

Econometric analysis of potential outcomes time series:
instruments, shocks, linearity and the causal response function*

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Abstract

Bojinov and Shephard (2019) defined potential outcome time series to nonparametrically measure dynamic causal effects in time series experiments. Four innovations are developed in this paper: “instrumental paths”, treatments which are “shocks”, “linear potential outcomes” and the “causal response function.” Potential outcome time series are then used to provide a nonparametric causal interpretation of impulse response functions, generalized impulse response functions, local projections and LP-IV.

Keywords: Dynamic causality, instrumental variables, linearity, potential outcomes, time series, shocks.

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1 Introduction

Bojinov and Shephard (2019) developed potential outcome time series to nonparametrically measure dynamic causal effects from time series randomized experiments conducted in financial markets. However, most time series data used in economics is observational. In this paper we develop the tools needed to use the potential outcome time series framework on observational data, yielding an observational, nonparametric framework for measuring dynamic causal effects. It provides a flexible foundation upon which to build new methods and interpret existing methods for causal inference on economic time series.

Our analysis is based on four new ideas beyond Bojinov and Shephard (2019). The first three are special cases of the potential outcome time series: “instrumented potential outcome time series”, treatments which are “shocks” and “linear potential outcomes”. The fourth innovation is the “causal response function,” which is a new dynamic causal estimand. To illustrate the power of these four ideas, we provide a nonparametric causal interpretation to four tools commonly used in the time series literature: impulse response functions, generalized impulse response functions, local projections and local projections with an instrumental variable (LP-IV). Our results show that a tightly parameterized model such as the structural moving average is not needed to provide a causal interpretation to these objects.

Of course, there is a storied history of economists trying to learn dynamic causal effects from time series data. Modern reviews include Ramey (2016) and Stock and Watson (2018). As this vast body of research emphasizes, conceptualizing and estimating dynamic causal effects is quite challenging. Dynamic feedback between treatments and observed outcomes makes it difficult to disentangle causes from effects. Additionally, in many important applications, only several hundred time series observations are available. Given these challenges, much of the literature on dynamic causal effects in time series relies on parameterized linear models. Canonical examples are structural vector autoregressions (Sims, 1980)¹, local projections (Jordá, 2005) and LP-IV (Jordá et al., 2015; Stock and Watson, 2018). However, there are exceptions such as Priestley (1988), Engle et al. (1990) and Gallant et al. (1993). Influentially, Koop et al. (1996) defined a “generalized impulse response function” for non-linear, non-causal models.

While tractable, the heavy emphasis on linear models has drawbacks. The role of particular sets of assumptions are often unclear in existing approaches. For example, it is common in economics to

¹Structural vector autoregressions are typically motivated as a linear approximation to an equilibrium arising from an underlying dynamic stochastic general equilibrium model such as Christiano et al. (1999, 2005), Smets and Wouters (2003, 2007).

restrict attention to the causal effects of “shocks” (Frisch, 1933; Slutsky, 1937; Ramey, 2016). Is this a convenient choice, or does it reflect something deeper? We address these questions by building upon the nonparametric potential outcome time series.

Our work does not appear in a vacuum. Over the last several decades, statisticians and economists have made enormous progress by defining causal effects nonparametrically as the comparison of potential outcomes (Imbens and Rubin, 2015; Angrist and Pischke, 2009), following a long history in econometrics and statistics (Neyman, 1923; Roy, 1951; Kempthorne, 1955; Cox, 1958; Rubin, 1974; Robins, 1986). These advances have been used to explore the nonparametric causal content of well-established empirical strategies, such as the LATE interpretation of instrumental variables (Imbens and Angrist, 1994; Angrist et al., 1996), as well as spur the development of new tools (Athey and Imbens, 2017; Abadie and Cattaneo, 2018). The potential outcome time series follows in this intellectual tradition.

The potential outcome time series is also related to the literature on dynamic treatment effects in small- T , large- N panels. The groundbreaking panel work of Robins (1986) led to an enormous literature on dynamic causal effects (Murphy et al., 2001; Murphy, 2003; Abbring and Heckman, 2007; Heckman and Navarro, 2007; Lechner, 2011; Heckman et al., 2016; Boruvka et al., 2018; Blackwell and Glynn, 2018; Hernan and Robins, 2019). However, the four new ideas in this paper are not the focus of those papers.

Inference on dynamic causal effects is one of the great themes of the broader time series literature. Researchers quantify causality in time series in a variety of ways such as using “Granger causality” (Wiener, 1956; Granger, 1969), highly structured models such as DSGE models (Herbst and Schorfheide, 2015), behavioral game theory (Toulis and Parkes, 2016), state space modelling (Harvey and Durbin, 1986; Harvey, 1996; Bondersen et al., 2015), Bayesian structural models (Brodersen et al., 2015) as well as intervention analysis (Box and Tiao, 1975) and regression discontinuity (Kuersteiner et al., 2018). The potential outcome time series is distinct from each of those approaches.

The closest work to the potential outcome time series framework is Angrist and Kuersteiner (2011) and Angrist et al. (2018), which also studies time series using potential outcomes (see also White and Lu (2010) and Lu et al. (2017)). That work is importantly different from Bojinov and Shephard (2019), as it avoids discussion of treatment paths, defining potential outcomes as a function of a single prior treatment — this difference will be detailed in Section 2. More importantly, Angrist and Kuersteiner (2011) and Angrist et al. (2018) do not discuss the main contribution of this paper which are the special cases of instruments, shocks and linear potential outcomes and the establishment of the causal response function. Also related to the framework is Robins et al. (1999) who used potential out-

come paths for binary time series and [Bondersen et al. \(2015\)](#) who used them for state space models. Recently, [Bojinov and Shephard \(2019\)](#) and [Blackwell and Glynn \(2018\)](#) used them in more general settings.

Overview of the paper: Section 2 recalls the definition of a potential outcome time series. Then two examples of this setup are given, before developing the three important special cases that deal with instrumental variables, shocks and linear potential outcomes. Section 3 defines causal effects, introducing a weighted causal effect and a causal response function. We provide definitions that allow us to link them with the economics literature and are more general than those in [Bojinov and Shephard \(2019\)](#). We show that the causal response function is closely related to the impulse response function. We also analyze the properties of these causal estimands under the assumptions of linear potential outcomes and shocked treatments. Section 3 finishes with a nonparametric causal interpretation of the local projection and its instrumental variables version. Section 4 concludes the paper. Longer proofs and a series of additional results are collected in our Web Appendix.

Notation: The mathematics of this paper is written using standard path notation: for a time series $\{A_t : t = 1, 2, \dots, T\}$, let $A_{1:t} := (A_1, \dots, A_t)$. Here $:=$ denotes a definition of the left hand side of the equation. Further, $A \perp\!\!\!\perp B$ generically means that the random variable A is stochastically independent of B . $A \not\perp\!\!\!\perp B$ denotes A and B not being independent, while $A \stackrel{L}{=} B$ means A and B have the same law or distribution. For a matrix A , A^\top is the transpose of A .

2 Potential outcome time series

2.1 Formal definition

We recall the definition of the potential outcome time series developed by [Bojinov and Shephard \(2019\)](#) in the context of time series experiments seen in financial economics. There is nothing novel in this first subsection.

There is a single unit that is observed over $t = 1, \dots, T$ periods. At each time period, the unit receives a new K -dimensional treatment W_t and we observe a scalar outcome Y_t . The potential outcome time series links treatments and outcomes using four foundation stones: (i) the definition of treatment and potential outcome paths, (ii) an assumption of non-anticipating outcomes, (iii) an assumption that generates outcomes by linking potential outcomes to treatments and (iv) an assumption of non-anticipating treatments.

A *potential outcome* describes what would be observed at time t for a particular path of treatments. Its formal definition is given below.

Definition 1. A *treatment path* $W_{1:T}$ is a stochastic process where each random variable W_t has compact support $\mathcal{W} \subset \mathbb{R}^K$. The *potential outcome path* is, for any deterministic $w_{1:T} \in \mathcal{W}^T$, the stochastic process

$$Y_{1:T}(w_{1:T}) := (Y_1(w_{1:T}), Y_2(w_{1:T}), \dots, Y_T(w_{1:T}))^\top,$$

where the time- t potential outcome $Y_t(w_{1:T}) : \mathcal{W}^T \rightarrow \mathbb{R}$.

In the definition above, the potential outcomes can depend on future treatments. Now, we employ our second foundation stone: restricting the potential outcomes to only depend on past and current treatments.

Assumption 1 (Non-anticipating potential outcomes). For each $t = 1, \dots, T$, $Y_t(w_{1:t}, w_{t+1:T}) = Y_t(w_{1:t}, w'_{t+1:T})$ almost surely, for all deterministic $w_{1:T} \in \mathcal{W}^T, w'_{t+1:T} \in \mathcal{W}^{T-t}$.

Assumption 1 is the time series analogue of SUTVA (Cox, 1958; Rubin, 1980). For convenience, we will drop references to future treatments and write the time- t potential outcome random variable $Y_t(w_{1:t}) : \mathcal{W}^t \rightarrow \mathbb{R}$, while the stochastic process version is written as

$$Y_{1:T}(w_{1:T}) = (Y_1(w_1), Y_2(w_{1:2}), \dots, Y_T(w_{1:T}))^\top.$$

We link the potential outcomes and treatments² to deliver outcomes through our third stone:

Assumption 2 (Outcomes). The time- t outcome is the random variable $Y_t := Y_t(W_{1:t})$, while the outcome stochastic process is

$$Y_{1:T} := (Y_1(W_1), Y_2(W_{1:2}), \dots, Y_T(W_{1:T}))^\top.$$

Let \mathcal{F}_t stand for the natural filtration generated by the observed stochastic process $\{Y_t, W_t\}$.

