1 Problem set 02

- Fill out the online cover sheet for each assignment to name your collaborators, list resources you used, and estimate the time you spent on the assignment.
- You're welcome to use Mathematica for as much or as little as you like of the assignment. If you do use Mathematica, please submit your notebook on Canvas when you submit your writeup. If you use another plotting system, acknowledge that system in the cover sheet and also in the part of the problem where you used it.
- Collaboration is encouraged on all assignments. At the same time, your individual written work for this class should be your own. You are expected not to consult outside solutions or solution manuals, not to read the completed solutions of your classmates, and not to copy your solutions directly from common work. You are encouraged to discuss the mathematics and to work out the math together. Then put away or erase joint work before writing up your solution. In addition, if you believe your work is incorrect, please do show it to your classmates and the teaching staff. If you believe your solution to be correct then go ahead and discuss or describe your solution, without actually showing your written work to others.

1. (How do solutions approach a fixed point?) Consider the system $\dot{x} = r - x^2$, where r is a parameter. *Show your work/mathematical steps for each of the parts below.*
   (a) Find the stable fixed point, $x^*(r)$ and the range of r for which it exists.
   (b) Keeping r as a parameter, rewrite the system in terms of $\eta = x - x^*$. This change of coordinates shifts the stable fixed point to $\eta = 0$ (where η is the greek letter "eta").
   (c) For the system $\dot{\eta} = g(\eta, r)$, find g' at the fixed point (g' is a constant at the point). Near the fixed point $(0, r)$, the system is approximately $\dot{\eta} = g'(0, r)\eta$. Find the exact solution (using separation of variables) to this approximate system to see how solutions $\eta(t)$ approach the fixed point.
   (d) How does the approach to the fixed point vary with r? In particular, think about what happens as r approaches the bifurcation value (sometimes called the critical point). Sketch curves for a few values of r between 0 and 1 (put the plots on the same x vs t axes so that you can compare their shape).
   (e) Now we will take a look at a solution curve approaching $x^* = 0$ right at the bifurcation point $r_c = 0$. We will stop approximating because our approximation breaks down when $r = -$. Instead of approximating, use separation of variables to solve the system $\dot{x} = -x^2$ exactly with initial conditions $x(0) = x_0$ with $x_0 > 0$. Compare a solution curve for $r = 0.01$ (use your approximation above for this) to the solution curve for $r = 0$ (make a plot as part of your comparison).
   (f) The behavior of solutions as $r \to r_c$ is sometimes referred to as *critical slowing down*. What do you think this term is referring to (where is the slowing)?

2. (Backwards time) Consider a system $\dot{x} = f(x)$. This is $\dfrac{dx}{dt} = f(x)$. We have been reasoning about what happens to solutions $x(t)$ as $t \to \infty$. This is sometimes referred to as the behavior of the variable x in *forward time*. You might also wonder what the behavior of x was before time 0. This would involve thinking about *backwards time*. We could redo all of our analyses for $t \to -\infty$.

   One way this is often approached mathematically is to do a change of variables. Create a new variable, $t_1 = -t$. Now rewrite the system in terms of t_1. When $t \to -\infty$ we have $t_1 \to \infty$, so using our standard analysis methods on the system we learn about the behavior of the system in backwards time.

   (a) For the generic system above, $\dfrac{dx}{dt} = f(x)$, do the change of variables suggested and rewrite the system in terms of t_1 (show your steps). Assume x^* is a fixed point of $\dfrac{dx}{dt} = f(x)$. Is it a fixed point of the new system (justify your answer)? If it is, identify its stability in the new system in terms of its stability in the original system.

Note: In a 1d system like this one, figuring out what is going on in backwards time is relatively straightforward, and will apply generally to all systems. It will be a bit harder to reverse time in 2d systems, so you can think of this problem as a warm-up for 2d ideas to come.

(b) For the system $\dfrac{dx}{dt} = 1 - x^2$ sketch solutions $x(t)$ for the qualitatively different cases. Include both positive and negative time on your sketch. Use arrows to indicate what happens as $t \to \infty$ and also what happens as $t \to -\infty$.

3. (Potential functions). Read section 2.7 of the text, about potential functions. When working with a potential function, we can think of a particle at position x as sliding down the landscape formed by the potential function. We need to remember to think of the particle as not having inertia, though, so it can fall down a hill, but it doesn't roll up the other side (it gets stuck at the bottom).

   (3.4.16c) (Potentials, subcritical pitchfork) Let $V(x)$ be the potential, in the sense that $\dot{x} = -\frac{dV}{dx}$. Consider the system $\dot{x} = rx + x^3 - x^5$. Sketch the bifurcation diagram (feel free to just use Mathematica code for this). In addition, find the potential function and sketch it (as a function of x) for all the qualitatively different cases (so for various values of r - include the bifurcation values of r as distinct cases). Mark the locations of fixed points on the x-axis of your potential function sketches.

4. (3.7.5) See the text for this question. Do the question in its entirety. In addition, for part (f), explain why the title of the problem was "A biochemical switch" (where was the switch?)

5. (4.4.4) See the text for this question. Show your work for each part.

   Some context: see section 4.4 for a diagram of the setup. We are applying a constant torque to the pendulum (a rotational version of force) and that torque is being balanced by gravity pulling the pendulum. In addition, once we add the spring, we have the spring helping gravity. Usually a pendulum equation involves angular position, angular velocity, and angular acceleration (where angular acceleration is $\ddot{\theta}$). We don't know how to deal with second derivatives (angular acceleration) at the moment, so we can only think about a case where $\dot{\theta}$ (the friction/damping/resistance to turning at the joint of the pendulum) is very large relative to $\ddot{\theta}$. This is called the *overdamped* limit.