

Online Optimization : Competing with Dynamic Comparators

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Abstract

Recent literature on online learning has focused on developing adaptive algorithms that take advantage of a *regularity* of the sequence of observations, yet retain worst-case performance guarantees. A complementary direction is to develop prediction methods that perform well against complex benchmarks. In this paper, we address these two directions together. We present a fully adaptive method that competes with dynamic benchmarks in which regret guarantee scales with regularity of the sequence of cost functions and comparators. Notably, the regret bound adapts to the smaller complexity measure in the problem environment. Finally, we apply our results to drifting zero-sum, two-player games where both players achieve no regret guarantees against best sequences of actions in hindsight.

I. INTRODUCTION

The focus of this paper is an online optimization problem in which a *learner* plays against an *adversary* or *nature*. At each round $t \in \{1, \dots, T\}$, the learner chooses an action x_t from some convex feasible set $\mathcal{X} \subseteq \mathbb{R}^d$. Then, nature reveals a convex function $f_t \in \mathcal{F}$ to the learner. As a result, the learner incurs the corresponding *loss* $f_t(x_t)$. A learner aims to minimize his *regret*, a comparison to a single best action in hindsight:

$$\mathbf{Reg}_T^s \triangleq \sum_{t=1}^T f_t(x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^T f_t(x). \quad (1)$$

Let us refer to this as *static* regret in the sense that the comparator is *time-invariant*. In the literature, there are numerous algorithms that guarantee a static regret rate of $\mathcal{O}(\sqrt{T})$ (see e.g. [1]–[3]). Moreover, when the loss functions are strongly convex, a rate of $\mathcal{O}(\log T)$ could be achieved [4]. Furthermore, minimax optimality of algorithms with respect to the worst-case adversary has been established (see e.g. [5]).

There are two major directions in which the above-mentioned results can be strengthened: (1) by exhibiting algorithms that compete with non-static comparator sequences (that is, making the benchmark harder), and (2)

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by proving regret guarantees that take advantage of *niceness* of nature’s sequence (that is, exploiting some non-adversarial quality of nature’s moves). Both of these distinct directions are important avenues of investigation. In the present paper, we attempt to address these two aspects by developing a single, adaptive algorithm with a regret bound that shows the interplay between the difficulty of the comparison sequence and niceness of the sequence of nature’s moves.

With respect to the first aspect, a more stringent benchmark is a *time-varying* comparator, a notion that can be termed *dynamic* regret [3], [6]–[8]:

$$\mathbf{Reg}_T^d \triangleq \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x_t^*), \quad (2)$$

where $x_t^* \triangleq \operatorname{argmin}_{x \in \mathcal{X}} f_t(x)$. More generally, dynamic regret against a comparator sequence $\{u_t\}_{t=1}^T$ is

$$\mathbf{Reg}_T^d(u_1, \dots, u_T) \triangleq \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u_t).$$

It is well-known that in the worst case, obtaining a bound on dynamic regret is not possible. However, it is possible to achieve worst-case bounds in terms of

$$C_T(u_1, \dots, u_T) \triangleq \sum_{t=1}^T \|u_t - u_{t-1}\|, \quad (3)$$

i.e., the *regularity* of the comparator sequence, interpolating between the static and dynamic regret notions. Furthermore, the authors in [9] introduce an algorithm which proposes a variant of C_T involving a dynamical model.

In terms of the second direction, there are several ways of incorporating potential regularity of nature’s sequence. The authors in [10], [11] bring forward the idea of predictable sequences – a generic way to incorporate some external knowledge about the gradients of the loss functions. Let $\{M_t\}_{t=1}^T$ be a *predictable* sequence computable by the learner at the beginning of round t . This sequence can then be used by an algorithm in order to achieve regret in terms of

$$D_T \triangleq \sum_{t=1}^T \|\nabla f_t(x_t) - M_t\|_*^2. \quad (4)$$

The framework of predictable sequences captures *variation* and *path-length* type regret bounds (see e.g. [12], [13]). Yet another way in which niceness of the adversarial sequence can be captured is through a notion of *temporal variability* studied in [14]:

$$V_T \triangleq \sum_{t=1}^T \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)|. \quad (5)$$

What is interesting—and intuitive—dynamic regret against the optimal sequence $\{x_t^*\}_{t=1}^T$ becomes a feasible objective when V_T is *small*. When only noisy versions of gradients are revealed to the algorithm, Besbes et al. in [14] show that using a restarted *Online Gradient Descent* (OGD) [3] algorithm, one can get a bound of form $T^{2/3}(V_T + 1)^{1/3}$ on the expected regret. However, the regret bounds attained in [14] are only valid when an upper bound on V_T is known to the learner before the game begins. For the full information online convex optimization

setting, when one receives exact gradients instead of noisy gradients, a bound of order V_T is trivially obtained by simply playing (at each round) the minimum of the previous round.

The three quantities we just introduced — C_T, D_T, V_T — measure distinct aspects of the online optimization problem, and their interplay is an interesting object of study. Our first contribution is to develop a fully adaptive method (without prior knowledge of these quantities) whose dynamic regret is given in terms of these three complexity measures. This is done for the full information online convex optimization setting, and augments the existing regret bounds in the literature which focus on only one of the three notions — C_T, D_T, V_T — (and not all the three together). To establish a sub-linear bound on the dynamic regret, we utilize a variant of the *Optimistic Mirror Descent (OMD)* algorithm [10].

When noiseless gradients are available and we can calculate variations at each round, we not only establish a regret bound in terms of V_T and T (without a priori knowledge of a bound on V_T), but also show how the bound can in fact be improved when deviation D_T is $o(T)$. We further also show how the bound can automatically adapt to C_T the length of sequence of comparators. Importantly, this avoids suboptimal bounds derived only in terms of one of the quantities — C_T, V_T — in an environment where the other one is small.

The second contribution of this paper is the technical analysis of the algorithm. The bound on the dynamic regret is derived by applying the *doubling trick* to a non-monotone quantity which results in a non-monotone step size sequence (which has not been investigated to the best of authors' knowledge).

We provide uncoupled strategies for two players playing a sequence of drifting zero sum games. We show how when the two players play the provided strategies, their pay offs converge to the average minimax value of the sequence of games (provided the games drift slowly). In this case, both players simultaneously enjoy no regret guarantees against best sequences of actions in hindsight that vary slowly. This is a generalization of the results by Daskalakis *et al.* [15], and Rakhlin *et al.* [11], both of which are for fixed games played repeatedly.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Notation

Throughout the paper, we assume that for any action $x \in \mathcal{X} \subset \mathbb{R}^d$ at any time t , it holds that

$$|f_t(x)| \leq G. \quad (6)$$

We denote by $\|\cdot\|_*$ the dual norm of $\|\cdot\|$, by $[T]$ the set of natural numbers $\{1, \dots, T\}$, and by $f_{1:t}$ the shorthand of f_1, \dots, f_t , respectively. Whenever C_T is written without arguments, it will refer to regularity $C_T(x_1^*, \dots, x_T^*)$ of the sequence of minimizers of the loss functions. We point out that our initial statements hold for the regularity of any sequence of comparators. However, for upper bounds involving $\sqrt{C_T}$, one needs to choose a computable quantity to tune the step size, and hence our main results are stated for $C_T(x_1^*, \dots, x_T^*)$.

The quantity D_T is defined with respect to an arbitrary predictable sequence $\{M_t\}_{t=1}^T$, but this dependence is omitted for brevity.

B. Comparing with existing regret bounds in the dynamic setting

We state and discuss relevant results from the literature on online learning in dynamic environments. For any comparator sequence $\{u_t\}_{t=1}^T$ and the specific minima sequence $\{x_t^*\}_{t=1}^T$ the following results are established in the literature:

Reference	Regret Notion
	Regret Rate
[3], [9]	$\sum_{t=1}^T f_t(x_t) - f_t(u_t)$
	$\mathcal{O}\left(\sqrt{T}(1 + C_T(u_1, \dots, u_T))\right)$
[14]	$\sum_{t=1}^T \mathbb{E}[f_t(x_t)] - f_t(x_t^*)$
	$\mathcal{O}(T^{2/3}(1 + V_T)^{1/3})$
[11]	$\sum_{t=1}^T f_t(x_t) - f_t(u)$
	$\mathcal{O}(\sqrt{D_T})$
Our work	$\sum_{t=1}^T f_t(x_t) - f_t(x_t^*)$
	$\tilde{\mathcal{O}}\left(\sqrt{D_T + 1} + \min\left\{\sqrt{(D_T + 1)C_T}, (D_T + 1)^{1/3}T^{1/3}V_T^{1/3}\right\}\right)$

where $\tilde{\mathcal{O}}(\cdot)$ hides the $\log T$ factor. Lemma 1 below also yields a rate of $\mathcal{O}(\sqrt{D_T + 1}(1 + C_T(u_1, \dots, u_T)))$ for any comparator sequence $\{u_t\}_{t=1}^T$. A detailed explanation of the bounds will be done after Theorem 3.

We remark that the authors in [14] consider a setting in which a *variation budget* (an upper bound on V_T) is known to the learner, but he/she only has noisy gradients available. Then, the restarted **OGD** guarantees the mentioned rate for convex functions; the rate is modified to $\sqrt{(V_T + 1)T}$ for strongly convex functions.