The final foundation stone is that $W_{1:T}$ is non-anticipating: the assignment of the treatment depends only on past outcomes and past treatments. This is a probabilistic assumption involving the

² Angrist and Kuersteiner (2011); Angrist et al. (2018) allow treatments to stochastically depend on past outcomes and treatments but define their potential outcomes as $\{Y_{t,p}(w), w \in \mathcal{W}\}$, for each lag $p \geq 0$. This latter step limits the dependence of the potential outcomes on the full treatment path, e.g. for $p = 1$, $\{Y_{t,1}(w), w \in \mathcal{W}\}$ only depends on the treatment assigned at period $t - 1$ but not on the treatment assigned at period t . In principle, the 1-step ahead causal effect of the treatment on the outcome may differ depending on what treatments are assigned at period t but this notation rules this out. As we next discuss in detail in Section 3, introducing explicit dependence on the full treatment path leads to a rich set of interesting causal estimands.

joint law of $\{W_t, Y_{t:T}(W_{1:t-1}, w_{t:T})\}|\mathcal{F}_{t-1}$. The associated (conditional) probability triple of this joint conditional distribution is written as $(\Omega, \mathcal{G}, \Pr)$, hiding the implicit dependence on $w_{t:T}$ and \mathcal{F}_{t-1} .

Assumption 3 (Non-anticipating treatment paths). *For each $t = 1, \dots, T$*

$$\{\{Y_{t:T}(W_{1:t-1}, w_{t:T}), w_{t:T} \in \mathcal{W}^{T-t+1}\} \perp\!\!\!\perp W_t\} | \mathcal{F}_{t-1}.$$

Assumption 3 is the time-series analogue of unconfoundedness.³ It says that the future potential outcomes $\{Y_{t:T}(W_{1:t-1}, w_{t:T}), w_{t:T} \in \mathcal{W}^{T-t+1}\}$ do not Granger-cause the current treatment W_t (Sims, 1972; Chamberlain, 1982; Engle et al., 1983; Kuersteiner, 2010; Lechner, 2011; Hendry, 2017).

With our four foundation stones in place we can now define a *potential outcome time series*.

Definition 2 (Potential outcome time series). *A stochastic process of potential outcomes and treatments $\{Y_{1:t}, W_{1:t}\}$ that satisfies Assumptions 1, 2 and 3 is a **potential outcome time series**.*

2.2 Two examples

We now illustrate the potential outcome time series through two examples.

Example 1 (Autoregression). *Consider a bivariate time series, where for all $w_{1:t} \in \mathcal{W}^t$,*

$$\begin{pmatrix} Y_t(w_{1:t}) \\ W_t \end{pmatrix} = \begin{pmatrix} \mu + \phi Y_{t-1}(w_{1:t-1}) + \beta_0 w_t \\ \gamma + \theta W_{t-1} + \delta Y_{t-1}(W_{1:t-1}) \end{pmatrix} + \begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix}, \quad \begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix} \stackrel{iid}{\sim} N \left(0, \begin{pmatrix} \sigma_\epsilon^2 & \rho \sigma_\epsilon \sigma_\eta \\ \rho \sigma_\epsilon \sigma_\eta & \sigma_\eta^2 \end{pmatrix} \right). \quad (1)$$

The resulting $\{Y_t, W_t\}$ is a Gaussian process. However, in general this system is not a potential outcome time series, as ϵ_t and η_t are contemporaneously correlated which disallows the use of Assumption 3. If $\rho = 0$ then this is a potential outcome time series. More generally, if we replace the assumption about the joint law of ϵ_t, η_t in (1) entirely with the assumption $\{\epsilon_t \perp\!\!\!\perp W_t\}|\mathcal{F}_{t-1}$, then this is a potential outcome time series.

Example 2 (Expectations of future treatments and non-anticipation). *In economics, consumers and firms are often modelled as forward-looking, with the distribution of futures outcomes influencing today's treatment choice. A simple version of this (e.g. in the tradition of Muth (1961), Lucas (1972),*

³In panel data settings, Robins (1994), Robins et al. (1999) and Abbring and van den Berg (2003) use this type of “selection on observables” assumption for the treatment paths $W_{1:T}$. When $T = 2$ this assumption is equivalent to the “latent sequential ignorability” assumption of Ricciardi et al. (2020). More broadly, Frangakis and Rubin (1999) call this type of assumption “latent ignorable”.

Sargent (1981)) is:

$$W_t = \arg \max_{w_t} \left(\max_{w_{t+1:T}} \mathbb{E}[U^*(Y_{t:T}(W_{1:t-1}, w_{t:T}), w_{t:T}) \mid \mathcal{F}_{t-1}] \right), \quad (2)$$

where U^* is a utility function of future outcomes and treatments. This decision rule delivers W_t and thus $Y_t(W_{1:t})$. This is a potential outcome time series.⁴

2.3 Three different special cases: instruments, linearity and shocks

We now focus on three, new special cases of the nonparametric potential outcome time series which allow the formal definitions of an instrumental path, a linear potential outcome and a nonparametric shock in causal time series models. These three cases were not in [Bojinov and Shephard \(2019\)](#).

The first special case of the potential outcome time series connects this framework to the literature on instrumental variables ([Angrist et al. \(1996\)](#); [Angrist and Krueger \(2001\)](#)).

Definition 3 (Instrumented potential outcome time series). *Partition the treatment path $V_t = (W'_t, Z'_t)'$, where $W_t \in \mathcal{W}_W$ and $Z_t \in \mathcal{W}_Z$. Assume $\{Y_t, V_t\}$ is a potential outcome time series and additionally that:*

1. *Exclusion condition: $Y_t(w_1, z_1, \dots, w_t, z_t) = Y_t(w_1, z'_1, \dots, w_t, z'_t)$ for all $w_{1:t} \in \mathcal{W}_W^t, z_{1:t}, z'_{1:t} \in \mathcal{W}_Z^t$.*
2. *Relevance condition: $Z_t \not\perp W_t \mid \mathcal{F}_{t-1}$.*

Then $\{Y_{1t}, V_t\}$ is an **instrumented potential outcome time series**, where $Z_{1:t}$ is labelled an **instrument path**.

The lack of dependence of the potential outcomes on the instrument means it is convenient to refer to it as $Y_t(w_{1:t}) : \mathcal{W}_W^t \rightarrow \mathbb{R}$, while $Y_{1:T}(w_{1:T}) = (Y_1(w_1), Y_2(w_{1:2}), \dots, Y_T(w_{1:T}))^\top$.

Example 1 (continuing from p. 6). *In economics, it is often difficult to measure accurately the treatment W_t , so instead, researchers use an estimator, \hat{W}_t , of the treatment. We take the instrument $Z_t = \hat{W}_t$, following the statistical measurement error tradition of [Durbin \(1954\)](#), which is used in the context of dynamic linear causal models by [Jordá et al. \(2015\)](#). An empirical example of this is [Stock and Watson \(2018\)](#) where W_t is a monetary policy movement and \hat{W}_t is an estimator of*

⁴The non-anticipation assumptions are similarly plausible if a different model for expectations is used. “Natural expectations” as in [Fuster et al. \(2010\)](#) or “diagnostic expectations” as in [Bordalo et al. \(2018\)](#) both only allow current decisions to depend on (possibly biased) beliefs about future outcomes, not the exact realizations along alternative paths $Y_t(w_{1:t})$.

W_t constructed from high-frequency movements in the rates on federal funds contracts around policy announcements.⁵ A simple time series example of this extends Example 1 with

$$\hat{W}_t = \alpha_0 + \alpha_1 W_t + \zeta_t, \quad \text{where} \quad \begin{pmatrix} \epsilon_t \\ \eta_t \\ \zeta_t \end{pmatrix} \stackrel{iid}{\sim} N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\epsilon^2 & 0 & 0 \\ 0 & \sigma_\eta^2 & 0 \\ 0 & 0 & \sigma_\zeta^2 \end{pmatrix} \right).$$

Hence the estimated treatment is biased but not independent of the treatment. As $\hat{W}_{1:t}$ does not move around the potential outcomes, this system is an instrumented potential outcome time series.

Remark 2.1. The non-anticipation of treatments means that instrumented potential outcome time series has $\{Z_{t-p} \perp\!\!\!\perp Y_t(W_{1:t-p-1}, w_{t-p:t}) \mid \mathcal{F}_{t-p-1}\}$ for all $w_{t-p:t} \in \mathcal{W}_W^{p+1}$.

Our second special case of the potential outcome time series bridges this framework to the literature on linear dynamic causal models (e.g. the survey of Ramey (2016)).

Definition 4 (Linear potential outcome time series). Assume a potential outcome time series. If, for every $w_{1:t} \in \mathcal{W}^t$,

$$Y_t(w_{1:t}) = U_t + \sum_{s=0}^{t-1} \beta_{t,s} w_{t-s}, \quad \text{almost surely,}$$

where $\beta_{t,s}$ are non-stochastic, then $\{Y_t, W_t\}$ is called a **linear potential outcome time series**. If $\beta_{t,s} = \beta_s$ for every t , then the linear potential outcome time series is **time-invariant**.

