# 1 Problem set 03

## 1.1 General problem set policies (identical to last week)

- Fill out the online cover sheet for each assignment to name your collaborators, list resources you used, and estimate the time you spent on the assignment.
- You're welcome to use Mathematica for as much or as little as you like of the assignment. If you do use Mathematica, please submit your notebook on Canvas when you submit your writeup. If you use another plotting system, acknowledge that system in the cover sheet and also in the part of the problem where you used it.
- Collaboration is encouraged on all assignments. At the same time, your individual written work for this class should be your own. You are expected not to consult outside solutions or solution manuals, not to read the completed solutions of your classmates, and not to copy your solutions directly from common work. You are encouraged to discuss the mathematics and to work out the math together. Then put away or erase joint work before writing up your solution. In addition, if you believe your work is incorrect, please do show it to your classmates and the teaching staff. If you believe your solution to be correct then go ahead and discuss or describe your solution, without actually showing your written work to others.

## 1.2 Problem set length (adding a feedback to the system)

Ideally the problem sets would take about 5-7 hours each week on the assumption that you'd then spend another 3-4 hours preparing for class, working to understand material, etc.

In order to reach a desired equilibrium, the system needs to have some sort of feedback, so that the length of the problem sets can change somewhat dynamically. If I know by Tuesday or Wednesday that the problem set is running long, I can then remove the last problem, indicate that you should skip parts of the writeup, provide the solution to some subparts, etc, so as to shorten the experience. Please email me at if you're working on the problem set and there are signs that it will likely run long.

## 1.3 Extra info for this problem set

This is a team problem set! Your teams are

- Group 1: Jacob, Brooke, Laura
- Group 2: Dominique, Elbert, Joey
- Group 3: Allison, Amos, Kevin
- Group 4: Eryk, Anna
- Group 5: Rachael, Taras, Jessica
- Group 6: Joel, Min, Catherine

If you head to Piazza and make a new post, you'll see an option to post to "entire class" or "class group". If you choose to post to a Class Group, the post will go to just your group. I have entered these groups (as numbered above) into Piazza.

- This is a team homework. You should complete it in the team that's assigned here and should submit one writeup together.
- Despite the fact that this problem set is being done in a group, it is expected that you each contribute to the mathematical discourse of every problem (even if you, personally, are not responsible for the key idea that solves a given problem). In other words, do your work in a group, by working out your own ideas and bringing them to group meetings. A divide-and-conquer strategy is not unacceptable, and is considered a violation of our academic integrity policy.
- Your writeups for this team homework should be in an ?essay? type format using complete sentences to describe your work.
- By editing a post within your Class Group, you are expected to use Piazza to write up the the solutions to at least three subparts of one of the problems (your choice of which problem).

- It is permissible to divide-and-conquer the write-ups of the solutions.
- See the participation page for the rubric for collaborative homework here: `https://docs.google.com/document/d/1yHLzm6QQ_8qhpmxhkxgi4As_kESFpHk09TiXt0RtSzo`. You will fill out a survey about yourself and your teammates based on this rubric.

1. (5.1.9) Consider the system $\dot{x} = -y, \dot{y} = -x$.
   (a) Sketch the vector field.
   (b) Solutions of the system are pairs of time-dependent functions, $(x(t), y(t))$. When plotted in the $xy$-plane, these solution curves form trajectories. Show that the function $x^2 - y^2$ is constant along every solution curve. This means that solution curves are curves of the form $x^2 - y^2 = c$.
   To do this, show that $\frac{d}{dt}\left(x^2 - y^2\right) = 0$ under the action of the vector field.
   (c) The origin is a saddle point. Find equations for its stable and unstable manifolds.
   The *stable manifold* of a saddle point $\underline{x}$ is defined as the set of initial conditions $\underline{x}_0$ such that $\underline{x}(t) \to \underline{x}$ as $t \to \infty$. So this is the line (or curve in a nonlinear system) along which trajectories would approach the fixed point.
   The *unstable manifold* is the set of initial conditions such that $\underline{x}(t) \to \underline{x}$ as $t \to -\infty$.
   (d) This system decouples under a change of coordinates, which leads to a solution method. Introduce new variables $u$ and $v$ where $u = x + y, v = x - y$. Rewrite the system in terms of $u$ and $v$. Solve for $u(t)$ and $v(t)$ starting from arbitrary initial conditions $(u_0, v_0)$.
   (e) What are the equations for the stable and unstable manifolds in terms of $u$ and $v$?
   (f) Use $u(t)$ and $v(t)$ to find a general solution for $x(t)$ and $y(t)$ when starting from initial conditions $(x_0, y_0)$.