For the case of *noiseless* gradients, we first aim to show that our algorithm is adaptive in the sense that the learner needs not know an upper bound on V_T in advance when he/she can calculate variations observed so far. Furthermore, we shall establish that our method recovers the known bounds for stationary settings (as well as cases where V_T does not change gradually along the time horizon)

C. Comparison of Regularity and Variability

We now show that V_T and C_T are not comparable in general. To this end, we consider the classical problem of prediction with expert advice. In this setting, the learner deals with the linear loss $f_t(x) = \langle f_t, x \rangle$ on the d -dimensional probability simplex. Assume that for any $t \geq 1$, we have the vector sequence

$$f_t = \begin{cases} (-\frac{1}{T}, 0, 0, \dots, 0) & , \text{ if } t \text{ even} \\ (0, -\frac{1}{T}, 0, \dots, 0) & , \text{ if } t \text{ odd} \end{cases} .$$

Setting u_t , the comparator of round t , to be the minimizer of f_t , i.e. $u_t = x_t^*$, we have

$$C_T = \sum_{t=1}^T \|x_t^* - x_{t-1}^*\|_1 = \Theta(T) \quad V_T = \sum_{t=1}^T \|f_t - f_{t-1}\|_\infty = \mathcal{O}(1),$$

according to (3) and (5), respectively. We see that V_T is considerably smaller than C_T in this scenario. On the other hand, consider prediction with expert advice with two experts. Let $f_t = (-1/2, 0)$ on even rounds and

$f_t = (0, 1/2)$ on odd rounds. Expert 1 remains to be the best throughout the game, and thus $C_T = \mathcal{O}(1)$, while variation $V_T = \Theta(T)$. Therefore, one can see that taking into account only one measure might lead us to suboptimal regret bounds. We show that both measures play a key role in our regret bound. Finally, we note that if $M_t = \nabla f_{t-1}(x_{t-1})$, the notion of D_T can be related to V_T in certain cases, yet we keep the predictable sequence arbitrary and thus as playing a role separate from V_T and C_T .

III. MAIN RESULTS

A. Optimistic Mirror Descent and Relation to Regularity

We now outline the **OMD** algorithm previously proposed in [10]. Let \mathcal{R} be a 1-strongly convex function with respect to a norm $\|\cdot\|$, and $\mathcal{D}_{\mathcal{R}}(\cdot, \cdot)$ represent the Bregman divergence with respect to \mathcal{R} . Also, let \mathcal{H}_t be the set containing all available information to the learner at the beginning of time t . Then, the learner can compute the vector $M_t : \mathcal{H}_t \rightarrow \mathbb{R}^d$, which we call the predictable process. Supposing that the learner has access to the side information $M_t \in \mathbb{R}^d$ from the outset of round t , the **OMD** algorithm is characterized via the following interleaved sequence,

$$x_t = \operatorname{argmin}_{x \in \mathcal{X}} \left\{ \eta_t \langle x, M_t \rangle + \mathcal{D}_{\mathcal{R}}(x, \hat{x}_{t-1}) \right\} \quad (7)$$

$$\hat{x}_t = \operatorname{argmin}_{x \in \mathcal{X}} \left\{ \eta_t \langle x, \nabla_t \rangle + \mathcal{D}_{\mathcal{R}}(x, \hat{x}_{t-1}) \right\}, \quad (8)$$

where $\nabla_t \triangleq \nabla f_t(x_t)$, and η_t is the *step size* that can be chosen adaptively to attain low regret. One could observe that for $M_t = 0$, the **OMD** algorithm amounts to the well-known *Mirror Descent* algorithm [16], [17]. On the other hand, the special case of $M_t = \nabla_{t-1}$ recovers the scheme proposed in [13]. It is shown in [10] that the static regret satisfies

$$\mathbf{Reg}_T^s \leq 4R_{\max} \left(\sqrt{D_T} + 1 \right),$$

using the step size

$$\eta_t = R_{\max} \min \left\{ \left(\sqrt{D_{t-1}} + \sqrt{D_{t-2}} \right)^{-1}, 1 \right\},$$

where $R_{\max}^2 \triangleq \sup_{x, y \in \mathcal{X}} \mathcal{D}_{\mathcal{R}}(x, y)$. The following lemma extends the result to arbitrary sequence of comparators $\{u_t\}_{t=1}^T$. Throughout, we assume that $\|\nabla_0 - M_0\|_*^2 = 1$ by convention.

Lemma 1. *Let \mathcal{X} be a convex set in a Banach space \mathcal{B} . Let $\mathcal{R} : \mathcal{B} \mapsto \mathbb{R}$ be a 1-strongly convex function on \mathcal{X} with respect to a norm $\|\cdot\|$, and let $\|\cdot\|_*$ denote the dual norm. For any $L > 0$, employing the time-varying step size*

$$\eta_t = \frac{L}{\sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} + \sqrt{\sum_{s=0}^{t-2} \|\nabla_s - M_s\|_*^2}},$$

and running the Optimistic Mirror Descent algorithm for any comparator sequence $\{u_t\}_{t=1}^T$, yields

$$\mathbf{Reg}_T^d(u_1, \dots, u_T) \leq 2\sqrt{1 + D_T}L + 2\sqrt{1 + D_T} \frac{\gamma C_T(u_1, \dots, u_T) + 4R_{\max}^2}{L},$$

so long as $\mathcal{D}_{\mathcal{R}}(x, z) - \mathcal{D}_{\mathcal{R}}(y, z) \leq \gamma \|x - y\|, \forall x, y, z \in \mathcal{X}$.

Lemma 1 underscores the fact that one can get a tighter bound for regret once the learner advances a sequence of conjectures $\{M_t\}_{t=1}^T$ well-aligned with the gradients. Moreover, if the learner has prior knowledge of C_T (or an upper bound on it), then the regret bound would be $\mathcal{O}\left(\sqrt{(D_T + 1)C_T}\right)$ by tuning L .

Note that when the function \mathcal{R} is Lipschitz on \mathcal{X} , the Lipschitz condition on the Bregman divergence is automatically satisfied. For the particular case of KL divergence this can be achieved via mixing a uniform distribution to stay away from boundaries (see e.g. section 4.2 of the paper in this regard). In this case, the constant γ is of $\mathcal{O}(\log T)$.

B. The Adaptive Optimistic Mirror Descent Algorithm

The main objective of the paper is to develop the *Adaptive Optimistic Mirror Descent (AOMD)* algorithm. The **AOMD** algorithm incorporates all notions of variation D_T , C_T and V_T to derive a comprehensive regret bound. The proposed method builds on the **OMD** algorithm with adaptive step size, combined with a *doubling trick* applied to a threshold growing non-monotonically (see e.g. [1], [10] for application of doubling trick on monotone quantities). The scheme is adaptive in the sense that no prior knowledge of D_T , C_T or V_T is necessary.

Observe that the prior knowledge of a variation budget (an upper bound on V_T) does not tell us how the changes between cost functions are distributed throughout the game. For instance, the variation can increase gradually along the time horizon, while it can also take place in the form of discrete switches. The learner does not have any information about the variation pattern. Therefore, she must adopt a flexible strategy that achieves low regret in the benign case of finite switches or shocks, while it is simultaneously able to compete with the worst-case of gradual change. Before describing the algorithm, let us first use Lemma 1 to bound the general dynamic regret in terms of D_T , C_T and V_T .

Lemma 2. *Let \mathcal{X} be a convex set in a Banach space \mathcal{B} . Let $\mathcal{R} : \mathcal{B} \mapsto \mathbb{R}$ be a 1-strongly convex function on \mathcal{X} with respect to a norm $\|\cdot\|$. Run the Optimistic Mirror Descent algorithm with the step size given in the statement of Lemma 1. Letting the comparator sequence be $\{u_t\}_{t=1}^T$, for any $L > 2R_{\max}$ we have*

$$\mathbf{Reg}_T^d(u_1, \dots, u_T) \leq 4\sqrt{1 + D_T}L + \mathbf{1}\{\gamma C_T(u_1, \dots, u_T) > L^2 - 4R_{\max}^2\} \frac{4\gamma R_{\max} T V_T}{L^2 - 4R_{\max}^2},$$

so long as $\mathcal{D}_{\mathcal{R}}(x, z) - \mathcal{D}_{\mathcal{R}}(y, z) \leq \gamma \|x - y\|, \forall x, y, z \in \mathcal{X}$.

We now describe **AOMD** algorithm shown in table 1, and prove that it automatically adapts to V_T , D_T and C_T . The algorithm can be cast as a repeated **OMD** using different step sizes. The learner sets the parameter $L = 3R_{\max}$ in Lemma 1, and runs the **OMD** algorithm. Along the process, the learner collects deviation, variation and regularity observed so far, and checks the doubling condition in table 1 after each round. Once the condition is satisfied, the learner doubles L , discards the accumulated deviation, variation and regularity, and runs a new **OMD** algorithm. Note importantly that the doubling condition results in a non-monotone sequence of step size during the learning process.