Here, $\{U_t\}$ is an arbitrary stochastic process whose only constraint is that it does not vary with $w_{1:t}$ and $(U_t \perp\!\!\!\perp W_t) \mid \mathcal{F}_{t-1}$. For example, $\{U_t\}$ may be an ARCH process, which is non-linear, or a random walk, which is non-stationary.

Our last special case bridges the potential outcome time series framework to the literature on shocks in economics (e.g. the surveys of Ramey (2016) and Stock and Watson (2018)).

Definition 5 (Shocked potential outcome). For a potential outcome time series, if,

$$\mathbb{E}[W_t \mid \mathcal{F}_{t-1}] = 0,$$

then W_t is called a **shock** and we label $\{Y_t, W_t\}$ a **shocked potential outcome time series**.

⁵Moreover, constructed measures of changes in government spending and tax policy have also recently been used as instruments to study the effects of fiscal policy on macroeconomic outcomes using time series data (Ramey and Zubairy, 2018; Fieldhouse et al., 2018; Mertens and Montiel Olea, 2018).

Here the non-anticipating treatments assumption is augmented, requiring the treatments to also be a martingale difference sequence with respect to the filtration generated by the potential outcome time series (e.g. [Hall and Heyde \(1980\)](#)).

Example 1 (continuing from p. 6). *If $Y_t(w_{1:t}) = \mu + \phi Y_{t-1}(w_{1:t-1}) + \beta_0 w_t + \epsilon_t$, where $\mathbb{E}(W_t | \mathcal{F}_{t-1}) = 0$, and $\{\epsilon_t \perp\!\!\!\perp W_t\} | \mathcal{F}_{t-1}$, then the system is a shocked potential outcome time series.*

The class of shocked potential outcome time series provides the formal definition of a sequence of nonparametric shocks within a causal framework. To our knowledge, this formalization of a causal shock is novel. Shocks are often described heuristically or precisely with respect to a model such as a structural moving average. For example, [Stock and Watson \(2018\)](#) describe macroeconomic shocks as “unanticipated structural disturbances” that produce “unexpected changes” in the macroeconomic outcomes of interest. [Ramey \(2016\)](#) also describes shocks as: (1) “exogenous with respect to the other current and lagged endogenous variables,” (2) “uncorrelated with other exogenous shocks” and (3) “either unanticipated movements in exogenous variables or news about future movements in exogenous variables.”

Shocks are central to modern macroeconomics and financial economics. Leading empirical examples of macroeconomic shocks include “oil price shocks,” ([Hamilton, 2003, 2013](#)) and sudden changes in national defense spending ([Ramey, 2011; Barro and Redlick, 2011; Ramey and Zubairy, 2018](#)). Examples of shocks in financial economics include “earnings surprises” ([Kothari, 2001; Kothari et al., 2006; Patton and Verardo, 2012](#)) and “news impact” ([Engle and Ng, 1993; Anatolyev and Petukhov, 2016](#)).

2.4 L^2 projections of potential outcomes

In economics, it is common to use best linear approximations or representations of potentially non-linear systems or expectations ([Rudd, 2000; Plagborg-Møller and Wolf, 2019](#)). That tradition generates two superpopulation L^2 projections of potential outcomes on lagged treatments.

Definition 6. *Suppose $\{Y_t, W_t\}$ is a shocked potential outcome time series where $K = 1$, $\mathbb{E}(Y_t^2) < \infty$, $0 < \mathbb{E}(W_{t-p}^2) < \infty$, and $p = 0, 1, 2, \dots$. Define the time- t projection*

$$\beta_{t,p}^L := \arg \min_{\beta} \left[\min_{\alpha} \mathbb{E}(Y_t - \alpha - \beta W_{t-p})^2 \right],$$

and the “universal”

$$\beta_p^U := \arg \min_{\beta} \left[\min_{\alpha} S_p(\alpha, \beta) \right] \quad \text{where} \quad S_p(\alpha, \beta) := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T-p} \sum_{t=p+1}^T (Y_t - \alpha - \beta W_{t-p})^2 \right].$$

Also define the L^2 projections of the time- t potential outcomes

$$Y_t^L(w_{1:t}) := \alpha_t + \sum_{s=0}^{t-1} \beta_{t,s}^L w_{t-s}, \quad \text{and} \quad Y_t^U(w_{1:t}) := \alpha + \sum_{s=0}^{t-1} \beta_s^L w_{t-s},$$

where $\alpha_t = \mathbb{E}(Y_t)$ and $\alpha = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E}(Y_t)$.

Then

$$\beta_{t,p}^L = \frac{\mathbb{E}(Y_t W_{t-p})}{\mathbb{E}(W_{t-p}^2)}, \quad \text{and} \quad \beta_p^U = \frac{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \mathbb{E}(Y_t W_{t-p})}{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \mathbb{E}(W_{t-p}^2)},$$

since the martingale difference treatments implies $\mathbb{E}(W_{t-p}) = 0$. The two terms are related to one another through

$$\beta_p^U = \frac{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \beta_{t,p}^L \mathbb{E}(W_{t-p}^2)}{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \mathbb{E}(W_{t-p}^2)}. \quad (3)$$

Hence β_p^U is a weighted average of $\{\beta_{t,p}^L\}$, where the weights are the time-varying variance of the treatments. If $\mathbb{E}(W_t^2)$ is time-invariant, then the simplification $\beta_p^U = \lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \beta_{t,p}^L$ holds.

These quantities are important in modern dynamic econometrics. In Section 3.5, we will show that β_p^U is the implicit estimand for the “Local Projection” estimator of the lag- p dynamic causal effect for a shocked potential outcome time series.

To link $\{\beta_{t,p}^L\}$ and β_p^U directly to definitional terms $\{\beta_{t,p}\}$ under the linear potential outcomes (Definition 4), we combine the shocked assumption with linearity.

Theorem 2.1. *If $\{Y_t, W_t\}$ is a shocked, linear potential outcome time series where $K = 1$, $\mathbb{E}(Y_t^2) < \infty$, $0 < \mathbb{E}(W_{t-p}^2) < \infty$, and $p = 0, 1, 2, \dots$, then $\beta_{t,p}^L = \beta_{t,p}$, and $\beta_p^U = \beta_p^{U*}$, where*

$$\beta_p^{U*} := \frac{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \beta_{t,p} \mathbb{E}(W_{t-p}^2)}{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \mathbb{E}(W_{t-p}^2)}. \quad (4)$$

Proof. Given in the Appendix A. □

3 Dynamic causal effects

3.1 Lag- p causal effects

Dynamic causal effects are comparisons of potential outcomes at a particular point in time along different treatment paths. In particular, for a potential outcome time series, the time- t causal effect on Y_t of treatment path $w_{1:t}$, compared to counterfactual path $w'_{1:t}$, is $Y_t(w_{1:t}) - Y_t(w'_{1:t})$.⁶

The time- t , lag- p causal effect measures how the outcome at time t changes if the treatment at time $t - p$ changes, where $p \geq 0$, fixing the treatment path up to time $t - p - 1$ at the observed $W_{1:t-p-1}$.

Definition 7 (Lag- p causal effect). *For a potential outcome time series and scalars w, w' , then*

$$\tau_{t,p}(w, w') := Y_t(W_{1:t-p-1}, w, w_{t-p+1:t}) - Y_t(W_{1:t-p-1}, w', w'_{t-p+1:t}),$$

is a lag- p , time- t causal effect for all $w_{t-p+1:t}, w'_{t-p+1:t} \in \mathcal{W}^p$.

Bojinov and Shephard (2019) introduced and studied the case where the treatment and counterfactual at time $t - p$ varies but $w'_{t-p+1:t} = w_{t-p+1:t}$. The more general time- t , lag- p $\tau_{t,p}(w, w')$ is new and our focus. This generalization is essential to link existing model-based dynamic causal methods developed in economics to the potential outcome framework. Its development below is the fourth main contribution of this paper.

We can similarly define the projection versions of the lag- p , time- t causal effect as $\tau_{t,p}^L(w, w') := Y_t^L(W_{1:t-p-1}, w, w_{t-p+1:t}) - Y_t^L(W_{1:t-p-1}, w', w'_{t-p+1:t})$ and $\tau_{t,p}^U(w, w') := Y_t^U(W_{1:t-p-1}, w, w_{t-p+1:t}) - Y_t^U(W_{1:t-p-1}, w', w'_{t-p+1:t})$.

Example 3. *Assume a linear potential outcome time series, then $\tau_{t,p}^L(w, w') = \beta_{t,p}(w - w')$, and*

$$\tau_{t,p}(w, w') = \beta_{t,p}(w - w') + \sum_{s=0}^{p-1} \beta_{t,s}(w_{t-s} - w'_{t-s}).$$

For a shocked potential outcome time series: $\tau_{t,p}^L(w, w') = \beta_{t,p}^L(w - w') + \sum_{s=0}^{p-1} \beta_{t,s}^L(w_{t-s} - w'_{t-s})$, and $\tau_{t,p}^U(w, w') = \beta_p^U(w - w') + \sum_{s=0}^{p-1} \beta_s^U(w_{t-s} - w'_{t-s})$ are, respectively, the time- t and universal L^2 projections of the lag- p , time- t causal effect. Under a shocked, linear potential outcomes, notice that

⁶Our approach follows the finite sample tradition that manipulates causal effects without reference to superpopulations (Imbens and Rubin, 2015). It contrasts with the superpopulation approach used by Robins (1986), Angrist and Kuersteiner (2011) and Boruvka et al. (2018) in the context of panel data and Angrist et al. (2018) for time series.

