For your reference (section 13.3):
Algebraic definition of the *dot product*: If we write $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.

Geometric definition of the *dot product*: $\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}||\,\cos\theta$, where θ is the angle between the two vectors, and we always choose θ between 0 and π.

Dot product facts: $\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$, $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$, $\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$, $\lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w} = \vec{v} \cdot (\lambda\vec{w})$.

The *equation for a plane* with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ containing the point (x_0, y_0, z_0) can be written $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. This is $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

Given a direction $\vec{w}$, a vector $\vec{v}$ can be written as the sum of a component of $\vec{v}$ parallel to the direction of $\vec{w}$ (denoted $\vec{v}_{\text{parallel}}$) and a component perpendicular to the direction of $\vec{w}$ (denoted $\vec{v}_{perp}$). $\vec{v}_{\text{parallel}} = \left(\vec{v} \cdot \frac{\vec{w}}{\|\vec{w}\|}\right) \frac{\vec{w}}{\|\vec{w}\|}$ and $\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}$.

Definitions are equivalent.

Use the geometric definition of the dot product to find the algebraic definition. Expand $(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \cdot (w_1\vec{i} + w_2\vec{j} + w_3\vec{k})$. Assume the properties $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ and $\lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w} = \vec{v} \cdot (\lambda\vec{w})$ to do the expansion.

Example. Find $\vec{u} \cdot \vec{v}$ where $\vec{u} = 4\vec{i} - 6\vec{k}$ and $\vec{v} = -\vec{i} + \vec{j} + \vec{k}$.

Example. Find $\vec{u} \cdot \vec{v}$ where $\vec{u} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{v}$ is a vector of length 2 oriented at an angle of $\pi/4$ away from the direction of $\vec{u}$.

For your reference (section 13.4):
Algebraic definition of the *cross product*: If we write $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then $\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}$.

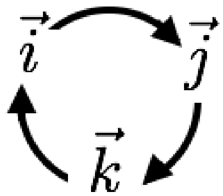
Geometric definition of the *cross product*: $\vec{v} \times \vec{w} = \begin{pmatrix} \text{area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w} \end{pmatrix} \vec{n}$, where $\vec{n}$ is a unit normal vector perpendicular to the parallelogram with direction given by the right hand rule.

The *parallelogram* formed by the vectors $\vec{v}$ and $\vec{w}$ has base length $\|\vec{v}\|$ and height $\|\vec{w}\| \sin\theta$ where θ is the angle between the vectors $(0 \leq \theta \leq \pi)$, so it has *area* $\|\vec{v}\|\|\vec{w}\| \sin\theta = \|\vec{v} \times \vec{w}\|$

There are two possible vector directions normal to the parallelogram formed by two vectors. The *right hand rule* is a convention for choosing one of the two normal vectors. Draw $\vec{v}$ and $\vec{w}$ with their tails at the same point. Point your index finger along vector $\vec{v}$ and your middle finger along vector $\vec{w}$ (you may have to turn your hand over to do this). The direction of your thumb is the direction of $\vec{v} \times \vec{w}$.

Example. Using the geometric definition, what is $\vec{i} \times \vec{j}$? What is $\vec{j} \times \vec{i}$?

One way people sometimes remember these relationships is with the following diagram:



Example. Find $\vec{v} \times \vec{v}$ using the geometric definition.

Example. Find $\vec{v} \times \vec{w}$ using the algebraic definition with $\vec{v} = 3\vec{i} - 2\vec{j} + 4\vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$

We can use *determinant* notation to remember the algebraic definition: $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k} = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_2)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}$

Construct a *normal vector* to a plane by finding the cross product of two vectors parallel to the plane.

An *area vector* for a planar shape is a vector that points normal to the plane containing the shape, with magnitude equal to the area of the shape.

The *volume of a parallelepiped* with base given by the parallelogram formed by $\vec{v}$ and $\vec{w}$, and with the other sides formed by $\vec{u}$ has volume $\|\vec{v} \times \vec{w}\| |(\|\vec{u}\| \cos\theta|) = |\vec{u} \cdot (\vec{v} \times \vec{w})| =$ abs($\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$).

Example. Let points $(0,1,2)$, $(2,-1,3)$ and $(0,0,1)$ form a triangle that lies in a plane.

- Find a normal vector to the plane and construct an equation for the plane
- Find the area of the triangle and construct an area vector for it.

Example. Find the volume of the parallelepiped with sides parallel to $\vec{u} = (3,4,5), \vec{v} = (5,4,3), \vec{w} = (1,1,1)$.

Matlab Examples:

```
vecu = [4,0,-6]; vecv = [-1,1,1];
dot(vecu,vecv) %example 1
vecu = [3,1,-1];
norm(vecu)*2*cos(pi/4) %example 2
cross([1,0,0],[0,1,0]) %example 3
cross([0,1,0],[1,0,0])
vecv = [3,-2,4]; vecw = [1,2,-1];
cross(vecv,vecw) %example 4
```

```
pt = [0,1,2] % Find the plane.
vecu = pt-[2,-1,3];
vecv = pt-[0,0,1];
vecn = cross(vecu,vecv)
syms x y z
g = @(x,y,z) dot(vecn, [x,y,z]-pt)
```

Command list

```
dot          cross
```