Algorithm 1 Adaptive Optimistic Mirror Descent Algorithm

 Parameter : R_{\max} , some arbitrary $x_0 \in \mathcal{X}$

 Initialize $N = 1$, $C_{(1)} = V_{(1)} = 0$, $D_{(1)} = 1$, $x_1 = x_0$, $L_1 = 3R_{\max}$, $\Delta_1 = 0$ and $k_1 = 1$.

for $t = 1$ to T **do**

% check doubling condition

if $L_N^2 < \gamma \min \left\{ C_{(N)}, V_{(N)}^{2/3} \Delta_N^{2/3} D_{(N)}^{-1/3} \right\} + 4R_{\max}^2$ **then**

 % increment N and double L_N
 $N = N + 1$
 $L_N = 3R_{\max} 2^{N-1}$, $C_{(N)} = V_{(N)} = 0$, $D_{(N)} = 1$ and $\Delta_N = 0$
 $k_N = t$
end if

 Play x_t and suffer loss $f_t(x_t)$

 Calculate M_{t+1} (predictable sequence) and gradient $\nabla_t = \nabla f_t(x_t)$

 % update $D_{(N)}, C_{(N)}, V_{(N)}$ and Δ_N
 $D_{(N)} = D_{(N)} + \|\nabla_t - M_t\|_*^2$
 $C_{(N)} = C_{(N)} + \|x_t^* - x_{t-1}^*\|$
 $V_{(N)} = V_{(N)} + \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)|$
 $\Delta_N = \Delta_N + 1$

% set step-size and perform optimistic mirror descent update

 $\eta_{t+1} = L_N \left(\sqrt{D_{(N)}} + \sqrt{D_{(N)} - \|\nabla_t - M_t\|_*^2} \right)^{-1}$
 $\hat{x}_t = \operatorname{argmin}_{x \in \mathcal{X}} \left\{ \eta_t \langle x, \nabla_t \rangle + \mathcal{D}_{\mathcal{R}}(x, \hat{x}_{t-1}) \right\}$
 $x_{t+1} = \operatorname{argmin}_{x \in \mathcal{X}} \left\{ \eta_{t+1} \langle x, M_{t+1} \rangle + \mathcal{D}_{\mathcal{R}}(x, \hat{x}_t) \right\}$
end for

Notice that once we have completed running the algorithm, N is the number of doubling epochs, Δ_i is the number of instances in epoch i , k_i and $k_{i+1} - 1$ are the start and end points of epoch i , $\sum_{i=1}^N \Delta_i = T$, $\sum_{i=1}^N C_{(i)} = C_T$, $\sum_{i=1}^N D_{(i)} = D_T + N$ and $\sum_{i=1}^N V_{(i)} = V_T$. Also, there is a technical reason for initialization choice of L which shall become clear in the proof of Lemma 2. Theorem 3 shows the bound enjoyed by the proposed **AOMD** algorithm.

Theorem 3. Assume that $\mathcal{D}_{\mathcal{R}}(x, z) - \mathcal{D}_{\mathcal{R}}(y, z) \leq \gamma \|x - y\|$, $\forall x, y, z \in \mathcal{X}$, and let $C_T = \sum_{t=1}^T \|x_t^* - x_{t-1}^*\|$. The **AOMD** algorithm enjoys the following bound on dynamic regret :

$$\mathbf{Reg}_T^d \leq \tilde{\mathcal{O}} \left(\sqrt{D_T + 1} \right) + \tilde{\mathcal{O}} \left(\min \left\{ \sqrt{(D_T + 1)C_T}, (D_T + 1)^{1/3} T^{1/3} V_T^{1/3} \right\} \right),$$

where $\tilde{\mathcal{O}}(\cdot)$ hides a $\log T$ factor.

Based on Theorem 3 we can obtain the following table that summarizes bounds on \mathbf{Reg}_T^d for various cases

(disregarding the first term $\tilde{\mathcal{O}}(\sqrt{D_T + 1})$ in the bound above):

Regime	Rate
$C_T \leq T^{2/3}(D_T + 1)^{-1/3}V_T^{2/3}$	$\tilde{\mathcal{O}}\left(\sqrt{C_T(D_T + 1)}\right)$
$V_T \leq D_T + 1$	$\tilde{\mathcal{O}}((D_T + 1)^{2/3}T^{1/3})$
$D_T \leq V_T - 1$	$\tilde{\mathcal{O}}(V_T^{2/3}T^{1/3})$
$D_T = \mathcal{O}(T)$	$\tilde{\mathcal{O}}(T^{2/3}V_T^{1/3})$

The following remarks are in order :

- In all cases, given the condition $V_T = o(T)$, the regret is sub-linear. When the gradients are bounded, the regime $D_T = \mathcal{O}(T)$ always holds, guaranteeing the worst-case bound of $\tilde{\mathcal{O}}(T^{2/3}V_T^{1/3})$.
- Theorem 3 allows us to recover $\tilde{\mathcal{O}}(1)$ regret for certain cases where $V_T = \mathcal{O}(1)$. Let nature divide the horizon into B batches, and play a smooth convex function $f_i(x)$ on each batch $i \in [B]$, that is for some $H_i > 0$ it holds that

$$\|\nabla f_i(x) - \nabla f_i(y)\|_* \leq H_i \|x - y\|, \quad (9)$$

$\forall i \in [B]$ and $\forall x, y \in \mathcal{X}$. Set $M_t = \nabla f_i(\hat{x}_{t-1})$ and note that the gradients are Lipschitz continuous. In this case, the **OMD** corresponding to each batch can be recognized as the *Mirror Prox* method [18], which results in $\tilde{\mathcal{O}}(1)$ regret during each period. Also, since $C_T = \mathcal{O}(1)$ the bound in Theorem 3 is of $\mathcal{O}(\log T)$.

IV. APPLICATIONS

A. Competing with Strategies

So far, we mainly considered dynamic regret \mathbf{Reg}_T^d defined in Equation 2. However, in many scenarios one might want to consider regret against a more specific set of strategies, defined as follows :

$$\mathbf{Reg}_T^\Pi \triangleq \sum_{t=1}^T f_t(x_t) - \inf_{\pi \in \Pi} \sum_{t=1}^T f_t(\pi_t(f_{1:t-1})),$$

where each $\pi \in \Pi$ is a sequence of mappings $\pi = (\pi_1, \dots, \pi_T)$ and $\pi_t : \mathcal{F}^{t-1} \rightarrow \mathcal{X}$. Notice that if Π is the set of all mappings then \mathbf{Reg}_T^Π corresponds to dynamic regret \mathbf{Reg}_T^d and if Π corresponds to set of constant history independent mappings, that is, each $\pi \in \Pi$ is indexed by some $x \in \mathcal{X}$ and $\pi_1^x(\cdot) = \dots = \pi_T^x(\cdot) = x$, then \mathbf{Reg}_T^Π corresponds to the static regret \mathbf{Reg}_T^s . We now define

$$C_T^\Pi = \sum_{t=1}^T \|\pi_t^*(f_{1:t-1}) - \pi_{t-1}^*(f_{1:t-2})\|,$$

where $\pi_t^* = \operatorname{arginf}_{\pi \in \Pi} \sum_{s=1}^t f_s(\pi_s(f_{1:s-1}))$. Assume that there exists sequence of mappings $\tilde{C}_1, \dots, \tilde{C}_T$ where \tilde{C}_t maps any f_1, \dots, f_t to reals and is such that for any t and any f_1, \dots, f_{t-1} ,

$$\tilde{C}_{t-1}(f_{1:t-1}) \leq \tilde{C}_t(f_{1:t}),$$

and further, for any T and any f_1, \dots, f_T ,

$$\sum_{t=1}^T \|\pi_t^*(f_{1:t-1}) - \pi_{t-1}^*(f_{1:t-2})\| \leq \tilde{C}_T(f_{1:T}).$$

In this case a simple modification of **AOMD** algorithm where $C_{(N)}$'s are replaced by $\tilde{C}_{\Delta_N}(f_{k_N:k_{N+1}-1})$ leads to the following corollary of Theorem 3.

Corollary 4. *Assume that $\mathcal{D}_{\mathcal{R}}(x, z) - \mathcal{D}_{\mathcal{R}}(y, z) \leq \gamma \|x - y\|, \forall x, y, z \in \mathcal{X}$. The **AOMD** algorithm with the modification mentioned above achieves the following bound on regret*

$$\mathbf{Reg}_T^{\Pi} \leq \tilde{O}\left(\sqrt{D_T + 1}\right) + \tilde{O}\left(\min\left\{\sqrt{(D_T + 1)\tilde{C}_T(f_{1:T})}, (D_T + 1)^{1/3}T^{1/3}V_T^{1/3}\right\}\right).$$

The corollary naturally interpolates between the static and dynamic regret. In other words, letting $\tilde{C}_T(f_{1:T}) = 0$ (which holds for constant mappings), we recover the result of [11] (up to logarithmic factors), whereas $\tilde{C}_T(f_{1:T}) = C_T$ simply recovers the regret bound in Theorem 3 corresponding to dynamic regret. The extra log factor is the cost of adaptivity of the algorithm as we assume no prior knowledge about the environment.

B. Switching Zero-sum Games with Uncoupled Dynamics

Consider two players playing T zero sum games defined by matrices $A_t \in [-1, 1]^{m \times n}$ for each $t \in [T]$. We would like to provide strategies for the two players such that, if both players honestly follow the prescribed strategies, the average payoffs of the players approach the average minimax value for the sequence of games at some fast rate. Furthermore, we would also like to guarantee that if one of the players (say the second) deviates from the prescribed strategy, then the first player still has small regret against sequence of actions that do not change drastically. To this end, one can use a simple modification of the **AOMD** algorithm for both players that uses KL divergence as $\mathcal{D}_{\mathcal{R}}$, and mixes in a bit of uniform distribution on each round, producing an algorithm similar to the one in [11] for unchanging uncoupled dynamic games. The following theorem provides bounds for when both players follow the strategy and bound on regret for player I when player II deviates from the strategy.

On round t , Player I performs

Play x_t and observe $f_t^\top A_t$

Update

$$\hat{x}_t(i) \propto \hat{x}'_{t-1}(i) \exp\{-\eta_t [f_t^\top A_t]_i\}$$

$$\hat{x}'_t = (1 - \beta) \hat{x}_t + (\beta/n) \mathbf{1}_n$$

$$x_{t+1}(i) \propto \hat{x}'_t(i) \exp\{-\eta_{t+1} [f_t^\top A_t]_i\}$$

and simultaneously Player II performs

Play f_t and observe $A_t x_t$

Update

$$\hat{f}_t(i) \propto \hat{f}'_{t-1}(i) \exp\{-\eta'_t [A_t x_t]_i\}$$

$$\hat{f}'_t = (1 - \beta) \hat{f}_t + (\beta/m) \mathbf{1}_m$$

$$f_{t+1}(i) \propto \hat{f}'_t(i) \exp\{-\eta'_{t+1} [A_t x_t]_i\}$$

Note that in the description of the algorithm as well as the following proposition and its proof, any letter with the prime symbol refers to Player II, and it is used to differentiate the letter from its counterpart for player I.