2. (6.3.9) Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.
   (a) Find the fixed points of this system. *Show your algebraic steps.*

   **Solution:** At fixed points, we have

   $$\begin{aligned} y^3 - 4x =& 0 \\ y^3 - y - 3x =& 0. \end{aligned}$$

   So $y^3 = 4x$ at fixed points. Substituting into the $\dot{y}$ equation, we find

   $$4x - y - 3x = 0. \Rightarrow x - y = 0.$$

   So $y = x$ at fixed points. We now have

   $$x^3 - 4x = x(x^2 - 4) = 0.$$

   The roots are $x = 0$ or $x^2 = 4$ so $x = 0, x = -2, x = 2$. Thus there are three fixed points: $(-2, -2), (0, 0), (2, 2)$.

   (b) For each fixed point, find the linearized system, identify $\tau$ and $\Delta$, and use this to classify the fixed point.
   (c) Show that the line $y = x$ is invariant, meaning that any trajectory that starts on this line stays on this line.
   The curve $g(x, y) = 0$ is invariant if $\frac{dg}{dt} = 0$ along the curve $g(x, y) = 0$. Using the chain rule, the curve is invariant if $g_x\dot{x} + g_y\dot{y} = 0$ along the curve.
   (d) Show that for trajectories not starting on the curve $x = y$ we have $|x(t) - y(t)| \to 0$ as $t \to \infty$, so that $x \to y$ as $t \to \infty$.
   To do this, form a differential equation for the evolution of $v = x - y$.
   Note that $x$ and $y$ increase (or decrease) with $y^3$, so they diverge towards infinity quickly. They converge to the line $x = y$ more slowly.

(e) Sketch the phase portrait for the system.

(f) Plot an accurate phase portrait (using numerical integration of solutions) on the square domain $-20 \leq x, y \leq 20$. Notice that the trajectories seem to approach a curve as $t \to -\infty$. Can you explain this behavior intuitively, and perhaps find an approximate equation for this curve?

*I will post code for the plotting, since this is nontrivial in Mathematica, and then I'll ask you to comment the code/identify what it is doing. For the curve that is approached, I will post a hint later in the week, but want to give you the chance to think about it.*

3. (Thinking about projects) As you know, our course will wrap up with a project. You'll be designing the project.

Projects can have many different flavors and still be dynamical systems projects. You could consider developing a model of a science or social science phenomenon you find interesting; you could take a model already in the literature and analyze its behavior; you could learn about an area of dynamics we aren't covering in depth (chaos, global bifurcations, manifold theory).

The ideal group size for a project is two or three people, but individual projects are possible. With more than three people interested in a topic, it's often more constructive to split into two related subgroups.

Our project design is gradual and we'll be in a preliminary phase until after Spring Break. During this phase, you'll be floating possible project ideas, you'll begin to look around for resources or references, and you'll connect with and give feedback to other students.

On problem set 01 I asked you to briefly describe a question you might be intrigued to explore.

For this problem set, talk with your group members about project ideas that interest you. Work to explain your thinking, including explaining any unfamiliar terms or ideas. Give feedback to each other - does the idea make sense to your classmates? Help each other work to pose the ideas as questions to which you don't currently know an answer.

Now head to Piazza. Create a new post for the class with a meaningful title. If someone has already made a post related to your interests it is fine to work within that post rather than creating a new one. Write a paragraph about what you might like to do your class project on. Include the details and definitions of terms that made your ideas understandable for your group. Make it clear whether you're thinking about building a model, analyzing an existing model, learning dynamical systems topics outside the scope of the course, or a mixture of these.

Give it some thought and make use of your group to help you brainstorm. You're going to work hard on this project, so the time spent workshopping ideas is worthwhile.