1 Problem set 01

- Fill out the online cover sheet for each assignment to name your collaborators, list resources you used, and estimate the time you spent on the assignment.
- Collaboration is encouraged on all assignments. At the same time, your individual written work for this class should be your own. You are expected not to consult outside solutions or solution manuals, not to read the completed solutions of your classmates, and not to copy your solutions directly from common work. You are encouraged to discuss the mathematics and to work out the math together. Then put away or erase joint work before writing up your solution. In addition, if you believe your work is incorrect, please do show it to your classmates and the teaching staff. If you believe your solution to be correct then go ahead and discuss or describe your solution, without actually showing your written work to others.

The following problems are copied (nearly verbatim) from the text.

1. Consider the differential equation $\dot{x} = e^x - \cos x$.
   (a) Sketch e^x and $\cos x$ on the same axes.
   Note: In general, label all plots with some kind of title identifying what is being plotted as well as axes labels and scale markings.
   (b) Use your sketch to approximately identify fixed points, determine their stability (using the sign of $f(x)$) and sketch the vector field on the real line (include arrows and filled or open circles for fixed points, depending on their stability).
   (c) For $x \ll 0$, find an approximate expression for all fixed points, along with their stability.
   (d) Sketch the graph of $x(t)$ for a few qualitatively different cases.
2. Formulate a differential equation that would result in an unstable fixed point at $x = -2$, a stable fixed point at $x = 1$ and a half stable fixed point at $x = 2$. Explain your thinking clearly.
3. Consider the model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 2X$$

   in which molecule A combines with molecule X to form two molecules of X. This means that the chemical X stimulates its own production, a process called *autocatalysis*. This positive feedback leads to a chain reaction, which eventually is limited by a "back reaction" in which $2X$ returns to $A + X$.

   According to the *law of mass action* of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants (this is called a "law" but is actually a model). We denote the concentration by lowercase $x =]X]$ and $a = [A]$. Assume that there's an enormous surplus of chemical A, so that its concentration a can be regarded as constant. Then the equation for the kinetics of x is

$$\dot{x} = k_1 a x - k_{-1} x^2$$

   where k_1 and k_{-1} are positive parameters called rate constants (these are found empirically).
   (a) Find all the fixed points of this equation and classify their stability. Show your calculation steps or title and label your graph clearly.
   (b) Sketch approximate graphs of $x(t)$ for various initial values x_0. Include all of the qualitatively different cases.
4. Let

$$\dot{x} = rx - \frac{x}{1 + x^2}.$$

   Show your mathematical steps or reasoning for each part.

(a) Compute the values of r at which bifurcations occur.

(b) Classify the bifurcations as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork.

(c) Sketch the bifurcation diagram of fixed points x^* vs r.

5. Consider the system $\dot{x} = rx - \sin x$.

   (a) For the case $r = 0$, find and classify all fixed points of the system, and sketch the vector field on the x-axis.

   (b) For $r > 1$ show that there is only one fixed point, and classify it.

   (c) As r decreases from ∞ to 0 classify *all* of the bifurcations that occur. *To think about this, plot* $\sin x$ *and* rx *on the same axes. Using a tool that allows you to manipulate* r *will allow you to see when bifurcations occur.* Remember to label all plots that you include in your write-up.

   (d) For $0 < r \ll 1$, find an approximate formulate for values of r at which bifurcations occur. *For small* r*, note that bifurcations occur with* $\sin x \approx 1$ *or* $\sin x \approx -1$*. This observation will allow you to approximate* x *and then* r*.*

   (e) Plot the bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

6. Head to Piazza

   (a) Find the 'pset01 LaTeX' post. Add a comment to the post. To enter math into Piazza, we can use LaTeX commands, a mark-up language for mathematics. In your post, include

   ```
   $$\dot{x} = r x - \frac{x}{1+x^2}$$
   ```

   In addition, head to `https://rpi.edu/dept/arc/training/latex/LaTeX_symbols.pdf` and choose a second LaTeX command to try out in your post. Feel free to post links to other LaTeX resources that you're familiar with if you like.

   (b) Find the 'pset 01 projects' post. Add a comment to the post. In your comment briefly describe a question that you think you might be intrigued to explore via a project. No need to reflect on whether it seems specifically relevant to this class; whatever you have on your mind as a possible question is fine.