Proposition 5. Define $\mathcal{F}_t \triangleq \sum_{i=1}^t \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2$, and let

$$\eta_t = \min \left\{ \log(T^2 n) \frac{L}{\sqrt{\mathcal{F}_{t-1}} + \sqrt{\mathcal{F}_{t-2}}}, \frac{1}{32L} \right\}.$$

Also define $\mathcal{A}_t \triangleq \sum_{i=1}^t \|A_i x_i - A_{i-1} x_{i-1}\|_\infty^2$, and let

$$\eta'_t = \min \left\{ \log(T^2 m) \frac{L}{\sqrt{\mathcal{A}_{t-1}} + \sqrt{\mathcal{A}_{t-2}}}, \frac{1}{32L} \right\}.$$

Let $\beta = 1/T^2$, $M_t = f_{t-1}^\top A_{t-1}$, and $M'_t = A_{t-1} x_{t-1}$. When Player I uses the prescribed strategy, irrespective of the actions of player II, the regret of Player I w.r.t. any sequence of actions u_1, \dots, u_T is bounded as :

$$\sum_{t=1}^T (f_t^\top A_t x_t - f_t^\top A_t u_t) \leq 2 \log(T^2 n) (C_T(u_1, \dots, u_T) + 2) \left(32L + \frac{2\sqrt{\mathcal{F}_T}}{\log(T^2 n)L} \right) + \log(T^2 n) \frac{L}{2} \sqrt{\mathcal{F}_T}.$$

Further if both players follow the prescribed strategies then, as long as

$$2L^2 > \max \{C_T, C'_T\} + 3, \tag{10}$$

we get,

$$\begin{aligned} \sum_{t=1}^T \sup_{f_t \in \Delta_m} f_t^\top A_t x_t &\leq \sum_{t=1}^T \inf_{x_t \in \Delta_n} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t + \frac{256L}{T} + \frac{1}{2L} + 4 \sum_{t=1}^T \|A_{t-1} - A_t\|_\infty \\ &\quad + 32L (\log(T^2 n) C_T + \log(T^2 m) C'_T + 2 \log(T^4 nm)) \\ &\quad + (C_T + C'_T + 4) \frac{20 + 4\sqrt{\sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2}}{L}. \end{aligned}$$

A simple consequence of the above proposition is that if for instance the game matrix A_t changes at most K times over the T rounds, and we knew this fact a priori, then by letting $L = \frac{1}{\sqrt{\log(T^2 n)}}$, we get that regret for Player I w.r.t. any sequence of actions that switches at most K times even when Player II deviates from the prescribed strategy is $\mathcal{O}\left((K+2)\sqrt{\log(T^2 n)T}\right)$. At the same time if both players follow the strategy, then average payoffs of the players converge to the average minimax equilibrium at the rate of $\mathcal{O}\left(L(K+2)\log(T^4 nm)\right)$ under the condition on L given in (10). This shows that if the game matrix only changes/switches a constant number of times, then players get $\sqrt{\log(T)T}$ regret bound against arbitrary sequences and comparator actions that switch at most K times while simultaneously get a convergence rate of $\mathcal{O}(\log(T))$ to average equilibrium when both players are honest. Also, when we let $K = 0$ and set L to some constant, the proposition recovers the rate in static setting [11] where the matrix sequence is time-invariant.

V. CONCLUSION

In this paper, we proposed an online learning algorithm for dynamic environments. We considered time-varying comparators to measure the dynamic regret of the algorithm. Our proposed method is fully adaptive in the sense that the learner needs no prior knowledge of the environment. We derive a comprehensive upper bound on the dynamic regret capturing the interplay of regularity in the function sequence versus the comparator sequence. Interestingly, the regret bound adapts to the smaller quantity among the two, and selects the best of both worlds. As an instance of dynamic regret, we considered drifting zero-sum, two-player games, and characterized the convergence rate to the average minimax equilibrium in terms of variability in the sequence of payoff matrices.

ACKNOWLEDGEMENTS

We gratefully acknowledge the support of ONR BRC Program on Decentralized, Online Optimization, NSF under grants CAREER DMS-0954737 and CCF-1116928, as well as Dean's Research Fund.

APPENDIX : PROOFS

Proof of Lemma 1. For any $u_t \in \mathcal{X}$, it holds that

$$\langle x_t - u_t, \nabla_t \rangle = \langle x_t - \hat{x}_t, \nabla_t - M_t \rangle + \langle x_t - \hat{x}_t, M_t \rangle + \langle \hat{x}_t - u_t, \nabla_t \rangle. \quad (11)$$

First, observe that for any primal-dual norm pair we have

$$\langle x_t - \hat{x}_t, \nabla_t - M_t \rangle \leq \|x_t - \hat{x}_t\| \|\nabla_t - M_t\|_*.$$

Any update of the form $a^* = \arg \min_{a \in \mathcal{X}} \langle a, x \rangle + \mathcal{D}_{\mathcal{R}}(a, c)$ satisfies for any $d \in \mathcal{X}$,

$$\langle a^* - d, x \rangle \leq \mathcal{D}_{\mathcal{R}}(d, c) - \mathcal{D}_{\mathcal{R}}(d, a^*) - \mathcal{D}_{\mathcal{R}}(a^*, c).$$

This entails

$$\langle x_t - \hat{x}_t, M_t \rangle \leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(\hat{x}_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(\hat{x}_t, x_t) - \mathcal{D}_{\mathcal{R}}(x_t, \hat{x}_{t-1}) \right\}$$

and

$$\langle \hat{x}_t - u_t, \nabla_t \rangle \leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \mathcal{D}_{\mathcal{R}}(\hat{x}_t, \hat{x}_{t-1}) \right\}.$$

Combining the preceding relations and returning to (11), we obtain

$$\begin{aligned} \langle x_t - u_t, \nabla_t \rangle &\leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \mathcal{D}_{\mathcal{R}}(\hat{x}_t, x_t) - \mathcal{D}_{\mathcal{R}}(x_t, \hat{x}_{t-1}) \right\} \\ &\quad + \|\nabla_t - M_t\|_* \|x_t - \hat{x}_t\| \\ &\leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \frac{1}{2} \|\hat{x}_t - x_t\|^2 - \frac{1}{2} \|\hat{x}_{t-1} - x_t\|^2 \right\} \\ &\quad + \|\nabla_t - M_t\|_* \|x_t - \hat{x}_t\|, \end{aligned} \tag{12}$$

where in the last step we appealed to strong convexity: $\mathcal{D}_{\mathcal{R}}(x, y) \geq \frac{1}{2} \|x - y\|^2$ for any $x, y \in \mathcal{X}$. Using the simple inequality $ab \leq \frac{\rho a^2}{2} + \frac{b^2}{2\rho}$ for any $\rho > 0$ to split the product term, we get

$$\begin{aligned} \langle x_t - u_t, \nabla_t \rangle &\leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \frac{1}{2} \|\hat{x}_t - x_t\|^2 - \frac{1}{2} \|\hat{x}_{t-1} - x_t\|^2 \right\} \\ &\quad + \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \frac{1}{2\eta_{t+1}} \|x_t - \hat{x}_t\|^2, \end{aligned}$$

Applying the bound

$$\frac{1}{2\eta_{t+1}} \|x_t - \hat{x}_t\|^2 - \frac{1}{2\eta_t} \|x_t - \hat{x}_t\|^2 \leq R_{\max}^2 \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right),$$

and summing over $t \in [T]$ yields ,

$$\begin{aligned} \sum_{t=1}^T \langle x_t - u_t, \nabla_t \rangle &\leq \sum_{t=1}^T \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \sum_{t=1}^T \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) \right\} + \frac{R_{\max}^2}{\eta_{T+1}} \\ &\leq \sum_{t=1}^T \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + R_{\max}^2 \left(\frac{1}{\eta_1} + \frac{1}{\eta_{T+1}} \right) \\ &\quad + \sum_{t=2}^T \left\{ \frac{\mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1})}{\eta_t} - \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_{t-1}} \right\} \\ &\leq \sum_{t=2}^T \left\{ \frac{\mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1})}{\eta_t} - \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_t} + \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_t} - \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_{t-1}} \right\} \\ &\quad + \sum_{t=1}^T \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \frac{2R_{\max}^2}{\eta_{T+1}} \\ &\leq \sum_{t=1}^T \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \gamma \sum_{t=2}^T \frac{\|u_t - u_{t-1}\|}{\eta_t} \\ &\quad + \sum_{t=2}^T \mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1}) \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{2R_{\max}^2}{\eta_{T+1}} \\ &\leq \sum_{t=1}^T \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \gamma \sum_{t=2}^T \frac{\|u_t - u_{t-1}\|}{\eta_t} + \frac{4R_{\max}^2}{\eta_{T+1}}, \end{aligned}$$

where we used the Lipschitz continuity of $\mathcal{D}_{\mathcal{R}}$ in the penultimate step. Now let us set

$$\eta_t = \frac{L}{\sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} + \sqrt{\sum_{s=0}^{t-2} \|\nabla_s - M_s\|_*^2}} = \frac{L \left(\sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} - \sqrt{\sum_{s=0}^{t-2} \|\nabla_s - M_s\|_*^2} \right)}{\|\nabla_{t-1} - M_{t-1}\|_*^2},$$

and $\|\nabla_0 - M_0\|_*^2 = 1$ to have

$$\begin{aligned} \sum_{t=1}^T \langle x_t - u_t, \nabla_t \rangle &\leq \frac{L}{2} \sum_{t=1}^T \left\{ \sqrt{\sum_{s=0}^t \|\nabla_s - M_s\|_*^2} - \sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} \right\} \\ &\quad + \frac{2\gamma \sqrt{1 + \sum_{t=1}^T \|\nabla_t - M_t\|_*^2}}{L} \sum_{t=2}^T \|u_t - u_{t-1}\| + \frac{8R_{\max}^2 \sqrt{1 + \sum_{t=1}^T \|\nabla_t - M_t\|_*^2}}{L} \\ &\leq 2 \sqrt{1 + \sum_{t=1}^T \|\nabla_t - M_t\|_*^2} \left(L + \frac{\gamma \sum_{t=1}^T \|u_t - u_{t-1}\| + 4R_{\max}^2}{L} \right). \end{aligned}$$

