Today:

- Compute the directional derivative of a function of two or three variables (given a function, an input point and a direction).
- Determine the sign of the directional derivative of a function of two variables at a point.
- Identify directions of fastest increase, of no change, and of fastest decrease of a function at a point.
- Construct tangent lines or planes based on the gradient vector.

Example (differential). The area of a disk is given by \( A = \pi r^2 \), so the differential is \( dA = 2\pi r \, dr \). If the radius is measured to be \( r = 2 \pm 0.1 \) meters, estimate the maximum error, \( \Delta A \) in the area calculation.

Example (directional derivative). The following code is computing a directional derivative.

```plaintext
1: expr = x^2 + 2*x*y;
2: pt = [-1,1];
3: vec = [1,2];
4: fx = diff(expr,x);
5: fxpt = subs(fx,[x,y],pt);
6: fy = diff(expr,y);
7: fypt = subs(fy,[x,y],pt);
8: gvec = [fxpt, fypt];
9: uvec = vec/norm(vec);
10: directderly = dot(gvec,uvec);
```

Identify the function, the point, and the direction. What is happening in line 9? pollQ

Example (directional derivative). The directional derivative \( f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 \). Let \( \text{grad} \ f(0,1) = 3\vec{i} - 5\vec{j} \). What is the sign of \( f_{\vec{u}}(0,1) \) for the directions drawn below? pollQ

For your reference (section 14.4):
The gradient of a function \( f \), denoted \( \text{grad} \ f \) or \( \nabla f \), is given by \( \nabla f = f_x\vec{i} + f_y\vec{j} \).

The gradient vector at a point \( (a,b) \) is \( \nabla f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j} \).

If \( \nabla f \) is defined and is nonzero then it points in the direction of the maximum rate of increase of \( f \).

The maximum rate of change of a function \( f \) at a point \( (a,b) \) is \( ||\nabla f|| \).
Example (gradient vector). The directional derivative, $f_{\vec{u}}(a, b)$, is maximum when the direction, $\vec{u}$, aligns with the gradient vector, $\vec{\nabla}f(a, b)$. Which of the vectors below could be $\vec{\nabla}f$ at the point at which the tail is attached for $f$ shown by the contours? pollQ

Example (tangent line). Given a normal vector to a line, $\vec{n} = ax + bj$, and a point on the line, $(x_0, y_0)$, an equation for a line through the point is given by $a(x - x_0) + b(y - y_0) = 0$.

The function $f(x, y) = x^2 + 2y^2$ has gradient vector $2\vec{i} + 4\vec{j}$ at the point $(1, 1)$. Is this enough information to construct an equation for the tangent line to the curve $x^2 + 2y^2 = 3$ at $(1, 1)$? pollQ

For your reference (section 14.5): For $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ a unit vector, we find the directional derivative $f_{\vec{u}}(a, b, c) = f_x(a, b, c)u_1 + f_y(a, b, c)u_2 + f_z(a, b, c)u_3 = \vec{\nabla}f \cdot \vec{u}$.

The gradient of a function of three variables is given by $\text{grad } f = \vec{\nabla}f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$. At a point $(a, b, c)$ we have $\text{grad } f(a, b, c) = \vec{\nabla}f(a, b, c) = f_x(a, b, c)\vec{i} + f_y(a, b, c)\vec{j} + f_z(a, b, c)\vec{k}$.

The gradient of a function of three variables is perpendicular to the level surface of $f$ at $(a, b, c)$, so it is a normal vector to the tangent plane of the level surface at that point.

Example (directional derivative). Let $f(x, y, z)$ represent the temperature in degrees C at the point $(x, y, z)$ with $x, y, z$ in meters. Assume you are moving at $\vec{v}$ meters per second through space. Identify the units for $\|\vec{\nabla}f\|$, $\vec{\nabla}f \cdot \vec{v}$, $\|\vec{\nabla}f\|\|\vec{v}\|$. 

Example (tangent plane). Let $f(x, y, z) = x^2 + y^2 + z^2$. At the point $(1, 2, 1)$, find an equation for the plane tangent to the level surface of $f$ at that point.

Extra: At $(1, 2, 1)$, what is the rate of change of $f$ in the direction perpendicular to the plane $x + 2y + 3z = 8$, and moving away from the origin?

Extra: Is there a point on the same level surface as $(1, 2, 1)$ where the tangent plane is parallel to the $xz$-plane? If so, find the point.
After class: Read sections 14.4 and 14.5

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For next time: Read section 14.6

Learning Objectives: Class 12 + pset 4

These objectives are associated with Class 12 + Workshop on Oct 1st + Section on Oct 1st and 2nd + Problem Set 4 questions + Office hours. There is no quiz associated with this material because of the holiday on October 8th.

Students will be asked to

- directional derivative
  - compute the directional derivative of a function of two or three variables (given a function, an input point and a direction)
  - determine the sign, and estimate the value, of the directional derivative of a function of two variables at a point (based on a contour diagram, a graph, or a table).
  - identify a relationship between the values of \( f_x(a, b) \) and \( f_y(a, b) \) and the value of the directional derivative at \((a, b)\) in a specified direction
- gradient
  - define gradient vector and use the notation \( \nabla f(a, b), \nabla f(a, b), \nabla f(a, b, c), \nabla f(a, b, c) \).
  - compute the gradient of a function of two or three variables
    §14.4: 1-14, §14.5: 1-18
  - identify directions of fastest increase, of no change, and of fastest decrease of a function at a point using properties of the gradient vector that arise from the geometric definition of the dot product.
    §14.4: 51, 54, 61-64, 68, 73-74, 84, §14.5: 51-52, 57, 60
  - construct tangent lines or planes based on the gradient vector (using the property that the gradient is normal to level curve of a function of two variables or to level surfaces of a function of three variables)
- recognize problems where methods related to the directional derivative or the gradient are relevant
- simplify problems using methods related to the directional derivative or the gradient