Example. Make a contour plot for \( f(x, y) = x^2 + 2y^2 \).
- \( x^2 + 2y^2 = c \) is the equation of the \( c \)-level set (in the \( xy \)-plane).
- Pick a value of \( c \) to make reasoning more concrete. Let's use 1.
- In the \( xy \)-plane this equation is similar to a circle - recognize that it is an ellipse.
- Find the \( x \)-axis and \( y \)-axis intercepts: when \( x = 0 \) we have \( y = \pm \frac{1}{\sqrt{2}} \) so \((0, \pm \frac{1}{\sqrt{2}})\) are points on the ellipse. Note: \( \frac{1}{\sqrt{2}} \approx 0.7 \).
- Plot the four points you found and link them together by drawing something plausibly elliptical.
- Choose a contour interval (say 2). Increment \( c \) (so now it is 3) and repeat the plotting process.
- Repeat the incrementing and plotting process twice more. Once you have four curves made using an even contour interval, label the curves with their \( c \)-value, and you have a contour plot.

Example. Make a contour plot for \( f(x, y) = y^2 - x^2 \).
Follow the steps above. For the \( x \)-axis and \( y \)-axis intercepts: when \( x = 0 \) we have \( y = \pm 1 \) so \((0, 1)\) and \((0, -1)\) are points on the hyperbola. When \( y = 0 \) we find no values of \( x \) that work. So the hyperbola does not cross the line \( y = 0 \) in the \( xy \)-plane.
Since the hyperbola does not cross \( y = 0 \) it must open up/down (rather than opening left/right).

**Topic:** Linear functions

**For your reference (section 12.4):**
The contour diagram for a linear function of two variables, \( f(x, y) = mx + ny + c \), consists of evenly spaced, parallel, lines.

Given points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) in a plane, we have \( z_2 = mx_2 + ny_2 + c \) and \( z_1 = mx_1 + ny_1 + c \). Subtracting these, \( \Delta z = m(\Delta x) + n(\Delta y) \). In a row of a table where \( \Delta y = 0 \), we have \( m = \frac{\Delta z}{\Delta x} \). In a column of a table where \( \Delta x = 0 \), we have \( n = \frac{\Delta z}{\Delta y} \).

**Example.** Consider the function \( f(x) = 2x + 3 \), with graph \( y = 2x + 3 \) (in 2-space). Construct a function \( g(x, y) \) such that \( y = 2x + 3 \) is its \( 5 \)-level set.

**Topic:** Functions of three variables.

**For your reference (section 12.5):**
A \( c \)-level set of a function of three variables, \( f(x, y, z) \), is the set of points \((x, y, z)\) such that \( f(x, y, z) = c \).

**Example.** Let \( f(x, y, z) = x^2 + y^2 + z \). The \( c \)-level set is the set of points that satisfy \( x^2 + y^2 + z = c \). We have \( z = c - x^2 - y^2 \). This is a circular paraboloid centered about the \( z \)-axis and opening
downward, with a maximum value of $c$.

**Example.** Are the level sets associated with $g(x, y, z) = e^{-(x^2+(y-2)^2+z^2)}$ points in 1-space, curves in 2-space, surfaces in 3-space, or solids in 4-space? poll/Q

**Example.** Construct a function $f(x, y, z)$ such that $z = xy$ is the 2-level set of the function. poll/Q

**Topic:** Vectors

For your reference (section 13.1):

Vectors have magnitude and direction. In this course they are described using two or three numbers (denoted as $\vec{v} = (1, 3, 4) = \vec{i} + 3\vec{j} + 4\vec{k}$).

$\vec{i}, \vec{j}, \vec{k}$ are vectors of length one with direction pointing along the $x$-, $y$-, and $z$-axes respectively. The quantities of our usual algebra are called scalars (to contrast with vectors). Two vectors are parallel if one can be written as a scalar multiple of another (so $\vec{v}$ and $-3\vec{v}$ are parallel).

More about J.W. Gibbs: [https://yalealumnimagazine.com/articles/4496-josiah-willard-gibbs](https://yalealumnimagazine.com/articles/4496-josiah-willard-gibbs)

Matlab Examples:

\begin{verbatim}
syms x y f = @(x,y) exp(-x^2-y^2); fsurf(x,y,f(x,y),[-3 3 -3 3]) xlabel('x'); ylabel('y'); zlabel('z'); title('product of Gaussians') set(gca,'fontsize',16)
syms x y z f = @(x,y,z) exp(-x^2-(y-2)^2-z^2); fimplicit3(f(x,y,z)-sym(exp(-2)),... [-3 3 -3 3]) xlabel('x'); ylabel('y'); zlabel('z'); title('level surfaces') set(gca,'fontsize',16) hold on fimplicit3(f(x,y,z)-sym(exp(-4)),... [-3 3 -3 3]) axis equal
\end{verbatim}

Command list

- `fsurf`
- `fimplicit3`