$\tau_{t,p}^L(w, w') = \tau_{t,p}(w, w') \neq \tau_{t,p}^U(w, w')$ and that $Y_t^L(w_{1:t})$, $Y_t^U(w_{1:t})$ and $Y_t(w_{1:t})$ all differ, recalling the definitions of $Y_t^L(w_{1:t})$, $Y_t^U(w_{1:t})$ from Definition 6 (e.g. α , α_t and U_t all differ).

3.2 Causal response function

We now introduce causal estimands built from the lag- p , time- t causal effect $\tau_{t,p}(w, w')$.

Many possible $w_{t-p:t}$ and $w'_{t-p:t}$ are consistent with passing through $w_{t-p} = w$ and $w'_{t-p} = w'$. Each possible path leads to a valid lag- p , time- t causal effect. We weight these different paths, selecting a weight function which will eventually lead to existing model-based dynamic causal methods developed in economics. The weights we choose will be generated by using distributions of $W_{t-p:t}$, $W'_{t-p:t}$ given past data.

Definition 8. Let $Y_t := Y_t(W_{1:t})$, $Y'_t := Y_t(W_{1:t-p-1}, W'_{t-p:t})$. Then, the **weighted causal effect** is

$$\tau_{t,p}^*(w, w') := \mathbb{E} \left[(Y_t - Y'_t) \mid \mathcal{F}_{t-p-1}, W_{t-p} = w, W'_{t-p} = w', \{Y_{t-p:t}(W_{1:t-p-1}, w_{t-p:t}), w_{t-p:t} \in \mathcal{W}^{p+1}\} \right], \quad (5)$$

if it exists, where the expectation is generated by $\{W_{t-p:t}, W'_{t-p:t}\} \mid \mathcal{F}_{t-p-1}, \{Y_{t-p:t}(W_{1:t-p-1}, w_{t-p:t}), w_{t-p:t} \in \mathcal{W}^{p+1}\}$. The **causal response function** is, if it exists,

$$CRF_{t,p}(w, w') := \mathbb{E} \left[(Y_t - Y'_t) \mid W_{t-p} = w, W'_{t-p} = w', \mathcal{F}_{t-p-1} \right], \quad (6)$$

where the expectation is generated by $\{Y_t, W_{t-p:t}, Y'_t, W'_{t-p:t}\} \mid \mathcal{F}_{t-p-1}$.

Temporally averaging these causal effects produces the estimands:

$$\bar{\tau}_p^*(w, w') = \frac{1}{T-p} \sum_{t=p+1}^T \tau_{t,p}^*(w, w'), \quad \overline{CRF}_p(w, w') = \frac{1}{T-p} \sum_{t=p+1}^T CRF_{t,p}(w, w'), \quad (7)$$

which we label the **lag- p average weighted causal effect** and the **lag- p average causal response function**, respectively.

The lag- p average weighted causal effect $\bar{\tau}_p^*(w, w')$ is a finite sample dynamic causal estimand, invoking no stochastic model for the potential outcomes. Intuitively, it describes the observed, historical causal effects. $\overline{CRF}_p(w, w')$ is a superpopulation quantity. The difference between superpopulation and finite sample causal estimands is subtle and increasingly emphasized in microeconomics (Aronow and Samii, 2016; Abadie et al., 2020). Here we introduce this distinction into time series.

We now make an additional assumption about the selected weights that places restrictions on the relationship between the counterfactual and treatment paths, enabling us to simplify the expressions for the weighted causal effect and the causal response function.

Assumption 4. *For a potential outcome time series assume that:*

1. $\{Y'_t, W'_{t-p:t}\} | \mathcal{F}_{t-p-1} \stackrel{L}{=} \{Y_t, W_{t-p:t}\} | \mathcal{F}_{t-p-1}$, where $Y_t := Y_t(W_{1:t})$, $Y'_t := Y_t(W_{1:t-p-1}, W'_{t-p:t})$,
2. $\{Y_t, W_{t-p:t}\} \perp\!\!\!\perp W'_{t-p} | \mathcal{F}_{t-p-1}$, and $\{Y'_t, W'_{t-p:t}\} \perp\!\!\!\perp W_{t-p} | \mathcal{F}_{t-p-1}$.

Assumption 4.2 means that the treatment path and outcome is independent from the $t - p$ counterfactual, given the past.

Lemma 3.1. *For a potential outcome time series, if Assumption 4 holds and the expectations exist, then*

$$\begin{aligned} \tau_{t,p}^*(w, w') &= \mathbb{E}[Y_t | \mathcal{F}_{t-p-1}, W_{t-p} = w, \{Y_{t-p:t}(W_{1:t-p-1}, w_{t-p:t}), w_{t-p:t} \in \mathcal{W}^{p+1}\}] \\ &\quad - \mathbb{E}[Y_t | \mathcal{F}_{t-p-1}, W_{t-p} = w', \{Y_{t-p:t}(W_{1:t-p-1}, w_{t-p:t}), w_{t-p:t} \in \mathcal{W}^{p+1}\}], \end{aligned}$$

where the expectations are from $W_{t-p:t} | \mathcal{F}_{t-p-1}, \{Y_{t-p:t}(W_{1:t-p-1}, w_{t-p:t}), w_{t-p:t} \in \mathcal{W}^{p+1}\}$. Likewise,

$$CRF_{t,p}(w, w') = \mathbb{E}[Y_t | \mathcal{F}_{t-p-1}, W_{t-p} = w] - \mathbb{E}[Y_t | \mathcal{F}_{t-p-1}, W_{t-p} = w'],$$

where the expectations are from the law of $\{Y_t(W_{1:t}), W_{t-p}\} | \mathcal{F}_{t-p-1}$.

Proof. Given in the Appendix A. □

Lemma 3.1 shows that under Assumption 4 the $CRF_{t,p}(w, w')$ is the same as the “generalized impulse response function” of Koop et al. (1996) when $w' = 0$, but those authors have no broad discussion of causality. The $CRF_{t,p}(w, w')$ is also similar in spirit to the “average policy effect” in Angrist et al. (2018) where w, w' are discrete. However, the “average policy effect” is not explicitly defined in terms of treatment paths.

A simple $T^{2/5}$ -consistent and asymptotically Gaussian kernel estimator of the finite sample, average weighted causal effect $\bar{\tau}_p^*(w, w')$ is developed in Appendix B for continuous w, w' . This means that the average dynamic causal effects can be nonparametrically identified solely from assuming a potential outcome time series. No further assumptions on the potential outcomes, such as stationarity, linearity or shocks, are needed. Those auxiliary assumptions on the potential outcomes may improve the

efficiency of estimation, but they are not fundamental to causal identification of the average weighted causal effect $\bar{\tau}_p^*(w, w')$. This is a conceptually important point.

3.3 Impulse response function

We now link the CRF to the impulse response function (IRF), which was introduced by Sims (1980) for vector autoregressions (Ramey, 2016; Stock and Watson, 2016; Kilian and Lutkepohl, 2017). We first give the IRF definition.

Definition 9 (Impulse response function). *Assume $\{Y_t, W_t\}$ is strictly stationary and $IRF_p(w, w') := \mathbb{E}[Y_t | W_{t-p} = w] - \mathbb{E}[Y_t | W_{t-p} = w']$, exists, where here $\mathbb{E}[\cdot]$ is calculated from the joint law of Y_t, W_{t-p} . Then, $IRF_p(w, w')$ is an **impulse response function** (IRF).*

The IRF is commonly viewed as tracing out the dynamic causal effect of the treatment on the outcome. However, the IRF does not have causal meaning without additional assumptions as it is just the difference of two conditional expectations. In contrast, a causal effect measures what would happen if W_{t-p} is *moved* from w to w' . This is well known, as IRFs are typically used in the context of parametrized causal models such as the structural vector moving average.

With that said, Theorem 3.1 gives the IRF a nonparametric causal meaning by linking it to the CRF.

Theorem 3.1. *Assume $\{Y_t, W_t\}$ is a stationary potential outcome time series and that Assumption 4 holds. Then, if the expectations exist,*

$$\mathbb{E}[CRF_{t,p}(w, w')] = IRF_p(w, w'),$$

where the expectation is generated by the stationary distribution of treatments and outcomes.

Proof. If the expectations exist, then $\mathbb{E}[CRF_{t,p}(w, w')] = \mathbb{E}[Y_t(W_{1:t}) | W_{t-p} = w] - \mathbb{E}[Y_t(W_{1:t}) | W_{t-p} = w']$, and the RHS is the IRF. \square

Here, \mathcal{F}_{t-p-1} is averaged out by stationarity, implying the causal measure holds universally. Hence, if we add stationarity to the potential outcome time series assumption, we can nonparametrically estimate the impulse response function by the difference of a kernel regression of Y_t on W_{t-p} (Robinson, 1983; Fan and Yao, 2006) evaluated at w and w' , respectively, converging at, again, $T^{2/5}$, the standard nonparametric rate. However, this rate is not an improvement over what could be obtained for the average weighted causal effect $\bar{\tau}_p^*(w, w')$ without stationarity.