Appealing to convexity of $\{f_t\}_{t=1}^T$, and replacing C_T (3) and D_T (4) in above, completes the proof. \blacksquare

Proof of Lemma 2. We define

$$U_T \triangleq \left\{ u_1, \dots, u_T \in \mathcal{X} : \gamma \sum_{t=1}^T \|u_t - u_{t-1}\| \leq L^2 - 4R_{\max}^2 \right\}, \quad (13)$$

and

$$(u_1^*, \dots, u_T^*) \triangleq \operatorname{argmin}_{u_1, \dots, u_T \in U_T} \sum_{t=1}^T f_t(u_t).$$

Our choice of $L > 2R_{\max}$ guarantees that any sequence of fixed comparators $u_t = u$ for $t \in [T]$ belongs to U_T , and hence, (u_1^*, \dots, u_T^*) exists. Noting that (u_1^*, \dots, u_T^*) is an element of U_T , we have $\gamma \sum_{t=1}^T \|u_t^* - u_{t-1}^*\| + 4R_{\max}^2 \leq L^2$. We now apply Lemma 1 to $\{u_t^*\}_{t=1}^T$ to bound the dynamic regret for arbitrary comparator sequence $\{u_t\}_{t=1}^T$ as follows,

$$\begin{aligned} \mathbf{Reg}_T^d(u_1, \dots, u_T) &= \sum_{t=1}^T \left\{ f_t(x_t) - f_t(u_t^*) \right\} + \sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} \\ &\leq 4\sqrt{1 + D_T}L + \sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} \\ &\leq 4\sqrt{1 + D_T}L + \mathbf{1} \left\{ \gamma \sum_{t=1}^T \|u_t - u_{t-1}\| > L^2 - 4R_{\max}^2 \right\} \left(\sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} \right), \quad (14) \end{aligned}$$

where the last step follows from the fact that

$$\sum_{t=1}^T f_t(u_t^*) - \sum_{t=1}^T f_t(u_t) \leq 0 \quad \text{if } (u_1, \dots, u_T) \in U_T.$$

Given the definition of R_{\max}^2 , by strong convexity of $\mathcal{D}_{\mathcal{R}}(x, y)$, we get that $\|x - y\| \leq \sqrt{2}R_{\max}$, for any $x, y \in \mathcal{X}$. This entails that once we divide the horizon into B number of batches and use a single, fixed point as a comparator

along each batch, we have

$$\sum_{t=1}^T \|u_t - u_{t-1}\| \leq B\sqrt{2}R_{\max}, \quad (15)$$

since there are at most B number of changes in the comparator sequence along the horizon. Now let $B = \frac{L^2 - 4R_{\max}^2}{\gamma\sqrt{2}R_{\max}}$, and for ease of notation, assume that T is divisible by B . Noting that $f_t(x_t^*) \leq f_t(u_t)$, we use an argument similar to that of [14] to get for any fixed $t_i \in [(i-1)(T/B) + 1, i(T/B)]$,

$$\sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} \leq \sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(x_t^*) \right\} \quad (16)$$

$$\begin{aligned} &= \sum_{i=1}^B \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \left\{ f_t(u_t^*) - f_t(x_t^*) \right\} \\ &\leq \sum_{i=1}^B \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \left\{ f_t(x_{t_i}^*) - f_t(x_t^*) \right\} \end{aligned} \quad (17)$$

$$\leq \left(\frac{T}{B} \right) \sum_{i=1}^B \max_{t \in [(i-1)(T/B)+1, i(T/B)]} \left\{ f_t(x_{t_i}^*) - f_t(x_t^*) \right\}. \quad (18)$$

Note that $x_{t_i}^*$ is fixed for each batch i . Substituting our choice of $B = \frac{L^2 - 4R_{\max}^2}{\gamma\sqrt{2}R_{\max}}$ in (15) implies that the comparator sequence $u_t = x_{t_i}^* \mathbf{1} \left\{ \frac{(i-1)T}{B} + 1 \leq t \leq \frac{iT}{B} \right\}$ belongs to U_T , and (17) follows by optimality of (u_1^*, \dots, u_T^*) . We now claim that for any $t \in [(i-1)(T/B) + 1, i(T/B)]$, we have,

$$f_t(x_{t_i}^*) - f_t(x_t^*) \leq 2 \sum_{s=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_s(x) - f_{s-1}(x)|. \quad (19)$$

Assuming otherwise, there must exist a $\hat{t}_i \in [(i-1)(T/B) + 1, i(T/B)]$ such that

$$f_{\hat{t}_i}(x_{t_i}^*) - f_{\hat{t}_i}(x_{\hat{t}_i}^*) > 2 \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)|,$$

which results in

$$\begin{aligned} f_t(x_{t_i}^*) &\leq f_{\hat{t}_i}(x_{t_i}^*) + \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)| \\ &< f_{\hat{t}_i}(x_{t_i}^*) - \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)| \leq f_t(x_{t_i}^*), \end{aligned}$$

The preceding relation for $t = t_i$ violates the optimality of $x_{t_i}^*$, which is a contradiction. Therefore, Equation (19) holds for any $t \in [(i-1)(T/B) + 1, i(T/B)]$. Combining (16), (18) and (19) we have

$$\begin{aligned} \sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} &\leq \frac{2T}{B} \sum_{i=1}^B \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)| \\ &= \frac{2TV_T}{B} = \frac{2\gamma\sqrt{2}R_{\max}TV_T}{L^2 - 4R_{\max}^2}. \end{aligned} \quad (20)$$

Using the above in Equation (14) we conclude the following upper bound

$$\mathbf{Reg}_T^d(u_1, \dots, u_T) \leq 4\sqrt{1 + D_T}L + \mathbf{1} \left\{ \gamma \sum_{t=1}^T \|u_t - u_{t-1}\| > L^2 - 4R_{\max}^2 \right\} \frac{4\gamma R_{\max} T V_T}{L^2 - 4R_{\max}^2},$$

thereby completing the proof. \blacksquare

Proof of Theorem 3. For the sake of clarity in presentation, we stick to the following notation for the proof

$$\begin{aligned} \underline{D}_{(i)} &\triangleq D_{(i)} - \|\nabla_{k_{i+1}-1} - M_{k_{i+1}-1}\|_*^2 \\ \underline{C}_{(i)} &\triangleq C_{(i)} - \|x_{k_{i+1}-1}^* - x_{k_{i+1}-2}^*\| \\ \underline{V}_{(i)} &\triangleq V_{(i)} - \sup_{x \in \mathcal{X}} |f_{k_{i+1}-1}(x) - f_{k_{i+1}-2}(x)| \\ \underline{\Delta}_{(i)} &\triangleq \Delta_i - 1, \end{aligned}$$

for any doubling epoch $i = 1, \dots, N$, where we recall that $k_{i+1} - 1$ is the last instance of epoch i . Therefore, any symbol with lower bar refers to its corresponding quantity removing only the value of the last instance of that interval.

Let the **AOMD** algorithm run with the step size given by Lemma 1 in the following form

$$\eta_t = \frac{L_i}{\sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} + \sqrt{\sum_{s=0}^{t-2} \|\nabla_s - M_s\|_*^2}},$$

and let L_i be tuned with a doubling condition explained in the algorithm. Once the condition stated in the algorithm fails, the following pair of identities must hold

$$\gamma \min\{\underline{C}_{(i)}, \underline{\Delta}_i^{2/3} \underline{V}_{(i)}^{2/3} \underline{D}_{(i)}^{-1/3}\} + 4R_{\max}^2 \leq L_i^2 \quad \gamma \min\{C_{(i)}, \Delta_i^{2/3} V_{(i)}^{2/3} D_{(i)}^{-1/3}\} + 4R_{\max}^2 > L_i^2. \quad (21)$$

Observe that the algorithm doubles L_i only after the condition fails, so at violation points we suffer at most $2G$ by boundedness (6). Then, under purview of Lemma 2, it holds that

$$\begin{aligned} \mathbf{Reg}_T^d &\leq \sum_{i=1}^N \left\{ 4\sqrt{\underline{D}_{(i)}}L_i + \mathbf{1} \left\{ \gamma \underline{C}_{(i)} > L_i^2 - 4R_{\max}^2 \right\} \frac{4\gamma R_{\max} \underline{\Delta}_i \underline{V}_{(i)}}{L_i^2 - 4R_{\max}^2} \right\} + 2NG \\ &\leq \sum_{i=1}^N \left\{ 4\sqrt{D_{(i)}}L_i + \mathbf{1} \left\{ \underline{C}_{(i)} > \underline{\Delta}_i^{2/3} \underline{V}_{(i)}^{2/3} \underline{D}_{(i)}^{-1/3} \right\} \frac{4\gamma R_{\max} \underline{\Delta}_i \underline{V}_{(i)}}{L_i^2 - 4R_{\max}^2} \right\} + 2NG, \end{aligned} \quad (22)$$

where the last step follows directly from (21) and the fact that $\underline{D}_{(i)} \leq D_{(i)}$. Bounding $\sqrt{\underline{D}_{(i)}}L_i$ in above, using the second inequality in (21), we get

