fourier transform of $\tilde{f}(\omega)=1$ gives a function $f(t)=\delta(t)$ which corresponds to an infinitely sharp pulse. For a pulse has no characteristic time associated with it, no frequency can be picked out. That's why white noise has all frequencies equally.

6.1 Some mathematics of $\delta(\omega)$ (optional)

For $\omega \neq 0$ the quickest way to evaluate $\delta(\omega)$ integral is by contour integration. If you've never seen any complex analysis, just ignore this section. If you have, consider the integral in the complex ω plane along the red contour:



(28)

The integral along the contour is equal to $2\pi i$ times the residues of poles within the contour.

$$\int_{-\infty}^{\infty} dt e^{-i\omega t} f(t) + \int_{\text{curve}} dt e^{-i\omega t} f(t) = 2\pi i \sum_{\text{poles } \omega_j} \text{Res}[f, \omega_j] \tag{29}$$

For the curved part of the contour, t has a negative imaginary part. Thus $e^{-i\omega t} \to 0$ as $|t| \to$ infinity and the integral along the curved part vanishes. There are no poles in $e^{-i\omega t}$, thus the right hand side of Eq. (29) vanishes. Therefore

$$\delta(\omega) = 0, \quad \omega \neq 0 \tag{30}$$

On the other hand, for $\omega = 0$,

$$\delta(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt = \infty \tag{31}$$

So

$$\delta(\omega) = \begin{cases} 0, \omega \neq 0 \\ \infty, \omega = 0 \end{cases} \tag{32}$$

Clearly $\delta(\omega)$ is no ordinary function. It is a distribution.

A practical way to define $\delta(x)$ is as a limit. There are lots of ways to do this. Here are three:

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}, \quad \delta(x) = \lim_{\varepsilon \to 0} \varepsilon \left(\frac{1}{x}\right)^{1-\varepsilon}, \quad \delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}}, \quad \cdots \tag{33}$$

To check these definitions, try integrating any of them against any test function $g(x)$ to see that Eq. (27) is reproduced.