1 Problem set 07

1.1 Problem set

1. **Project** It's time to form teams for the project. Form a group of 3, 2, or 1 students. Working in groups is strongly encouraged and is likely to result in a better project, but it is not required.

   In your group you have each generated at least one possible question for a project, and perhaps more. In addition, there are a number of other possible questions you can browse on Piazza. Your group should select two possible topics / questions to explore. You'll submit two paragraphs (via Canvas) as a team:

   - Write one paragraph about each of the two possibilities you are exploring.
   - In each paragraph, briefly identify the project. Where would you start if you were to choose this project? What do you anticipate finding hard about working on this project idea? What math do you think you will need to learn to make progress on this? What else do you anticipate needing to learn how to do for this project?
   - Length guideline: You may submit up to 1 page of typewritten work for this.

2. (working with polar, 7.1.5, 7.3.1)

   (a) (7.1.5) Show that the system $\dot{r} = r(1-r^2), \dot{\theta} = 1$ is equivalent to

   $$\dot{x} = x - y - x(x^2+y^2)$$
   $$\dot{y} = x + y - y(x^2+y^2)$$

   where $x = r\cos\theta$ and $y = r\sin\theta$. No need to analyze the system.
   Recall the multivariable chain rule: $\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta$.
   You are welcome to use Mathematica for this; be sure to submit your code in that case.

   Solution:
   Using the chain rule, we have

   $$\frac{dx}{dt} = r(1-r^2)\cos\theta - r\sin\theta, \quad \frac{dy}{dt} = r(1-r^2)\sin\theta + r\cos\theta.$$

   We need to convert these equations to Cartesian coordinates. Rearranging the equations we are given, we have $\sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}$. Substituting,

   $$\frac{dx}{dt} = r(1-r^2)\frac{x}{r} - r\frac{y}{r}, \quad \frac{dy}{dt} = r(1-r^2)\frac{y}{r} + r\frac{x}{r}.$$

   Simplifying, we have

   $$\frac{dx}{dt} = x(1-r^2) - y, \quad \frac{dy}{dt} = y(1-r^2) + x.$$

   Substituting $r^2 = x^2 + y^2$ this becomes

   $$\frac{dx}{dt} = x - y - x(x^2+y^2), \quad \frac{dy}{dt} = x + y - y(x^2+y^2),$$

   which is what we expected to find.

   (b) (7.3.3) Consider the system $\dot{x} = x - y - x^3, \dot{y} = x + y - y^3$. Rewrite the system in polar coordinates, where $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}$. Then use the Poincaré-Bendixson Theorem to show that the system has a closed trajectory.
   From the chain rule: $r\dot{r} = x\dot{x} + y\dot{y}$ *and* $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.
   You are welcome to use Mathematica for this; be sure to submit your code in that case.

Solution:

Via Mathematica, I find

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dxdt = x - y - x (x^2);
dydt = x + y - y (y^2);
drdt = Expand[
  Simplify[(x dxdt + y dydt) /. x -> r Cos[\[Theta]] /.
    y -> r Sin[\[Theta]]]]
Simplify[((x dydt - y dxdt)/r^2) /. x -> r Cos[\[Theta]] /.
  y -> r Sin[\[Theta]]]
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$$\frac{dr}{dt} = r^2\left(1 - \frac{1}{4}r^2(3+\cos 4\theta)\right), \quad \frac{d\theta}{dt} = 1 + \frac{1}{4}r^2 \sin 4\theta.$$

What does the variation with θ do to the system?

When $\cos 4\theta = -1$ we have $\frac{dr}{dt} = r^2(1 - \frac{1}{2}r^2)$, so $dr/dt = 0$ when $r = 0$ or when $r = \sqrt{2}$.

When $\cos 4\theta = 1$ we have $\frac{dr}{dt} = r^2(1 - r^2)$ so $dr/dt = 0$ when $r = 0$ or when $r = 1$.

I want to create a trapping region.

Consider the circles $r_1 = \frac{1}{2}$ and $r_2 = 2$.

3. (7.2.12) Show that $\dot{x} = -x + 2y^3 - 2y^4$, $\dot{y} = -x - y + xy$ has no periodic solutions.

   To do this, construct a function $V(x, y)$ such that $V(x, y) > 0$, except at the fixed point, (x^*, y^*) where $V(x^*, y^*) = 0$. In addition, construct V such that $\frac{dV}{dt} < 0$ on all trajectories (excepting the fixed point). Since $\frac{dV}{dt} < 0$ on trajectories, there cannot be a closed orbit in the system. Such a function is called a Liapunov function.

   Try $V(x, y) = x^m + ay^n$, and choose a, m, n so that it is a Liapunov function for this system.

   You are welcome to use Mathematica for this; be sure to submit your code in that case.

4. (van der Pol system and excitability 7.5.6 / 8.2.1 / 4.5.3a) Consider the biased van der Pol system $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. This system is biased by a constant force, a, where a can be positive, negative, or zero. Assume $\mu > 0$ as usual.

   (a) (8.2.1) Find the fixed points and classify them as stable, unstable, or saddle points. There are Hopf bifurcations in this system. Sketch the bifurcation curves in the μa-plane to make a stability plot.

   (b) (7.5.6) Using the Liénard transformed system, plot the nullclines. Show that if (and only if) they intersect on the middle branch of the cubic nullcline, the corresponding fixed point is unstable (reference your work in part a).

   (c) Assume $\mu \gg 1$. Show that the system has a stable limit cycle if and only if $|a| < 1$. What types of Hopf bifurcations are occurring in this system?

   (d) Choose a slightly greater than 1. Show that the system is *excitable*: it has a globally attracting fixed point, but certain disturbances can send the system on a long excursion through phase space before returning to the fixed point. Plot a phase portrait of the system with a trajectory showing the long excursion superimposed.

   (e) (4.5.3a) For a simple caricature of an excitable system, consider the system $\dot{\theta} = \mu + \sin\theta$ where μ is slightly less than 1. Show that this system is excitable. What is playing the role of the rest state? What sets the threshold for disturbances that lead to long excursions?

   Steve relates these models of excitability to neural systems. Notice that the spike size (plot x or θ with its excursion) doesn't depend on the size of the stimulus (so long as the stimulus exceeds some threshold). The forced van der Pol model is related to the Fitzhugh-Nagumo model of neural activity. (See the corresponding textbook questions for references).