$$\begin{aligned} \sqrt{\underline{D}_{(i)}}L_i &\leq \sqrt{\gamma \min\left\{ D_{(i)} C_{(i)}, \Delta_i^{2/3} V_{(i)}^{2/3} D_{(i)}^{2/3} \right\} + 4R_{\max}^2 D_{(i)}} \\ &\leq 2R_{\max} \sqrt{D_{(i)}} + \sqrt{\gamma} \min\left\{ \sqrt{D_{(i)} C_{(i)}}, \Delta_i^{1/3} V_{(i)}^{1/3} D_{(i)}^{1/3} \right\}, \end{aligned}$$

by the simple inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. Plugging the bound above into (22) and noting that

$$\sum_{i=1}^N \sqrt{D_{(i)}} = N \sum_{i=1}^N \frac{1}{N} \sqrt{D_{(i)}} \leq N \sqrt{\frac{1}{N} \sum_{i=1}^N D_{(i)}} = \sqrt{N D_T + N},$$

by Jensen's inequality, we obtain

$$\begin{aligned} \mathbf{Reg}_T^d &\leq 2NG + 8R_{\max} \sqrt{ND_T + N} + 4\sqrt{\gamma} \sum_{i=1}^N \min \left\{ \sqrt{D_{(i)} C_{(i)}}, D_{(i)}^{1/3} \Delta_i^{1/3} V_{(i)}^{1/3} \right\} \\ &\quad + \sum_{i=1}^N \mathbf{1} \left\{ \underline{C}_{(i)} > \underline{\Delta}_i^{2/3} \underline{V}_{(i)}^{2/3} \underline{D}_{(i)}^{-1/3} \right\} \frac{4R_{\max} \underline{\Delta}_i \underline{V}_{(i)}}{\min \left\{ \underline{C}_{(i)}, \underline{\Delta}_i^{2/3} \underline{V}_{(i)}^{2/3} \underline{D}_{(i)}^{-1/3} \right\}}, \end{aligned}$$

where we used the first inequality in (21) to bound the last term. Given the condition in the indicator function $\mathbf{1}\{\cdot\}$, we can simplify above to derive,

$$\begin{aligned} \mathbf{Reg}_T^d &\leq 2NG + 8R_{\max} \sqrt{ND_T + N} + 4\sqrt{\gamma} \sum_{i=1}^N \min \left\{ \sqrt{D_{(i)} C_{(i)}}, D_{(i)}^{1/3} \Delta_i^{1/3} V_{(i)}^{1/3} \right\} \\ &\quad + 4R_{\max} \sum_{i=1}^N \mathbf{1} \left\{ \underline{C}_{(i)} > \underline{\Delta}_i^{2/3} \underline{V}_{(i)}^{2/3} \underline{D}_{(i)}^{-1/3} \right\} \underline{D}_{(i)}^{1/3} \underline{V}_{(i)}^{1/3} \underline{\Delta}_i^{1/3} \\ &= 2NG + 8R_{\max} \sqrt{ND_T + N} + 4\sqrt{\gamma} \sum_{i=1}^N \min \left\{ \sqrt{D_{(i)} C_{(i)}}, D_{(i)}^{1/3} \Delta_i^{1/3} V_{(i)}^{1/3} \right\} \\ &\quad + 4R_{\max} \sum_{i=1}^N \mathbf{1} \left\{ \sqrt{\underline{D}_{(i)} \underline{C}_{(i)}} > \underline{\Delta}_i^{1/3} \underline{V}_{(i)}^{1/3} \underline{D}_{(i)}^{1/3} \right\} \underline{D}_{(i)}^{1/3} \underline{V}_{(i)}^{1/3} \underline{\Delta}_i^{1/3} \\ &\leq 2NG + 8R_{\max} \sqrt{ND_T + N} + 4\sqrt{\gamma} \sum_{i=1}^N \min \left\{ \sqrt{D_{(i)} C_{(i)}}, D_{(i)}^{1/3} \Delta_i^{1/3} V_{(i)}^{1/3} \right\} \\ &\quad + 4R_{\max} \sum_{i=1}^N \min \left\{ \sqrt{\underline{D}_{(i)} \underline{C}_{(i)}}, \underline{D}_{(i)}^{1/3} \underline{V}_{(i)}^{1/3} \underline{\Delta}_i^{1/3} \right\}. \end{aligned} \tag{23}$$

Given the fact that

$$\underline{C}_{(i)} \leq C_{(i)} \quad \underline{D}_{(i)} \leq D_{(i)} \quad \underline{V}_{(i)} \leq V_{(i)} \quad \underline{\Delta}_i \leq \Delta_i,$$

we return to (23) to derive

$$\begin{aligned} \mathbf{Reg}_T^d &\leq 2NG + 8R_{\max} \sqrt{ND_T + N} + (4\sqrt{\gamma} + 4R_{\max}) \sum_{i=1}^N \min \left\{ \sqrt{D_{(i)} C_{(i)}}, D_{(i)}^{1/3} \Delta_i^{1/3} V_{(i)}^{1/3} \right\} \\ &\leq 2NG + 8R_{\max} \sqrt{ND_T + N} + (4\sqrt{\gamma} + 4R_{\max}) \min \left\{ \sum_{i=1}^N \sqrt{D_{(i)} C_{(i)}}, \sum_{i=1}^N D_{(i)}^{1/3} \Delta_i^{1/3} V_{(i)}^{1/3} \right\} \\ &\leq 2N \left(G + 4R_{\max} \sqrt{D_T + 1} + (2\sqrt{\gamma} + 2R_{\max}) \min \left\{ \sqrt{(D_T + 1)C_T}, (D_T + 1)^{1/3} T^{1/3} V_T^{1/3} \right\} \right). \end{aligned} \tag{24}$$

where we bounded the sums using the following fact about the summands

$$C_{(i)} \leq C_T \quad D_{(i)} \leq D_T + 1 \quad V_{(i)} \leq V_T \quad \Delta_i \leq T.$$

To bound the number of batches N , we recall that $L_i = 3R_{\max} 2^{i-1}$, and use the second inequality in (21) to bound

L_{N-1} as follows

$$\begin{aligned}
N &= 2 + \log_2(2^{N-2}) = 2 + \log_2(L_{N-1}) - \log_2(3R_{\max}) \\
&\leq 2 + \frac{1}{2} \log_2 \left(\gamma \min \left\{ C_{(N-1)}, \Delta_{N-1}^{2/3} V_{(N-1)}^{2/3} D_{(N-1)}^{-1/3} \right\} + 4R_{\max}^2 \right) - \log_2(3R_{\max}) \\
&\leq 2 + \frac{1}{2} \log_2 (\gamma C_{(N-1)} + 4R_{\max}^2) - \log_2(3R_{\max}) \\
&\leq 2 + \frac{1}{2} \log_2 (2\gamma R_{\max} T + 4R_{\max}^2) - \log_2(3R_{\max}).
\end{aligned}$$

In view of the preceding relation and (24), we have

$$\mathbf{Reg}_T^d \leq \kappa \left(G + 4R_{\max} \sqrt{D_T + 1} + (2\sqrt{\gamma} + 2R_{\max}) \min \left\{ \sqrt{(D_T + 1)C_T}, (D_T + 1)^{1/3} T^{1/3} V_T^{1/3} \right\} \right),$$

where $\kappa \triangleq 4 + \log_2 (2\gamma R_{\max} T + 4R_{\max}^2) - 2 \log_2(3R_{\max})$, thereby completing the proof. \blacksquare

Proof of Proposition 5. Assume that the player I uses the prescribed strategy. This corresponds to using the optimistic mirror descent update with $\mathcal{R}(x) = \sum_{i=1}^n x_i \log(x_i)$ as the function that is strongly convex w.r.t. $\|\cdot\|_1$. Correspondingly, $\nabla_t = f_t^\top A_t$ and $M_t = f_{t-1}^\top A_{t-1}$. Following the line of proof in Lemma 1, in particular, using Equation 12 for the specific case with $\mathcal{D}_{\mathcal{R}}$ as KL divergence, we get that for any t and any $u_t \in \Delta_n$,

$$\begin{aligned}
f_t^\top A_t x_t - f_t^\top A_t u_t &\leq \frac{1}{\eta_t} \left\{ \sum_{i=1}^n u_t[i] \log \left(\frac{\hat{x}_t[i]}{\hat{x}'_{t-1}[i]} \right) - \frac{1}{2} \|\hat{x}_t - x_t\|_1^2 - \frac{1}{2} \|\hat{x}'_{t-1} - x_t\|_1^2 \right\} \\
&\quad + \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 \\
&\leq \frac{1}{\eta_t} \left\{ \sum_{i=1}^n u_t[i] \log \left(\frac{\hat{x}_t[i]}{\hat{x}'_{t-1}[i]} \right) - \frac{1}{2} \|\hat{x}_t - x_t\|_1^2 - \frac{1}{2} \|\hat{x}'_{t-1} - x_t\|_1^2 \right\} \\
&\quad + \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 + \frac{1}{\eta_t} \max_{i \in [n]} \log \left(\frac{\hat{x}_t[i]}{\hat{x}'_{t-1}[i]} \right).
\end{aligned}$$

Now let us bound for some i the term, $\log \left(\frac{\hat{x}_t[i]}{\hat{x}'_{t-1}[i]} \right)$. Notice that if $\hat{x}_t[i] \leq \hat{x}'_{t-1}[i]$ then the term is anyway bounded by 0. Now assume $\hat{x}_t[i] > \hat{x}'_{t-1}[i]$. Letting $\beta = 1/T^2$, since $\hat{x}'_{t-1}[i] = (1 - T^{-2})\hat{x}_t[i] + 1/(nT^2)$, we can have $\hat{x}_t[i] > \hat{x}'_{t-1}[i]$ only when $\hat{x}_t[i] > 1/n$. Hence,

$$\log \left(\frac{\hat{x}_t[i]}{\hat{x}'_{t-1}[i]} \right) = \log \left(\frac{\hat{x}_t[i]}{(1 - T^{-2})\hat{x}_t[i] + 1/(nT^2)} \right) \leq \frac{2}{T^2}.$$