3.4 Example: linear potential outcomes and shocked treatments

Here we detail the properties of the weighted causal effect and the causal response function under special features such as a linear potential outcomes and shocked treatments. These two assumptions are crucial, as most empirical dynamic causal work in economics is carried out using linear models under the assumption that treatments are shocks. It is this restriction that will eventually allow a parametric rate of convergence.

Example 3 (continuing from p. 11). *Under the linear potential outcomes, then the weighted causal effect becomes*

$$\tau_{t,p}^*(w, w') = \beta_{t,p}(w - w') + \sum_{s=0}^{p-1} \beta_{t,s} \{ \mu_{t-s|t-p-1}(w) - \mu_{t-s|t-p-1}(w') \},$$

where $\mu_{t-s|t-p-1}(w) = \mathbb{E} [W_{t-s} | \mathcal{F}_{t-p-1}, W_{t-p} = w, \{Y_{t-p:t}(W_{1:t-p-1}, w_{t-p:t}), w_{t-p:t} \in \mathcal{W}^{p+1}\}]$ and the causal response function becomes

$$CRF_{t,p}(w, w') = \beta_{t,p}(w - w') + \sum_{s=0}^{p-1} \beta_{t,s} \{ \mathbb{E}[W_{t-s} | \mathcal{F}_{t-p-1}, W_{t-p} = w] - \mathbb{E}[W_{t-s} | \mathcal{F}_{t-p-1}, W_{t-p} = w'] \},$$

assuming all the relevant moments exist. Likewise for a linear, shocked potential outcome time series

$$\tau_{t,p}^*(w, w') = CRF_{t,p}(w, w') = \beta_{t,p}(w - w') = \tau_{t,p}^I(w, w').$$

Under time-invariant, linear, stationary potential outcome time series $IRF_p(w, w') = \beta_p(w - w') + \sum_{s=0}^{p-1} \beta_s \{ \mathbb{E}[W_{t-s} | W_{t-p} = w] - \mathbb{E}[W_{t-s} | W_{t-p} = w'] \}$. For a time-invariant, linear, stationary, shocked potential outcome time series, then $IRF_p(w, w') = \tau_{t,p}^*(w, w') = CRF_{t,p}(w, w') = \beta_p(w - w')$.

This example shows that if treatments are shocks and potential outcomes are linear, then

$$\overline{CRF}_p(w, w') = \bar{\tau}_p^*(w, w') = (w - w') \frac{1}{T-p} \sum_{t=p+1}^T \beta_{t,p}.$$

Thus estimating $\overline{CRF}_p(w, w')$ or $\bar{\tau}_p^*(w, w')$ will be estimating the temporal average of $\beta_{t,p}$. The time series properties of the outcomes (which includes $\{U_t\}$) do not drive this result, it is the properties of the treatments and the linear potential outcomes which determine it.

3.5 Local projection estimator of causal estimands

Here we use the shocked potential outcome time series to provide a formal, causal interpretation to the “local projections” estimator, which is commonly used in economics. This estimator directly regresses the observed outcome on the observed treatment at a variety of lags, interpreting the coefficients on the lagged treatments as estimates of dynamic causal effects (Jordá, 2005; Ramey, 2016; Stock and Watson, 2018).⁷

Theorem 3.2 (Local projection). *Assume $\{Y_t, W_t\}$ is a shocked potential outcome time series where $K = 1$, $\mathbb{E}(Y_t^2) < \infty$, $0 < \mathbb{E}(W_{t-p}^2) < \infty$, $p = 0, 1, 2, \dots$. Construct $\beta_{t,p}^L = \mathbb{E}(Y_t W_{t-p}) / \mathbb{E}(W_{t-p}^2)$ and the mean-zero error $U_t^L := Y_t W_{t-p} - \beta_{t,p}^L W_{t-p}^2$. Assume that $\{U_t^L\}$, $\{W_{t-p}^2\}$ are ergodic processes and β_p^U (Definition 6) exists. If $T \rightarrow \infty$, then*

$$\hat{\beta}_p^{OLS} = \frac{\sum_{t=p+1}^T Y_t W_{t-p}}{\sum_{t=p+1}^T W_{t-p}^2} \xrightarrow{p} \beta_p^U.$$

If $T^{-1/2} \sum_{t=p+1}^T U_t^L = O_p(1)$, $T^{-1} \sum_{t=p+1}^T W_{t-p}^2 \xrightarrow{p} \sigma_W^2 > 0$, then $\hat{\beta}_p^{OLS}$ is $T^{1/2}$ -consistent for β_p^U .

Proof. The probability limit is by construction. The convergence rate is a standard calculation. \square

By construction, $\hat{\beta}_p^{OLS}$ estimates, at the parametric rate, the universal β_p^U from Definition 6. However, β_p^U only has indirect causal meaning, through the definition $\tau_{t,p}^U(w, w')$ in Example 3. If we further assume a linear potential outcome time series, then this has a direct causal meaning.

Corollary 3.1. *Maintain the same conditions as Theorem 3.2 and strengthen $\{Y_t, W_t\}$ to a shocked, linear potential outcome time series. If $T \rightarrow \infty$, then*

$$\hat{\beta}_p^{OLS} \xrightarrow{p} \beta_p^{U*} = \frac{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T CRF_{t,p}(1, 0) \mathbb{E}(W_{t-p}^2)}{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \mathbb{E}(W_{t-p}^2)}.$$

If $T^{-1/2} \sum_{t=p+1}^T U_t^L = O_p(1)$, $T^{-1} \sum_{t=p+1}^T W_{t-p}^2 \xrightarrow{p} \sigma_W^2 > 0$, then $\hat{\beta}_p^{OLS}$ is $T^{1/2}$ -consistent for β_p^{U*} .

Proof. The strenghtening to linearity implies $\beta_{t,p}^L = \beta_{t,p} = CRF_{t,p}(1, 0)$ and $\beta_p^U = \beta_p^{U*}$, so result follows from Theorem 3.2. \square

Under a shocked, linear potential outcome time series β_p^{U*} is the temporal weighted average of

⁷This is related to, but different from, the literature on direct forecasting, which forecasts Y_t by regressing on Y_{t-p} rather than iterating one step ahead forecasts p times (Cox, 1961; Marcellino et al., 2006).

$CRF_{t,p}(1, 0)$, weighting by the $\mathbb{E}(W_{t-p}^2)$. It is $\lim_{T \rightarrow \infty} \overline{CRF}_p(1, 0)$ if $\mathbb{E}(W_{t-p}^2)$ is time-invariant. If $\beta_{t,p} = \beta_p$, then $\hat{\beta}_p^{OLS} \xrightarrow{p} \beta_p$ irrespective of the variation of $\mathbb{E}(W_{t-p}^2)$.

3.6 Local projection with instrumental variables

A major concern is that precisely measuring the treatment may be very difficult (Jordá et al., 2015; Stock and Watson, 2018; Plagborg-Møller and Wolf, 2018). Here, we use the instrumented potential outcome time series to provide a causal interpretation of LP-IV.

Theorem 3.3 (LP-IV). *Suppose $\{Y_t, V_t\}$ is a shocked, linear, instrumented potential outcome times series, where for each $t = 1, 2, \dots, T$, that $V_t = (W_t, \hat{W}_t)$, $\mathbb{E}(Y_t^2) < \infty$, $0 < \mathbb{E}(W_t^2) < \infty$, $0 < \mathbb{E}(\hat{W}_t^2) < \infty$. For each $t = 1, 2, \dots, T$, and $p = 0, 1, \dots, t - 1$ construct $\beta_{t,p}^L = \mathbb{E}(Y_t W_{t-p}) / \mathbb{E}(W_{t-p}^2)$, $\eta_t^L := (Y_t - \beta_{t,p}^L W_{t-p}) \hat{W}_{t-p}$ and $\zeta_t^L := \beta_{t,p}^L \{W_{t-p} \hat{W}_{t-p} - \mathbb{E}(W_{t-p} \hat{W}_{t-p})\}$ and assume that $\beta_0^\gamma := \lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T \beta_{t,p}^L \mathbb{E}(W_{t-p} \hat{W}_{t-p})$ exists. If $\{\eta_t^L\}$ and $\{\zeta_t^L\}$ are ergodic and $\beta_0^\gamma \neq 0$, then*

$$\hat{\beta}_p^{IV} = \frac{\sum_{t=p+1}^T Y_t \hat{W}_{t-p}}{\sum_{t=p+1}^T Y_{t-p} \hat{W}_{t-p}} \xrightarrow{p} \beta_p^{IV} := \frac{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T CRF_{t,p}(1, 0) \mathbb{E}(W_{t-p} \hat{W}_{t-p})}{\lim_{T \rightarrow \infty} \frac{1}{T-p} \sum_{t=p+1}^T CRF_{t,0}(1, 0) \mathbb{E}(W_{t-p} \hat{W}_{t-p})}.$$

If, additionally, $T^{-1/2} \sum_{t=p+1}^T \eta_t^L = O_p(1)$, $T^{-1} \sum_{t=p+1}^T Y_{t-p} \hat{W}_{t-p} \xrightarrow{p} \beta_0^\gamma$, then $\hat{\beta}_p^{IV}$ is $T^{1/2}$ -consistent for β_p^{IV} .

