1 Problem set 09

1.1 Problem set

1. (Project) As part of your problem set this week, submit an individual update on your project work. It can be brief bullet points.
   - Log any group meetings this week. (When did they happen? Who was there?)
   - Log any references you consulted (log them here, not on the cover sheet).
   - Log what you worked on (any math analyses, reading, or data-analysis you attempted).

2. (8.6.9) This question is not in the first edition of the book so I will post screenshots of its (very long) setup to Piazza.

   We are exploring a model of tree frogs. We want the model to explain experimental results that are observed with two tree frogs and with three tree frogs.

   (a) With two tree frogs, the observation is that they alternate their croak rhythms to croak a half-cycle apart. This is called *antiphase synchronization.*
   Assume the frogs have identical natural frequencies and the same response function to hearing other frogs.
   A model of the interaction is

$$\begin{aligned}\dot{\theta}_1 &= \omega + H(\theta_2 - \theta_1)\\ \dot{\theta}_2 &= \omega + H(\theta_1 - \theta_2),\end{aligned}$$

   where H is the coupling function. Assume it is odd, smooth, and 2π-periodic.
   Rewrite this system in terms of the phase difference $\phi = \theta_1 - \theta_2$. To explain the experimental results you would want $\phi \to \pi$ or $-\pi$ at $t \to \infty$.
   Recall that an odd function is a function where $f(-x) = -f(x)$.

   Solution:
   The phase difference $\phi = \theta_1 - \theta_2$. Taking the time derivative, $\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2$.

$$\begin{aligned}\dot{\phi} &= \omega + H(-\phi) - (\omega + H(\phi))\\ &= H(-\phi) - H(\phi)\\ &= -2H(\phi)\end{aligned}$$

   for $H(\phi)$ an odd function. So $\dot{\phi} = -2H(\phi)$.

   (b) Show that the experimental results for two frogs are consistent with the simple interaction function $H(x) = a\sin x$, if the sign of a is chosen appropriately.

   Solution:
   We want the long term behavior of the system to lead to a phase difference of π.

   (c) With three of the frogs interacting, they cannot each be half a cycle away from the others. They have been observed to settle into one of two distinctive patterns. One stable pattern involves a pair calling in unison with the third half a cycle out of phase of both. The other stable pattern has the three frogs maximally out of sync, with each calling one-third of a cycle apart from the other two.
   A model of the interaction for three frogs is

$$\begin{aligned}\dot{\theta}_1 &= \omega + H(\theta_2 - \theta_1) + H(\theta_3 - \theta_1)\\ \dot{\theta}_2 &= \omega + H(\theta_3 - \theta_2) + H(\theta_1 - \theta_2)\\ \dot{\theta}_3 &= \omega + H(\theta_1 - \theta_3) + H(\theta_2 - \theta_3).\end{aligned}$$

The interaction function is the same as for the two-frog model (since it is the same frogs).
Rewrite this system in terms of the phase differences $\phi = \theta_1 - \theta_2$ and $\psi = \theta_2 - \theta_3$.
To explain the experimental results you would want $\phi \to 0$ with $\psi \to \pi$ or $\phi \to \pi$ with $\psi \to 0$ or $\phi \to 2\pi/3$ and $\psi \to 2\pi/3$ as $t \to \infty$.

Solution:
We still have $\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2$. We also have $\dot{\psi} = \dot{\theta}_2 - \dot{\theta}_3$.
Note that $\phi = \theta_1 - \theta_2$, $\psi = \theta_2 - \theta_3$ and $\phi + \psi = \theta_1 - \theta_3$.

$$\begin{aligned}\dot{\phi} &= \omega + H(-\phi) + H(-\phi - \psi) - (\omega + H(-\psi) + H(\phi)) \\ &= H(-\phi) + H(-\phi - \psi) - H(-\psi) - H(\phi) \\ &= -2H(\phi) + H(\psi) - H(\phi + \psi).\end{aligned}$$

for H an odd function.

$$\begin{aligned}\dot{\psi} &= \omega + H(-\psi) + H(\phi) - (\omega + H(\phi + \psi) + H(\psi)) \\ &= H(-\psi) + H(\phi) - H(\phi + \psi) - H(\psi) \\ &= -2H(\psi) + H(\phi) - H(\phi + \psi).\end{aligned}$$

If I swap ϕ and ψ then the equations are completely symmetric, which is something I expect the system to be, so that serves as a modest check on the algebra.

(d) Show that $H(x) = a \sin x$ cannot account for the three-frog results.
You can use Mathematica for your analysis of the fixed points.

(e) Consider a family of more complicated interaction functions of the form $H(x) = a \sin x + \sin 2x$. Use Mathematica to plot phase portraits in the (ϕ, ψ)-plane for various values of a. Show that for a suitable choice of a you can explain the experimental results for two frogs and for three frogs.

3. (9.2.6) Consider the system

$$\begin{aligned}\dot{x} &= -\nu x + yz \\ \dot{y} &= -\nu y + (z - a)x \\ \dot{z} &= 1 - xy\end{aligned}$$

where $a, \nu > 0$ are parameters. (Note: the parameter ν is the Greek letter nu.) This is the Rikitake model of geomagnetic reversals.

(a) Show that the fixed points may be written in parametric form as $x^* = \pm k, y^* = \pm k^{-1}, z^* = \nu k^2$ where $\nu(k^2 - k^{-2}) = a$.
It is not quite enough to substitute into the equation and show that this works for the fixed points, because that leaves open the possibility of other fixed points not of this form. Instead, work to find the fixed points of the system. I found it natural to find the fixed points in terms of z^* (i.e., x^* and y^* in terms of z^*). Then I let $k = \sqrt{z^*/\nu}$ and worked to show the constraint $\nu(k^2 - k^{-2}) = a$.

(b) Classify the fixed points.
We have left 2D systems, so τ and Δ no longer are sufficient to classify a fixed point. You will need to find the eigenvalues of the Jacobian evaluated at the fixed points, and then can try to classify based on how many of the eigenvalues have positive or negative real part. You may want to use Mathematica to aid you.