Using this we can conclude that :

$$\begin{aligned}
f_t^\top A_t x_t - f_t^\top A_t u_t &\leq \frac{1}{\eta_t} \left\{ \sum_{i=1}^n u_t[i] \log \left(\frac{\hat{x}_t[i]}{\hat{x}'_{t-1}[i]} \right) - \frac{1}{2} \|\hat{x}_t - x_t\|_1^2 - \frac{1}{2} \|\hat{x}'_{t-1} - x_t\|_1^2 \right\} \\
&\quad + \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 + \frac{2}{T^2} \frac{1}{\eta_t}.
\end{aligned}$$

Summing over $t \in [T]$ we obtain that :

$$\begin{aligned} \sum_{t=1}^T (f_t^\top A_t x_t - f_t^\top A_t u_t) &\leq \sum_{t=1}^T \frac{1}{\eta_t} \left\{ \sum_{i=1}^n u_t[i] \log \left(\frac{\hat{x}'_t[i]}{\hat{x}'_{t-1}[i]} \right) - \frac{1}{2} \|\hat{x}_t - x_t\|_1^2 - \frac{1}{2} \|\hat{x}'_{t-1} - x_t\|_1^2 \right\} \\ &\quad + \sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 + \frac{2}{T^2} \sum_{t=1}^T \frac{1}{\eta_t}. \end{aligned}$$

Note that $\frac{1}{\eta_t} \leq \mathcal{O}(\sqrt{T})$ and so assuming T is large enough, $\frac{1}{T^2} \sum_{t=1}^T \frac{1}{\eta_t} \leq 1$ and so,

$$\begin{aligned} \sum_{t=1}^T (f_t^\top A_t x_t - f_t^\top A_t u_t) &\leq \sum_{t=1}^T \frac{1}{\eta_t} \left\{ \sum_{i=1}^n u_t[i] \log \left(\frac{\hat{x}'_t[i]}{\hat{x}'_{t-1}[i]} \right) - \frac{1}{2} \|\hat{x}_t - x_t\|_1^2 - \frac{1}{2} \|\hat{x}'_{t-1} - x_t\|_1^2 \right\} \\ &\quad + \sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 + 1. \end{aligned} \quad (25)$$

Now note that we can rewrite the first sum in the above bound and get :

$$\begin{aligned} \sum_{t=1}^T \frac{1}{\eta_t} \sum_{i=1}^n u_t[i] \log \left(\frac{\hat{x}'_t[i]}{\hat{x}'_{t-1}[i]} \right) &\leq \sum_{t=2}^T \frac{\sum_{i=1}^n u_t[i] \log \left(\frac{1}{\hat{x}'_{t-1}[i]} \right)}{\eta_t} - \frac{\sum_{i=1}^n u_{t-1}[i] \log \left(\frac{1}{\hat{x}'_{t-1}[i]} \right)}{\eta_{t-1}} + \frac{\log(T^2 n)}{\eta_1} \\ &\leq \sum_{t=2}^T \frac{\sum_{i=1}^n (u_t[i] - u_{t-1}[i]) \log \left(\frac{1}{\hat{x}'_{t-1}[i]} \right)}{\eta_t} \\ &\quad + \sum_{t=2}^T \sum_{i=1}^n u_{t-1}[i] \log \left(\frac{1}{\hat{x}'_{t-1}[i]} \right) \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{\log(T^2 n)}{\eta_1}. \end{aligned}$$

Since by definition of \hat{x}'_{t-1} , we are mixing in $1/T^2$ of the uniform distribution we have that for any i , $\hat{x}'_{t-1}[i] > \frac{1}{T^2 n}$ and, since η_t 's are non-increasing, we continue bounding above as

$$\begin{aligned} \sum_{t=1}^T \frac{1}{\eta_t} \sum_{i=1}^n u_t[i] \log \left(\frac{\hat{x}'_t[i]}{\hat{x}'_{t-1}[i]} \right) &\leq \log(T^2 n) \sum_{t=2}^T \frac{\|u_{t-1} - u_t\|_1}{\eta_t} + \log(T^2 n) \sum_{t=2}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{\log(T^2 n)}{\eta_1} \\ &\leq \log(T^2 n) \left(\sum_{t=2}^T \frac{\|u_{t-1} - u_t\|_1}{\eta_t} + \frac{1}{\eta_T} - \frac{1}{\eta_1} \right) + \frac{\log(T^2 n)}{\eta_1} \\ &\leq \log(T^2 n) \left(\sum_{t=2}^T \frac{\|u_{t-1} - u_t\|_1}{\eta_t} + \frac{1}{\eta_T} \right), \end{aligned}$$

using the above in Equation 25 we get

$$\begin{aligned}
& \sum_{t=1}^T f_t^\top A_t x_t - f_t^\top A_t u_t \\
& \leq \log(T^2 n) \sum_{t=2}^T \frac{\|u_{t-1} - u_t\|_1}{\eta_t} - \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_t} \|\hat{x}_t - x_t\|_1^2 - \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_t} \|\hat{x}'_{t-1} - x_t\|_1^2 + 1 \\
& \quad + \sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 + \frac{\log(T^2 n)}{\eta_T} \\
& \leq \frac{\log(T^2 n) (C_T(u_1, \dots, u_T) + 2)}{\eta_T} - \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_t} \|\hat{x}_t - x_t\|_1^2 - \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_t} \|\hat{x}'_{t-1} - x_t\|_1^2 \\
& \quad + \sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1. \tag{26}
\end{aligned}$$

Notice that our choice of step size given by,

$$\begin{aligned}
\eta_t &= \min \left\{ \log(T^2 n) \frac{L}{\sqrt{\sum_{i=1}^{t-1} \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2} + \sqrt{\sum_{i=1}^{t-2} \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2}}, \frac{1}{32L} \right\} \\
&= \min \left\{ \log(T^2 n) \frac{L \left(\sqrt{\sum_{i=1}^{t-1} \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2} - \sqrt{\sum_{i=1}^{t-2} \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2} \right)}{\|f_{t-1}^\top A_{t-1} - f_{t-2}^\top A_{t-2}\|_\infty^2}, \frac{1}{32L} \right\}, \tag{27}
\end{aligned}$$

guarantees that

$$\eta_t^{-1} = \max \left\{ \frac{\sqrt{\sum_{i=1}^{t-1} \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2} + \sqrt{\sum_{i=1}^{t-2} \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2}}{\log(T^2 n) L}, 32L \right\}.$$

Using the step-size specified above in the bound 26, we get

$$\begin{aligned}
& \sum_{t=1}^T f_t^\top A_t x_t - \sum_{t=1}^T f_t^\top A_t u_t \\
& \leq \log(T^2 n) (C_T(u_1, \dots, u_T) + 2) \left(\frac{2\sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2}}{\log(T^2 n) L} + 32L \right) \\
& \quad + \sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 - 16L \sum_{t=1}^T \|\hat{x}_t - x_t\|_1^2 - 16L \sum_{t=1}^T \|\hat{x}'_{t-1} - x_t\|_1^2. \tag{28}
\end{aligned}$$

Now note that by triangle inequality, we have

$$\begin{aligned}
\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty &= \|f_t^\top A_t - f_t^\top A_{t-1} + f_t^\top A_{t-1} - f_{t-1}^\top A_{t-1}\|_\infty \\
&\leq \|A_{t-1} - A_t\|_\infty + \|f_t - f_{t-1}\|_1 \\
&\leq \|A_{t-1} - A_t\|_\infty + \|f_t - \hat{f}_{t-1}\|_1 + \|\hat{f}_{t-1} - f_{t-1}\|_1,
\end{aligned}$$

since the entries of matrix sequence $\{A_t\}_{t=1}^T$ are bounded by one. Using the bound above in (28) and splitting the

product term, we see that

$$\begin{aligned}
\sum_{t=1}^T \left(f_t^\top A_t x_t - f_t^\top A_t u_t \right) &\leq \log(T^2 n) (C_T(u_1, \dots, u_T) + 2) \left(\frac{2\sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2}}{\log(T^2 n)L} + 32L \right) \\
&\quad + 2 \sum_{t=1}^T \|A_t - A_{t-1}\|_\infty - 8L \sum_{t=1}^T \|\hat{x}_t - x_t\|_1^2 - 16L \sum_{t=1}^T \|\hat{x}'_{t-1} - x_t\|_1^2 \\
&\quad + \frac{1}{16L} \sum_{t=1}^T \|f_t - \hat{f}_{t-1}\|_1^2 + \frac{1}{16L} \sum_{t=1}^T \|\hat{f}_{t-1} - f_{t-1}\|_1^2, \tag{29}
\end{aligned}$$

where we used the simple inequality $ab \leq \frac{\rho}{2}a^2 + \frac{1}{2\rho}b^2$ for $\rho > 0$.

a) *When Player II follows prescribed strategy:* In this case we would like to get convergence of payoffs to the average value of the games. To get this, using the notation $x_t^* = \operatorname{argmin}_{x_t \in \Delta_n} f_t^\top A_t x_t$ and denoting the corresponding sequence regularity for Player I by C_T , we get

$$\begin{aligned}
\sum_{t=1}^T \left(f_t^\top A_t x_t - f_t^\top A_t x_t^* \right) &\leq \log(T^2 n) (C_T + 2) \left(\frac{2\sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2}}{\log(T^2 n)L} + 32L \right) \\
&\quad + 2 \sum_{t=1}^T \|A_t - A_{t-1}\|_\infty - 8L \sum_{t=1}^T \|\hat{x}_t - x_t\|_1^2 - 16L \sum_{t=1}^T \|\hat{x}'_{t-1} - x_t\|_1^2 \\
&\quad + \frac{1}{16L} \sum_{t=1}^T \|f_t - \hat{f}_{t-1}\|_1^2 + \frac{1}{16L} \sum_{t=1}^T \|\hat{f}_t - f_t\|_1^2 + \frac{1}{4L},
\end{aligned}$$

where the term $\frac{1}{4L}$ appeared in the last line comparing to (29) is due to

$$\frac{1}{16L} \sum_{t=1}^T \|\hat{f}_{t-1} - f_{t-1}\|_1^2 - \frac{1}{16L} \sum_{t=1}^T \|\hat{f}_t - f_t\|_1^2 \leq \frac{1}{4L}.$$