Proof. Given in the Appendix A. □

Under a shocked, linear, instrumented potential outcome time series β_p^{IV} is the ratio of the weighted-average of the $CRF_{t,p}(1, 0)$, where the weights depend on $\mathbb{E}(W_{t-p} \hat{W}_{t-p})$ to the weighted average of $CRF_{t,0}(1, 0)$. If additionally $\mathbb{E}(W_{t-p} \hat{W}_{t-p})$ is time-invariant, then $\beta_p^{IV} = \lim_{T \rightarrow \infty} \overline{CRF}_p(1, 0) / \lim_{T \rightarrow \infty} \overline{CRF}_0(1, 0)$. In the LP-IV literature it is conventional to take $\beta_{t,0} = 1$ (e.g. Stock and Watson (2018)), which would mean that $\beta_p^{IV} = \lim_{T \rightarrow \infty} \overline{CRF}_p(1, 0)$.

If the correlation between the treatment and the instrument temporally changes signs, then some of these weights will have opposite signs, complicating the causal interpretation of β_p^{IV} . A sufficient condition to rule out such behavior is $\mathbb{E}(W_{t-p} \hat{W}_{t-p}) \geq 0$ for all time periods t , which is a sign restriction and is similar in spirit to the “monotonicity” assumption found in the LATE literature on cross-sectional instrumental variables (Imbens and Angrist, 1994; Angrist et al., 1996). Whether such a restriction is reasonable will depend on the empirical application.

Remark 2.1 says that the instrumented potential outcome times series implies \hat{W}_{t-p} is uncorrelated from the counterfactual. This lack of correlation is needed for the LP-IV to be causal. Otherwise,

$$\hat{\beta}_p^{IV} \xrightarrow{p} \frac{\beta_p^\gamma + \beta_p'}{\beta_0^\gamma},$$

where $\frac{1}{T-p} \sum_{t=p+1}^T \mathbb{E}(U_t \hat{W}_{t-p}) \rightarrow \beta_p'$.

Remark 3.1 (Lead-lag exogeneity). *The need for the condition that $Cov(\hat{W}_{t-p}, Y_t(W_{1:t-p-1}, w_{t-p:t})) = 0$ is seemingly missing from the LP-IV literature. Instead the existing literature typically uses a “lead-lag exogeneity” assumption that $Cov(W_t, \hat{W}_s) = 0$ for all $t \neq s$. Unfortunately, lead-lag exogeneity plus the assumption that $\{Y_t, W_t\}$ is a shocked potential outcome time series does not imply that $Cov(\hat{W}_{t-p}, Y_t(W_{1:t-p-1}, w_{t-p:t})) = 0$. This assumption is implied by lead-lag exogeneity assumption in the tightly parameterized setting studied by existing literature on LP-IV (i.e., outcomes that are generated by a structural moving average in the treatment, where the treatments are white noise). Our analysis shows that lead-lag exogeneity is not sufficient in more general causal models.*

4 Conclusion

In this paper, we adapted the nonparametric potential outcomes time series framework for experiments to formalize dynamic causal effects in observational time series data. We did so by introducing three crucial special cases of the potential outcome time series: instruments, shocks and linearity. Further, we deepened our understanding of dynamic causal effects by developing a fourth idea: the finite sample weighted causal effect and its superpopulation analogue, the causal response function.

These four ideas give nonparametric causal meaning to the impulse response function, which is a major device for economists to measure dynamic causal effects. Further, we used this framework to provide a causal interpretation to the implicit estimand of the local projections estimator. Finally, we made two important contributions to literature on LP-IV. We showed that the LP-IV estimator identifies a weighted average of dynamic causal effects, where the weights depend on the possibly time-varying relationship between the instrument and the treatment. We also showed that typical assumptions (i.e. lead-lag exogeneity) are not sufficient to identify a causally interpretable estimand because it does not enforce that the instrument is independent of the counterfactual given the past.

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Econometric analysis of potential outcomes time series: instruments, shocks, linearity and the causal response function

Online Appendix

Ashesh Rambachan Neil Shephard

A Appendix: a collection of proofs

Proof of Theorem 2.1. Under a linear potential outcome time series

$$Y_t(w_{1:t}) = U_t + \sum_{s=0}^{t-1} \beta_{t,s} w_{t-s},$$

so if $\{W_t\}$ is a MD sequence, then

$$\mathbb{E}(Y_t W_{t-p}) = \mathbb{E}(U_t W_{t-p}) \quad \mathbb{E}(Y_t) = \mathbb{E}(U_t) = \alpha_t.$$

By non-anticipating treatments of the potential outcome time series, $E(U_t W_{t-p}) = 0$ so long as the moment exists. This delivers the required result using conventional arguments. \square

Proof of Lemma 3.1. As the moments exist, so $CRF_{t,p}(w, w')$ simplifies to

$$\begin{aligned} & \mathbb{E}[\{Y_t(W_{1:t-p-1}, w, W_{t-p+1:t}) \mid (W_{t-p} = w, W'_{t-p} = w', \mathcal{F}_{t-p-1})\}] \\ & - \mathbb{E}[\{Y_t(W_{1:t-p-1}, w', W'_{t-p+1:t}) \mid (W_{t-p} = w, W'_{t-p} = w', \mathcal{F}_{t-p-1})\}]. \end{aligned}$$

Due to property 2 of the causal predictive weight,

$$\mathbb{E}[Y_t(W_{1:t}) \mid \mathcal{F}_{t-p-1}, W_{t-p} = w, W'_{t-p} = w'] = \mathbb{E}[Y_t(W_{1:t}) \mid \mathcal{F}_{t-p-1}, W_{t-p} = w].$$

Due to property 1 of the causal predictive weights,

$$\mathbb{E}[Y_t(W_{1:t-p-1}, w', W'_{t-p+1:t}) \mid \mathcal{F}_{t-p-1}, W'_{t-p} = w'] = \mathbb{E}[Y_t(W_{1:t}) \mid \mathcal{F}_{t-p-1}, W_{t-p} = w'].$$

The corresponding results for the weighted causal effect follow using the same logical arguments. \square

Proof of Theorem 3.3. Define $\epsilon_{t,p}^L := Y_t - \beta_{t,p}^L W_{t-p}$, then by the shock and instrument property of the time series, $\mathbb{E}(\epsilon_{t,p}^L \hat{W}_{t-p}) = 0$. So construct the zero mean time series $\eta_t^L := \epsilon_{t,p}^L \hat{W}_{t-p}$ and $\zeta_t^L := \beta_{t,p}^L \{W_{t-p} \hat{W}_{t-p} - \mathbb{E}(W_{t-p} \hat{W}_{t-p})\}$. Then

$$\frac{1}{T-p} \sum_{t=p+1}^T Y_t \hat{W}_{t-p} = \frac{1}{T-p} \sum_{t=p+1}^T \beta_{t,p}^L W_{t-p} \hat{W}_{t-p} + \frac{1}{T-p} \sum_{t=p+1}^T \eta_t^L.$$

If $\{\eta_t^L\}$ is ergodic, the latter sum disappears, while if $\{\zeta_t^L\}$ is ergodic then the former term converges to the limit of the expectations as expected. Shocks plus linearity implies $\beta_{t,p}^L = \beta_{t,p} = CRF_{t,p}(1, 0)$. \square

B Appendix: estimation of $\bar{\tau}_p^*(w, w')$

B.1 Conditioning on the potential outcomes

Throughout this Section $\mathcal{F}_{T,t}$ denotes the triangular filtration (pg. 53 of Hall and Heyde (1980)) generated by

$$\{W_{1:t}, Y_{1:t}, \{Y_{t+1:T}(W_{1:t}, w_{t+1:T}), w_{t+1:T} \in \mathcal{W}^{T-t}\}\}.$$

Recall

$$\bar{\tau}_p^*(w, w') = \frac{1}{T-p} \sum_{t=p+1}^T \tau_{t,p}^*(w, w')$$

where

$$\tau_{t,p}^*(w, w') = \mathbb{E}[Y_t | \mathcal{F}_{T,t-p-1}, W_{t-p} = w] - \mathbb{E}[Y_t | \mathcal{F}_{T,t-p-1}, W_{t-p} = w'].$$

The expectations are over the treatment path, holding fixed the potential outcomes. Fixing the potential outcomes follows the microeconometrics tradition discussed by Imbens and Rubin (2015), Abadie et al. (2017, 2020) and traces back to Fisher (1925, 1935) and Cox (1958). Bojinov and Shephard (2019) first introduced this type of approach into time series experiments.

Our task is to estimate $\tau_{t,p}^*(w, w')$ and $\bar{\tau}_p^*(w, w')$.

B.2 When \mathcal{W} is discrete

B.2.1 Estimator

We start by assuming that \mathcal{W} is discrete and that the treatment is *probabilistic*.

Assumption 5 (Probabilistic treatment). *For all $t \geq 1$, $\mathcal{F}_{T,t-1}$ and $w \in \mathcal{W}$,*

$$p_t(w) := \Pr(W_t = w | \mathcal{F}_{T,t-1}) > 0.$$

Assumption 5 is the analogue of the “overlap” assumption made in cross-sectional settings. Throughout we will regard $p_t(w)$ as known, which will be true in experimental settings and unlikely in observational ones where $p_t(w)$ would need to be estimated.

