1 Problem set 11

1.1 Problem set

This is a team-optional problem set. You are welcome to collaborate with others for your work and write-up, or to do your write up on your own. It is fine for you to work on it in a team of up to 3 people. Register yourself (if you are working alone) or your team in the PSet 11 groups area of the People tab on Canvas.

1. **Project (not a problem set question)** You have slides due on Monday Apr 23rd at 9am. I'll post more info on Monday. Those slides are for the (6 minute) progress report on Wednesday Apr 25th. During class on Monday you'll give and receive feedback around the slides, and can begin revising them for Wednesday.

   `https://hbr.org/2013/06/how-to-give-a-killer-presentation`

   - A few tips from the article: you are telling a story. (see "Frame Your Story")

     You don't have to share everything you've done in this presentation (that's one thing the annotations you'll submit with your final presentation are for - to tell me about other things you tried that didn't make the presentation).

     Instead, find an angle through your question and your project work that tells us a story. You are familiar with the dynamical systems background of your classmates, and it should be story that is accessible to them.
   - Your presentation slides are there to be illustrations. (see "Plan the Multimedia"). Nothing should be written on the slides that you plan to talk about out loud. Related images that illustrate what you're talking about are great!

     When I practice a talk, I watch for moments when I am describing something for the audience to imagine, or when I am gesturing a lot. Then I'll add an image to the slides (often hand-drawn) so that I can point to the image at that moment in the talk.

2. (10.1.12-13 Newton's method). Suppose you want to find the roots of an equation $g(x) = 0$. *Newton's method* says you should consider the map $x_{n+1} = f(x_n)$ where

$$f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}.$$

   (a) Write down the "Newton map" $x_{n+1} = f(x_n)$ for the equation $g(x) = x^2 - 4 = 0$.

   (b) Show that the Newton map has fixed points at $x^* = \pm 2$.

   (c) Show that these fixed points are *superstable* (multiplier is zero).

   (d) Iterate the map numerically starting from $x_0 = 1$, giving iterates $x_0, x_1, ...$ until you consider it close to the right answer (notes the fast convergence!).

   (e) Show that (under appropriate circumstances, see below) the roots of an equation $g(x) = 0$ always correspond to superstable fixed points of the Newton map.

   (f) What are the appropriate circumstances?

3. Consider the 1d dynamical system $\dot{x} = -10 \sin x$. (This is a flow, not a map).

   (a) What is the long term behavior of the system for initial conditions in the interval $(-\pi, \pi)$?

   (b) A simple numerical scheme to solve a differential equation is Euler's method. It replaces $\dot{x}$ with $(x_{n+1} - x_n)/\Delta t$, where $x_n = x(n\Delta t)$ and Δt is called the time step. Find the map $x_{n+1} = f(x_n)$ corresponding to the dynamical system above.

   (c) For $\Delta t = 0.2$, plot the map for the range $(-\pi, \pi)$.

   (d) Consider an initial condition of $x_0 = 1$ and draw the cobweb diagram.

   (e) Does this "numerical solution" to the differential equation behave the way you expect from part (a)? Critique the method. If you are happy with this method, could you use an even larger time step Δt to speed it up? If you aren't happy with it, is it improved by just decreasing the time step?

4. (10.3.6 abd, cubic map) Consider the cubic map $x_{n+1} = f(x_n)$ where $f(x_n) = rx_n - x_n^3$.

(a) Find the fixed points. For which values of r do they exist? For which values are they stable?

(b) To find the two-cycles of the map, suppose that $f(p) = q$ and $f(q) = p$. Show that p, q are roots of the equation $x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) = 0$ and use this to find all the two-cycles.

(c) Plot a partial bifurcation diagram (or orbit diagram, either is acceptable) based on the information obtained. *Label the diagram to identify whether it is a bifurcation diagram or an orbit diagram.*