Using the same bound for Player 2 (using loss as $-f_t^\top A_t x_t$ on round t), as well as using $f_t^* = \operatorname{argmin}_{f_t \in \Delta_m} -f_t^\top A_t x_t$ and denoting the corresponding sequence regularity by C'_T , we have that

$$\begin{aligned}
\sum_{t=1}^T \left(f_t^\top A_t x_t - f_t^{*\top} A_t x_t \right) &\geq -\log(T^2 m) (C'_T + 2) \left(\frac{2\sqrt{\sum_{t=1}^T \|A_t x_t - A_{t-1} x_{t-1}\|_\infty^2}}{\log(T^2 m)L} + 32L \right) \\
&\quad - 2 \sum_{t=1}^T \|A_t - A_{t-1}\|_\infty + 8L \sum_{t=1}^T \|\hat{f}_t - f_t\|_1^2 + 16L \sum_{t=1}^T \|\hat{f}'_{t-1} - f_t\|_1^2 \\
&\quad - \frac{1}{16L} \sum_{t=1}^T \|x_t - \hat{x}_{t-1}\|_1^2 - \frac{1}{16L} \sum_{t=1}^T \|\hat{x}_t - x_t\|_1^2 - \frac{1}{4L}.
\end{aligned}$$

Combining the two and noting that

$$\begin{aligned}
f_t^{*\top} A_t x_t &= \sup_{f_t \in \Delta_m} f_t^\top A_t x_t \geq \inf_{x_t \in \Delta_n} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t \\
&= \sup_{f_t \in \Delta_m} \inf_{x_t \in \Delta_n} f_t^\top A_t x_t \geq \inf_{x_t \in \Delta_n} f_t^\top A_t x_t = f_t^\top A_t x_t^*,
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{t=1}^T \sup_{f_t \in \Delta_m} f_t^\top A_t x_t &\leq \sum_{t=1}^T \inf_{x_t \in \Delta_n} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t + \frac{256L}{T} + \frac{1}{2L} + 4 \sum_{t=1}^T \|A_t - A_{t-1}\|_\infty \\
&\quad + \log(T^2 n) (C_T + 2) \left(\frac{2\sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2}}{\log(T^2 n)L} + 32L \right) \\
&\quad + \log(T^2 m) (C'_T + 2) \left(\frac{2\sqrt{\sum_{t=1}^T \|A_t x_t - A_{t-1} x_{t-1}\|_\infty^2}}{\log(T^2 m)L} + 32L \right) \\
&\quad + \left(\frac{1}{16L} - 8L \right) \sum_{t=1}^T \|\hat{x}_t - x_t\|_1^2 + \left(\frac{1}{16L} - 16L \right) \sum_{t=1}^T \|\hat{x}_{t-1} - x_t\|_1^2 \\
&\quad + \left(\frac{1}{16L} - 8L \right) \sum_{t=1}^T \|\hat{f}_t - f_t\|_1^2 + \left(\frac{1}{16L} - 16L \right) \sum_{t=1}^T \|\hat{f}_{t-1} - f_t\|_1^2, \tag{30}
\end{aligned}$$

where the constant $256L/T$ appeared in the first line accounts for the identities

$$\|\hat{x}_{t-1} - x_t\|_1^2 - \|\hat{x}'_{t-1} - x_t\|_1^2 \leq \frac{8}{T^2} \quad \|\hat{f}_{t-1} - f_t\|_1^2 - \|\hat{f}'_{t-1} - f_t\|_1^2 \leq \frac{8}{T^2}.$$

Using the triangle inequality again,

$$\begin{aligned}
\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2 &= \sum_{t=1}^T \|f_t^\top A_t - f_t^\top A_{t-1} + f_t^\top A_{t-1} - f_{t-1}^\top A_{t-1}\|_\infty^2 \\
&\leq 2 \sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2 + 2 \sum_{t=1}^T \|f_t - f_{t-1}\|_1^2 \\
&\leq 2 \sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2 + 4 \sum_{t=1}^T \|f_t - \hat{f}_{t-1}\|_1^2 + 4 \sum_{t=1}^T \|\hat{f}_{t-1} - f_{t-1}\|_1^2, \tag{31}
\end{aligned}$$

which also implies

$$\begin{aligned}
\sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2} &\leq \sqrt{2 \sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2 + 4 \sum_{t=1}^T \|f_t - \hat{f}_{t-1}\|_1^2 + 4 \sum_{t=1}^T \|\hat{f}_{t-1} - f_{t-1}\|_1^2} \\
&\leq 2 \sqrt{\sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2} + 2 \sqrt{\sum_{t=1}^T \|f_t - \hat{f}_{t-1}\|_1^2 + \sum_{t=1}^T \|\hat{f}_{t-1} - f_{t-1}\|_1^2} \\
&\leq 2 \sqrt{\sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2} + 2 + 2 \sum_{t=1}^T \|f_t - \hat{f}_{t-1}\|_1 + 2 \sum_{t=1}^T \|\hat{f}_{t-1} - f_{t-1}\|_1 \\
&\leq 2 \sqrt{\sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2} + 10 + 2 \sum_{t=1}^T \|f_t - \hat{f}_{t-1}\|_1 + 2 \sum_{t=1}^T \|\hat{f}_t - f_t\|_1, \tag{32}
\end{aligned}$$

where we used the bound $\sqrt{c} \leq c + 1$ for any $c \geq 0$ in the penultimate line. Similar bounds as Equations (31) and (32) hold for the other player as well. Using them in Equation 30 after some calculations, we conclude that

$$\begin{aligned} \sum_{t=1}^T \sup_{f_t \in \Delta_m} f_t^\top A_t x_t &\leq \sum_{t=1}^T \inf_{x_t \in \Delta_n} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t + \frac{256L}{T} + \frac{1}{2L} + 4 \sum_{t=1}^T \|A_{t-1} - A_t\|_\infty \\ &+ 32L(\log(T^2 n)C_T + \log(T^2 m)C'_T + 2 \log(T^4 nm)) + (C_T + C'_T + 4) \frac{20 + 4\sqrt{\sum_{t=1}^T \|A_{t-1} - A_t\|_\infty^2}}{L} \\ &+ 4 \left(\frac{C_T + 3}{L} - 2L \right) \left(\sum_{t=1}^T \|\hat{f}_t - f_t\|_1^2 + 2 \sum_{t=1}^T \|\hat{f}_{t-1} - f_t\|_1^2 \right) \\ &+ 4 \left(\frac{C'_T + 3}{L} - 2L \right) \left(\sum_{t=1}^T \|\hat{x}_t - x_t\|_1^2 + 2 \sum_{t=1}^T \|\hat{x}_{t-1} - x_t\|_1^2 \right). \end{aligned}$$

b) *When Player II is dishonest:* In this case we would like to bound Player I's regret regardless of the strategy adopted by Player II. Dropping one of the negative terms in Equation 26, we get :

$$\begin{aligned} \sum_{t=1}^T (f_t^\top A_t x_t - f_t^\top A_t u_t) &\leq \frac{\log(T^2 n) (C_T(u_1, \dots, u_T) + 2)}{\eta_T} - \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_t} \|\hat{x}_t - x_t\|_1^2 \\ &+ \sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty \|x_t - \hat{x}_t\|_1 \\ &\leq \frac{\log(T^2 n) (C_T(u_1, \dots, u_T) + 2)}{\eta_T} - \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_t} \|\hat{x}_t - x_t\|_1^2 \\ &+ \sum_{t=1}^T \frac{\eta_{t+1}}{2} \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2 + \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_{t+1}} \|x_t - \hat{x}_t\|_1^2. \end{aligned} \quad (33)$$

Noting to the telescoping sum

$$\frac{1}{2} \sum_{t=1}^T \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \|x_t - \hat{x}_t\|_1^2 \leq 2 \sum_{t=1}^T \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \leq \frac{2}{\eta_{T+1}},$$

as well as the choice of step-size (27) which entails

$$\begin{aligned} \sum_{t=1}^T \frac{\eta_{t+1}}{2} \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2 &\leq \log(T^2 n) \frac{L}{2} \sum_{t=1}^T \sqrt{\sum_{i=1}^t \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2} - \sqrt{\sum_{i=1}^{t-1} \|f_i^\top A_i - f_{i-1}^\top A_{i-1}\|_\infty^2} \\ &\leq \log(T^2 n) \frac{L}{2} \sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2}, \end{aligned}$$

we bound (33) to obtain

$$\begin{aligned} \sum_{t=1}^T (f_t^\top A_t x_t - f_t^\top A_t u_t) &\leq \frac{\log(T^2 n) (C_T(u_1, \dots, u_T) + 2)}{\eta_T} + \frac{2}{\eta_{T+1}} + \log(T^2 n) \frac{L}{2} \sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2} \\ &\leq 2 \log(T^2 n) (C_T(u_1, \dots, u_T) + 2) \left(32L + \frac{2\sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2}}{\log(T^2 n)L} \right) \\ &+ \log(T^2 n) \frac{L}{2} \sqrt{\sum_{t=1}^T \|f_t^\top A_t - f_{t-1}^\top A_{t-1}\|_\infty^2}. \end{aligned}$$

A similar statement holds for Player II that her/his pay off converges at the provided rate to the average minimax equilibrium value. ■

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