Define a time series version of the classic [Horvitz and Thompson \(1952\)](#) style estimator

$$\hat{\tau}_p^*(w, w') := \frac{1}{T-p} \sum_{t=p+1}^T \hat{\tau}_{t,p}^*(w, w'), \quad \hat{\tau}_{t,p}^*(w, w') := \frac{Y_t \left\{ \mathbb{1}(W_{t-p} = w) - \mathbb{1}(W_{t-p} = w') \right\}}{p_{t-p}(W_{t-p})}. \quad (8)$$

This estimator appears in [Angrist et al. \(2018\)](#), but for a superpopulation estimand. [Bojinov and Shephard \(2019\)](#) also use a [Horvitz and Thompson \(1952\)](#) style estimator, but differently setup and for a different finite sample estimand. The results which follow are roughly inline with those in [Bojinov and Shephard \(2019\)](#), although the details differ. No new ideas are needed to generate the results.

B.2.2 Properties of $\hat{\tau}_{t,p}^*(w, w')$ and $\hat{\tau}_p^*(w, w')$

The following theorem shows that $\hat{\tau}_{t,p}^*(w, w') - \tau_{t,p}^*(w, w')$ has martingale difference errors and hence $\hat{\tau}_p^*(w, w')$ is unbiased, conditional on the potential outcomes.

Theorem B.1 (Properties of $\hat{\tau}_{t,p}^*(w, w')$). *Assume a potential outcome time series and Assumption 5. Let $u_{t-p}(w, w') := \hat{\tau}_{t,p}^*(w, w') - \tau_{t,p}^*(w, w')$. Then, over the non-anticipating treatment path,*

$$\mathbb{E}[u_{t-p}(w, w') | \mathcal{F}_{T,t-p-1}] = 0, \quad \text{and} \quad \mathbb{E}[\hat{\tau}_p^*(w, w')] = \hat{\tau}_p^*(w, w'). \quad (9)$$

Further $\eta_{t-p}^2(w, w') := \text{Var}[u_{t-p}(w, w') | \mathcal{F}_{T,t-p-1}]$, is

$$\mathbb{E} \left(\frac{Y_t^2(W_{1:t-p-1}, w, W_{t-p+1:t})}{p_{t-p}(w)} \mid \mathcal{F}_{T,t-p-1}, W_{t-p} = w \right) \quad (10)$$

$$+ \mathbb{E} \left(\frac{Y_t^2(W_{1:t-p-1}, w', W_{t-p+1:t})}{p_{t-p}(w')} \mid \mathcal{F}_{T,t-p-1}, W_{t-p} = w' \right) - \tau_{t,p}^{*2}. \quad (11)$$

Proof. We produce equation (9) by noting that

$$\begin{aligned} & \mathbb{E} \left(\frac{Y_t(W_{1:t-p-1}, w, W_{t-p+1:t}) \mathbb{1}(W_{t-p} = w)}{p_{t-p}(w)} \mid \mathcal{F}_{T,t-p-1} \right) \\ &= \mathbb{E} \{ Y_t(W_{1:t-p-1}, w, W_{t-p+1:t}) \mid \mathcal{F}_{T,t-p-1}, W_{t-p} = w \} \\ &= \mathbb{E} \{ Y_t \mid \mathcal{F}_{T,t-p-1}, W_{t-p} = w \} \end{aligned}$$

The form of $\eta_{t-p}^2(w, w')$ is expected from the cross-sectional literature, and can be derived using the variance of a Bernoulli trial. \square

Thus, over the treatment path, conditioning on the entire path of all potential outcomes,

$$(T - p)Var(\hat{\tau}_p^*(w, w') - \bar{\tau}_p^*(w, w') | \{Y_{1:T}(w_{1:T}), w_{1:T} \in \mathcal{W}^T\}) = \bar{\eta}_T(w, w') \quad (12)$$

where

$$\bar{\eta}_T(w, w') = \frac{1}{T - p} \sum_{t=p+1}^T \mathbb{E}(\eta_{t-p}^2(w, w') | \{Y_{1:T}(w_{1:T}), w_{1:T} \in \mathcal{W}^T\}).$$

So long as the conditional mean of $\eta_{t-p}^2(w, w')$ is bounded, then this the conditional variance of $\hat{\tau}_p^*(w, w')$ will contract with T .

The following Theorem, which just applies a triangular martingale difference central limit theorem, extends these results to where $T \rightarrow \infty$. It shows that $\hat{\tau}_{t,p}^*(w, w')$ is consistent for $\bar{\tau}_p^*(w, w')$ and the estimator's error is asymptotically normal under weak conditions.

Theorem B.2. *Under the conditions of Theorem B.1, additionally assume that $\lim_{T \rightarrow \infty} \bar{\eta}_T(w, w') < \infty$. Then $\hat{\tau}_p^*(w, w') - \bar{\tau}_p^*(w, w') \xrightarrow{P} 0$ as $T \rightarrow \infty$. Finally, if $\frac{1}{T-p} \sum_{t=p+1}^T \eta_{t-p}^2(w, w') \xrightarrow{P} \eta^2(w, w') > 0$, then, over the non-anticipating treatment path, as $T \rightarrow \infty$,*

$$\sqrt{T} \frac{\{\hat{\tau}_p^*(w, w') - \bar{\tau}_p^*(w, w')\}}{\eta(w, w')} \xrightarrow{d} N(0, 1). \quad (13)$$

Proof. The first result follows from (12) as $\bar{\eta}_T(w, w')$ is bounded. The second follows from a martingale array CLT of Theorem 3.2 in Hall and Heyde (1980) as the potential outcomes are bounded which means the Lindeberg condition hold. \square

Again, the only source of randomness here is the path of the treatments.

B.3 When \mathcal{W} is continuous

B.3.1 Estimator

There is a modest literature on the nonparametric estimation of causal effects when treatments are continuous in cross-sectional and panel settings. For example, Hirano and Imbens (2004) study continuous treatments using “generalized propensity scores.” Marginal structural models of Robins et al. (2000) provide parametric and series based nonparametric strategies to deal with continuous treatments. Cattaneo (2010) provides an extensive discussion of the multivalued case and the related literature. Yang et al. (2016) is a recent paper on this topic.

Write

$$F_t(w) := \Pr(W_t \leq w | \mathcal{F}_{T,t-1}), \quad \text{and} \quad f_t(w) := \partial F_t(w) / \partial w.$$

Then, using a bandwidth $h > 0$, define the time- t kernel regression estimator

$$\hat{\tau}_{t,p}^*(w, w') := \hat{g}_{t,p}(w) - \hat{g}_{t,p}(w'), \quad \text{and} \quad \hat{g}_{t,p}(w) := Y_t \frac{k_h(W_{t-p} - w)}{f_{t-p}(W_{t-p})}, \quad (14)$$

where $k_h(u) = h^{-1}k(u/h)$ is a kernel weight function, and the estimand is

$$\tau_{t,p}^*(w, w') = g_{t,p}(w) - g_{t,p}(w'), \quad \text{where} \quad g_{t,p}(W_{t-p}) := \mathbb{E}(Y_t | \mathcal{F}_{T,t-p-1}, W_{t-p}).$$

In a moment we will use the definitions $g_{t,p}^{[2]}(W_{t-p}) = \mathbb{E}(Y_t^2 | \mathcal{F}_{T,t-p-1}, W_{t-p})$, $\kappa_j = \int u^j k(u) du$, $b = \int k(u)^2 du$ and $k^*(x) = \int k(u)k(x+u)du$.

Theorem B.3 quantifies the variance and bias terms of the time- t kernel regression estimator, holding the potential outcomes as fixed. The derivation of the result is entirely conventional from the kernel literature.

Theorem B.3. *Assume $h > 0$ and $f_{t-p}(w) > 0$ for all w . Define*

$$\begin{aligned} \mu_{t,p}(w) &:= \mathbb{E} \left[\hat{g}_{t,p}(w) | \mathcal{F}_{T,t-p-1} \right], \quad \sigma_{t,p}^2(w) := h \times \text{Var} \left[\hat{g}_{t,p}(w) | \mathcal{F}_{T,t-p-1} \right], \\ c_{t,p}(w, w') &:= h \times \text{Cov}(\hat{g}_{t,p}(w), \hat{g}_{t,p}(w') | \mathcal{F}_{T,t-p-1}), \end{aligned}$$

where the expectations are over the treatment process $W_{t-p:t} | \mathcal{F}_{T,t-p-1}$, holding the potential outcomes fixed. If $u_{t,p}(w) := \hat{g}_{t,p}(w) - \mu_{t,p}(w)$ then

$$\begin{aligned} \mathbb{E}(u_{t,p}(w) | \mathcal{F}_{T,t-p-1}) &= 0, \quad \text{Var}(u_{t,p}(w) | \mathcal{F}_{T,t-p-1}) = h^{-1} \sigma_{t,p}^2(w) \\ \text{Cov}(u_{t,p}(w), u_{t,p}(w') | \mathcal{F}_{T,t-p-1}) &= h^{-1} c_{t,p}(w, w'). \end{aligned}$$

Further, if $g_{t,p}(w)$ is twice continuously differentiable in w , $\kappa_0 = 1$, $\kappa_1 = 0$ and $h \downarrow 0$, then

$$\begin{aligned} \mu_{t,p}(w) &\simeq g_{t,p}(w) + 0.5h^2 g_{t,p}''(w) \kappa_2, \quad \sigma_{t,p}^2(w) \simeq \frac{g_{t,p}^{[2]}(w)}{f_{t-p}(w)} b, \\ c_{t,p}(w, w') &\simeq \frac{1}{2} \left(\frac{g_{t,p}^{[2]}(w)}{f_{t-p}(w)} + \frac{g_{t,p}^{[2]}(w')}{f_{t-p}(w')} \right) k^*((w - w')/h). \end{aligned}$$

Finally, if as $|x| \rightarrow \infty$, $k^*(x) = o(1)$, then $c_{t,p}(w, w) = o(1)$, if $w \neq w'$.

Proof. All but the last 3 results are by definition. Now

$$h_{t,p}(w) := \mathbb{E} \left[Y_t \frac{k_h(W_{t-p} - w)}{f_{t-p}(W_{t-p})} \middle| \mathcal{F}_{T,t-p-1} \right] = h^{-1} \int g_{t,p}(x) k((x-w)/h) dx.$$

Transforming to $u = (x-w)/h$, so $x = w + hu$, we have

$$h_{t,p}(w) = \int g_{t,p}(w + hu) k(u) du \simeq g_{t,p}(w) + 0.5h^2 g_{t,p}''(w) \kappa_2,$$

as $\kappa_0 = 1$ and $\kappa_1 = 0$. Likewise

$$\begin{aligned} \mathbb{E} \left[Y_t^2 \frac{k_h(W_{t-p} - w)^2}{f_{t-p}(W_{t-p})^2} \middle| \mathcal{F}_{T,t-p-1} \right] &= h^{-2} \int \frac{g_{t,p}^{[2]}(x)}{f_{t-p}(x)} k((x-w)/h)^2 dx \\ &= h^{-1} \int \frac{g_{t,p}^{[2]}(w + hu)}{f_{t-p}(w + hu)} k(u)^2 du \simeq h^{-1} \frac{g_{t,p}^{[2]}(w)}{f_{t-p}(w)} b, \end{aligned}$$

while

$$\begin{aligned} \mathbb{E} \left[Y_t^2 \frac{k_h(W_{t-p} - w)}{f_{t-p}(W_{t-p})} \frac{k_h(W_{t-p} - w')}{f_{t-p}(W_{t-p})} \middle| \mathcal{F}_{T,t-p-1} \right] &= h^{-2} \int \frac{g_{t,p}^{[2]}(x)}{f_{t-p}(x)} k((x-w)/h) k((x-w')/h) dx \\ &= h^{-2} \frac{1}{2} \int \frac{g_{t,p}^{[2]}(x)}{f_{t-p}(x)} k((x-w)/h) k((x-w')/h) dx + h^{-2} \frac{1}{2} \int \frac{g_{t,p}^{[2]}(x)}{f_{t-p}(x)} k((x-w)/h) k((x-w')/h) dx \\ &= h^{-2} \frac{1}{2} \int \frac{g_{t,p}^{[2]}(w + hu)}{f_{t-p}(w + hu)} k(u) k(u + (w-w')/h) du + h^{-2} \frac{1}{2} \int \frac{g_{t,p}^{[2]}(w' + hu)}{f_{t-p}(w' + hu)} k(u + (w-w')/h) k(u) du \\ &\simeq h^{-1} \frac{1}{2} \left(\frac{g_{t,p}^{[2]}(w)}{f_{t-p}(w)} + \frac{g_{t,p}^{[2]}(w')}{f_{t-p}(w')} \right) \int k(u) k(u + (w-w')/h) du \\ &= h^{-1} \frac{1}{2} \left(\frac{g_{t,p}^{[2]}(w)}{f_{t-p}(w)} + \frac{g_{t,p}^{[2]}(w')}{f_{t-p}(w')} \right) k^*((w-w')/h). \end{aligned}$$

Then the result follows by the assumed property of k^* . \square

For each T fix the bandwidth as h_T . For each T , the estimation error $\{u_{t,p}(w)\}$ is a martingale difference sequence but centered at $\mu_{t,p}(w)$ not $g_t(w)$. Now assume that $\frac{1}{T-p} \sum_{t=p+1}^T \sigma_{t,p}^2(w) \xrightarrow{p} \sigma_p^2(w)$, then for $h_T > 0$ the triangular array central limit theorem implies that for the kernel regression estimator

$$\sqrt{h(T-p)} \frac{\{\hat{\tau}_p^*(w) - \hat{\tau}_p^*(w')\} - \{\bar{\mu}_p(w) - \bar{\mu}_p(w')\}}{\sqrt{\sigma_p^2(w) + \sigma_p^2(w')}} \xrightarrow{d} N(0, 1),$$

where $\bar{\mu}_p(w) = \frac{1}{T-p} \sum_{t=p+1}^T \mu_{t,p}(w)$. Of course

$$\bar{\mu}_p(w) - \bar{\mu}_p(w') \simeq \{\bar{g}_p(w) - \bar{g}_p(w')\} + 0.5h^2 \kappa_2 \{\bar{g}_p''(w) - \bar{g}_p''(w')\},$$

where $\bar{g}_p''(w) = \frac{1}{T-p} \sum_{t=p+1}^T g_{t,p}''(w)$. Notice that the bias involves the difference of two second derivatives of $\bar{g}_p(w)$ evaluated at w and w' .

Remark B.1. *The corresponding results when the regression kernels for $\hat{g}_p(w)$ and $\hat{g}_p(w')$ use different bandwidth, h_w and $h_{w'}$, is straightforward to write out. However, in practice this has the disadvantage that the bias term becomes $0.5\kappa_2\{h_w^2\bar{g}_p''(w) - h_{w'}^2\bar{g}_p''(w')\}$, which shows no sign of cancelling.*

Further, if we aggregate period of period mean square error, then

$$\begin{aligned} & \frac{1}{T-p} \sum_{t=p+1}^T \mathbb{E} \left[\left(\{\hat{\tau}_{t,p}^*(w) - \hat{\tau}_{t,p}^*(w')\} - \{g_{t,p}(w) - g_{t,p}(w')\} \right)^2 \middle| \mathcal{F}_{T,t-p-1} \right] \\ & \simeq \frac{1}{h(T-p)} \{\sigma_p^2(w) + \sigma_p^2(w')\} + 0.25h^4 \kappa_2 \frac{1}{T-p} \sum_{t=p+1}^T \left(g_{t,p}''(w) - g_{t,p}''(w') \right)^2, \end{aligned}$$

which is minimized by selecting $h \propto (T-p)^{-1/5}$ so the mean square error declines at the usual nonparametric rate $T^{-4/5}$, which does not vary with p . None of these results are surprising from the vast nonparametric literature.

Remark B.2. *At a fundamental level it would be convenient to be able to estimate individual finite sample terms like $Y_t(W_{1:t-p-1}, w) - Y_t(W_{1:t-p-1}, w')$, where $w, w' \in \mathcal{W}^{p+1}$, or their temporal average. Can these terms be nonparametrically identified just using the structure of the potential outcome time series, conditioning on all of the potential outcomes? We sketch out below that the answer to this is yes, but that the result is of little immediate practice use due to the slow rate of convergence. Write the intermediate estimand as $g_{t,p}(w) := Y_t(W_{1:t-p-1}, w)$, where $w \in \mathcal{W}^{p+1}$, while write $g_{t,p}''^i(w) := \partial^2 g_{t,p} / \partial w_i^2$, $k_{h,r}(u) := h^{-r} k(u_1) \dots k(u_r)$, and $F_{t-p:t}(w) := \Pr(W_{t-p:t} \leq w | \mathcal{F}_{T,t-p-1})$, and $f_{t-p:t}(w) := \partial F_{t-p:t}(w) / \partial w$. The corresponding intermediate estimator is*

$$\hat{g}_{t,p}(w) := \frac{Y_t}{f_{t-p:t}(W_{t-p:t})} k_{h,p+1}(W_{t-p:t} - w).$$

The eventual goal is to use $\hat{g}_{t,p}(w) - \hat{g}_{t,p}(w')$ to estimate $g_{t,p}(w) - g_{t,p}(w')$. Now

$$\mathbb{E}(\hat{g}_{t,p}(w, w') | \mathcal{F}_{T,t-p-1}) = h^{-(p+1)} \int g_{t,p}(x) k_{h,r}((x-w)/h) dx_1 \dots dx_{p+1}$$

$$\begin{aligned}
&= \int g_{t,p}(w + hu)k(u_1)\dots k(u_{p+1})du_1\dots du_{p+1} \\
&= g_{t,p}(w) + h^2(p+1)0.5\kappa_2\frac{1}{p+1}\sum_{i=1}^{p+1}g_{t,p}''(w),
\end{aligned}$$

while

$$\mathbb{E}(\hat{g}_{t,p}(w, w')^2 | \mathcal{F}_{T, t-p-1}) = h^{-2(p+1)} \int \frac{g_{t,p}^2(x)}{f_{t-p:t}(x)} K((x-w)/h)^2 dx_1 \dots dx_{p+1} \simeq h^{-(p+1)} \frac{g_{t,p}^2(w)}{f_{t-p:t}(w)}.$$

As before the covariance between $\hat{g}_{t,p}(w)$ and $\hat{g}_{t,p}(w')$ is comparatively unimportant. Hence, averaging over T data points, in terms of mean square the best bandwidth choice would be $h \propto T^{-1/(p+5)}$ so the mean square error declines at the usual multivariate rate of $T^{(p+4)/(p+5)}$. Hence $g_{t,p}(w) - g_{t,p}(w')$ is nonparametrically identified, but at its core it is a very nasty result empirically. As the length of the lags increases the rate of convergence slows.